Bollettino U. M. I.
(9) **II** (2009), 529-533

Reidemeister Torsion and Analytic Torsion of Discs

T. de Melo - L. Hartmann - M. Spreafico

Abstract. – *We study the Reidemeister torsion and the analytic torsion of the m-dimensional disc in the Euclidean m-dimensional space, using the base for the homology defined by Ray and Singer in* [10]. *We prove that the Reidemeister torsion coincides with the square root of the volume of the disc. We study the additional terms arising in the analytic torsion due to the boundary, using generalizations of the Cheeger-Müller theorem. We use a formula proved by Brüning and Ma* [1], *that predicts a new anomaly boundary term beside the known term proportional to the Euler characteristic of the boundary* [6]. *Some of our results extend to the case of the cone over a sphere, in particular we evaluate directly the analytic torsion for a cone over the circle and over the 2-sphere. We compare the results obtained in the low dimensional cases. We also consider a different formula for the boundary term given by Dai and Fang* [4], *and we show that the result obtained using this formula is inconsistent with the direct calculation of the analytic torsion.*

The analytic torsion is an important geometric invariant of Riemannian manifolds introduced by Ray and Singer [10], as an analytic version of the Reidemeister torsion [12]. The equivalence of the two torsions, conjectured by Ray and Singer, was proved independently by J. Cheeger [2] and W. Müller [9], for manifolds without boundary. In the case of a manifold with boundary, some further boundary terms appear. These boundary terms were first given by Lück [6], in the case of a product metric near the boundary, and recently by Dai and Fang [4], and by Brüning and Ma [1], in the general case.

Beside the intensive investigation, explicit evaluations of the torsion were exhibited in comparably few cases [11] [16]. The mentioned recent results for the anomaly boundary term permit to study one of the more natural, simpler still open cases: namely the case of a disc in the Euclidean space $\mathbb{R}^m$. The aim of this note is to announce formulas for the Reidemeister torsion and for the analytic torsion of the m-dimensional disc. As a by product of our analysis, we also give formulas for the analytic torsion of the geometric cone over the circle and over the 2-sphere. A detailed analysis and complete proofs can be found in [5].

Let (M, g) be a compact connected simply connected Riemannian m-manifold with metric g. Let denote by $\tau(M)$, and by $T(M)$ the Reidemeister torsion and the analytic torsion of M, respectively. If M is not acyclic, assume the base for the homology is fixed by the choice of an orthonormal base of harmonic forms, as in [10]. Let $S^n_l = \{x \in \mathbb{R}^{m=n+1} \mid |x| = l\}$, $D^m_l = \{x \in \mathbb{R}^m \mid |x| \leq l\}$, $l > 0$, with the standard Riemannian metric g induced by the immersion in $\mathbb{R}^m$. Using the standard cellular decomposition of the disc with one top cell, one $(m-1)$-cell and one 0-cell, we construct the chain complex of real vector spaces:

$$C : 0 \longrightarrow \mathbb{R}[c^1_{n+1}] \longrightarrow \mathbb{R}[c^1_n] \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{R}[c^1_0] \longrightarrow 0.$$

Applying the definition of the Reidemeister torsion given for example in [8], since a base for harmonic m-forms is $\sqrt{|\det g|}dvol(x)$, we prove the following formulas for the Reidemeister torsion, where $\mathrm{Vol}(D^m_l)$ denotes the volume of the disc.

THEOREM 1. – *The Reidemeister torsion of the disc D^m_l of radius $l > 0$ in $\mathbb{R}^m$ with the standard metric induced by the immersion in the Euclidean space, is:*

$$\tau(D^m_l) = \sqrt{\mathrm{Vol}(D^m_l)}.$$

In the same situation, the Reidemeister torsion of the pair (D^m_l, S^{m-1}_l) is:

$$\tau(D^m_l, S^{m-1}) = \left(\sqrt{\mathrm{Vol}(D^m_l)}\right)^{(-1)^{m-1}}.$$

Applying the formula of [1], the anomaly boundary term appearing in the analytic torsion is given by some local quantities constructed from the curvature tensor:

$$\frac{1}{2}\int_{S^{m-1}_l}\left(B(\nabla_1^{TD^m_l}) - B(\nabla_0^{TD^m_l})\right).$$

We prove that in the present case, the forms $B\big(\nabla_j^{TD^m_l}\big)$ are given by the Berezin integral of some power of a one-form $\mathcal{S}$ related to the Chern-Simon form,

$$B\big(\nabla_1^{TD^m_l}\big) = \frac{1}{2\Gamma\left(\frac{m+1}{2}\right)}\sum_{j=0}^{\left[\frac{m}{2}-1\right]}\frac{1}{m-2j-1}\int^B \mathcal{S}_1^{m-1},$$

and by evaluating this integral, we obtain the anomaly boundary term as follows.

