THE ZETA DETERMINANT OF A CONE

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Abstract. We apply a new method introduced in [16] to study the analytic properties of the zeta function of the Laplace operator on the metric cone over a compact connected Riemannian manifold. We obtain an explicit formula for the derivative of the zeta function at zero.

Spectral geometry of spaces with conical singularity has been studied by Cheeger in a series of works in the early '80 [4] [5] [6], where in particular the heat kernel was investigated in details. Consequently, many works appeared, where particular instances, applications to physics and generalizations have been considered (see for example [7] or [2] for a complete list of references). The more relevant results concerning the analytic properties of the zeta function at zero, are in works by Cognola and Zerbini [7], who study the regularity of the zeta function at zero, and give a formula for the residue, and by Bordag, Kirsten and Dowker [2], who obtain the heat kernel expansion, and therefore a formula for the finite part of the zeta function at zero, and also address the computation of the zeta determinant (see also [1] where the case of the m-dimensional ball is solved).

The aim of this letter is to complete the analysis performed in these works, giving an explicit formula for the finite part of the derivative of the zeta function at zero, and therefore for the zeta determinant. As well known, this is a much harder problem, since in general the derivative of the zeta function can not be obtained by applying heat kernels methods. Consequently, a new approach is necessary. Such approach is based, on one side, on a new technique introduced in series of works of the author [11] [13] [14] to study the analytic properties at zero of the zeta functions associated to a class of sequences, called of spectral type, that includes the sequences of the eigenvalues of the metric Laplacian on a compact manifold, and on the other side, on some recent results on the spectral decomposition's properties of the zeta function associated to some class of double sequences [16]. The main point in this new approach is that, instead of the heat kernel, it employs an other spectral function, called logarithmic Gamma function and strictly related with the Fredholm determinant (see also [17]) in order to provide a formula that gives the analytic continuation of the zeta function at zero (see equation (2) below). As a side effect of our approach, we also reobtain the known results for the residua and the finite part of the zeta function at zero. Complete proofs and applications will appear some where else.

Let (M, g_M) be a compact connected Riemannian manifold of dimension m without boundary and with metric g_M (we chose M without boundary for simplicity, but all our results extend to the case $\partial M \neq \emptyset$). Let Δ_M be the metric Laplacian on M, and $Sp\Delta_M = \{\lambda_n\}_{n=1}^{\infty}$, the spectrum of Δ_M. Let $C_\nu M$ be the metric cone

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Date: September 8, 2007.

2000 *MATHEMATICS SUBJECT CLASSIFICATION: 11M41, ?58G25, 81T16?* .

over M as defined by Cheeger in [4], but with the further parameter ν appearing in the metric. Namely, $C_\nu M$ is the space $(0,1]\times M$ with metric

$$g=(dx)^2+\frac{x^2}{\nu^2}g_M,$$

where ν is a positive constant. Note that, whenever the metric g can be obtained as induced metric by some embedding of $C_\nu M$ in some Euclidean space, then the constant ν measures the singularity of $C_\nu M$. In fact, by passing to Cartesian coordinates, it is clear that $C_1 M$ is a smooth manifold in this case. Particular instances of this situations have been studied in [1] (m-ball) [13] (cone over a circle).

The induced metric Laplacian on $C_\nu M$ is

$$\Delta_{C_\nu M}=-d_x^2-\frac{m}{x}d_x+\frac{\nu^2}{x^2}\Delta_M,$$

that can be written as (by the opportune Liouville transformation)

$$\Delta_{C_\nu M}=-d_x^2+\frac{1}{x^2}\left(\nu^2\Delta_M-\frac{m}{4}\right).$$

Decomposing on the eigenspaces of Δ_M, as in [3] or [13], we obtain the family of singular Sturm operators

$$L_{\nu\lambda_n}=-d_x^2+\frac{1}{x^2}\left(\nu^2\lambda_n+\frac{(m-1)^2}{4}-\frac{1}{4}\right),$$

that can be solved in term of Bessel functions (see [13]).

