NON-HOLONOMIC SYSTEMS WITH SYMMETRY
ALLOWING A CONFORMALLY SYMPLECTIC REDUCTION

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ABSTRACT. Non-holonomic mechanical systems can be described by a degenerate almost-Poisson structure [10] (dropping the Jacobi identity) in the constrained space. If enough symmetries transversal to the constraints are present, the system reduces to a nondegenerate almost-Poisson structure on a "compressed" space. Here we show, in the simplest non-holonomic systems, that in favorable circumnstances the compressed system is conformally symplectic, although the "non-compressed" constrained system never admits a Jacobi structure (in the sense of Marle et al. [4][9]).

1. INTRODUCTION

We adopt in this work the view of writing the equations of a non-holonomic systems in terms of an almost-Poisson bracket on a constrained manifold $P \subset T^*Q$, introduced by van der Shaft and Mashke [10]. See also Cantrijn et al. [3] for more recent developments.

In this note we add a new twist to the simplest example of a non-holonomic system, the contact system in $Q = \mathbf{R}^3$ (see eg. [2]), namely:

After performing the reduction by the transversal $\mathbf{R}^1-$symmetry, we observe that, for some metrics, the reduced ("compressed") almost-Poisson bivector admits a conformal symplectic structure, ie, a special Jacobi structure [4][9]. In contrast, the "non-compressed" constrained system never admits a Jacobi structure.

The examples point to the fact that, in quite favorable circumstances, the reduced system can be studied by symplectic techniques (for instance, when internal symmetries are present, integrability can be achieved by Marsden-Weinstein procedure, which holds in the conformally symplectic setting, see [5]). The examples also show that, generally, non-holonomic systems are non-Jacobi systems (in the sense of Marle et al. [4][9]).

Hopefully our observations can help attracting more interest to investigations on the geometrical properties of almost-Poisson bivectors which naturally describe non-holonomic dynamics on the constrained submanifold of the original cotangent bundle.

2. THE CONTACT NON-HOLONOMIC SYSTEM

Consider a non-holonomic systems in $Q = \mathbf{R}^3$ having the constraint

$$\dot{z} - x\dot{y} = 0 \ . \tag{1}$$

The admissible sub-bundle E is the union of the horizontal spaces for the connection 1-form $\omega = dz - xdy$ on the (trivial) bundle $G = \mathbf{R}^1 \hookrightarrow Q = \mathbf{R}^3 \to S = \mathbf{R}^2$, with curvature $d\omega = -dx \wedge dy$, where the $\mathbf{R}^1$ action is the usual translation on the z-fibers.

For motivation, consider the javelin, a rod of mass m and moment of inertia I moving on a vertical plane (y, z) in such a way that it always remains tangent to its trajectory. If φ is the angle with the horizontal, then $dz = \tan(\varphi)\, dy$, and if we introduce the change of

variables $x = \tan\varphi$, then $T = \frac{1}{2}\left[m(\dot{y}^2 + \dot{z}^2) + I\dot{\varphi}^2\right] = \frac{1}{2}\left[m(\dot{y}^2 + \dot{z}^2) + I\frac{\dot{x}^2}{(1+x^2)^2}\right]$. Choose units so that $m = I = 1$. Under the assumption of small angles, $\varphi \approx x$, we get

$$(2) \qquad\qquad L = T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \;.$$

According to our previous work ([6]) system (1,2) is "z-Caplygin" and thus can be reduced to $TS = \{(x, y, \dot{x}, \dot{y})\}$ with lagrangian

$$(3) \qquad\qquad \overline{L} = \frac{1}{2}\left(\dot{x}^2 + (1 + x^2)\,\dot{y}^2\right) \;.$$

with an external gyroscopic force added to it. Actually it can be also interesting to regard it as a y-Caplygin system $dy = \cot(\varphi)\,dz = (1/x)dz$, since we may want to add the gravitational potential $V = gz$.

