

Zeta invariants for sequences of spectral type, special functions and the Lerch formula

Mauro Spreafico

ICMC USP, São Carlos, Brazil, mauros@icmc.usp.br

(MS received ;)

We discuss the spectral properties of a class of sequences that we call of spectral type. We introduce an effective method to calculate the zeta invariants for this type of sequences. Such invariants are given in terms of some new and old special functions, and we consider a number of examples where we study the properties of these special functions.

1. Introduction

Since it was introduced in the early '70 [1] [25] as a tool to deal with determinants of elliptic operators and analytic torsion, the zeta function has played a fundamental role in the new discipline of geometric analysis, where analytic methods are used to study geometrical objects. This is in fact an older topic, and probably the Euler Gauss Bonnet theorem is the first example. The zeta function technique in this context is a new tool to provide information. Such information is given in terms of particular values of the zeta function, that consequently are called zeta invariants. The first example is indeed the Euler characteristic of a closed surface, given by the value of the associated zeta function at the origin. In fact, it turns out that the more interesting zeta invariants are precisely the values of the zeta function and that of its first derivative at the origin, the latter being the so called regularized determinant. An enormous amount of work has been produced since then in this area, and very important results have been achieved (just two examples are [11] [18]). In particular the zeta function technique to regularize the determinant of an elliptic operator became of primary importance in mathematical physics [16]. In this direction, many works [9] [21] [35] [33] [28] appeared in the literature where the effort was to give complete results for a number of cases where it was possible to find out a relationship with some known special functions. These works are all related with the case of compact connected riemannian manifold, where the zeta function can always be defined in a subset of the complex plane, by the sum

$$\zeta_M(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

of the negative complex powers of the eigenvalues λ_n of the Laplace operator induced by the riemannian metric [26]. In particular, when a constant term is added

2000 Mathematics Subject Classification: 58C40 33E99 58G30.

to the Laplace operator the zeta invariants of the non homogeneous resulting zeta function are related to those of the homogeneous zeta function by a relation that is (for the derivative) a generalization of the classical Lerch formula relating the derivatives of the Riemann and Hurwitz zeta functions:

$$\zeta'_H(0, q) = \zeta'_R(0) + \ln \Gamma(q).$$

Under a more geometrical point of view, this is a relation between the regularized determinants of the non homogeneous and the homogeneous Laplacian operators on the circle (see Section 3.1). This is the point of view that we will take in these notes, and in fact one of our result is to give a series of generalization of the above Lerch formula for some interesting cases (see Section 3). With this purpose in mind, the approach adopted in these notes consists first in providing a general abstract setting for the problem: namely, we introduce in Section 2 the zeta function associated to a particular kind of sequences of complex numbers, that we call a *sequence of spectral type* and that covers all the cases of interest. Beside the zeta function, we introduce for such sequences other spectral functions in terms of which we give a general Lerch formula (Proposition 2.7). Under this geometrical point of view, the main results of our analysis are an effective tool (that generalize a technique introduced in [32]) to calculate the zeta invariants that applies for any compact connected manifolds (Corollary 2.2), an explanation to the appearance of the spectral functions in the generalized Lerch formula, and a generalization of the last on the two sphere, answering a question introduced in [30].

Taking an other point of view, the Lerch formula is a good tool to provide information (and in particular the constant term of the asymptotic expansion) for some classes of special functions. In the Lerch formula for the circle, the derivative of the homogeneous and non homogeneous Riemann zeta function evaluated at $s = 0$ are related by the logarithm of the Gamma function. When considering different types of zeta functions, in the Lerch formula different type of special functions will appear. Under this point of view, the zeta function technique gives a very useful device to study the properties of these special functions [33] [29] [5] [19] [31]. A class of zeta functions that has been investigated since very long ago is that of the multiple zeta functions [3]. As an application of our methods, we analyze this class of zeta functions and the associated special functions, obtaining some new interesting interpretations and results. This is in Section 3.2. We conclude by observing that this analysis is also very important in physics. In fact, adding a constant to the Laplacian on a compact manifold means to add a constant potential term m , usually a mass term, to the associated physical theory, and it is of great importance in physics to know the behavior for large and small m [6].

2. Sequences of spectral type

2.1. Definition and spectral functions

Let S be a sequence¹ of complex numbers whose only accumulation point is the point at infinity. For the purposes we are concerned here, we can assume without

¹By a sequence of elements of a set X we mean a mapping from the natural numbers to X .

loss of generality that $0 \notin S$. We can order the elements of S using the natural numbers, and we write $S = \{a_n\}_{n \in \mathbb{N}_0}$, with $0 < |a_1| \leq |a_2| \leq \dots$. The number

$$\alpha = \limsup_{n \rightarrow \infty} \frac{\ln n}{\ln |a_n|},$$

is the *exponent of convergence* of the sequence S . If α is finite, the sequence is of *finite exponent*. For a sequence S of finite exponent α , the series $\sum_{n=1}^{\infty} a_n^{-s}$ converges (locally) uniformly and absolutely for $\operatorname{Re}(s) > \alpha$, and if we assume $\operatorname{Re}(a_n)$ bounded below, the series $\sum_{n=1}^{\infty} e^{-a_n t}$ also converges (locally) uniformly and absolutely for $t > 0$. In geometry the most natural situation where the above setting applies is when $S = \sigma_p P - \operatorname{Ker} P = \{\lambda_n\}_{n=1}^{\infty}$ is the sequence of the positive eigenvalues of a linear operator P . The operator P can be an elliptic self adjoint operator on a compact manifold², but more general settings are important. In particular, studying topology and geometry of manifolds, P is the Laplace operator Δ_M on a compact manifold M , and the associated zeta function $\zeta(s, \Delta_M) = \sum_{n=1}^{\infty} \lambda_n^{-s}$ detects many information on the geometry and the topology of the space [26]. When P is self adjoint, positive definite and elliptic, there exists an asymptotic expansion for the eigenvalues $\lambda_n = Kn^\epsilon + o(n^\epsilon)$, for large n , with some $K, \epsilon > 0$, the function $\sum_{n=0}^{\infty} e^{-\lambda_n t}$ is the trace of the heat operator and $\det P = e^{-\zeta'(s, P)}$ is the *functional determinant* of P [1]. A full asymptotic expansion [20] [24] for the trace of the heat kernel

$$\sum_{n=0}^{\infty} e^{-\lambda_n t} \sim t^{-\frac{m}{2}} \sum_{j=0}^{\infty} e_j t^{\frac{j}{2}},$$

can be obtained by using classical methods in pseudo differential operator theory (see for example [14]). As a consequence, the analytic continuation of the associated zeta function is regular at $s = 0$ with $\zeta(0, \Delta_M) = e_m - \dim \operatorname{Ker} \Delta_M$, and has possible poles at $s = \frac{m-j}{2}$, $j = 0, 1, 2, \dots$, that are non negative integers, with residua

$$\operatorname{Res}_1 \left(\zeta(s, \Delta_M), s = \frac{m-j}{2} \right) = \frac{e_j}{\Gamma\left(\frac{m-j}{2}\right)}.$$

This can be obtained using the Mellin transform and the expansion for the trace of the heat kernel. In particular, all the coefficients of odd index vanish if the manifold has no boundary [7]. The growth estimate for the eigenvalues can be deduced applying Ikehara's Tauberian theorem [36] from the residua at the first pole of the zeta function. Furthermore, in the case of the Laplace operator Δ_M on a manifold of dimension m , a better estimate for the growth of the eigenvalues is available. That is to say we have the Weyl formula [17] [22] [27] (where the coefficients are known)

$$\lambda_n = K_0 n^{\frac{2}{m}} + O(n^{\frac{1}{m}}),$$

or even

$$\lambda_n = K_0 n^{\frac{2}{m}} + K_1 n^{\frac{1}{m}} + o(n^{\frac{1}{m}}),$$

if some suitable geometric conditions are satisfied. The above estimates hold also if a constant (potential) term is added to the Laplacian. This suggests to start

²When M has a boundary some suitable boundary conditions must be imposed [14] 1.11.

the analysis from the properties of the sequence itself, with the final purpose of getting properties and relationships among the zeta invariants directly from some information about the sequence. This is the point of view of these notes. Our main result is to show that the main the zeta invariants are determined by the knowledge of the asymptotic expansion of a spectral function associated to S (Proposition 2.4). Also, we can introduce a sufficient condition on the sequence to ensure the existence of such an expansion (Lemma 2.3). To obtain these results we need to restrict the class of allowed sequences. This we do by requiring two further conditions. Before we need to introduce three spectral functions associated to S .

Given a sequence $S = \{a_n\}_{n=1}^{\infty}$ with finite exponent α , the *zeta function* associated to S is the function of the complex variable s defined by the series

$$\zeta(s, S) = \sum_{n=1}^{\infty} a_n^{-s},$$

when $\operatorname{Re}(s) > \alpha$, and by analytic continuation elsewhere; if $\operatorname{Re}(a_n)$ is bounded below, the *heat function* associated to S is the function of the real positive variable t defined by the series

$$f(t, S) = \sum_{n=0}^{\infty} e^{-a_n t},$$

with $a_0 = 0$. For a sequence S as above, there exists a least integer p such that the series $\sum_{n=1}^{\infty} a_n^{-1-p}$ is convergent. If α is not an integer, p is the greatest integer less than α ; if α is an integer, p may be either α or $\alpha - 1$. In any case $\alpha - 1 \leq p \leq \alpha$. The Weierstrass canonical product

$$F(z, S) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right) \exp\left(\sum_{j=1}^p \frac{(-1)^j z^j}{j a_n^j}\right),$$

converges uniformly and absolutely in any bounded closed region of the plane, and $F(z, S)$ is an integral function of finite order³ $\rho = \alpha$ (this is the first Borel theorem) which vanishes if and only if $z = -a_n$ for some n . The integer p is called the *genus* of the canonical product, and F the *Fredholm determinant* associated to S .