With the opportune boundary conditions, that generalize standard Dirichlet conditions (see [3] [13] [5]), the positive spectrum of the metric Laplacian on the cone is

$$Sp^+\Delta_{C_\nu M}=\left\{j^2_{\nu\mu_n,k}\right\}_{n=0,k=1}^{\infty},\qquad \mu_n=\sqrt{\nu^2\lambda_n+\frac{(m-1)^2}{4}}.$$

where the $j^2_{\nu,k}$ are the positive zeros of the Bessel function J_ν.

Next, we consider the associated zeta functions. Proceeding as in [14], we can associate to a sequence of spectral type $S=\{a_n\}_{n=1}^\infty$, with genus p, some spectral functions, and in particular the zeta function defined by the series $\zeta(s,S)=\sum_{n=1}^\infty a_n^{-s}$, for $\mathrm{Re}(s)>p$, and by analytic extension elsewhere. We will be concerned here with the following two zeta functions:

$$\zeta\left(s,Sp\left(\nu^2\Delta_M+\frac{(m-1)^2}{4}\right)\right)=\sum_{n=0}^\infty\left(\nu^2\lambda_n+\frac{(m-1)^2}{4}\right)^{-s},$$

$$\zeta(s,Sp^+\Delta_{C\nu M})=\sum_{n=0,k=1}^\infty j_{\mu_n,k}^{-2s}.$$

The first, is the zeta function on the setion of the cone, twisted by the parameter ν and shifted by the constant $\frac{(m-1)^2}{4}$. Note that, we must omit the zero mode when $m=1$ in order to have a proper definition. Since the one dimensional case as been completely treated in [13], we will assume $m>1$ in the following; however, the approach presented here covers also the case $m=1$. The second zeta function is the zeta function on the metric cone, and is the object of our analysis. We claim that the sequence $S=Sp^+\Delta_{C\nu M}$ is spectrally decomposable over the sequence $U=Sp\left(\nu^2\Delta_M+\frac{(m-1)^2}{4}\right)$, as in Definition 3.1 of [16]. For, it is easy to see that S is a sequence of spectral type (using classical estimates for the zeros of the Bessel functions), and has relative genus $(p_0, p_1, p_2) = \left(\left[\frac{m+1}{2}\right], 0, \left[\frac{m}{2}\right)\right]$, and also that U is a totally regular sequence of spectral type, as are all linear combinations of sequences of eigenvalues of metric Laplacians on compact manifolds (by Weil formula), has genus $\left[\frac{m}{2}\right]$ and infinite order. The key point, in order to prove decomposability of S over U, is to show that the Fredholm determinant $\log\Gamma(\lambda, S_n)$ [14] [17] (called logarithmic Gamma function in [16]) associated to the sequence $S_n = \{S/\mu_n\}$, has a uniform asymptotic expansion for large μ_n. This is of course the key point in all development of spectral analysis on spaces with conical singularities, as can be seen reading the works of Cheeger. Such an expansion is known from asymptotic theory of special functions (see for example [9] 10.7). We obtain (see also [13])

$$\begin{aligned}
\log\Gamma(\lambda, \tilde S_n) &= -\sum_{k=1}^{\infty} \log\left(1 + \frac{\mu_n^2(-\lambda)}{j^2_{\mu_n,k}}\right)\\
&= \left(1 - \log 2 + \log(1+\sqrt{1-\lambda}) - \sqrt{1-\lambda}\right)\mu_n\\
&\quad + \frac{1}{4}\log(1-\lambda)\\
&\quad + \sum_{j=1}^{\infty}\left(\frac{(-1)^j}{j}\left(\sum_{k=1}^{\infty}\frac{U_k(1/\sqrt{1-\lambda})}{\mu_n^k}\right)^j - \frac{B_{2j}}{2j(2j-1)}\mu_n^{2j-1}\right),
\end{aligned}$$

where the B_k are the Bernoulli numbers, and the $U_k(z)$ are polynomials in z of order $3k$. The first polynomials are given in [9], where we can also find a recursive formula for the general ones. A further expansion gives