3. Almost-Poisson brackets via moving frames

The giroscopic force can be concealed in an almost Poisson bracket in the constrained manifold $P \subset T^*Q$ via the dynamical equation

$$\dot{x} = \{x, H\}_P$$

where H is the hamiltonian, and $P \subset T^*Q$ is the Legendre transform of the constraint subbundle $E \subset TQ$, as defined below.

Let $e_J = e_{LJ}\partial/\partial q_L$ be a moving frame on Q such that the first m vectors (labelled by lowercase latin indices) generate E. By a direct calculation ([10], equation(19)) van der Shaft and Maschke verified that the brackets are given by

$$\{q_I, q_J\} = 0\,,\; \{q_I, \tilde{p}_J\} = e_{LJ}(q) = dq_I \cdot e_J \quad (e_J = e_{LJ}\partial/\partial q_L)$$

$$(4) \qquad\qquad \{\tilde{p}_I, \tilde{p}_J\} = -p_q \cdot [e_I, e_J] \equiv R_{IJ}$$

where p_q is evaluated on P. For a geometric interpretation and simple derivation of these formulas using the moving frame method see our paper [7]. Here's an outline:

Choose an *adapted* moving frame to the constrained distribution $E \subset TQ$ that is, consider a complete set of vector fields e_i , e_α where $e_i(q) \in E_q$. The greek labels α denote $e_\alpha \notin E_q$. Denote the dual 1-forms by ϵ_i , ϵ_α. We shall denote by uppercase latin indices e_I , ϵ_I the complete dual set in TQ and T^*Q. The canonical 1-form on T^*Q writes as

$$(5) \qquad\qquad p\,dq = u_I\epsilon_I$$

where u_I define new coordinates on each T_q^*Q. Now,

$$d(pdq) = du_I \wedge \epsilon_I + u_I d\epsilon_I \;.$$

The last term vanishes on vertical vectors. Moreover

$$u_I d\epsilon_I(e_J, e_K) = u_I e_J(\epsilon_I(e_K)) - u_I e_K(\epsilon_I(e_J)) - u_I\epsilon_I[e_J, e_K] = -u_I\epsilon_I[e_J, e_K]$$

which gives us the symplectic matrix on T^*Q as

$$(6) \qquad\qquad [\hat{\Omega}]_{\text{moving frame}} = \begin{pmatrix} R & -I_n \\ I_n & 0_n \end{pmatrix}$$

and its inverse, the Poisson matrix on T^*Q as $[\hat{\Lambda}]_{\text{moving coframe}} = [\hat{\Omega}]^{-1}_{\text{moving coframe}}$, where

$$(7) \qquad\qquad R_{JK} = -u_I\epsilon_I[e_J, e_K] = -p_q \cdot [e_J, e_K] \;.$$

The moving coframe for $[\hat{\Lambda}]_{\text{moving coframe}}$ is ϵ_I, du_I. The moving frame for (6) is its dual. Caveat: this basis contains the vertical vectors $\partial/\partial u_I$ and lifted $e_I^* = e_I + vertical$ (see [7]). From this we obtain the matrix of almost-Poisson brackets in $P \subset T^*Q$ by directly substituting the u_α by the u_i (and q's) via the defining equation for P :

(8) $$\partial H/\partial u_\alpha = 0$$

which is a consistency requirement for any constrained (holonomic or non-holonomic) dynamical system on Q , see [7].

4. THE CONTACT ALMOST-POISSON STRUCTURE

In the case of the contact system, let us begin by taking as moving co-frame the set

(9) $$\epsilon_1 = dx \,, \ \epsilon_2 = dy \,, \ \epsilon_3 = \omega = dz - xdy$$

that is dual to

(10) $$e_1 = \partial/\partial x \,, \ e_2 = \partial/\partial y + x\partial/\partial z \,, \ e_3 = \partial/\partial z \,.$$

Note that

(11) $$[e_1, e_2] = e_3 \,, \ [e_1, e_3] = [e_2, e_3] = 0 \,,$$

so that we have the 3-dimensional Heisenberg algebra. The identity (5) yields

(12) $$u_1 = p_x \,, \ u_2 = p_y + xp_z \,, \ u_3 = p_z \,,$$

hence the Hamiltonian is

(13) $$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) = \frac{1}{2}(u_1^2 + (u_2 - xu_3)^2 + u_3^2) \,.$$