THEOREM 2. – *The analytic torsion of the disc D^m_l of radius $l > 0$ in $\mathbb{R}^m$ with the standard metric induced by the immersion, and absolute boundary conditions, is:*

$$\log T_{\mathrm{abs}}(D^{2p-1}_l) = \frac{1}{2}\log \mathrm{Vol}(D^{2p-1}_l) + \frac{1}{2}\log 2 + \frac{1}{4}\sum_{n=1}^{p-1}\frac{1}{n},$$

$$\log T_{\mathrm{abs}}(D^{2p}_l) = \frac{1}{2}\log \mathrm{Vol}(D^{2p}_l) + \frac{1}{2}\sum_{n=1}^{p}\frac{1}{2n-1}.$$

Beside our main results concern the case of the discs, that are smooth manifolds, our technique extends to the case of the cone. Consider the sphere $S^n_{l\sin a}$ embedded in the hyperplane $\mathbb{R}^{n+1}\times\{l\cos a\}$ of $\mathbb{R}^{n+2}$, with center in $\{0\}\times\{l\cos a\}$. Denote by $C_aS^n_{l\sin a}$ the $m=n+1$-dimensional surface in $\mathbb{R}^{n+2}$ defined by the segments joining the origin with the points of the sphere. This is a cone of angle a, vertex in the origin, and length l. The induced metric is

$$g = dr\otimes dr + r^2\sin^2 a g_{S^n_1},$$

where $r\in[0,l]$ is the geodesic distance from the origin, and $g_{S^n_1}$ the standard metric on the sphere. Using the extension of the classical Hodge theory and the functional calculus on the cone given by Cheeger in [3], we obtain complete systems of eigenvalues and L^2-eigenfunctions for the Laplace operator $\Delta^{(q)}$ on forms on $C_aS^1_{l\sin a}$ and $C_aS^2_{l\sin a}$. By definition [10], the analytic torsion is

$$\log T(M) = \frac{1}{2}\sum_{q=1}^{m}(-1)^q q\zeta'(0,\Delta^{(q)}),$$

where the zeta function is defined by

$$\zeta(s,\Delta^{(q)}) = \sum_{\lambda\in \mathrm{Sp}_+\Delta^{(q)}}\lambda^{-s}.$$

The eigenvalues λ of $\Delta^{(q)}$ are given by the zeros $z_{\mu,k}$ of some linear combinations of Bessel functions J_μ and their derivatives, where μ denotes the eigenvalues of the Laplacian on forms on the section. The zeta functions appearing are therefore some double zeta functions and can be analyzed using the method developed by Spreafico in [13] [14] [15]. This permits to evaluate the derivative at zero, and consequentely to prove the following formulas for the analytic torsion.

THEOREM 3. – *The analytic torsion of the cone $C_aS^1_{l\sin a}$, of angle a, and length $l>0$, over the circle, with the standard metric induced by the immersion, and absolute/relative boundary conditions, is:*

$$\log T_{\mathrm{abs}}(C_aS^1_{l\sin a}) = -\log T_{\mathrm{rel}}(C_aS^1_{l\sin a}) = \frac{1}{2}\log(\pi l^2\sin a) + \frac{1}{2}\sin a,$$

in particular, for the disc D^2_l $\left(a=\frac{\pi}{2}\right)$ we have:

$$\log T_{\mathrm{abs}}(D^2_l) = -\log T_{\mathrm{rel}}(D^2_l) = \frac{1}{2}\log \pi l^2 + \frac{1}{2}.$$

THEOREM 4. – *The analytic torsion of the cone $C_aS^2_{l\sin a}$, of angle a, and length $l>0$, over the sphere, with the standard metric induced by the immersion, and* *absolute/relative boundary conditions, is:*

$$\log T_{\rm abs}(C_a S^2_{l\sin a}) = \log T_{\rm rel}(C_a S^2_{l\sin a}) = \frac{1}{2}\log\frac{4}{3}l^3 - \frac{1}{2}F(0,\csc a) + \frac{1}{4}\sin^2 a,$$

in particular, for the disc D^3_l $\left(a = \frac{\pi}{2}\right)$ *we have:*

$$\log T_{\rm abs}(D^3_l) = \log T_{\rm rel}(D^3_l) = \frac{1}{2}\log\frac{4\pi l^3}{3} + \frac{1}{2}\log 2 + \frac{1}{4}.$$

The function $F(0,x)$ has a known power series expansion in x, for $x > 1$:

$$F(0,x) = \sum_{j=-1}^{\infty} c_j x^j,$$

where the coefficients are particular values of the zeta function of the Laplacian on function on the 2-sphere (see Appendix B of [5]). We conclude with some remarks.

