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FAST TRACK COMMUNICATION

A semiclassically entangled puzzle

Pedro de M Rios[1]

Department of Mathematics, University of California, Berkeley, CA, 94720, USA

E-mail: prios@icmc.usp.br

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Abstract
For a maximally entangled eigenstate of a system of two non-interacting identical one-dimensional harmonic oscillators, at the semiclassical level, it is not obviously true that a nonlinear interaction with one of the subsystems leaves the reduced semiclassical Wigner function of the other subsystem unaffected. Once stated, we advance various speculations regarding this puzzle.

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1. Introduction

When Einstein, Podolsky and Rosen [7] explored the 'strange' behavior of entangled states, their main goal was to set forth an argument concerning the ontology of quantum mechanics. Only quite later, did Bell [2] show that some of the discussion had a definite empirical bearing, namely on matters related to the nonlocal character of quantum mechanics. After the empirical confirmation of this latter [1], emphasis has been placed on the 'peaceful coexistence' [20] of this kind of nonlocality and the partial order of spacetime events derived from the principle of relativity [6].

In terms of the Wigner [23] function W, the Cartesian Weyl [22] representation of the density operator $\hat{\rho}$ as a real function on Euclidean phase space, this question is addressed in the following way: let x_1 and x_2 denote phase-space points of two subsystems whose total system is in an entangled eigenstate of the Hamiltonian $\hat{H}_1 + \hat{H}_2$, with corresponding Wigner function $W_0(x_1, x_2)$. Also, let $\hat{H}_2'$ be a new Hamiltonian, whose Weyl symbol H_2' generates a flow $x_2(0) \mapsto x_2(t)$, and let W_t denote the evolution of W_0 effected by this new Hamiltonian. Further, let $W_0^1(x_1) \equiv \int W_0(x_1, x_2)\,\mathrm{d}x_2$ and $W_t^1(x_1) \equiv \int W_t(x_1, x_2)\,\mathrm{d}x_2$ be their reduced Wigner functions, which encode all the empirical information of subsystem 1 alone, and denote their difference by $\delta W^1(x_1, t)$.

In this context, conservation of the relativistic partial order of events is implied by

$$\delta W^1 \equiv 0, \tag{1}$$

[1] Present address: Departamento de Matemática, ICMC, Universidade de São Paulo, Cx Postal 668, São Carlos, SP, 13560-970, Brazil.

which is always true in full quantum mechanics ($\hbar = 1$), due to its linear unitary structure [9]. In a classical setting ($\hbar = 0$), on the other hand, a sufficient condition for the validity of (1) is given by

$$W_0(x_1, x_2(0)) \longrightarrow W_t(x_1, x_2(t)) = W_0(x_1, x_2(0)). \tag{2}$$

Years ago, Heller [10] noted that the generic expression corresponding to (2) is not always valid at the semiclassical level ($\hbar \to 0^+$), a fact which has recently been reinterpreted in a more geometrical way [19]. Accordingly, for a generic semiclassical Wigner function $\mathcal{W}$, (2) is generally true only if H_2' is quadractic. For a nonlinear interaction, (2) is valid only for points x_2 close to the Lagrangian leaf corresponding to the classical limit of W, whenever its semiclassical expression $\mathcal{W}$ is an oscillatory function of x_2, as is usually the case for pure states [4, 10, 19].

The purpose of this communication is to state the following puzzle: at the semiclassical level, it is *not* obvious whether (1) is valid in general. To see this puzzle, we focus on a particularly simple semiclassically entangled state, namely the (anti)symmetric state of two identical one-dimensional harmonic oscillators, under a small nonlinear interaction H_2'. We apply the results of [19] to ask whether, in a suitable limit, (1) might not hold. Finally, we emphasize the speculative nature of this communication.