We can now introduce the definition of the type of sequences we will work with. First, we use the notation $\Sigma_{\theta, c}$ to denote the closed sector of the positive complex plane of angle θ translated by the real constant c , namely $\Sigma_{\theta, c} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - c)| \leq \frac{\theta}{2}\}$. We call a sequence $S = \{a_n\}_{n=1}^{\infty}$ of finite exponent α a *sequence of spectral type* if the following two properties hold:

1. the points a_n are all contained in a translated closed sector $\Sigma_{\theta, c}$ of angle $\theta < \pi$ and shift $c > 0$ of the complex plane;
2. the logarithm of the Fredholm determinant $\ln F(z, S)$ associated to S has an asymptotic expansion in terms of powers and logarithms for large z along any

³Recall the *order* of an integral function of finite order f is $\rho = \limsup_{r \rightarrow \infty} \frac{\ln \ln M(r)}{\ln r}$, where $M(r)$ is the maximum modulus of $f(z)$ on the circle $|z| = r$.

ray in the complex plane disjoint from $\Sigma_{\theta,c}$ at least up to the constant term, namely

$$\ln F(z, S) = \sum_{k=0}^K \sum_{j=0}^J a_{j,k} z^{\alpha_j} \ln^k z + o(1),$$

where the α_j are real and $\alpha_0 > \alpha_1 > \dots > \alpha_J \geq 0$.

The choice of characterizing a sequence of spectral type by a property of one of the associated spectral functions is suggested by the main application, i.e. the Laplacian over compact manifolds, where this property is always satisfied as we will show in Corollary 2.2. First, we show that the existence of an asymptotic expansion for the Fredholm determinant and for the heat function imply each other. For consider the asymptotic sequences of functions (see [20] for terminology) $f_{\delta,j}(t) = t^{-\delta} \log^j t$ and $g_{\alpha,k}(z) = z^\alpha \log^k z$, where $\{\delta\}$ and $\{\alpha\}$ are any two sets of complex numbers with $\operatorname{Re}(\delta), \operatorname{Re}(\alpha) \rightarrow -\infty$, and $j/k = 0, 1, \dots, J_\delta/K_\alpha \in \mathbb{N}$, for each δ and α respectively (see also [8] pg. 372).

Lemma 2.1. *Let $S = \{a_n\}_{n=1}^\infty$ be a sequence with finite exponent and suppose the points a_n are all contained in a sector $\Sigma_{\theta,c}$ ($\theta < \pi$, $c > 0$). Then, the heat function associated to S has a full⁴ asymptotic expansion with respect to the asymptotic sequence $\{f_{\delta,j}(t) = t^{-\delta} \log^j t\}$ as $t \rightarrow 0^+$ if and only if the logarithm of the Fredholm determinant associated to S has a full asymptotic expansion with respect to the asymptotic sequence $\{g_{\alpha,k}(z) = z^\alpha \log^k z\}$ as $z \rightarrow \infty$ along any ray disjoint from $\Sigma_{\theta,c}$.*

Proof. Before to start, we must point out the following notational remark. We will often use the complex variable $-\lambda = z$, defined by cutting the complex plane along the real positive axis and setting $\arg(-\lambda) = 0$ on the line $(-\infty, 0]$, as usual to deal with contour integrals of Hankel type, see for example [37] for more details.

First, suppose the heat function has a full expansion. By classical asymptotic analysis (see [20] or [24]), to prove that $F(z, S)$ has an expansion it is enough to prove that one of its derivative has such an expansion. Thus, we introduce the function

$$R(\lambda, S) = \frac{d}{d\lambda} \ln F(-\lambda, S) = \sum_{n=1}^{\infty} \left[\frac{1}{\lambda - a_n} + \sum_{j=0}^{p-1} \frac{\lambda^j}{a_n^{j+1}} \right],$$

that we call resolvent function associated to S , together with its derivatives

$$R^{(k)}(\lambda, S) = \frac{d^k}{d\lambda^k} R(\lambda, S).$$

In particular, notice that

$$R^{(p)}(\lambda, S) = \frac{d^p}{d\lambda^p} R(\lambda, S) = -p! \zeta(p+1, S - \lambda) = -p! \sum_{n=1}^{\infty} (a_n - \lambda)^{-p-1},$$

⁴By a similar proof equivalence of expansions of finite order can also be showed.

since the last series is uniformly convergent, and by applying the Mellin transform

$$R^{(p)}(\lambda, S) = - \int_0^\infty t^p e^{\lambda t} (f(t, S) - 1) dt.$$

Thus, we have expressed $R^{(p)}(\lambda, S)$ as the Laplace transform of the heat function and we use the Watson lemma [24] to get the expansion for $R^{(p)}$ from the one of f .

For the other implication, suppose that $\ln F(z, S)$ has a full expansion. Let $\Lambda_{\theta, c} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda - c)| = \frac{\theta}{2}\}$ be the boundary of the sector $\Sigma_{\theta, c}$ appearing in the definition of the sequence (where $c < |a_1|$). Then, we can write

$$f(t, S) - 1 = \frac{1}{2\pi i} \int_{\Lambda_{\theta, c}} e^{-\lambda t} R(\lambda, S) d\lambda.$$

In fact, both the series are uniformly and absolutely convergent, and

$$\frac{1}{2\pi i} \int_{\Lambda_{\theta, c}} e^{-\lambda t} R(\lambda, S) d\lambda = \sum_{n=1}^{\infty} e^{-a_n t},$$

since the integrals of all the non negative integer powers of $(-\lambda)$ vanish, as it can be immediately seen by modifying the contour to the usual Hankel contour and using the Hankel formula for the Gamma function. Next, we integrate by parts to get

$$f(t, S) - 1 = \frac{t}{2\pi i} \int_{\Lambda_{\theta, c}} e^{-\lambda t} \ln F(-\lambda, S) d\lambda,$$

and we can use the expansion of the logarithm of the Fredholm determinant in the above integral when t is small, changing the variable in λ/t . In fact, modifying the contour in a suitable way, we will get integrals of Hankel type in each term. \square

Corollary 2.2. *The sequence of the eigenvalues of the Laplace operator with a regular potential on a compact connected manifold is a sequence of spectral type.*

Next, we give a sufficient condition for a sequence to be of spectral type. This is a quite well-know result but we present a direct simple proof for completeness.

Lemma 2.3. *Let $S = \{a_n\}_{n=1}^\infty$ be a sequence of finite exponent α , and suppose the following expansion for the general term of S holds for large n*

$$a_n = \sum_{j=0}^J K_j n^{\beta_j} + o\left(\frac{1}{n}\right),$$

with real K_j and β_j , and $\frac{1}{\alpha} = \beta_0 > \beta_1 > \dots > \beta_J \geq 0$. Then, S is of spectral type.

Proof. It is clear that the point of S are all contained in the real positive axis, up to a finite number of cases. Then, by a similar argument as in Lemma 2.1, we just have to prove that the associated heat function has an asymptotic expansion for small t . We do not loose in generality assuming $J = K_0 = K_1 = 1$. Then,

$$\sum_{n=1}^{\infty} e^{-(n^{\beta_0} + n^{\beta_1})t} \left[e^{-o(n^{-1})t} - 1 \right],$$

is uniformly convergent to 0 for $t \rightarrow 0^+$. Therefore, to prove that the heat function has an asymptotic expansion for small t , it is enough to prove that the function

$$f_0(t) = \sum_{n=0}^{\infty} e^{-(n^{\beta_0} + n^{\beta_1})t},$$

has such an expansion. To do that, we apply the Plana theorem [37] to the sum $f_0(t) = \sum_{n=0}^{\infty} \phi(n, t)$, where $\phi(z, t) = e^{-(z^{\beta_0} + z^{\beta_1})t}$ is a function of the complex variable z and of the real non negative variable t . To give a precise definition to the complex powers, we cut the complex plane along the negative axis, and set $z^\beta = e^{\beta \ln x + i\beta\theta} = x^\beta e^{i\beta\theta}$. If $-\frac{\pi}{2} \leq \beta_0\theta \leq \frac{\pi}{2}$, ϕ is analytic and bounded uniformly for all $t \geq 0$. Let first assume $\beta_0 \leq 1$, then $(\beta_0) = \beta_0$ and $[\beta_0] = 0$, ϕ is analytic and bounded in the strip $0 \leq \text{Re}(z) \leq M$, for all $M > 0$, and Plana theorem gives

$$\begin{aligned} \sum_{n=0}^M \phi(n, t) &= \frac{1}{2} [\phi(0, t) - \phi(M, t)] + \int_0^M \phi(x, t) dx + \\ &+ i \int_0^\infty \frac{\phi(iy, t) - \phi(-iy, t) + \phi(M + iy, t) - \phi(M - iy, t)}{e^{2\pi y} - 1} dy, \end{aligned}$$

where the last integral is uniformly convergent. The limit for $M \rightarrow +\infty$ gives

$$\begin{aligned} f_0(t) &= \frac{1}{2} + \int_0^\infty e^{-(x^{\beta_0} + x^{\beta_1})t} dx + \\ &+ 2 \int_0^\infty \frac{e^{-(y^{\beta_0} \cos \frac{\pi}{2} \beta_0 + y^{\beta_1} \cos \frac{\pi}{2} \beta_1)t} \sin [(y^{\beta_0} \sin \frac{\pi}{2} \beta_0 + y^{\beta_1} \sin \frac{\pi}{2} \beta_1)t]}{e^{2\pi y} - 1} dy. \end{aligned}$$