Now, the Legendre transform of the admissible sub-bundle $\mathcal{E} \subset TQ$ is given by (8) as $\partial H/\partial u_3 = 0$ so that the constrained manifold P is

(14) $$P = \{(x, y, z, u_1, u_2, u_3) \,|\, u_3 = \frac{xu_2}{1 + x^2} \} = \{(x, y, z, p_x, p_y, p_z) \,|\, p_z = xp_y \} \,.$$

As moving basis for T^*P we take $dx, dy, dz - xdy, du_1, du_2$ and the corresponding 5×5 matrix of Poisson Brackets is given by

(15) $$[\Lambda]_{\text{moving}} = \begin{pmatrix} 0_{2\times2} & 0_{2\times1} & I_{2\times2} \\ 0_{1\times2} & 0_{1\times1} & 0_{1\times2} \\ -I_{2\times2} & 0_{2\times1} & R_{2\times2} \end{pmatrix}$$

where $R_{2\times2}$ is the antisymmetric matrix with

(16) $$R_{12} = -p_q[e_1, e_2] = -u_3 = -\frac{xu_2}{1 + x^2} \,.$$

Here, the vanishing of the middle row and column means of course that

(17) $$\{dz - xdy, H\} = 0 \implies \dot{z} - x\dot{y} = 0$$

which gives the differential equation for z.

5. The compressed system

Deleting the middle column and row of (15), we obtain a non-degenerate matrix

$$(18) \qquad \overline{[\Lambda]} = [\Lambda]_{compressed} = \begin{pmatrix} 0_{2\times 2} & I_{2\times 2} \\ -I_{2\times 2} & R_{2\times 2} \end{pmatrix}$$

which characterizes the almost-Poisson structure in $\overline{P} = P_{compressed} = \{(x, y, u_1, u_2)\}$ with

$$(19) \qquad \overline{H} = H_{compressed} = \frac{1}{2}\left(u_1^2 + \frac{u_2^2}{1 + x^2}\right)$$

as the reduced Hamiltonian. It follows that the reduced equations of motion are

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{pmatrix} 0_{2\times 2} & I_{2\times 2} \\ -I_{2\times 2} & R_{2x2} \end{pmatrix} \begin{bmatrix} \overline{H}_x \\ \overline{H}_y \\ \overline{H}_{u_1} \\ \overline{H}_{u_2} \end{bmatrix}$$

or

$$\dot{x} = u_1 \,, \; \dot{y} = \frac{u_2}{1 + x^2}\,, \; \dot{u}_1 = -\overline{H}_x + R_{12}\overline{H}_{u_2}\,, \; \dot{u}_2 = -\overline{H}_y - R_{12}\overline{H}_{u_1}$$

A "miraculous" cancelation takes place in the $\dot{u}_1$ equation, and the system is actually

$$(20) \qquad \dot{x} = u_1 \,, \; \dot{y} = \frac{u_2}{1 + x^2}\,, \; \dot{u}_1 = 0\,, \; \dot{u}_2 = \frac{x}{1 + x^2}u_1 u_2$$

This cancelation has a reason. The Lagrangian is invariant under the 1-parameter group $(x, y, z) \to (x + \epsilon, y, z)$ and the generator $\partial/\partial x$ is an admissible vectorfield. By the non-holonomic Noether theorem [1], the momentum $p_x = u_1 = \dot{x}$ is conserved. The 2 degrees of freedom compressed system separates and can be integrated by quadratures:

$$\ln(u_2) = \int a\frac{x_o + at}{1 + (x_o + at)^2}\,dt = \frac{1}{2}\ln(1 + (x_o + at)^2) + \text{const.} \quad \text{Thus,}$$

$$u_2 = A\sqrt{1 + (x_o + at)^2} \quad, \quad y = y_o + \int_0^t A\frac{\sqrt{1 + (x_o + at)^2}}{1 + (x_o + at)^2}\,dt$$

and we reconstruct the z-fiber dynamics via the constraint equation:

$$z = z_o + \int_0^t (x_o + at)\frac{A\sqrt{1 + (x_o + at)^2}}{1 + (x_o + at)^2}dt \;.$$

The almost-symplectic form in the compressed space T^*S, $S = \mathbf{R}^2 = \{x, y\}$, is the 2-form $\overline{\Omega} = du_1 \wedge dx + du_2 \wedge dy + R_{12}dx \wedge dy$, $R_{12} = -\frac{xu_2}{1+x^2}$ and so $d\overline{\Omega} = -\frac{x}{1+x^2}\,du_2 dx dy$.

