The extra-nice dimensions

Maria Aparecida Soares Ruas
ICMC-USP

I Encontro de Singularidades do Triângulo Mineiro
26-28 de fevereiro de 2018

Introduction

The extra-nice dimensions, with Raul Oset-Sinha & Roberta Wik Atique

- Stable mappings $f : N^n \to P^p$, where N and P are smooth manifolds,

- Stable families of mappings $F : N \times [0, 1] \to P$

Let $C^\infty(N, P) = \{f : N \to P,\ f \in C^\infty\}$ with the Whitney topology.

$\mathcal{A}$-equivalence:

$f \sim_{\mathcal{A}} g$

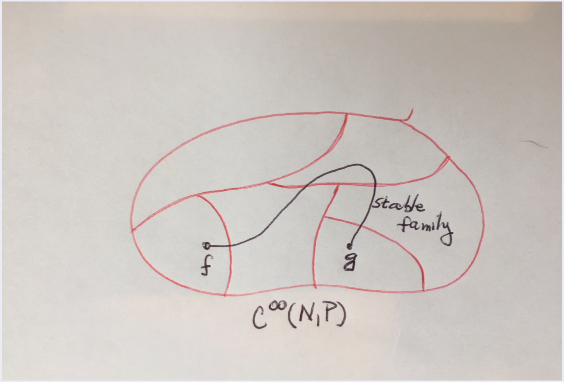
$$\begin{array}{ccc} N & \xrightarrow{f} & P \\ h\downarrow & \circlearrowleft & \downarrow k \\ N & \xrightarrow{g} & P \end{array},$$

h, k C^∞ diffeomorphisms, $g = k \circ f \circ h^{-1}$.

$f \in C^\infty(N, P)$ is stable if there exists a neighbourhood U of f such that for every $g \in U$, it follows that $f \sim_{\mathcal{A}} g$.

A family $F : N \times [0,1] \to P$ is a stable one parameter family if F_t is stable for all $t \in [0,1] \setminus \{t_1, \ldots, t_k\}$ and at each point t_i, the family F is transversal to the orbits in jet space.

E.Chíncaro (n,2), J. Rieger (2,2), Goryunov, Mond and Marar (2,3). Cerf, Igusa- Pseudo-isotopies

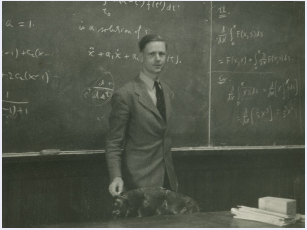


Goal: The set of stable families is a residual set in $C^\infty(N^n \times [0,1], P^p)$ if and only the pair (n,p) is in the extra-nice dimensions.

Hassler Whitney

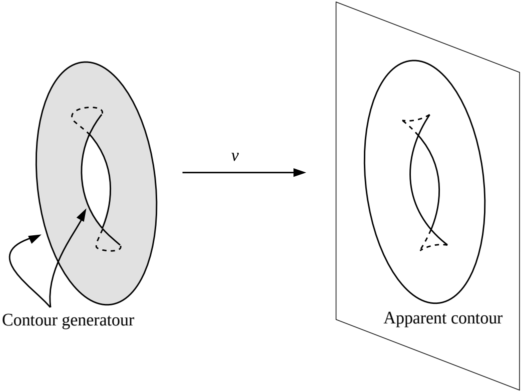
Singularity theory began with the work of Hassler Whitney in the decades of 40 and 50's of century XX.

Hassler Whitney

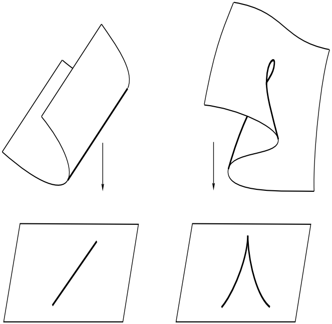


Mappings of the plane into the plane, Whitney 1955

Ann. of Math. (2) 62 (1955), 374-410.



Whitney observed that “generically" only two types of singularities are persistent under deformations: folds and cusps. All the others desintegrate by small perturbations.