The expansion of the integral in the middle term can be calculated by expanding one of the exponential and making the substitution $u = x^{\beta_0}$:

$$\begin{aligned} \int_0^\infty e^{-x^{\beta_0}t} e^{-x^{\beta_1}t} dx &= \int_0^\infty e^{-x^{\beta_0}t} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{\beta_1 k} t^k dx = \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^k \int_0^\infty e^{-ut} u^{\frac{\beta_1}{\beta_0} k + \frac{1}{\beta_0} - 1} du \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma\left(\frac{\beta_1}{\beta_0} k + \frac{1}{\beta_0}\right) t^{k - \frac{\beta_1}{\beta_0} k - \frac{1}{\beta_0}}. \end{aligned}$$

In the integral in the last term we can expand the factors, due to uniform convergence, and after rearranging the terms in the product of the series we get the thesis when $\beta \leq 1$. When $\beta > 1$, $\beta = [\beta] + (\beta)$, and $z^\beta = x^\beta e^{i(\beta)\theta}$. \square

Notice that the proof of the existence of the asymptotic expansion given for Lemma 2.3 provides also a method to determinate the coefficients in the expansion when the terms in the expansion of the sequence are all known. An application is in [31]. Notice also that applying Lemma 2.3 to the first term in the growth of the eigenvalues we get the first term in the expansion of the heat function explicitly.

2.2. Zeta invariants for homogeneous sequences

Next, we generalize a technique introduced in [32] and we give an effective method to determine the main zeta invariants for a sequence of spectral type when there is some knowledge on the asymptotic expansion of the associated Fredholm determinant. To deal with the applications that we have in mind we need only a particular kind of sequences of spectral type, namely the regular ones as defined below.

A sequence of spectral type S is called *regular* if the coefficients $a_{j,k}$ in the expansion of the logarithm of the associated Fredholm determinant $\ln F(z, S)$ vanish for all $k \neq 0, 1$. A result similar to Proposition 2.4 for a larger class of sequences of spectral type can be proved by precisely the same means, but is more complicate. In particular, notice that for regular sequences of spectral type, the point $s = 0$ is a regular point for the associated zeta function $\zeta(s, S)$. In the more general situation, a generalization of Proposition 2.4 does not state that $s = 0$ is a regular point, but just gives some coefficients of the Laurant expansion of the zeta function $\zeta(s, S)$ near $s = 0$. In fact, possible terms of the type $\ln^{k+1}(-\lambda)$ with $k > 0$ in the asymptotic expansion of $\ln F(\lambda, S)$, should imply a pole of order k at $s = 0$ for the zeta function. In particular, such terms appear when studying the spectrum of the Laplace operator on manifolds with singularities of conical type [12] [8] [32].

Proposition 2.4. *If S is a regular sequence of spectral type then the associated zeta function $\zeta(s, S)$ is regular at $s = 0$ and*

$$\text{Res}_0(\zeta(s, S), s = 0) = F_1, \quad \text{Res}_0(\zeta'(s, S), s = 0) = F_0,$$

where F_0 and F_1 are the coefficients of the constant term and of the logarithmic term in the asymptotic expansion of the logarithm of the Fredholm determinant associated to S , namely, for large z :

$$\ln F(z, S) \sim \dots + F_1 \ln z + F_0 + \dots$$

Moreover, $\zeta(s, S)$ is regular in the positive complex half-plane $\text{Re}(s) > -\epsilon$, up to a finite set of poles. The poles are at most $J+1$, are located at $s = \alpha_J, \alpha_{J-1}, \dots, \alpha_0 = \alpha$, and are of order at most 2. The residua can be explicitly calculated from the coefficients $a_{j,k}$ in the asymptotic expansion of $\ln F(z, S)$.

Proof. Using the Mellin transform, we obtain the following analytic representation of the zeta function

$$\zeta(s, S) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} [f(t, S) - 1] dt = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta,c}} e^{-\lambda t} R(\lambda, S) d\lambda dt,$$

where the contour and the resolvent function $R(\lambda, S)$ were introduced in the proof of Lemma 2.1. The behavior of $R(\lambda, S)$ for large λ along some ray in the complex plane disjoint from $\Sigma_{\theta,c}$ is at most $o((-\lambda)^p)$. In fact, taking the p -th derivative or $R(\lambda, S)$ with respect to λ , we get a uniformly convergent series of functions of λ (outside $\Sigma_{\theta,c}$), vanishing for large λ . Next, observe that

$$R(\lambda, S) = -\frac{d}{d\lambda} T(\lambda, S),$$

where

$$T(\lambda, S) = -\ln \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{a_n}\right) \exp\left(\sum_{j=1}^p \frac{(-1)^j (-\lambda)^j}{j (a_n)^j}\right) + T_0.$$

Actually, since we have seen that the possible presence of any further term that is a polynomial in $-\lambda$ in $T(\lambda, S)$ does not affect the analytic representation of the zeta function, we can take the more general possible form for $T(\lambda, S)$:

$$T(\lambda, S) = T(\lambda, S) + P_k(\lambda),$$

where $P_k(\lambda)$ is a polynomial of degree k in $-\lambda$ with $P_k(0) = T_0$. The introduction of this more general form for T is due to the fact that, in applications, we find often that T can be expressed in terms of some known special functions. Since $e^{-T(\lambda, S)}$ is an integral function of $-\lambda$ of finite order α with zeros at a_n , that does not vanish at the origin, by the Hadamard factorization theorem

$$e^{-T(\lambda, S)} = e^{Q_p(-\lambda)} F(-\lambda, S),$$

where Q_p is a polynomial of degree p . Also, due to the known behavior of $R(\lambda, S)$, we have that for large λ outside $\Sigma_{\theta, c}$, $T(\lambda, S) = o((-\lambda)^{p+1})$. In particular, $T(\lambda, S)$ has an asymptotic expansion since S is of spectral type, and let

$$\begin{aligned} T(\lambda, S) &= -\ln F(-\lambda, S) - Q_p(\lambda) = \sum_{k=0,1}^J \sum_{j=0}^k a_{j,k} (-\lambda)^{\alpha_j} \ln^k(-\lambda) + o(1) + Q_p(\lambda) = \\ &= \dots + B + A \ln(-\lambda) + \dots \end{aligned}$$

The introduction of the function $T(\lambda, S)$, allows to integrate by parts first in the λ and then in the t integrals, in the analytic representation of the zeta function

$$\zeta(s, S) = \frac{s}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta, c}} \frac{e^{-\lambda t}}{-\lambda} T(\lambda, S) d\lambda dt.$$

The presence of a factor with a zero of second order at $s = 0$, allows to use the standard technique (see for example [14]) employed to calculate the residua also to treat the derivative at $s = 0$. In fact, splitting the t integral at $t = 1$, the integral \int_1^{∞} represents a regular function of s near $s = 0$, and hence gives no contribution to the derivative. We can write

$$\zeta(s, S) = \frac{s}{\Gamma(s)} \int_0^1 t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta, c}} \frac{e^{-\lambda t}}{-\lambda} T(\lambda, S) d\lambda dt + O(s^2).$$

For what concerns the integral \int_0^1 , we can change the variable λ to λ/t , and use the expansion of $T(\lambda, S)$ for large λ , provided we can modify the contour without crossing the pole at $\lambda = 0$. A way to do that, is to split the contour as $\Lambda_{\theta, c} = \Lambda_{\theta, -c} - C_c$, where C_c is a circle of ray c and center in the origin. Thus

$$\zeta(s, S) = \frac{1}{\Gamma(s)} \left[\dots - \frac{A}{s} + \gamma A - B + T(0, S) + \dots \right] + O(s^2),$$

$$\operatorname{Res}_0(\zeta(s, S), s = 0) = -A, \quad \operatorname{Res}_0(\zeta'(s, S), s = 0) = -B + T(0, S),$$

and it is clear that $-A = F_1$ and $-B + T(0, S) = F_0$. The same method applies to prove the second part of the proposition, using the complete expansion given above and recalling that

$$\int_0^1 t^x \ln t dt = -\frac{1}{(x+1)^2}.$$

□

Proposition 2.4 suggests to call a regular sequence of spectral type *simple* if the associated zeta function has only simple poles. Here are conditions for this.