Let us investigate if there is a function $f : T^*S \to \mathbf{R}$ such that $f\overline{\Omega}$ is closed. It looks simpler to try the *ansatz* $f = f(x)$ only. The condition $df \wedge \overline{\Omega} + f d\overline{\Omega} = 0$ leads to

$$(21) \qquad [f'(x) + f\frac{x}{1 + x^2}]dxdu_2dy = 0 \,, \; \text{so that} \;\; f = \frac{A}{\sqrt{1 + x^2}} \;.$$

The significance of this observation is that the compressed system (20) is Hamiltonian in the time scale s such that $dt/ds = \sqrt{1 + x^2}$, with the same Hamiltonian $\overline{H} = H_{compressed}$ and bona-fide symplectic form $f\overline{\Omega}$. Now, the corresponding Poisson structure in T^*S is $\frac{1}{f}[\Lambda]_{compressed}$ (refer to (18)) and one is tempted to guess that the conformally changed bivector in P given by $\frac{1}{f}[\Lambda]_{moving}$ (refer to (15)) satisfies Jacobi. But as we shall see below, *this is not the case !*

6. NON-JACOBI FOR THE CONSTRAINED ALMOST-POISSON

Let us now take a closer look on the algebraic properties of the almost-Poisson structure on the constrained space P. In order to compute the Schouten-Nijenhuis bracket of the almost-Poisson bivector Λ with itself, we first rewrite Λ given by (15) as a matrix in a coordinate basis. Choosing $\{x, y, z, u_1, u_2\}$ as coordinates, we write $\Lambda_{\{x,y,z,u_1,u_2\}}$ as

$$
\begin{pmatrix}
0_{2\times 2} & 0_{2\times 1} & I_2 \\
0_{1\times 2} & 0_{1\times 1} & L_{1\times 2} \\
-I_{2\times 2} & -L_{2\times 1}^T & R_{2\times 2}
\end{pmatrix}
$$

where, as before, $R_{2\times 2}$ is the antisymmetric matrix with $R_{12} = -xv/(1+x^2)$ and now

$$
L_{1\times 2} = (\ 0\ \ x\ )
$$

in such a way that we can compute the self S-N bracket $[\Lambda, \Lambda]$ using:

$$
(22) \qquad\qquad [\Lambda, \Lambda]^{IJK} = \Lambda^{LK}\partial_L\Lambda^{IJ} + \Lambda^{LI}\partial_L\Lambda^{JK} + \Lambda^{LJ}\partial_L\Lambda^{KI}
$$

where the summation convention is subtended. We then get

$$
(23) \qquad\qquad [\Lambda, \Lambda] = \left(\frac{2}{1+x^2}\right)(x\partial_y - \partial_z) \wedge \partial_{u_1} \wedge \partial_{u_2}\ .
$$

One can check explicitly that there is no vector field E satisfying the first of the equations for the existence of an associated Jacobi structure [4] (the second one is given by (40)) :

$$
(24) \qquad\qquad [\Lambda, \Lambda] = 2E \wedge \Lambda\ .
$$

Thus no Jacobi structure, in the sense of Marle, exists for this bi-vector (or any conformal one as well). Notice that we can rewrite $[\Lambda, \Lambda] = \left(\frac{2}{1+x^2}\right)e^\perp \wedge \partial_{u_1} \wedge \partial_{u_2}$, where $e^\perp = x\partial_y - \partial_z$ is a vector orthogonal to e_1 and e_2 and hence to the distribution. Remark that Λ/f, where f is given by (21), is really "z-almost" Poisson. A calculation in the same lines shows that the only nonvanishing entries of $[\Lambda/f, \Lambda/f]$ are the permutations of $(3, 4, 5)$, namely $-2(1+x^2)\partial_z \wedge \partial_{u_1} \wedge \partial_{u_2}$. ("almost" is really almost)!