2. The setup

We study the semiclassical dynamics of an eigenstate of $\hat{H} = \hat{H}_1 + \hat{H}_2$, where $\hat{H}_i = \left(\hat{p}_i^2 + q_i^2\right)/2$, and of $\hat{P}_{12} : 1 \leftrightarrow 2$. This state is $\sqrt{2}|\Psi\rangle = |e_m, 1\rangle \otimes |e_n, 2\rangle \pm |e_n, 1\rangle \otimes |e_m, 2\rangle$, where the ± 1 refers to the eigenvalue of $\hat{P}_{12}$ and where $e_{m+n} = e_m + e_n$ is the eigenvalue of $\hat{H}$, with $e_m = (m + 1/2)\hbar$, $e_n = (n + 1/2)\hbar$ and where the energy difference is $\lambda = e_m - e_n = l\hbar$, with $m - n = l > 0$.

The corresponding Wigner function factors as $2W_{m+n}(x_1, x_2) = W_m(x_1)W_n(x_2) + W_n(x_1)W_m(x_2) \pm M_m^n(x_1)M_n^m(x_2) \pm M_n^m(x_1)M_m^n(x_2)$, where M_m^n is the Moyal [16] function, or Cartesian Weyl representation of the transition operator $|e_m\rangle\langle e_n|$, satisfying $M_m^n = \overline{M_n^m}$. We write this function in polar $\{x = (r, \theta)\}$ coordinates but emphasize that we are not considering an action-angle Weyl representation, which is not equivalent [3]. In taking the semiclassical limit[2], we formally let $\hbar \to 0^+$. This is equivalent to letting $m, n \to \infty$, while keeping $e_m, e_n < \infty$, or to varying an 'effective' $\hbar$.

Our analysis is based on the WKB-type semiclassical expression for the Moyal function which is not corrected by uniform approximations on the caustics. Ours is a particularly simple case of the general expression carefully treated in [18] and is given by (we use curly letters to indicate semiclassical expressions)

$$\mathcal{M}_m^n(x) = \frac{\mathrm{e}^{\mathrm{i}l\theta}\cos(\phi_{m,n}(r)/\hbar - \pi/4)}{\sqrt{\pi^3\hbar/2}D_{m,n}(r)}, \tag{3}$$

where $\phi_{m,n}(r)$ is half of the symplectic area between the circle with radius $\sqrt{(2m+1)\hbar}$ centered at the origin and the circle with radius $\sqrt{(2n+1)\hbar}$ centered at $(2r, \theta)$, and $D_{m,n}(r) = \left|\dot{x}_m^+ \wedge \dot{x}_n^-\right|^{1/2}$, where $\dot{x}_m^+, \dot{x}_n^-$ are the phase-space velocity vectors of the intersecting circles. These intersections are real as long as $r_- < r < r_+$, where $r_\pm = (\sqrt{(2m+1)\hbar} \pm \sqrt{(2n+1)\hbar})/2$, and the denominator goes to zero when the circles become tangent, i.e. $r = r_\pm$, which are

[2] There are different phenomenological ways of approaching the classical limit. The one considered here, large quantum numbers or small effective Planck's constant, should not be identified with the limit of many degrees of freedom, or large complexity.

the caustic lines. Outside of this annulus, $\mathcal{M}_m^n \equiv 0$, in this crude expression. Naturally, the corresponding formula for $\mathcal{W}_m$ is obtained from (3) with $m = n$. By performing uniform approximations as the intersection points coalesce, $\mathcal{M}_m^n$ can be expressed in terms of Airy functions which do not blow up and decay exponentially outside the annulus. Still using (3), the sum of the cross terms in $\mathcal{W}_{m+n}$ is expressed as

$$\mathrm{Re}\left\{\mathcal{M}_m^n(x_1)\mathcal{M}_n^m(x_2)\right\} = \frac{\cos(l(\theta_1-\theta_2))\cos(\phi_{m,n}(r_1)/\hbar-\pi/4)\cos(\phi_{m,n}(r_2)/\hbar-\pi/4)}{(\pi^3\hbar/2)D_{m,n}(r_1)D_{m,n}(r_2)}.$$

Of course, $\int \mathrm{Re}\left\{\mathcal{M}_m^n(x_1)\mathcal{M}_n^m(x_2)\right\}\mathrm{d}x_2 = 0$, so $\mathcal{W}_{m+n}^1 \equiv (\mathcal{W}_m + \mathcal{W}_n)/2$, as it should be.