$$\begin{aligned}
\log\Gamma(\lambda, \tilde S_n) &= \left(1 - \log 2 + \log(1+\sqrt{1-\lambda}) - \sqrt{1-\lambda}\right)\mu_n\\
&\quad + \frac{1}{4}\log(1-\lambda)\\
&\quad + \sum_{h=1}^{\infty}\left(D_h(1/\sqrt{1-\lambda}) - \frac{B_{h+1}}{h(h+1)}\right)\mu_n^{-h}\\
&= \sum_{h=-1}^{m}\phi_{\frac{h}{2}}(\lambda)\mu_n^{-\frac{h}{2}} + O(\mu_n^{-m-1}),
\end{aligned}$$

where observe that $D_h(1) = -\frac{\zeta_R(-h)}{h}$. The polynomial U_h or D_h, represent a set of invariants that completely characterize the problem, namely the geometry of the cone, at least for what is concerned with the heat kernel and the zeta function. This emerges clearly from the results of [7] and [2], as well as from the original results of Cheeger. In fact, all the formulas related to the analytic properties of the zeta function are given using information on the zeta function on the section and information contained in the above polynomials. We will now show how we can complete this information in order to obtain also the determinant. For, we need to consider from one side, not just the polynomial, but the functions $\phi_{\frac{h}{2}}(\lambda)$ defined above (always from the uniform expansion of the Bessel function $I_\nu(\nu z)$). From the other side, we must add the two new functions with non positive index. Resuming, we state that the determinant on the cone $C_\nu M$ is characterized by the following set of invariant functions:

$$\begin{aligned}
\phi_{-\frac{1}{2}}(\lambda) &= 1 - \log 2 + \log(1+\sqrt{1-\lambda}) - \sqrt{1-\lambda},\\
\phi_0(\lambda) &= \frac{1}{4}\log(1-\lambda),\\
\phi_{\frac{h}{2}}(\lambda) &= D_h(1/\sqrt{1-\lambda}) - \frac{B_{h+1}}{h(h+1)}, \qquad 1 \leq h \leq m,
\end{aligned} \tag{1}$$

plus information on the zeta function on the section.

In fact, by Lemma 3.4 of [16], the analytic extantion of the zeta function on the cone can be obtained from the following complex representation

$$\zeta(s, Sp^+\Delta_{C\nu M}) = \frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}\mathcal{T}(s,\lambda, Sp^+\Delta_{C\nu M})d\lambda, \tag{2}$$

where

$$\mathcal{T}(s,\lambda, Sp^+\Delta_{C\nu M}) = \sum_{n=1}^\infty u_n^{-\kappa s}\log\Gamma(\lambda, \tilde{S}_n).$$

Furthermore, the above representation splits in two terms (see Lemmas 3.5 and 3.6 of [16]) as follows:

$$\begin{aligned}
\zeta(s, Sp^+\Delta_{C\nu M}) =& \frac{s}{\Gamma(s)}\int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}\mathcal{P}(s,\lambda, Sp^+\Delta_{C\nu M})d\lambda dt\\
&+ s\sum_{h=-1}^{m}\Phi_{\frac{h}{2}}(s)\zeta(s+h/2, U),
\end{aligned} \tag{3}$$

where

$$\Phi_{\frac{h}{2}}(s) = \int_0^\infty t^{s-1}\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda}\phi_{\frac{h}{2}}(\lambda)d\lambda dt, \tag{4}$$

while $\mathcal{P}(s,\lambda, Sp^+\Delta_{C\nu M})$ is the regular part of $\mathcal{T}(s,\lambda, Sp^+\Delta_{C\nu M})$ at $s=0$ (see Lemma 3.5 of [16]). The point is that we can deal with the analytic expansion of the first integral in equation (3) near $s=0$, using the expansion of the function $\mathcal{P}$ for large λ, and the latter is available from the expansion of the logarithmic Gamma function $\log\Gamma(\lambda,\tilde{S}_n)$ for large λ and fixed n, by classical expansions of Bessel functions (see Lemma 3.6 of [16]). Equations (3) and (4) also show how the invariant functions $\phi_{\frac{h}{2}}(\lambda)$ enter in the game.

We can now state our main result, where we express the first coefficients in the expansion of the zeta function on the cone near $s=0$ as functions of quantities computed from the invariant functions $\phi_{\frac{h}{2}}(\lambda)$ introduced above, and invariants associated to the zeta function on the section.