In order to better appreciate the Non-Jacobi result for the almost-Poisson structure, let us now choose a different moving frame and co-frame adapted to the contact distribution. Specifically, we choose an orthonormal set with respect to the euclidean metric:

$$
(25) \qquad\qquad e_1 = \partial_x\ ,\ \ e_2 = \frac{\partial_y + x\partial_z}{\sqrt{1+x^2}}\ ,\ \ e_3 = \frac{\partial_z - x\partial_y}{\sqrt{1+x^2}}
$$

whose dual set is "itself":

$$
(26) \qquad\qquad \epsilon_1 = dx\ ,\ \ \epsilon_2 = \frac{dy + xdz}{\sqrt{1+x^2}}\ ,\ \ \epsilon_3 = \frac{dz - xdy}{\sqrt{1+x^2}} = \frac{\omega}{\sqrt{1+x^2}}\ .
$$

This frame does not respect the natural z-fibration, but it's still true that $\omega(e_1) = \omega(e_2) = 0$. However, the Lie algebra is now modified : $[e_1, e_2] = \left(\frac{1}{1+x^2}\right)e_3$, $[e_2, e_3] = 0$, $[e_3, e_1] = \left(\frac{1}{1+x^2}\right)e_2$, but the hamiltonian H corresponding to the euclidean metric is again euclidean: $H = (u_1^2 + u_2^2 + u_3^2)/2$, so that the condition $\partial H/\partial u_3 = 0$ yields the simpler equation

$$
(27) \qquad\qquad u_3 = 0
$$

for the definition of P (generally, we have $u_\alpha = 0$ for orthonormal frames).

From the theory of moving frames [7], the Poisson bi-vector on T^*Q can be written as

$$(28) \qquad \hat{\Lambda} = e_I^* \wedge \partial_{u_I} + R_{IJ}\partial_{u_I} \wedge \partial_{u_J}$$

where R_{IJ} is given by (4), ∂_{u_I} is the "vertical dual" to du_I and e_I^* is the correct lift of the base vector e_I to $T(TQ)$ [7]. Alternatively, we can rewrite this full bi-vector using Darboux quasi-coordinates as $\hat{\Lambda} = e_I \wedge \partial_{u_I}$. For an orthonormal adapted moving frame, from (5) we get that the almost-Poisson bivector on the constrained space P is:

$$(29) \qquad \Lambda = e_i \wedge \partial_{u_i} + \tilde{R}_{ij}\partial_{u_i} \wedge \partial_{u_j}$$

where $\tilde{\ }$ means evaluating at P, in this case:

$$(30) \qquad \tilde{R}_{IJ} = -u_i\epsilon_i[e_I, e_J] \ .$$

For the contact system with orthonormal moving frame, we have simply

$$(31) \qquad \Lambda = e_1 \wedge \partial_{u_1} + e_2 \wedge \partial_{u_2}$$

where ∂_{u_i} is the "vertical dual" to ϵ_i : $\partial_{u_1} = \partial_{p_x}$, $\partial_{u_2} = \frac{\partial_{p_y} + x\partial_{p_z}}{\sqrt{1+x^2}}$, $\partial_{u_3} = \frac{\partial_{p_z} - x\partial_{p_y}}{\sqrt{1+x^2}}$. The self S-N bracket of Λ can be easily computed using the decomposition formulas for the S-N bracket of bivectors. We get :

$$[e_1 \wedge \partial_{u_1} + e_2 \wedge \partial_{u_2}, e_1 \wedge \partial_{u_1} + e_2 \wedge \partial_{u_2}] = [e_1 \wedge \partial_{u_1}, e_1 \wedge \partial_{u_1}] + [e_2 \wedge \partial_{u_2}, e_2 \wedge \partial_{u_2}] + 2[e_1 \wedge \partial_{u_1}, e_2 \wedge \partial_{u_2}]$$