Stable singularities

Type 1: **Fold points**

$$(x, y) \longrightarrow (x, y^2)$$

Type 2: **Cusp points**

$$(x, y) \longrightarrow (x, y^3 - xy)$$

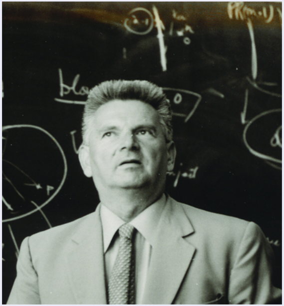
Whitney Theorem

The set of stable mappings is open and dense in $C^{\infty}(N^2, P^2)$

Whitney conjecture

The set of stable mappings is a dense set for all pairs (n, p). ..

René Thom



R. Thom, in his 1959 lectures *Singularities of differentiable mappings, I,* Bonn, 1959, sketched a proof that stable maps are not dense when $(n, p) = 9$, and formulated conjectures on density of C^0-stable maps that led to great developments in singularity theory.

There is a one-parameter family of germs
$f_t : (\mathbb{K}^9, 0) \to (\mathbb{K}^9, 0),\ \mathbb{K} = \mathbb{R}, \mathbb{C}$ corank$f_t = 3$ such that

- $\mathcal{A}_e -$ codimension $f_t = 1$
- f_t are not simple.

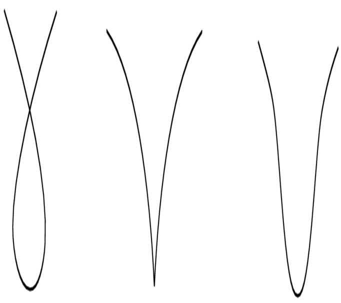
The $\mathcal{A}_e$ codimension is a measure of how degenerate is a singularity.

If f is stable, it follows that (the germ at any point) $\mathcal{A}_e - \text{cod}(f) = 0$.

Definition

$$\mathcal{A}_e - cod(f) = dim_{\mathbb{K}} \frac{\Theta_f}{tf(\Theta_n) + wf(\Theta_p)}$$

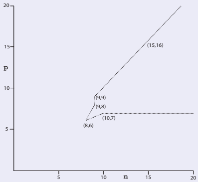
$\mathcal{A}_e - \text{cod}(f) = \#$ parameters of a versal unfolding of f.



A singularity is simple if there is a finite number of orbits in a neighbourhood of f.

John Mather, Stable mappings, 1968 to 1971

Stable mappins are dense $\Leftrightarrow$ (n, p) is in the nice dimensions.



Simplicity

In this work we obtain a refinement of the nice-dimensons, the extra-nice dimensions.

We first discuss the characterization of them in terms of simplicity of germs with low $\mathcal{A}_e$-codimensions. The following condition is clear.

Proposition

(n, p) is in nice-dimensions $\Rightarrow$ all $\mathcal{A}_e$-codimension 1 singularities $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ are simple.

T. Cooper,D. Mond and R. Wik Atique, *Compositio Math.* 131 (2002), no. 2, 121–160.

Are $\mathcal{A}_e$-cod 2 germs in nice-dimensions simple?

Theorem

All corank 1 $\mathcal{A}_e$-codimension 2 germs are simple.

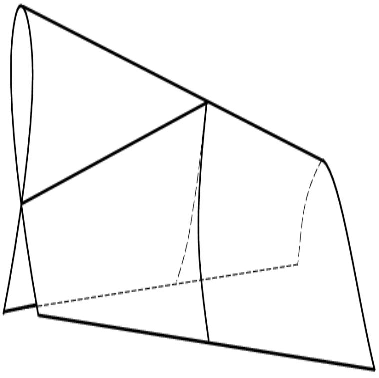
$$\begin{array}{ccc} (\mathbb{K}^{n+s},0) & \stackrel{F}{\longrightarrow} & \Delta \subset (\mathbb{K}^{p+s},0) \\ \uparrow & & i\uparrow \\ (\mathbb{K}^{n},0) & \stackrel{f}{\longrightarrow} & (\mathbb{K}^{p},0) \end{array}$$

When $s = 1$, we say that $i(\mathbb{K}^p)$ is a hyperplane section.