Theorem 0.1. *The zeta function on the cone $\zeta(s, Sp^+\Delta_{C\nu M})$ has an analytic extension to the complex s-plane with at most a simple pole at $s=0$ and:*

$$\operatorname*{Res_1}_{s=0} \zeta(s, Sp^+\Delta_{C_\nu M}) = -\frac{1}{2} \operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s, U), \tag{5}$$

$$\begin{aligned} \operatorname*{Res_0}_{s=0} \zeta(s, Sp^+\Delta_{C_\nu M}) = &-\frac{1}{2} \operatorname*{Res_0}_{s=-\frac{1}{2}} \zeta(s, U) - \frac{1}{4} \operatorname*{Res_0}_{s=0} \zeta(s, U) \\ &+ (\log 2 - 1) \operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s, U) - \sum_{h=1}^{m} \frac{\zeta_R(-h)}{h} \operatorname*{Res_1}_{s=\frac{h}{2}} \zeta(s, U), \end{aligned} \tag{6}$$

$$\begin{aligned} \operatorname*{Res_0}_{s=0} \zeta'(s, Sp^+\Delta_{C_\nu M}) = &-\left(\frac{\pi^2}{12} + (\log 2 - 1)^2 + 1\right) \operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s, U) \\ &+ (\log 2 - 1) \operatorname*{Res_0}_{s=-\frac{1}{2}} \zeta(s, U) - \frac{1}{2} \operatorname*{Res_0}_{s=-\frac{1}{2}} \zeta'(s, U) \\ &- \frac{1}{4}\zeta'(0, U) - \sum_{h=1}^{m} \frac{\zeta_R(-h)}{h} \operatorname*{Res_0}_{s=\frac{h}{2}} \zeta(s, U) \\ &+ \sum_{h=1}^{m} \sum_{j=0}^{h} c_j(h) \left(\gamma + \psi\left(\frac{h}{2} + j\right)\right) \operatorname*{Res_1}_{s=\frac{h}{2}} \zeta(s, U) \\ &+ \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{j\zeta_H(j+1, \mu_n + 1)}{2(j+1)(j+2)} - \sum_{h=1}^{m} \frac{B_{h+1}}{h(h+1)\mu_n^h} \right). \end{aligned} \tag{7}$$

where

$$\begin{aligned} \zeta(s, U) =& \zeta\left(s, Sp^+\left(\nu^2\Delta_M + \frac{(m-1)^2}{4}\right)\right), \\ \mu_n =& \sqrt{\nu^2\lambda_n + \frac{(m-1)^2}{4}}, \qquad \lambda_n \in Sp\Delta_M, \\ D_h(z) =& \sum_{j=0}^{h} c_j(h) z^{h+2j}. \end{aligned}$$

Some remarks on this result are in order.

- As observed in at the beginning, equation (5) is contained in [7] (equation (12)), and equation (6) in an incomplete form in [7] (equations (12) and (19)), and in complete form in [2] (equation (4.5)). Also, an attempt to obtain equation (7) was performed in Section 9 of [2], compare in particular with equation (9.8).
- If m is even, $\zeta(s, Sp^+\Delta_{C_\nu M})$ is regular at $s=0$. In fact, we can apply for example Proposition 1 of [10] to write

  $$\operatorname*{Res_1}_{s=-\frac{1}{2}} \zeta(s, U) = -\frac{\nu}{2\sqrt{\pi}} \sum_{j,k\geq 0, j+2k=m+1} \frac{(-1)^k}{k!} e_j(\Delta_M) \left(\frac{m-1}{2\nu}\right)^k,$$

  where the $e_j(\Delta_M)$ are the coefficients in the heat kernel expansion of Δ_M, namely $\sum_{n=0}^{\infty} \mathrm{e}^{-\lambda_n t} = \sum_{j=0}^{\infty} e_j(\Delta_M) t^{\frac{m-j}{2}}$ [8]. Whenever m is odd, since $j + 2k = m + 1$, we have that in the above sum all the indices j of the heat coefficients $e_j(\Delta_M)$ are odd, and therefore the coefficients vanish (recall M has no boundary).