It's easy to see that the first two terms on the right vanish. The third one decomposes as

$$[e_1 \wedge \partial_{u_1}, e_2 \wedge \partial_{u_2}] = [e_1, e_2] \wedge \partial_{u_1} \wedge \partial_{u_2} - e_1 \wedge [e_2, \partial_{u_1}] \wedge \partial_{u_2} - e_2 \wedge [e_1, \partial_{u_2}] \wedge \partial_{u_1} + e_1 \wedge e_2 \wedge [\partial_{u_1}, \partial_{u_2}]$$

Since the only nontrivial Lie bracket is $[e_1, e_2] = \left(\frac{1}{1+x^2}\right)e_3$ we are left with

$$(32) \qquad [\Lambda, \Lambda] = \left(\frac{2}{1 + x^2}\right) e_3 \wedge \partial_{u_1} \wedge \partial_{u_2} \ ,$$

which is equivalent to the result previously obtained (23). It's now even simpler to verify that no Jacobi structure [4][9] is possible for Λ, just compare (24), (31) and (32).

Let us consider a still simpler, even more symmetrical case. Namely, on the contact system, we consider a metric which is invariant under the full Heisenberg group. Again, taking (10) as moving frame we have the Heisenberg algebra (11). The simplest Heisenberg-invariant metric is thus given by the kinetic energy $T = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2)$, where v_i is given by the identification $v_1 e_1 + v_2 e_2 + v_3 e_3 = \dot{x}\partial_x + \dot{y}\partial_y + \dot{z}\partial_z$, which gives:

$$(33) \qquad T = \frac{1}{2}(\dot{x}^2 + (1 + x^2)\dot{y}^2 + \dot{z}^2 - 2x\dot{y}\dot{z}) \ .$$

In other words, for such metric $2T$, the Heisenberg moving frame (10) is orthonormal. In terms of the hamiltonian, we have that the co-frame (9) is also orthonormal, and thus

$$(34) \qquad H = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2) = \frac{1}{2}(p_x^2 + p_y^2 + (1 + x^2)p_z^2 + 2xp_yp_z)$$

where the relation between the u's and the p's is the one given earlier in (12), section 3. Once again, the almost-Poisson bivector is given by

$$(35) \qquad \Lambda = e_1 \wedge \partial_{u_1} + e_2 \wedge \partial_{u_2} \ , \quad \text{and thus} \quad [\Lambda, \Lambda] = 2e_3 \wedge \partial_{u_1} \wedge \partial_{u_2} \ .$$

Therefore, no Jacobi structure [4][9] exists on P for this Schaft-Maschke [10] bivector.

7. The compressed almost-Poisson structure
is not conformally symplectic in general

We've seen earlier in section 5 that the compressed system for the contact nonholonomic system with euclidean metric is conformally symplectic. However such conformal symplectic structure (or any Jacobi structure) on the compressed system does not exist for generic metrics. To see this, consider a z-invariant hamiltonian of the general form

$$(36) \qquad H = \gamma_{ij} u_i u_j + V(x, y) \ , \ \ \gamma_{ij} \equiv \gamma_{ij}(x, y) = \gamma_{ji}(x, y) \ .$$

The condition $\partial H / \partial u_3 = 0$ yields $u_3 = -(\gamma_{13} u_1 + \gamma_{23} u_2)/\gamma_{33} = -R_{12}$, so that the compressed almost-Poisson bivector can be written as

$$(37) \qquad \overline{\Lambda} = \partial_x \wedge \partial_{u_1} + \partial_y \wedge \partial_{u_2} + \left(\frac{\gamma_{13} u_1 + \gamma_{23} u_2}{\gamma_{33}} \right) \partial_{u_1} \wedge \partial_{u_2} \ .$$