Lemma 2.5. *A regular sequence S of spectral type is simple if one of the following conditions is satisfied:*

1. *the unique logarithmic terms appearing in the expansion of $\ln F(z, S)$ are of the form $\ln(-\lambda)$ or $(-\lambda) \ln(-\lambda)$;*
2. *there are no logarithmic terms in the expansion of the heat function $f(t, S)$.*

2.3. Non homogeneous sequences

In this section we analyze the problem of the contribution to the zeta invariants of the addition of a constant term to the general term of the sequence S . Namely, we consider a shift of the sequence S to the sequence $S_a = \{a_n + a\}_{n=1}^{\infty}$, with real $a \geq 0$, and $S_0 = S$, and consequently the non homogeneous zeta function

$$\zeta(s, S_a) = \sum_{n=1}^{\infty} (a_n + a)^{-s}.$$

We assume the sequences are all regular and simples. It turns out that a general form of the Lerch formula still holds (see also [35]), and this can be proved as follows. First, expand the power of the binomial in the definition of the zeta function

$$\zeta(s, S_a) = \sum_{j=0}^{\infty} \binom{-s}{j} \zeta(s + j, S_0) a^j.$$

Now, if we explicit the first p terms, we get

$$\zeta(s, S_a) = \sum_{j=0}^p \binom{-s}{j} \zeta(s + j, S_0) a^j + \sum_{j=p+1}^{\infty} \binom{-s}{j} \zeta(s + j, S_0) a^j.$$

We can calculate the s -derivative of the first term at $s = 0$ using the expansion

$$\binom{-s}{j} \zeta(s + j, S_0) = \frac{(-1)^j}{j} \{R_1(j) + [R_0(j) + (\gamma + \psi(j))R_1(j)] s\} + O(s^2),$$

where $R_k(j) = \operatorname{Res}_k(\zeta(s, S_0), s = j)$, for $j > 0$. For the second term, notice that

$$\left. \frac{d}{ds} \binom{-s}{j} \right|_{s=0} = \frac{(-1)^j}{j},$$

and hence taking the s -derivative at $s = 0$ we get

$$\sum_{j=p+1}^{\infty} \frac{(-1)^j}{j} \zeta(j, S_0) a^j = -\ln \prod_{n=1}^{\infty} \left(1 + \frac{a}{a_n}\right) e^{\sum_{j=1}^p \frac{(-1)^j}{j} \frac{a^j}{a_n^j}}.$$

We have then proved the following facts:

Lemma 2.6. For $|a| < |a_1|$,

$$\ln F(a, S_0) = \sum_{j=p+1}^{\infty} \frac{(-1)^{j+1}}{j} \zeta(j, S_0) a^j.$$

Proposition 2.7. (Lerch formula) For all a ,

$$\zeta(0, S_a) = \zeta(0, S_0) + \sum_{j=1}^p \frac{(-1)^j}{j} R_1(j) a^j,$$

$$\zeta'(0, S_a) = \zeta'(0, S_0) + \sum_{j=1}^p \frac{(-1)^j}{j} [R_0(j) + (\gamma + \psi(j)) R_1(j)] a^j - \ln F(a, S_0).$$

This gives a relation between the functional determinant and the Fredholm determinant [35] [29]. Notice that, when considering the zeta function associated to the Laplace operator Δ_M , the residua at the poles are expressible in terms of the coefficients of the heat kernel expansion, namely

$$R_1(j) = \frac{e_{m-2j}}{\Gamma(j)}.$$

We will apply the formula in Proposition 2.7 to some interesting situations in the next section. As a first example, we show here how it can be used to prove the following factorization lemma of Choi and Quine [13].

Corollary 2.8. Let $S = \{a_n\}_{n=1}^{\infty}$ be a regular simple sequence of spectral type with finite exponent α and genus p . Let $L = \{a_n^2\}_{n=1}^{\infty}$, and a any real number. Then,

$$\zeta(0, L_{a^2}) = \frac{1}{2} [\zeta(0, S_{ia}) + \zeta(0, S_{-ia})],$$

$$\zeta'(0, L_{a^2}) = \zeta'(0, S_{ia}) + \zeta'(0, S_{-ia}) - \sum_{j=1}^{\lfloor \frac{p}{2} \rfloor} \frac{(-1)^j}{j} \sum_{k=1}^j \frac{1}{2k-1} \text{Res}_1(\zeta(s, S_0), s=2j) a^{2j}.$$

Proof. It is clear that L_a has exponent $\frac{\alpha}{2}$ and genus $\lfloor \frac{p}{2} \rfloor$. Since $\zeta(s, L_0) = \zeta(2s, S_0)$,

$$\text{Res}_k(\zeta(s, L_0), s=j) = 2^{-k} \text{Res}_k(\zeta(s, S_0), s=2j),$$

and the thesis follows now from Proposition 2.7, using the relation

$$\psi(2k) = \frac{1}{2} \psi(k) - \frac{\gamma}{2} + \sum_{j=1}^k \frac{1}{2j-1}.$$

□

3. Applications and examples

In this section, we apply the devices provided in Section 2 to different examples. In each case, we put in evidence the relative Lerch formula. In particular, in 3.1 we review the classical case of the circle, while in 3.3 we deal with the two sphere.

3.1. The zeta function on the circle

The aim of this section is to show how the method outlined in Section 2 works on the simplest case: this is the one dimensional case with $S = \{a_n = n\}$ and $L = \{a_n = n^2\}$. The first, corresponding to the positive part of the operator $i \frac{d}{dx}$ and the second related to the Laplace operator on the circle. In the first case, $\alpha = p = 1$, the zeta function has a unique simple pole at $s = 1$ with residuum 1, and regular part γ ;

$$f(t, S) - 1 = \sum_{n=1}^{\infty} e^{-nt} = \frac{1}{e^t - 1} = \frac{1}{t} + \sum_{j=1}^{\infty} \frac{1}{j!} B_j t^{j-1},$$

where B_j are the Bernoulli numbers (see [15], $B_2 = \frac{1}{6}$),

$$R(\lambda, S) = \gamma + \psi(-\lambda + 1);$$

a possible choice for $T(\lambda, S) = -\ln F(-\lambda, S)$ is $\gamma(-\lambda) + \ln \Gamma(-\lambda + 1)$, so

$$\begin{aligned} T(\lambda, S) &= -\ln F(-\lambda, S) = -\ln \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{n}\right) e^{-\frac{-\lambda}{n}} = \\ &= (-\lambda) \ln(-\lambda) + (\gamma - 1)(-\lambda) + \frac{1}{2} \ln(-\lambda) + \frac{1}{2} \ln 2\pi + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j(2j-1)} (-\lambda)^{1-2j}, \end{aligned}$$

and $T(0, S) = 0$. Applying Proposition 2.4,

$$\zeta(0, S) = -\frac{1}{2}, \quad \zeta'(0, S) = -\frac{1}{2} \ln 2\pi.$$

In the second case, $\alpha = \frac{1}{2}$, $p = 0$, the zeta function has a unique simple pole at $s = \frac{1}{2}$ with residuum $\frac{1}{2}$, and regular part γ ;

$$f(t, L) - 1 = \sum_{n=1}^{\infty} e^{-n^2 t} = \frac{1}{2} \sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{+\infty} e^{-\frac{\pi^2 n^2}{t}} - \frac{1}{2} = \frac{1}{2} \sqrt{\frac{\pi}{t}} - \frac{1}{2} + O(e^{-\frac{1}{t}})$$

by the Poisson summation formula (the Fourier expansion for the theta function),

$$R(\lambda, L) = \frac{1}{2(-\lambda)} - \frac{\pi \operatorname{cth} \pi \sqrt{-\lambda}}{2 \sqrt{-\lambda}},$$

$$T(\lambda, L) = -\ln F(-\lambda, L) = -\ln \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{n^2}\right) = -\ln \frac{\operatorname{sh} \pi \sqrt{-\lambda}}{\pi \sqrt{-\lambda}} =$$

$$= -\pi\sqrt{-\lambda} + \frac{1}{2} \ln(-\lambda) + \ln 2\pi + O(e^{-2\pi\sqrt{-\lambda}}),$$

and $T(0, L) = 0$. Applying Proposition 2.4,

$$\zeta_{S^1}(0, 0) = 2\zeta(0, L) = -1, \quad \zeta'_{S^1}(0, 0) = 2\zeta'(0, L_0) = -2 \ln 2\pi,$$

where we introduce the notation $\zeta_{S^n}(s, q)$ for the zeta function associated to the non homogeneous Laplace operator on S^n (thus $\zeta_{S^1}(s, a^2) = 2\zeta(s, L_{a^2})$). Applying Proposition 2.7 we can also deal with the non homogeneous associated problems, namely the zeta functions

$$\zeta(s, S_a) = \sum_{n=1}^{\infty} (n+a)^{-s}, \quad \zeta(s, L_{a^2}) = \sum_{n=1}^{\infty} (n^2 + a^2)^{-s},$$

getting

$$\zeta(0, S_a) = -\frac{1}{2} - a, \quad \zeta(0, L_{a^2}) = -\frac{1}{2},$$

$$\zeta'(0, S_a) = \zeta'(0, S_0) - \gamma a - \ln F(a, S) = \zeta'(0, S_0) + \ln \Gamma(a+1),$$

Lerch formula $\zeta'(0, S_a) = \zeta'(0, S_0) + \ln \Gamma(a+1)$,

that is the classical Lerch formula, and

$$\zeta'(0, L_{a^2}) = \zeta'(0, L_0) - \ln F(a^2, S) = -\ln 2\pi - \ln \frac{\text{sh}\pi a}{\pi a},$$

Lerch formula $\zeta'_{S^1}(0, a^2) = \zeta'_{S^1}(0, 0) - 2 \ln \frac{\text{sh}\pi a}{\pi a}$,

that is a first generalization of such formula. In particular, notice also that

$$\zeta'(0, L_{a^2}) = \zeta'(0, S_{ia}) + \zeta'(0, S_{-ia}),$$

as expected from Corollary 2.8.

