LARGE TIME BEHAVIOUR OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The aim of this paper is to show that the spectral theory for linear autonomous and periodic functional differential equations yields explicit formulas for the large time behaviour of solutions. Our results are based on resolvent computations and Dunford calculus and yield insight, new proofs and generalizations of results that have recently appeared in the literature.

1. Introduction

Several aspects of the theory of functional differential equations can be understood as a proper generalization of the theory of ordinary differential equations. However, the fact that the state space for functional differential equations is infinite dimensional requires the development of methods and techniques from functional analysis and operator theory. The application of the theory of semigroups of operators on a Banach space allows one to use methods from dynamical systems in an infinite dimensional context. In particular, the perturbation theory, including a variation-of-constants formula, gives rise to a complete theory of invariant manifolds [1, 4]. The explicit computation of the flow on the unstable or center manifold requires precise information about the underlying unstable or center subspace of the linearized equation. In this paper it is our aim to show how resolvent computations and Dunford calculus yield explicit formulas for the spectral projection on the unstable or center subspace and, in particular, direct insight in the large time behaviour of both autonomous and periodic functional differential equations.

We begin to introduce the precise class of equations studied in this paper. After we have introduced the notion of transposed equation, we present the duality theory between the original and the transposed equation based on a bilinear form first introduced by Hale [4]. In Appendix A, we derive a functional analytic foundation of the Hale bilinear form. We show how to use the bilinear form to explicitly compute spectral projections for simple eigenvalues of the equation. We then continue with the computation of the spectral projections using resolvent computations and Dunford calculus. The advantage of this second approach is that the computations not only become much simpler, but also easily generalize to periodic equations. In Section 4, we present our main results on the large time behaviour of solutions of autonomous equations. In Section 5 we present our main results for periodic equations and in Section 6, we present applications of our main results to autonomous

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and periodic equations. Finally, in Appendix B, we collect some basic properties of characteristic equations needed in the examples.

2. Background on FDE

2.1. **Notation and definitions.** In order to recall the basic spectral theory for linear functional differential equations, we first need some preparations. Let $\mathcal{C} = \mathcal{C}\big([-r,0],\mathbb{C}^n\big)$ denote the Banach space of continuous functions endowed with the supremum norm. From the Riesz representation theorem it follows that every bounded linear mapping $L:\mathcal{C}\to\mathbb{C}^n$ can be represented by

$$L\varphi = \int_{-r}^{0} d\eta(\theta)\varphi(\theta)$$

where $\eta(\theta)$, $-r\leqslant\theta\leqslant 0$, is an $n\times n$-matrix whose elements are of bounded variation, normalized so that η is continuous from the left on $(-r,0)$ and $\eta(0)=0$. The same approach yields a representation for the dual space of $\mathcal{C}$. Every bounded linear functional $f:\mathcal{C}\to\mathbb{C}$ can be written as

$$f(\varphi) = \int_{-r}^{0} d\psi(\theta)\varphi(\theta) \stackrel{\text{def}}{=} \langle\psi,\varphi\rangle,$$

where $\psi(\theta)$, $-r\leqslant\theta\leqslant 0$, is an n-column vector whose elements are complex-valued functions of bounded variation, normalized so that ψ is continuous from the left on $(-r,0)$ and $\psi(0)=0$. This gives a representation for the dual space $\mathcal{C}^*$ as a set of $\mathbb{C}^n$-valued functions of normalized bounded variation.

For a function $x:[-r,\infty)\to\mathbb{C}^n$, we define $x_t\in\mathcal{C}$ by $x_t(\theta)=x(t+\theta)$, $-r\leqslant\theta\leqslant 0$ and $t\geqslant 0$.