3. The puzzle

The puzzle to be laid forth below follows from the fact that $\mathcal{M}_m^n$ is an *oscillatory* function, so that (3) has an essential singularity at $\hbar = 0$. This is the reason why (2) fails to hold when $\hbar \to 0^+$ as can be seen by directly plugging (3) into the Moyal bracket. Its successive derivatives then bring negative powers of $\hbar$ which alter the familiar $\hbar$-expansion obtained for smooth functions. Recently [19], a simple geometrical prescription for a more correct semiclassical propagation of such functions was obtained, in agreement with another old result derived by Berry and Balazs [4] according to which one should propagate the whole family of curves, or Lagrangian submanifolds, associated with a quantum state and then re-evaluate the semiclassical expressions accordingly.

The new result obtained in [19] somehow simplifies this procedure by showing that, equivalently, one can obtain a more correct semiclassical propagation of an oscillatory function like $\mathcal{M}_m^n(x)$ by classically propagating the *tips* of the chord centered on x instead of classically propagating the argument x itself, as in the Liouvillian case (2). Of course, Liovillian propagation is still semiclassically correct for smooth functions, when all chords are null, or if the flow is linear, as discussed in more detail in [19]. Since $\mathcal{M}_m^n(x)$ is not smooth, we might expect surprises when under nonlinear interactions[3].

Therefore, let us now suppose that subsystem 2 interacts nonlinearly for an interval of time t. For simplicity, we state the puzzle when the interaction is either cubic $H_2' = \alpha q_2^3/3$ or quartic $H_2' = \alpha q_2^4/4$ and denote by $\epsilon = \alpha t$ the strength of this interaction. After rewriting (4) as a sum of oscillatory functions of $\phi_{m,n}^{\pm}(r_2,\theta_2) = \phi_{m,n}(r_2) \pm \lambda\theta_2$, it is not too difficult to see that, within the stationary phase regime ($\hbar \to 0^+$), the effect of a small interaction is, in a first approximation, equivalent to having only phase shifts $\phi_{m,n}^{\pm}(r_2,\theta_2) \to \phi_{m,n}^{\pm}(r_2,\theta_2) + \delta\phi_{m,n}^{\pm}(r_2,\theta_2)$.

More precisely, to get these phase shifts we first apply equation (16) in [19] to the chord $\xi_{m,n}^{\pm} = -J\left(\partial\phi_{m,n}^{\pm}(x_2)/\partial x_2\right)$, centered on x_2, and get the 'midpoint phase difference' $\tilde{\phi}_{m,n}^{\pm}\left(\tilde{x}_2^t\right) - \phi_{m,n}^{\pm}(x_2)$, where $\tilde{x}_2^t$ is the midpoint of the Hamiltonian flow of the tips of the chord ξ. Then we note that, for small ϵ, the total phase difference acquired for the Hamiltonian flow of x_2 is approximately half of $\tilde{\phi}_{m,n}^{\pm}\left(\tilde{x}_2^t\right) - \phi_{m,n}^{\pm}(x_2)$. The amplitude being approximately covariant under the Hamiltonian flow, for small ϵ, this determines the looked-for phase shifts as

$$\delta\phi_{m,n}^{\pm}(r_2,\theta_2) \approx (2\epsilon/3)\left\{\sin(\theta_2)\left[e_{m+n} - (\lambda/2r_2)^2 - r_2^2\right]^{1/2} \pm \cos(\theta_2)[\lambda/2r_2]\right\}^3,$$