3.2. Multiple zeta and Gamma functions

Multiple Gamma functions, and in particular the double Gamma or G function, were introduced long time ago by Barnes as natural generalizations of the Euler Gamma function [3], and multiple zeta functions appeared in the same context as generalizations of the Riemann zeta function. Consequently, different approaches and various generalizations appeared with different aims and applications: in [33] multiple Gamma functions are introduced to study the functional determinant on the spheres, while a generalization of the Barnes G function [4] is considered in [29]. We study the two classes of zeta functions associated to these two classes of special functions, and we show how they are related in a very natural algebraic way. We introduce a slightly different definition for the multiple Gamma function, that will turn out to be a more natural generalization of the Euler function.

Let $n = (n_0, \dots, n_m)$ be a positive integer vector in \mathbb{N}_0^{m+1} , where $m = 0, 1, \dots$ is fixed, and a a non negative real constant. We prefer to avoid the zero vectors in the definition since this allows more compact formulas in the results. Then consider

the sequence $S_{1,a}$ where the general term $n+a$, with $n \in \mathbb{N}_0$, appears n^m times and $S_{2,a} = \{n_0 + \dots + n_m + a\}_{n \in \mathbb{N}_0^{m+1}}$. The associated multiple zeta functions are

$$z_m(s, a) = \zeta(s, S_{1,a}) = \sum_{n=1}^{\infty} n^m (n+a)^{-s},$$

$$\zeta_m(s, a) = \zeta(s, S_{2,a}) = \sum_{n \in \mathbb{N}_0^{m+1}} (n_0 + n_1 + \dots + n_m + a)^{-s},$$

for $\operatorname{Re}(s) > m+1$, while the associated special functions are⁵

$$\begin{aligned} G_m(z+1) &= e^{\frac{(-1)^m}{m+1}(2\gamma+\psi(m+1))z^{m+1}-\sum_{j=1}^m \frac{(-1)^j}{j} \zeta_R(j-m)z^j + \sum_{j=1}^m (-1)^j \binom{m}{j} \zeta'_R(j-m)z^j} \times \\ &\quad \times \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{n^m} e^{n^m \sum_{j=1}^{m+1} \frac{(-1)^j}{j} \frac{z^j}{n^j}}, \\ \frac{1}{\Gamma_m(z+1)} &= e^{\frac{(-1)^m}{m!} \sum_{j=0}^m [(2\gamma+\psi(j+1))s_{m+1,j+1} + \sum_{k=0, k \neq j}^m (-1)^{k+j} s_{m+1,k+1} \zeta_R(j+1-k)] \frac{z^{j+1}}{j+1}} \\ &\quad \times \prod_{n \in \mathbb{N}_0^{m+1}} \left(1 + \frac{z}{n_0 + \dots + n_m}\right) e^{\sum_{j=1}^{m+1} \frac{(-1)^j}{j} \frac{z^j}{(n_0 + \dots + n_m)^j}}, \end{aligned}$$

where the constants $s_{j,k}$ are the Stirling numbers; their appearance will be clarified in the following. Some remarks on these definitions are in order. The definition of the function G_m is the one given by Shuster in [29], and it is in fact the natural generalization of the definition given by Barnes for the G function [4]. Actually, it is the most natural generalization of the G function at least for the following reasons: it reduces to the Barnes G function for $m=1$, it is normalized by $G_m(1) = 1$, and it satisfies a functional equation similar to the one characterizing the G function. The definition of the function Γ_m is essentially the one given by Barnes, and then the one considered by Vardi [33]. However, here we prefer to stress the relation with the multi linear zeta function ζ_m , and therefore we choose normalization in order to give a nicer Lerch formula. Our definition also allows to introduce further multiple special functions related to the quadratic version of the multiple zeta functions z_m and ζ_m , as in Corollary 2.8. We will give the main properties of all these new special functions at the end of this section.

We first introduce a natural algebraic relationship between the two sets of multiple zeta functions, that shows their common nature. Recall the rising factorial [30] $R(x, k) = x(x+1)(x+2) \dots (x+k-1)$, is a polynomial in x of degree $k > 0$

$$R(x, k) = \prod_{j=0}^{k-1} (x+j) = \sum_{j=0}^k (-1)^{k+j} s_{k,j} x^j,$$

whose coefficients $s_{k,j}$ are the Stirling numbers ($s_{k,0} = 0, \forall k > 0$). Now,

$$\zeta_m(s, a) = \sum_{n=1}^{\infty} \binom{n+m}{m} (n+a)^{-s},$$

⁵ $G_0(z+1) = \frac{1}{\Gamma(z+1)}$.

and

$$\begin{aligned} \binom{x+m}{m} &= \frac{1}{m!} \prod_{j=0}^m (x+j) = \frac{1}{m!} \frac{1}{x} R(x, m+1) = \\ &= \frac{1}{m!} \sum_{j=0}^m (-1)^{m+j} s_{m+1, j+1} x^j = \frac{1}{m!} B_m(x), \end{aligned}$$

the last equivalence defining the polynomial $B_m(x)$ of degree m . In particular

$$B_0(x) = 1, \quad B_1(x) = x + 1, \quad B_2(x) = x^2 + 3x + 2,$$

and all the $B_m(x)$ have positive integer coefficients. It is also easy to see that the set $\{B_0(x), \dots, B_m(x)\}$ form a basis for the algebra of the polynomial of maximum degree m , $\mathcal{P}_m(x)$, since the matrix A_m to pass to the standard basis $\{1, x, x^2, \dots, x^m\}$ is upper triangular, with all 1s on the diagonal. Thus, $\det A_m = 1$, and hence actually $A_m \in Sl_m(\mathbb{Z})$. This gives the following relation for our zeta functions. Introducing the variables⁶ $z_{s,a} = z_0(s, a)$, $z'_{s,a} = z'_0(s, a)$, $\zeta_{s,a} = \zeta_0(s, a)$ and $\zeta'_{s,a} = \zeta'_0(s, a)$, and the formal powers $z_{s,a}^m = z_m(s, a)$, and similar for the others, we can write

$$m! \zeta_{s,a}^m = B_m(z_{s,a}), \quad m! (\zeta'_{s,a})^m = B_m(z'_{s,a}),$$

and these formulas can be inverted in $Sl_m(\mathbb{Z})$. For example, $\zeta_{s,0}^m = \zeta_m(s, 0) = \frac{1}{m!} B_m(z_{s,0}) = \frac{1}{m!} \sum_{j=0}^m (-1)^{m+j} s_{m+1, j+1} z_{s,0}^j = \frac{1}{m!} \sum_{j=0}^m (-1)^{m+j} s_{m+1, j+1} z_j(s, 0)$.

We now study the zeta functions z_m using Proposition 2.4. For $S_{1,a}$, $\alpha = p = m + 1$, $z_m(s, 0) = \zeta_R(s - m)$ has a unique simple pole at $s = m + 1$ with $R_0 = \gamma$ and $R_1 = 1$. The associated Fredholm determinant is

$$F(z, S_{1,0}) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{n^m} e^{n^m \sum_{j=1}^{m+1} \frac{(-1)^j}{j} \frac{z^j}{n^j}},$$

and we can take

$$T(\lambda, S_{1,0}) = -\ln G_m(-\lambda + 1) = \dots - \frac{(-1)^m}{m+1} B_{m+1} \ln(-\lambda) - \zeta'_R(-m) + \dots,$$

where the expansion for large λ is given by [29]. Propositions 2.4 and 2.7 give

$$\begin{aligned} z_m(0, a) &= \frac{(-1)^{m+1}}{m+1} [a^{m+1} - B_{m+1}], \\ z'_m(0, a) &= \zeta'_R(-m) + \frac{(-1)^{m+1}}{m+1} [2\gamma + \psi(m+1)] a^{m+1} + \\ &+ \sum_{j=1}^m \frac{(-1)^j}{j} \zeta_R(j-m) a^j - \ln F(a, S_{1,0}), \end{aligned}$$

Lerch formula $z'_m(0, a) = \sum_{j=0}^m (-1)^j \binom{m}{j} \zeta'_R(j-m) a^j - \ln G_m(a+1)$.