An initial value problem for a linear autonomous functional differential equation (FDE) is given by the following relation

$$\begin{cases} \frac{d}{dt}Dx_t = Lx_t, & t\geqslant 0,\\ x_0=\varphi, & \varphi\in\mathcal{C}, \end{cases} \tag{2.1}$$

where $D:\mathcal{C}\to\mathbb{C}^n$ is continuous, linear and atomic at zero, $L:\mathcal{C}\to\mathbb{C}^n$ is linear and continuous and both operators are, respectively, given by

$$L\varphi = \int_{-r}^{0} d\eta(\theta)\varphi(\theta), \qquad D\varphi = \varphi(0) - \int_{-r}^{0} d\mu(\theta)\varphi(\theta), \tag{2.2}$$

where η and μ are $n\times n$ matrix functions of bounded variation, and μ is continuous at zero.

As an example, the following equation

$$\frac{dx}{dt}(t) = Ax(t) + Bx(t-1), \quad t\geqslant 0, \tag{2.3}$$

where A and B are $n\times n$-matrices can be written in the form (2.1) with μ,η given by $\mu\equiv 0$ and $\eta(\theta)=0$ for $\theta\geqslant 0$, $\eta(\theta)=-A$ for $-1<\theta<0$ and $\eta(\theta)=-A-B$ for $\theta\leqslant -1$.

Although it is possible to develop the adjoint theory for functional equations, it turns out to be simpler to introduce the *transposed equation* and study its duality with the original equation.

There are two equivalent ways to deal with the transposed equation. One is to consider the transpose μ^{T} and η^{T} of the matrices μ and η used in (2.1) and (2.2). In this case the transposed equation takes the form

$$\begin{cases} \dfrac{d}{dt}\Big[z(t) - \displaystyle\int_{-r}^{0} d\mu^{\mathrm{T}}(\theta)z(t+\theta)\Big] = \displaystyle\int_{-r}^{0} d\eta^{\mathrm{T}}(\theta)z(t+\theta), & t \geqslant 0, \\ \qquad\qquad z_0 = \varphi, & \varphi \in \mathcal{C}. \end{cases}$$

This form has $\mathcal{C}$ as it state space too. The second approach is slightly more natural and will be considered in this article.

Define $\mathcal{C}' = \mathcal{C}\big([0,r], \mathbb{C}^{n*}\big)$, where $\mathbb{C}^{n*}$ denotes the row n-vectors with complex entries, and consider $y : (-\infty, r] \to \mathbb{C}^{n*}$. For each $s \in [0,\infty)$ let y^s designate the element in $\mathcal{C}'$ defined by $y^s(\xi) = y(-s+\xi)$, $0 \leqslant \xi \leqslant r$. We define operators D', $L' : \mathcal{C}' \to \mathcal{C}'$ by

$$D'\psi = \psi(0) - \int_{-r}^{0} \psi(-\xi)d\mu(\xi), \qquad L'\psi = \int_{-r}^{0} \psi(-\xi)d\eta(\xi), \tag{2.4}$$

where μ and η are the same measures as in the definition of D and L in (2.2). The transpose of (2.1) is defined to be

$$\begin{cases} \frac{d}{dt}\big[D'y(t+\,\cdot\,)\big] = -L'y(t+\,\cdot\,), & t \leqslant 0, \\ \qquad y^0 = \psi, & \psi \in \mathcal{C}'. \end{cases} \tag{2.5}$$

2.2. **Spectral theory for functional differential equations.** It is standard to view (2.1) as an evolutionary system describing the evolution of the state x_t in the Banach space $\mathcal{C}$. In order to do so, we associate with (2.1) a semigroup of solution operators in $\mathcal{C}$. The semigroup is strongly continuous and given by translation along the solution of (2.1)

$$T(t)\varphi = x_t(\,\cdot\,;\varphi),$$

where $x(\,\cdot\,;\varphi)$ denotes the solution of (2.1). See [4] for further details and more information. The infinitesimal generator A of the semigroup $T(t)$ is given by

$$\begin{cases} \mathcal{D}(A) = \big\{\varphi \in \mathcal{C} \mid \frac{d\varphi}{d\theta} \in \mathcal{C},\ D\frac{d\varphi}{d\theta} = L\varphi\big\} \\ \quad A\varphi = \frac{d\varphi}{d\theta}. \end{cases} \tag{2.6}$$