- If $M = S^m$, the sphere of dimension m and $\nu = 1$, then $\zeta(s, Sp^+\Delta_{C_1 S^m}) = \zeta(s, \Delta_{B^{m+1}})$ is regular at $s = 0$, where B^k is the ball of dimension k. This is an expected result, since the k-dimensional ball is a compact connected Riemannian manifold. A throughfull analysis of this case was performed in [1]. However, this can also be deduced from the above formula as follows. It is known that the spectrum of the metric Laplacian on the sphere is $Sp^+ = \Big\{ n(n+m-1) \Big\}_{n=1}^{\infty}$. Therefore, when $\nu = 1$,
$$Sp^+\left(\nu^2 \Delta_M + \frac{(m-1)^2}{4}\right) = \left(n + \frac{m-1}{2}\right)^2,$$
i.e. the eigenvalues are squares of natural numbers plus a constant. For general theoretical argument in zeta function theory (see for example [14]), the zeta functions associated to these series are regular at $s = -\frac{1}{2}$.
- The series appearing in the last term of equation (7) is a convergent series, as can be easily checked, the divergent part having been cancelled by the adding of the proper regularization's terms, namely the terms with the Bernoulli numbers. It should also been observed that, even if a complex integral representation of the same function of ν is possible in general, applying techniques that generalize the one used in [12] or [13] (see also [15]), the series representation given in equation (7) is much more convenient due to its fast convergence speed.
- The coefficients appearing in the equations given in the theorem, are obtained from the residues at zero of the functions $\Phi_{\frac{h}{2}}(s)$ defined in equation (4). Substituting in equation (4) the explicit formulas for the functions $\phi_{\frac{h}{2}}(\lambda)$ given in equation (1), we find some tricky complex countour integrals that can be computed explicitely. The result is given as a product of powers of s and Euler Gamma functions, and shows that the functions $\Phi_{\frac{h}{2}}(s)$ can have at most a double pole at $s = 0$, with explicit formulas for the residues. The contour integrals for $h \geq 0$ are computed in [13] and [16]. For completness we sketch here the missing case, namely $h = -\frac{1}{2}$. We need to compute the integral:
$$f(s) = \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} \log(1 + \sqrt{1-\lambda}) d\lambda dt.$$
First, consider the integral
$$h(a,t) = \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-\lambda t}}{-\lambda} (1 + \sqrt{1-\lambda})^a d\lambda;$$
changing variable $-z = 1 - \lambda$,
$$\begin{aligned} h(a,t) &= -\frac{\mathrm{e}^{-t}}{2\pi i} \int_{\Lambda_{\theta,c}} \frac{\mathrm{e}^{-zt}}{z+1} (1 + \sqrt{-z})^a dz \\ &= -\frac{\mathrm{e}^{-t}}{2\pi i} \int_0^\infty \frac{\mathrm{e}^{-xt}}{x+1} \left((1 + i\sqrt{x})^a - (1 - i\sqrt{x})^a\right) dx \\ &= -\frac{\mathrm{e}^{-t}}{\pi} \int_0^\infty \frac{\mathrm{e}^{-xt}}{x+1} (1+x)^{\frac{a}{2}} \sin\left(a \arctan\sqrt{x}\right) dx. \end{aligned}$$

Since

$$\frac{1}{2\pi i}\int_{\Lambda_{\theta,c}}\frac{\mathrm{e}^{-\lambda t}}{-\lambda}\log(1+\sqrt{1-\lambda})d\lambda = \left.\frac{d}{da}h(a,t)\right|_{a=0} = -\frac{\mathrm{e}^{-t}}{\pi}\int_0^\infty \frac{\mathrm{e}^{-xt}}{x+1}\arctan\sqrt{x}dx;$$

we have, assuming $\mathrm{Re}(s)$ large and using the definition of the Beta function

$$\begin{aligned} f(s) &= -\frac{1}{\pi}\int_0^\infty t^{s-1}\mathrm{e}^{-t}\int_0^\infty \frac{\mathrm{e}^{-xt}}{x+1}\arctan\sqrt{x}dxdt \\ &= -\frac{1}{2\pi s}\Gamma(s)\int_0^\infty (1+x)^{-1-s}\frac{1}{\sqrt{x}}dx \\ &= -\frac{1}{2\sqrt{\pi}}\frac{\Gamma\left(s+\frac{1}{2}\right)}{s^2}. \end{aligned}$$

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