(1) The topological approach proves to be much more effective and natural for evaluating the analytic torsion with respect to a direct calculation starting from the definition of the analytic torsion, and gives a clear interpretation of the different terms appearing in the final result (see also [7] for a similar result for spheres).

(2) In the formula for the analytic torsion of the disc D^2_l given in Theorem 2, two terms appear. The first is the topological term, corresponding to the volume of the disc, and in fact comes from the Reidemeister torsion, computed in Theorem 1. The second term comes from the boundary contribution. In this case, the unique boundary contribution is of the type described by Dai and Fang [4] and Brüning and Ma [1], since the manifold is not a product near the boundary, and the Euler characteristic of the boundary is trivial. The result given in Theorem 2 is obtained by using the formula of Brüning and Ma, and is consistent with the result obtained by direct calculation of the analytic torsion, given in Theorem 4. However, in this case the formula of Dai and Fang gives the same result.

(3) In the formula for the analytic torsion of the disc D^3_l given in Theorem 2, three terms appear. The first is the topological term, as in the case of D^2_l. The second term comes from the boundary contribution, and is precisely the term depending on the Euler characteristic as predicted by the formula of Lück [6]. The last term also must come from the boundary, and therefore comes from the fact that the manifold is not a product near the boundary. The result given in Theorem 2 is obtained by using the formula of Brüning and Ma, and is consistent with the result obtained by direct calculation of the analytic torsion, given in Theorem 4. In this case, it is possible to check that all the contributions arising from the formula in Theorem 1 of [4] vanish. Therefore, this result furnishes a counter example to the theorem of Dai and Fang, at least in the case of an even dimensional boundary.

REFERENCES

[1] J. Brüning - Xiaonan Ma, *An anomaly formula for Ray-Singer metrics on manifolds with boundary*, GAFA, **16** (2006) 767-873.

[2] J. Cheeger, *Analytic torsion and the heat equation*, Ann. Math., **109** (1979) 259-322.

[3] J. Cheeger, *Spectral geometry of singular Riemannian spaces*, J. Diff. Geom., **18** (1983) 575-657.

[4] X. Dai - H. Fang, *Analytic torsion and R-torsion for manifolds with boundary*, Asian J. Math., **4** (2000) 695-714.

[5] L. Hartmann - T. de Melo - M. Spreafico, *Reidemeister torsion and analytic torsion of discs*, preprint (2008), arXiv:0811.3196v1.

[6] W. Lück, *Analytic and topological torsion for manifolds with boundary and symmetry*, J. Differential Geom., **37** (1993) 263-322.

[7] T. de Melo - M. Spreafico, *Reidemeister torsion and analytic torsions of spheres*, preprint 2008.

[8] J. Milnor, *Whitehead torsion*, Bull. AMS, **72** (1966) 358-426.

[9] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. Math., **28** (1978) 233-305.

[10] D. B. Ray - I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. Math., **7** (1971) 145-210.

[11] D. B. Ray, *Reidemeister torsion and the Laplacian on lens spaces*, Adv. Math., **4** (1970) 109-126.

[12] K. Reidemeister, *Homotopieringe und Linseräume*, Hamburger Abhandl., **11** (1935) 102-109.

[13] M. Spreafico, *Zeta function and regularized determinant on a disc and on a cone*, J. Geo. Phys., **54** (2005) 355-371.

[14] M. Spreafico, *Zeta invariants for Dirichlet series*, Pacific. J. Math., **224** (2006) 180-199.

[15] M. Spreafico, *Zeta invariants for double sequences of spectral type and a generalization of the Kronecker first limit formula*, preprint (2006).

[16] L. Weng - Y. You, *Analytic torsions of spheres*, Int. J. Math., **7** (1996) 109-125.

ICMC, Universidade São Paulo, São Carlos, Brazil.
E-mail: mauros@icmc.usp.br

Received November 16, 2008 and in revised form March 11, 2009