⁶In particular, $z_0(s, a) = \zeta_H(s, a+1)$

Next, consider the sequence $S_{2,a}$. We have again $\alpha = p = m + 1$, but now the homogeneous zeta function has simple poles at $s = 1, 2, \dots, m + 1$. The residua are

$$\begin{aligned} \text{Res}_0(\zeta_m(s, 0), s = k) &= \frac{(-1)^{m+k+1}}{m!} \gamma s_{m+1,k} + \sum_{j=0, j \neq k-1}^m \frac{(-1)^{m+j}}{m!} s_{m+1,j+1} \zeta_R(k-j), \\ \text{Res}_1(\zeta_m(s, 0), s = k) &= \frac{(-1)^{m+k+1}}{m!} s_{m+1,k}, \end{aligned}$$

for $k = 1, \dots, m + 1$. The Fredholm determinant is

$$F(z, S_{2,0}) = \prod_{n \in \mathbb{N}_0^{m+1}} \left(1 + \frac{z}{n_0 + \dots + n_m} \right) e^{\sum_{j=1}^{m+1} \frac{(-1)^j}{j} \frac{z^j}{(n_0 + \dots + n_m)^j}}.$$

We can calculate $\zeta_m(0, a)$ and $\zeta'_m(0, a)$ using the know values for the z_m functions and the relation introduced above, explicitly

$$\begin{aligned} \zeta_m(0, 0) &= \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j s_{m+1,j+1} \zeta_R(-j), \\ \zeta'_m(0, 0) &= \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j s_{m+1,j+1} \zeta'_R(-j), \end{aligned}$$

Then, using Proposition 2.7

$$\begin{aligned} \zeta_m(0, a) &= \zeta_m(0, 0) - \frac{(-1)^m}{m!} \sum_{j=1}^{m+1} \frac{1}{j} s_{m+1,j} a^j, \\ \zeta'_m(0, a) &= \zeta'_m(0, 0) - \ln F(a, S_{2,0}) + \\ &\quad - \frac{(-1)^m}{m!} \sum_{j=1}^{m+1} \frac{1}{j} \left[(2\gamma + \psi(j)) s_{m+1,j} - \sum_{k=0, k \neq j-1}^m (-1)^{k+j} s_{m+1,k+1} \zeta_R(j-k) \right] a^j, \end{aligned}$$

Lerch formula $\zeta'_m(0, a) = \zeta'_m(0, 0) + \ln \Gamma_m(a + 1)$.

We closed the discussion on these linear multiple zeta functions by giving the functional equation for the homogeneous z_m . This can be deduced from the classical reflection formula for the Riemann zeta function (where $s_m = 2m + 1$)

$$\pi^{\frac{s-s_m}{2}} \Gamma\left(\frac{s_m - s}{2} - \frac{m}{2}\right) z_m(s_m - s, 0) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2} - \frac{m}{2}\right) z_m(s, 0).$$

Next step is to consider the related quadratic sequences, $L_{1,a^2} = \{n^m(n^2 + a^2)\}_{n=1}^{\infty}$ and $L_{2,a^2} = \{(n_0 + \dots + n_m)^2 + a^2\}_{n \in \mathbb{N}_0^{m+1}}$. For the first, $\alpha = \frac{m+1}{2}$, $p = p_m = [\alpha]$ (the integer part), and the associated zeta function is

$$Z_m(s, a^2) = \zeta(s, L_{1,a^2}) = \sum_{n=1}^{\infty} n^m (n^2 + a^2)^{-s}.$$

The homogeneous zeta function is $Z_m(s, 0) = z_m(2s, 0) = \zeta_R(2s - m)$, with a simple pole at $s = \frac{m+1}{2}$ with residua $R_0 = \gamma$ and $R_1 = \frac{1}{2}$.

$$Z_m(0, 0) = \zeta_R(-m), \quad Z'_m(0, 0) = 2\zeta'_R(-m),$$

$$F(a^2, L_{1,0}) = \prod_{n=1}^{\infty} \left(1 + \frac{a^2}{n^2}\right)^{n^m} e^{n^m \sum_{j=1}^{p_m} \frac{(-1)^j}{j} \frac{a^{2j}}{n^{2j}}}.$$

We can use both Proposition 2.7 or Corollary 2.8 to compute $Z'_m(0, a^2)$. Now,

$$\begin{aligned} G_m(1+z)G_m(1-z) &= e^{\sum_{j=1}^{p_m-1} [2\binom{m}{2j}\zeta'_R(2j-m) - \frac{1}{j}\zeta_R(2j-m)]z^{2j}} \times \\ &\times e^{-2\frac{p(m)}{m+1}(2\gamma+\psi(m+1))z^{m+1}} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)^{n^m} e^{n^m \sum_{j=1}^{p_m} \frac{1}{j} \frac{z^{2j}}{n^{2j}}}, \end{aligned}$$

(where $p(n) = 0/1$ if n is even/odd) suggests to introduce the function (and its dual Sh_m of purely imaginary argument as usual)

$$\begin{aligned} S_m(\pi z) &= \pi z e^{-2\frac{p(m)}{m+1}(2\gamma+\psi(m+1))z^{m+1}} e^{-\sum_{j=1}^{p_m-1} \frac{1}{j}\zeta_R(2j-m)z^{2j}} \times \\ &\times \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)^{n^m} e^{n^m \sum_{j=1}^{p_m} \frac{1}{j} \frac{z^{2j}}{n^{2j}}}, \end{aligned}$$

such that

$$G_m(1+z)G_m(1-z) = e^{2\sum_{j=1}^{p_m-1} \binom{m}{2j}\zeta'_R(2j-m)z^{2j}} \frac{S_m(\pi z)}{\pi z},$$

Lerch formula $Z'_m(0, a^2) = Z'_m(0, 0) - \ln \frac{Sh_m(\pi a)}{\pi a} - 2 \frac{(-1)^{p_m} p(m)}{m+1} \sum_{k=1}^{p_m} \frac{a^{m+1}}{2k-1}$.

The functions S_m and Sh_m can be thought as multiple sine and hyperbolic sine functions, and in fact they share with the classical sine and hyperbolic sine functions some important properties as we will show at the end of this section (see also [19]). Before, we consider the last sequence, namely L_{2,a^2} . In this case, $\alpha = \frac{m+1}{2}$, $p = p_m = [\alpha]$, and the associated zeta function is

$$\mathcal{Z}_m(s, a^2) = \zeta(s, L_{2,a^2}) = \sum_{n \in \mathbb{N}_0^{m+1}} [(n_0 + \dots + n_m)^2 + a^2]^{-s};$$

the function $\mathcal{Z}_m(s, 0) = \zeta_m(2s, 0)$ has simple poles at $s = \frac{1}{2}, 1, \frac{3}{2}, \dots, \frac{m+1}{2}$ with

$$\text{Res}_0(\mathcal{Z}_m(s, 0), s = k) = \text{Res}_0(\zeta_m(s, 0), s = 2k),$$

$$\text{Res}_1(\mathcal{Z}_m(s, 0), s = k) = \frac{1}{2} \text{Res}_1(\zeta_m(s, 0), s = 2k),$$

and

$$F(a^2, L_{2,0}) = \prod_{n \in \mathbb{N}_0^{m+1}} \left(1 + \frac{a^2}{(n_0 + \dots + n_m)^2}\right) e^{\sum_{j=1}^{p_m} \frac{(-1)^j}{j} \frac{a^{2j}}{(n_0 + \dots + n_m)^{2j}}}.$$

As before, we can introduce the functions \mathcal{S}_m and $\mathcal{S}h_m$ by

$$\begin{aligned} \mathcal{S}_m(\pi z) &= \\ &= \pi z e^{\frac{(-1)^m}{m!} \sum_{j=1}^{p_m} \frac{1}{j} [(2\gamma + \psi(2j) - \sum_{k=1}^j \frac{1}{2k-1}) s_{m+1,2j} + \sum_{k=0, k \neq 2j-1}^m s_{m+1,k+1} \zeta_R(2j-k)]} z^{2j} \times \\ &\quad \times \prod_{n \in \mathbb{N}_0^{m+1}} \left(1 - \frac{z^2}{(n_0 + \dots + n_m)^2} \right) e^{\sum_{j=1}^{p_m} \frac{1}{j} \frac{z^{2j}}{(n_0 + \dots + n_m)^{2j}}}, \end{aligned}$$

so that

$$\frac{1}{\Gamma_m(1+z)\Gamma_m(1-z)} = e^{\frac{(-1)^m}{m!} \sum_{j=1}^{p_m} \frac{1}{j} \sum_{k=1}^j \frac{1}{2k-1} s_{m+1,2j} z^{2j}} \frac{\mathcal{S}_m(\pi z)}{\pi z},$$

and **Lerch formula** $\mathcal{Z}'_m(0, a^2) = \mathcal{Z}'_m(0, 0) - \ln \frac{\mathcal{S}h_m(\pi a)}{\pi a}$

that should be compared with the one obtained for the circle in 3.1.

We conclude this section by giving some properties of the multiple special functions. Let's start with the multiple Gamma function Γ_m . We get from the definition that $\Gamma_0(z) = \Gamma(z)$ and $\Gamma_m(1) = 1$ for all m . Next, since

$$\binom{m+n+1}{m+1} - \binom{m+n}{m} = \binom{m+n}{m+1},$$

we find that

$$\zeta_{m+1}(s, a+1) = \zeta_{m+1}(s, a) - \zeta_m(s, a) - (a+1)^{-s}.$$

This can be used to prove the following functional equation for the Γ_m :

Proposition 3.1. (Functional equation)

$$\Gamma_{m+1}(z+1) = e^{-\zeta'_m(0,0)} \frac{z \Gamma_{m+1}(z)}{\Gamma_m(z)}.$$

Thanks to these properties, we find that the definition of the multiple Gamma function Γ_m actually coincides with the one given by Vardi in [33] up to a constant factor (depending on m), that is to say it is the unique function satisfying the above properties⁷ and some regularity assumption as shown by Vignères in [34] (see also [2] for the case $m = 0$). With our normalization, not only the Lerch formula, but also the following duplication formula take a nicer form.