Since $\{x, y, u_1, u_2\}$ is a coordinate system for T^*S, we can apply formula (22) directly to obtain $[\ \overline{\Lambda}, \overline{\Lambda}\ ] = -2 \left(\frac{\gamma_{13}}{\gamma_{33}} \right) \partial_x \wedge \partial_{u_1} \wedge \partial_{u_2} - 2 \left(\frac{\gamma_{23}}{\gamma_{33}} \right) \partial_y \wedge \partial_{u_1} \wedge \partial_{u_2}$. On the other hand, a general vector field on T^*S has the form :

$$(38) \qquad E = \alpha \partial_x + \beta \partial_y + \mu \partial_{u_1} + \nu \partial_{u_2}$$

so that the condition $[\ \overline{\Lambda}, \overline{\Lambda}\ ] = 2E \wedge \overline{\Lambda}$ is satisfied for

$$(39) \qquad \nu = -\gamma_{13}/\gamma_{33} \ , \ \ \mu = \gamma_{23}/\gamma_{33} \ .$$

For E to be the vector field of a Jacobi structure [4][9], it is also necessary that

$$(40) \qquad [E, \overline{\Lambda}] \equiv \mathcal{L}_E(\overline{\Lambda}) = 0 \ .$$

A simple computation gives $[E, \overline{\Lambda}] = - \left(\frac{\partial}{\partial x} \left(\frac{\gamma_{13}}{\gamma_{33}} \right) + \frac{\partial}{\partial y} \left(\frac{\gamma_{23}}{\gamma_{33}} \right) \right) \partial_{u_1} \wedge \partial_{u_2}$ and thus

$$(41) \qquad \frac{\partial}{\partial x} \left(\frac{\gamma_{13}}{\gamma_{33}} \right) + \frac{\partial}{\partial y} \left(\frac{\gamma_{23}}{\gamma_{33}} \right) = 0$$

is a necessary condition for the existence of a Jacobi structure on the compressed system in T^*S. Of course, the hamiltonians (13) and (34), obtained from the euclidean and the Heisenberg metrics, (2) and (33) respectively, satisfy the above condition. More generally, in order to verify this condition, one can substitute for the original metric elements $g_{ij}(x_1, x_2)$ in a basis $x_1 = x, x_2 = y, x_3 = z$, with the relations

$$(42) \qquad \frac{\gamma_{13}}{\gamma_{33}} = \frac{g_{12} g_{23} - g_{13} g_{22} - x_1(g_{12} g_{33} - g_{13} g_{23})}{g_{11} g_{22} - g_{12}^2 - 2 x_1(g_{11} g_{23} - g_{12} g_{13}) + x_1^2(g_{11} g_{33} - g_{13}^2)} \ ,$$

$$(43) \qquad \frac{\gamma_{23}}{\gamma_{33}} = \frac{g_{11} g_{23} - g_{12} g_{13} - x_1(g_{11} g_{33} - g_{13}^2)}{g_{11} g_{22} - g_{12}^2 - 2 x_1(g_{11} g_{23} - g_{12} g_{13}) + x_1^2(g_{11} g_{33} - g_{13}^2)} \ .$$

Clearly, (38), (39) and (41) mean that any possible Jacobi structure on this compressed system comes from a conformal symplectic structure.

8. Conclusions

We've seen that the failure of the Jacobi identity for the Schaft-Maschke [10] almost-Poisson structure on the constrained submanifold $P \subset T^*Q$ of a nonholonomic system cannot be cirvumvented by the introduction of an associated Jacobi structure [4][9], even in the simplest cases. It remains to be confirmed whether this "no-go" result is generic.

These examples also suggest that pehaps the almost-Poisson structure can be most useful when there is a principal bundle structure $G \to Q \to S$, and all data are equivariant with respect to the group G. In the contact case, z-invariant Lagrangians. In this case, the degenerate almost-Poisson bracket in P projects over a non-degenerate almost-Poisson bracket in T^*S, which in favorable cases (not all) is conformally symplectic, or Jacobi. It remains to be found a geometrical interpretation for these cases.

More generally, it also remains to be determined whether or when such a conformally symplectic reduction can be applied for less simple nonholonomic systems. Work in this direction is under way and shall be reported elsewhere.

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