[3] For example, if a nonlinear interaction is first performed in the quantum regime and then the (semi)classical limit is taken, generically the long-time result differs from first taking the (semi)classical limit and then interacting. This was originally stated for nonintegrable interactions of oscillatory Wigner functions [4], but recently for integrable interactions of non-oscillatory Wigner functions [5], as well.

for the cubic case, while for the quartic case we get

$$\delta\phi^{\pm}_{m,n}(r_2,\theta_2) \approx (2\epsilon) r_2 \cos(\theta_2)\{\sin(\theta_2)\left[e_{m+n} - (\lambda/2r_2)^2 - r_2^2\right]^{1/2} \pm \cos(\theta_2)[\lambda/2r_2]\}^3.$$

The next step is to compute $\int \mathrm{Re}\left\{\mathcal{M}^n_m(x_1)\mathcal{M}^m_n(x_2)\right\} \mathrm{d}x_2$ and check if it vanishes, as before. But note that the phase shifts have broken the symmetry of $\mathrm{Re}\left\{\mathcal{M}^n_m(x_1)\mathcal{M}^m_n(x_2)\right\}$ with respect to θ_2 and it is not obvious now whether the integral vanishes or not. In fact, we must now perform the new integration $\int \mathrm{Re}\left\{\mathcal{M}^n_m(x_1)\mathcal{M}^m_n(x_2)\right\} \mathrm{d}x_2$ by stationary phase and it is not difficult to see that the critical points of $\phi^{\pm}_{m,n}(r_2,\theta_2) + \delta\phi^{\pm}_{m,n}(r_2,\theta_2)$ are generally complex. This means that we must generally perform a two-dimensional steepest descent evaluation of the integral, carefully deforming the whole annulus into the complex 2-plane, counting all the significant contributions, adding them all up .... This is a rather delicate computation, but it is not unreasonable to suppose that, instead of zero, we could get an exponentially small result [11], somewhat reminding of what happens in 'tunneling' effects[4].

This being an asymptotic analysis, however, we still need to make sure whether the result of this steepest descent computation implies that $\delta\mathcal{W}^1$ is *really* different from zero exponentially in $-1/\hbar$ (remember that (3) is only the first term of a WKB-type $\hbar$-expansion), in some region of the positive $(\lambda, \epsilon, \hbar)$-space close to the $\hbar = 0$ plane. Also in this region, we must analyze the behavior of $\int \mathcal{W}_m(x_2)\,\mathrm{d}x_2$ and $\int \mathcal{W}_n(x_2)\,\mathrm{d}x_2$ under similar phase shifts to find out how norm preserving such a propagation of $\mathcal{W}^1$ actually is (note that $\int\int \mathrm{Re}\left\{\mathcal{M}^n_m(x_1)\mathcal{M}^m_n(x_2)\right\} \mathrm{d}x_1\,\mathrm{d}x_2$ vanishes). Then we must compare all the above results and further analyze and interpret them (for instance, in possible relation to Stokes phenomenon, or other exponential asymptotic phenomena [15]). Finally, to properly account for stronger interactions (ϵ not so small), closer resonance (λ small), or to actually get more precise estimates for $\delta\mathcal{W}^1$ (any ϵ or λ), we must leave the crude expression (3) and proceed much more carefully with the uniform approximations (this is a more difficult task), since non-Liouvillian propagation of the amplitude is so much more relevant near caustics.

We emphasize that any possible nontrivial answer for $\delta\mathcal{W}^1$ would only be an asymptotic one (and exponentially small). Such a possibility seems due to the θ_2-symmetry breaking of the whole semiclassical Wigner function $\mathcal{W}_{m+n}(x_1, x_2)$, effected by the nonlinear interaction with H_2', and the consequences of this symmetry breaking should become more explicit as the amplitude corrections are included. But it is possible that only a careful analysis based on more accurate semiclassical approximations can produce clearer answers, even because the accuracy of the chord expression (3), from which we started, is in fact worse than exponential. Moreover, as the relevant contribution can only come from complex critical points, a fully complexified dynamical analysis of the propagation may be imperative.