Proposition 3.2. (Duplication formula)

$$\Gamma_m(2z) = 2^{-\zeta_m(0, 2z-1)} e^{\zeta'_m(0,0)} \Gamma_m(z) \Gamma_m\left(z + \frac{1}{2}\right).$$

⁷Here there is also the extra factor z due to the fact that we have excluded the zero modes in our sum.

Proof. By definition

$$\begin{aligned} \zeta_m(s, x) + \zeta_m\left(s, x - \frac{1}{2}\right) &= 2^s \sum_{n \in \mathbb{N}_0^{m+1}} (2n_0 + \cdots + 2n_m + 2x)^{-s} + \\ &+ 2^s \sum_{n \in \mathbb{N}_0^{m+1}} (2n_0 + \cdots + 2n_m + 2x - 1)^{-s} = 2^s \zeta_m(s, 2x). \end{aligned}$$

Then,

$$\zeta'_m(0, x) + \zeta'_m\left(0, x - \frac{1}{2}\right) = \zeta_m(0, 2x) \ln 2 + \zeta'_m(0, 2x),$$

and we can use the Lerch formula with $2x + 1 = 2z$ to get the thesis. \square

The Taylor expansion of $\ln \Gamma_m(z + 1)$ for small z can be immediately obtained from Lemma 2.6 and the definition, while the asymptotic expansion can be obtained from the one given by [29] for the G_m function, and using the fact that

$$\begin{aligned} -\ln \Gamma_m(z + 1) &= \\ &= \frac{(-1)^m}{m!} \sum_{j=0}^m (-1)^j s_{m+1, j+1} \left[\ln G_j(z + 1) - \sum_{k=1}^j (-1)^k \binom{j}{k} \zeta'_R(k - j) z^k \right]. \end{aligned}$$

Proposition 3.3. For large z ,

$$\begin{aligned} \ln \Gamma_m(z + 1) &= \\ &= \frac{(-1)^{m+1}}{m!} \left[- \sum_{j=0}^m \frac{s_{m+1, j+1}}{j+1} z^{j+1} \ln z + \sum_{j=0}^m \frac{s_{m+1, j+1}}{j+1} (\gamma + \psi(j+2)) z^{j+1} + \right. \\ &\quad \left. + \sum_{j=0}^m (-1)^j s_{m+1, j+1} \zeta'_R(-j) + \sum_{j=0}^m \frac{1}{j+1} s_{m+1, j+1} B_{j+1} \ln z \right] + O(z^{-1}). \end{aligned}$$

As a final remark, a particular value of the multiple gamma function. Applying Proposition 3.2 with $z = \frac{1}{2}$ and using the values calculated for the zeta functions

$$\begin{aligned} \Gamma_m\left(\frac{1}{2}\right) &= 2^{\zeta_m(0,0)} e^{-\zeta'_m(0,0)} = \\ &= 2^{\frac{(-1)^m}{m!} \sum_{j=0}^m s_{m+1, j+1} \zeta_R(-j)} e^{-\frac{(-1)^m}{m!} \sum_{j=0}^m s_{m+1, j+1} \zeta'_R(-j)}. \end{aligned}$$

This is a remarkable formula, since it express the value of the Γ_m at $\frac{1}{2}$ as a function of the dimension m ; it seems the most natural generalization of the classical

$$\Gamma\left(\frac{1}{2}\right) = 2^{\zeta_R(0)} e^{-\zeta'_R(0)} = \sqrt{\pi},$$

and then gives an other answer to the question considered in [33].

Eventually, we give some properties of the multiple sine functions. It is clear that the two sets of functions (S and Sh versus \mathcal{S} and $\mathcal{S}h$) are related by the same algebraic relation introduced above for the zeta functions; thus it is enough to study one set of functions. It is easier to consider the functions $S_m(x)$, that are more explicitly related to the Barnes functions. Using the known expansions for the G_m functions, we get:

Proposition 3.4. *For large real x ,*

$$\begin{aligned} \ln Sh_m(\pi x) &= \\ &= 2 \frac{(-1)^{p_m} p(m)}{m+1} x^{m+1} \ln x + \frac{(-1)^{p_m}}{m+1} [2p(m)(\gamma + \psi(m+2)) + \pi p(m+1)] x^{m+1} + \\ &\quad + \ln \pi x - \frac{2B_{m+1}}{m+1} \ln x + 2\zeta'_R(-m) + O(x^{-1}). \end{aligned}$$

As for the function Γ_m , the Taylor expansion for $\ln Sh_m$ can be obtained from Lemma 2.3. We prefer to conclude by considering explicitly the case $m = 1$. In such a case we introduce the dual cosine double function that will turn out to be useful in the next section. With $m = 1$

$$\begin{aligned} S(z) = S_1(z) &= z e^{-(\gamma+1)\frac{z^2}{\pi^2}} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)^n e^{\frac{z^2}{\pi^2 n}}, \\ C(z) = C_1(z) &= e^{-(\gamma+1)\frac{z^2}{\pi^2}} \prod_{n=1}^{\infty} \left(1 - \frac{4z^2}{\pi^2(2n-1)^2}\right)^n e^{\frac{z^2}{\pi^2 n}}, \end{aligned}$$

and the following relations hold

$$\begin{aligned} S\left(\frac{\pi}{2} \pm z\right) &= \pi \Gamma\left(\frac{3}{2} \pm \frac{z}{\pi}\right) \left[G\left(\frac{1}{2}\right)\right]^2 C(z), \\ G(1+z)G(1-z) &= \frac{S(\pi z)}{\pi z}, \quad G\left(\frac{1}{2}-z\right)G\left(\frac{1}{2}+z\right) = \left[G\left(\frac{1}{2}\right)\right]^2 C(\pi z). \end{aligned}$$

3.3. The zeta function on the two spheres

In this section we consider the sequence $S = \{(2n+1)[n(n+1)]^{-s}\}_{n=1}^{\infty}$, that is to say the spectrum of the standard Laplace operator on the two sphere of unit ray. Now, $\alpha = p = 1$ and the associated zeta function is

$$\zeta_{S^2}(s, 0) = \zeta(s, S) = \sum_{n=1}^{\infty} (2n+1)[n(n+1)]^{-s},$$

for $\text{Re}(s) > 1$. The non homogeneous zeta function $\zeta_{S^2}(s, a^2) = \zeta(s, S_{a^2})$ is

$$\zeta_{S^2}(s, a^2) = \sum_{n=1}^{\infty} (2n+1)[n(n+1) + a^2]^{-s} = \sum_{n=1}^{\infty} (2n+1) \left[\left(n + \frac{1}{2}\right)^2 + q \right]^{-s},$$

where $q = a^2 - \frac{1}{4}$. This suggests to consider instead the function [35]

$$\zeta_b(s) = \sum_{n=1}^{\infty} (2n+b) \left(n + \frac{b}{2}\right)^{-2s} = 2 \sum_{n=1}^{\infty} \left(n + \frac{b}{2}\right)^{-2s+1},$$

with $b = 1$ as the (non homogeneous) zeta function associated to the problem. In fact, $\zeta_1(s)$ factors through the Riemann zeta function

$$\zeta_1(s) = 2^{2s} \sum_{n=1}^{\infty} (2n+1)^{-2s+1} = (2^{2s} - 2)\zeta_R(2s-1) - 2^{2s},$$

and in this way we get the zeta invariants for $\zeta_1(s) = \zeta_{S^2}(s, \frac{1}{4})$ and using Proposition 2.7 those for $\zeta_{S^2}(s, 0)$. In particular, this means that $\zeta_{S^2}(s, 0)$ has only one simple pole at $s = 1$ (see also [30]) with

$$\text{Res}_1(\zeta_{S^2}(s, 0), s = 1) = 1, \quad \text{Res}_0(\zeta_{S^2}(s, 0), s = 1) = 2\gamma.$$

Despite that, the function $\zeta_1(s)$ seems completely unrelated with the geometry, as can be seen by considering that $\zeta_1(0) = -\frac{11}{12}$, while the value at $s = 0$ of the zeta function associated to the Laplace operator over a closed surface is

$$\zeta_M(0) = \frac{1}{6}\chi(M) - \dim \ker \Delta_M,$$

that gives $-\frac{2}{3}$ for $M = S^2$, where $\chi(M)$ is the Euler characteristic of M . Because of this reason, we investigate directly the zeta function $\zeta_{S^2}(s, 0)$, naturally associated to the geometry of the problem. First, consider the heat function

$$f_{S^2}(t) = f(t, S) = \sum_{n=1}^{\infty} (2n+1)e^{-n(n+1)t} + 1.$$

We give here a simple method to obtain the full asymptotic expansion of $f_{S^2}(t)$ for small t using special functions. This result was originally obtained by Mulholland [23] by direct calculation, and is indeed a very useful result since it allows to compute asymptotic expansions for the heat kernel not just for all spheres, but for all symmetric spaces of rank one [10]. Consider the function

$$f_b(t) = e^{\frac{b^2}{4}t} \sum_{n=1}^{\infty} (2n+b)e^{-(n+\frac{b}{2})^2 t} = -\frac{2}{t} e^{\frac{b^2}{4}t} \frac{d}{db} \sum_{n=1}^{\infty} e^{-(n+\frac{b}{2})^2 t},$$

where b is a real parameter. Then, $f_{S^2}(t) = f_1(t) + 1$, and we get the asymptotic of $f_{S^2}(t)$, providing the one for $f_b(t)$. After some computation, we can write

$$\begin{aligned} f_b(t) &= \\ &= \frac{e^{\frac{b^2}{4}t}}{2\pi i} \int_{\Lambda_{c,\theta}} e^{-\lambda t} \left[\psi \left(1 + \frac{b}{2} + i\sqrt{-\lambda} \right) + \psi \left(1 + \frac{b}{2} - i\sqrt{-\lambda} \right) - 2\psi \left(1 + \frac{b}{2} \right) \right] d\lambda. \end{aligned}$$