Proposition

If all stable germs $F : (\mathbb{K}^{n+1},0) \longrightarrow (\mathbb{K}^{p+1},0)$ admit a $\mathcal{A}_e$-cod 1 hyperplane section, then all $\mathcal{A}_e$- cod 2 germs $f : (\mathbb{K}^n,0) \longrightarrow (\mathbb{K}^p,0)$ are simple.

the cross-cap



$$F : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)\ F(x, y) = (x, y^2, xy + y^3),$$

A non simple germ of $\mathcal{A}_e$-codimension 2.

Example

The stable corank 2 germ $F : (\mathbb{R}^6, 0) \to (\mathbb{R}^6, 0)$ *given by*

$$F(x, y, u) = (x^3 + y^3 + u_1 x + u_2 y + u_3 x^2 + u_4 y^2, xy, u) =$$

where $u = (u_1, u_2, u_3, u_4)$.

It does not admit a codimension 1 *hyperplane section, but it admits a section of codimension* 2, $U_3 + \lambda U_4 + U_4^2 = 0$.

Then the corank 2 germ $f : (\mathbb{R}^5, 0) \to (\mathbb{R}^5, 0)$ *given by*

$$f(x, y, u) = (x^3 + y^3 + u_1 x + u_2 y + (-\lambda u_4 - u_4^2)x^2 + u_4 y^2, xy, u),$$

with $\lambda \neq 0, -1$, *has* $\mathcal{A}_e$*-codimension 2 and is not simple.*

The extra-nice dimensions

Definition

(n, p) is in the extra-nice dimensions if for large enough l, there exists a Zarisky closed subset $\Lambda \subset J^l(n, p)$, $\mathcal{A}$-invariant, cod $(\Lambda) \geq n + 2$ such that its complement $J^l(n, p) \setminus \Lambda$ is a finite union of $\mathcal{A}$-orbits.

Extra-nice $\Rightarrow$ nice.

Theorem

(n, p) extra-nice $\iff \forall$ stable $F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$ admit a hyperplane section $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ of $\mathcal{A}_e$-codimension 1.

Corollary

(n, p) extra-nice $\Longrightarrow \mathcal{A}_e$-codimension 2 germs are simple.

- How to determine the extra-nice dimensions?
- How to relate the extra-nice dimensions and the density of one-parameter families?

Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $\Delta(f)$ be the discriminant of f if $n \geq p$ or the image of f if $n < p$

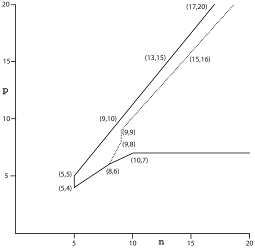
Definition

$Derlog(f) = \{\eta \in \theta_p$, *tangent vector field to* $\Delta(f)\}$

Proposition

$\exists\, \mathcal{A}_e$*-cod 1 hyperplane section* $\iff \exists\, L : \mathbb{K}^p \times \mathbb{K} \to \mathbb{K}$, *such that* $\Delta(f) = L^{-1}(0)$ *and* $\mathcal{M}_{p+1} \subset tL(Derlog(\Delta(f))) + \langle L \rangle$

the boundary of the extra-nice dimensions



Density of stable families

Theorem

The subset of 1-parameter stable families in $C^\infty(N^n \times [0,1], P^p)$, N compact, is dense $\Longleftrightarrow$ (n,p) is in the extra-nice dimensions.

Sketch of proof:
We consider an $\mathcal{A}$-invariant stratification of $J^k(n,p)$:

Let $\Lambda^k(n,p) = \{\sigma \in J^k(n,p) | \mathcal{A}^k - \text{cod}(\sigma) \geq n+2\}$.

- Λ is Zariski closed.
- Λ is $\mathcal{A}^k$-invariant.

When $J^k(n,p) \setminus \Lambda^k(n,p)$ has finite number of $\mathcal{A}^k$-orbits, the stratification in $J^k(n,p)$ induces a stratification $\mathcal{S}(N,P)$ in $J^k(N,P)$.

We can go to global: $F; N \times [0,1] \to P$ is a stable 1-parameter family $\Leftrightarrow J^k_1 F : N \times [0,1] \to J^k(n,p)$ is transversal to the stratification $\mathcal{S}(N,P)$ in $J^k(N,P)$.

The result follows from the Transversality theorem for families of mappings.

Muito obrigada !