Then, if such a possibility is indeed correct, we could conjecture that there would be an optimal way of approaching the classical limit, in the positive $(\lambda, \epsilon, \hbar)$-space, leading to a maximal $\delta\mathcal{W}^1$. Combined with the singular nature of the semiclassical limit, the presence of so many independent parameters is what seems to open this problem in such a manner. Furthermore, very significant roles could be played by issues from classical non-integrability of an interaction (for more degrees of freedom, or time-dependent Hamiltonian), or even more by issues from non-Hamiltonian quantum interactions (as for open systems), which are not unitary to begin with [13].

[4] For a much simpler analogy, consider the integral $\mathcal{I} = \int_0^{2\pi} \cos(l\theta)\,\mathrm{d}\theta$. Writing $l\theta = \phi(\theta)/\hbar$, where $\phi(\theta) = \lambda\theta$, then under the phase shift $\phi \to \tilde{\phi} = \phi + \delta\phi$, where $\delta\phi = \epsilon\theta^3/3$, we have that $\mathcal{I} \to \widetilde{\mathcal{I}}$ is asymptotically ($\hbar \to 0^+$) mapped into the Airy integral, that is, $\widetilde{\mathcal{I}} \approx \pi(\hbar/\epsilon)^{1/3}\mathcal{A}_i(y)$, where $y = \lambda\epsilon^{-1/3}\hbar^{-2/3}$.

4. Speculations

Final words on this puzzle from the phenomenological, empirical and philosophical points of view.

Phenomenologically, how could it ever be possible for (1) to fail at all, since it is valid in both the classical and the full quantum regimes?[5] Well, we could argue that (1) is valid in both regimes for different reasons. In the classical case, (2) translates the fact that no quantum correlations have 'survived'. But this can be seen as an effect of the correlations becoming 'random', which is well expressed by the oscillations in the Wigner function becoming so high that they average to zero everywhere but on the classical leaf. In the full quantum case, by contrast, the correlations are fully structured, in other words, they have a definite 'form', which is usually expressed by the concept of a definite normalized vector in the Hilbert space. Such a vector can only be preserved by linear unitary transformations, from which (1) follows.

Therefore, the puzzle amounts to a question of whether, at the semiclassical level, the linear unitary structure of quantum mechanics could feel some disturbing effects of a nonlinear classical dynamics[6] and, if so, how these effects could show up[7].

Empirically, if a suitable limit where $\delta\mathcal{W}^1 \neq 0$ could ever be found, would this mean the possibility of faster-than-light-communication signals? Here, we should stress that these sub-hypothetical '*falico*' signals would be fundamentally threshold phenomena and so a deeper understanding of this threshold between the quantum and the classical regimes could be required. On the other hand, even if granted a suitable limit of nontrivial $\delta\mathcal{W}^1$, it could still be a formidable task to actually measure such signals, although some direct measurements of Wigner functions have been claimed [14].

Philosophically, what would be at stake here is, among other things, the principle of relativity. But in this respect, it seems interesting to point out the usually overlooked fact that, as originally formulated [6], the relativistic partial order presupposes events which are correlated by an exchange of energy. In other words, which are correlated by 'material' causes. Such is not the case for the quantum correlations treated here, which could perhaps be described as 'formal' correlations. Maybe, these two kinds of correlations being essentially distinct, their principles need not be the same.

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[5] Examples of phenomena that appear in one or the other but not in both regimes are easier to come by, but the non-geometric glory [17] can be seen as a phenomenon that vanishes both 'classically' (ray regime) and 'fully quantum' (long-wave regime) but not 'semiclassically' (short-wave asymptotics).

[6] Questions on the possibility of corresponding a linear unitary transformation on quantum Hilbert space to any canonical transformation on ordinary classical phase space are not new [8, 12, 21].

[7] In a certain way, the two important factors for a Bell's telephone, namely the distinction between pure and mixed states and between linear and nonlinear evolution, show up in the semiclassical propagation of Wigner functions.

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