Changing the variable to λ/t and expanding the factors by using the know expansion for the digamma function, we get

$$f_{S^2}(t) = f_1(t) + 1 = \frac{1}{t} + \sum_{j=0}^{\infty} a_j t^j = \frac{1}{t} + \frac{1}{3} + \frac{t}{15} + \dots,$$

$$a_j = \frac{1}{2^{2j+2}} \left[\frac{1}{(j+1)!} + \sum_{k=0}^j \frac{(-1)^k (2^{2k+2} - 2)}{(k+1)!(j-k)!} B_{2k+2} \right].$$

Next, we give formulas for the other spectral functions on the sphere in terms of special functions. First, considering the expression $\psi(c) - \psi(a+b) + \psi(d) - \psi(a-b)$, we obtain

$$R(\lambda, S) = \psi\left(\frac{3}{2} + z\right) + \psi\left(\frac{3}{2} - z\right) - \psi(2) - \psi(1),$$

where $z = \sqrt{\lambda + \frac{1}{4}}$. The important point is that this expression can be integrated in term of the Barnes G function. After some manipulation, we get

$$\begin{aligned} T(\lambda, S) &= -\ln F(-\lambda, S) = -\ln \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{n(n+1)}\right)^{2n+1} e^{\frac{(2n+1)\lambda}{n(n+1)}} = \\ &= -2 \ln G\left(\frac{1}{2} + z\right) G\left(\frac{1}{2} - z\right) - \ln \Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) + (2\gamma+1)(-\lambda) + \ln(-\lambda) = \end{aligned}$$

and this can be written using the multiple cosine function introduced in Section 3.2

$$\begin{aligned} &= -2 \ln C(\pi z) + \ln \frac{\cos \pi z}{\pi} + (2\gamma+1)(-\lambda) + \ln(-\lambda) - 4 \ln G\left(\frac{1}{2}\right) = \\ &= \dots + \left(B_1 + \frac{1}{2}\right) \ln(-\lambda) + \frac{1}{2} - 4\zeta'_R(-1) + \dots \end{aligned}$$

This gives

$$\zeta_{S^2}(0) = -\frac{2}{3}, \quad \zeta'_{S^2}(0) = 4\zeta'_R(-1) - \frac{1}{2}.$$

The non homogeneous case $\zeta_{S^2}(s, a^2)$ follows from Proposition 2.7. We get (compare with [35])

$$\zeta_{S^2}(0, a^2) = -\frac{2}{3} - a^2,$$

$$\begin{aligned} \zeta'_{S^2}(0, a^2) &= \zeta'_{S^2}(0, 0) - 2\gamma a^2 - \ln F(a^2, S) = 4\zeta'_R(-1) - \frac{1}{2} + a^2 + \ln a^2 + \\ &- 2 \ln G\left(\frac{1}{2} + \sqrt{\frac{1}{4} - a^2}\right) G\left(\frac{1}{2} - \sqrt{\frac{1}{4} - a^2}\right) + \\ &- \ln \Gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} - a^2}\right) \Gamma\left(\frac{1}{2} - \sqrt{\frac{1}{4} - a^2}\right), \end{aligned}$$

Lerch formula $\zeta'_{S^2}(0, a^2) = \zeta'_{S^2}(0, 0) - 4 \ln G\left(\frac{1}{2}\right) + a^2 + \ln a^2 +$

$$-2 \ln C \left(\pi \sqrt{\frac{1}{4} - a^2} \right) + \ln \frac{\cos \pi \sqrt{\frac{1}{4} - a^2}}{\pi}.$$

This is a further generalization of the Lerch formula and answer a question of [30] in terms of special functions.

References

- 1 M. Atiyah, R. Bott and V.K. Patodi, *On the Heat Equation and the Index Theorem*, *Inventiones math.* 19 (1973) 279-330;
- 2 E. Artin, *The Gamma function*, (1964) Holt, Rinhehart and Winston;
- 3 E.W. Barnes, *The theory of the multiple Gamma function*, *Trans. Cambridge Phil. Soc.* 19 (1904) 374-425;
- 4 E.W. Barnes, *The theory of the G function*, *Quart. J. Math.* 31 (1899) 264-314;
- 5 16. J. Billingham and A.C. King, *Uniform asymptotic expansions for the Barnes double gamma function*, *Proc. R. Soc. Lond. A* 453 (1997) 1817-1829;
- 6 A.A. Bytsenko, E. Elizalde, S.D. Odintsov, A. Romeo and S. Zerbini, *Zeta regularization techniques with applications*, World Scientific (1994);
- 7 T. Branson and P. Gilkey, *The asymptotics of the Laplacian on a manifold with boundary*, *Part. Diff. Eq.* 15 (1990) 245-272;
- 8 J. Brüning and R. Seeley, *The resolvent expansion for second order regular singular operators*, *J. Funct. Anal.* 73 (1987) 369-429;
- 9 D. Burghelea, L. Friedlander and T. Kappeler, *On the determinant of elliptic boundary value problems on a line segment*, *Proc. Am. Math. Soc.* (1995) 3027-3028;
- 10 R.S. Cahn and J.A. Wolf, *Zeta functions and their asymptotic expansions for compact symmetric spaces of rank one*, *Comment. Math. Helv.* 51 (1976) 1-21;
- 11 J. Cheeger, *Analytic torsion and the heat equation*, *Ann. Math.* 109 (1979) 259-322;
- 12 J. Cheeger, *Spectral geometry of singular Riemannian spaces*, *J. Diff. Geom.* 18 (1984) 575-657;
- 13 J. Choi and J.R. Quine, *Zeta regularized products and functional determinants on spheres*, *Rocky Mount. Jour. Math.* 26 (1996) 719-729;
- 14 P.B. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, *Studies in Adv. Math.* CRC Press, (1995);
- 15 I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series and products*, Ac. Press (1980);
- 16 S.W. Hawking, *Zeta function regularization of path integrals in curved space time*, *Comm. Math. Phys.* 55 (1977) 139-170;
- 17 L. Hörmander, *The analysis of linear partial differential operators III*, Springer-Verlag (1984);
- 18 M. Kontsevich and S. Vishik, *Geometry of determinants of elliptic operators*, *Functional analysis on the eve of the 21st century*, *Progr. Math.* 131 (1995) 173-197;
- 19 N. Kurokawa, *Multiple sine functions and Selberg zeta functions*, *Proc. Japan Acad. A* 67 (1991) 61-64;
- 20 H. A. Lauwrier, *Asymptotic Analysis*, Mathematisch Centrum (1977);
- 21 M. Lesch, *Determinants of regular singular Sturm-Liouville operators*, *Math. Nachr.* 194 (1998) 139-170;
- 22 R.B. Melrose, *Weyl's conjecture for manifolds with concave boundary*, in *the Geometry of the Laplace operator*, *Proc. of Symp. in Pure Math.* XXXVI AMS (1980) 257-274;
- 23 H.P. Mulholland, *An asymptotic expansion for $\sum_{n=1}^{\infty} (2n+1)e^{(n+\frac{1}{2})^2 t}$* , *Proc. Cambridge Phil. Soc.* 24 (1928) 280-289;
- 24 F.W.J. Olver, *Asymptotics and special functions*, AKP Classic (1997);
- 25 D.B. Ray and I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, *Adv. Math.* 7 (1974) 145-210;

- 26 S. Rosenberg, *The Laplacian on a Riemannian manifold. An introduction to analysis on manifolds*, London Math. Soc. Student Texts. 31. Cambridge University Press. (1997);
- 27 Y. Safarov and D. Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, Transl. of Math. Monographs 155 AMS (1997);
- 28 P. Sarnak, *Determinants of Laplacians*, Comm. Math. Phys. 110 (1987) 113-120;
- 29 R. Shuster, *A generalized Barnes G-function*, Zeit. An. Anw. 11 (1992) 229-236;
- 30 M. Spreafico, *Zeta function and regularized determinant on projective spaces*, Rocky Mount. J. Maths 33 (2003) 1499-1512;
- 31 M. Spreafico, *A generalization of the Euler Gamma function*, Funk. An. Pri. 39 (2005) 87-91;
- 32 M. Spreafico, *On the non homogeneous Bessel zeta function*, Mathematika 51 (2004);
- 33 I. Vardi, *Determinants of Laplacians and multiple Gamma functions*, SIAM J. Math. Anal. 19 (1988) 493-507;
- 34 M.F. Vignéras, *L'équation fonctionnelle de la fonction zéta de Selberg du group modulaire $SL(2, \mathbb{Z})$* , Astérisque, 61 (1979) 235-249;
- 35 A. Voros, *Spectral functions, special functions and the Selberg zeta function*, Comm. Math. Phys. 110 (1987) 439-465;
- 36 N. Wiener, *Tauberian theorems*, Ann. of Math. 33 (1932) 461-472;
- 37 E.T. Whittaker and G.N. Watson, *A course in modern analysis*, Cambridge Univ. Press (1946).