Let $\lambda \in \sigma(A)$ be an eigenvalue of A. The kernel $\mathcal{N}\big(\lambda I - A\big)$ is called the eigenspace at λ and its dimension d_λ, the *geometric multiplicity*. The generalized eigenspace $\mathcal{M}_\lambda$ is the smallest closed subspace that contains all $\mathcal{N}\big((\lambda I - A)^j\big)$, $j = 1, 2, \ldots$ and its dimension m_λ is called the *algebraic multiplicity*. It is known that there is a close connection between the spectral properties of the infinitesimal generator A and the characteristic matrix $\Delta(z)$, associated with (2.1), given by

$$\Delta(z) = z\Big[I - \int_{-r}^{0} e^{zt}d\mu(t)\Big] - \int_{-r}^{0} e^{zt}d\eta(t). \tag{2.7}$$

See [1] and [6]. In particular, the geometric multiplicity d_λ equals the dimension of the null space of $\Delta(z)$ at λ and the algebraic multiplicity m_λ is equal to the multiplicity of $z = \lambda$ as a zero of $\det \Delta(z)$. Furthermore, the generalized eigenspace at λ is given by

$$\mathcal{M}_\lambda = \mathcal{N}\big((\lambda I - A)^{k_\lambda}\big), \tag{2.8}$$

where k_λ is the order of $z = \lambda$ as a pole of $\Delta(z)^{-1}$. Using the matrix of cofactors $\operatorname{adj}\Delta(z)$ of $\Delta(z)$, we have the representation

$$\Delta(z)^{-1} = \frac{1}{\det \Delta(z)} \operatorname{adj}\Delta(z). \tag{2.9}$$

From representation (3.10), we immediately derive that the spectrum of A consists of point spectrum only, and is given by the zero set of an entire function

$$\sigma(A) = \big\{\lambda \in \mathbb{C} \;\big|\; \det \Delta(\lambda) = 0\big\}.$$

The zero set of the function $\det \Delta(\lambda)$ is contained in a left half plane $\{z \mid \operatorname{Re} z < \gamma\}$ in the complex plane. For retarded equations (i.e., $D\varphi = \varphi(0)$), the function $\det \Delta(\lambda)$ has finitely many zeros in strips of the form $S_{\alpha,\beta} = \{z \mid \alpha < \operatorname{Re} z < \beta\}$, where $\alpha, \beta \in \mathbb{R}$. However, in general, for neutral functional differential equations, $\det \Delta(z)$ can have infinitely many zeros in $S_{\alpha,\beta}$.

An eigenvalue λ of A is called *simple* if $m_\lambda = 1$. So simple eigenvalues of A correspond to the simple roots of the characteristic equation

$$\det \Delta(\lambda) = 0.$$

For $k_\lambda = 1$, in particular if λ is simple, it is known that

$$\mathcal{M}_\lambda = \big\{\theta \mapsto e^{\lambda\theta} v \mid v \in \mathcal{N}\big(\Delta(\lambda)\big)\big\}. \tag{2.10}$$

We refer to Chapter 7 of [4]. In [6] and Section IV.3 of [1] a systematic procedure has been developed to construct a canonical basis for $\mathcal{M}_\lambda$ using Jordan chains for generic $\lambda \in \sigma(A)$. For the transposed system (2.5), we have similar notions.

Let $y(\cdot) \in \mathbb{C}^{n*}$ be a solution of Equation (2.5) on the interval $(-\infty, r]$. Similarly as before, we can write (2.5) as an evolutionary system for y^s, $s \geqslant 0$, in the Banach space $\mathcal{C}'$. In order to do so, we associate, by translation along the solution, a $\mathcal{C}_0$-semigroup $T^{\mathrm{T}}(s)$ with Equation (2.5), the *transposed semigroup*, defined by

$$T^{\mathrm{T}}(s)\psi = y^s(\,\cdot\,;\psi), \qquad s \geqslant 0. \tag{2.11}$$

The infinitesimal generator A^{T} associated with $T^{\mathrm{T}}(t)$ is given by (see Lemma 1.4 of Chapter 7 and Lemma 2.3 of Chapter 9 in [4])

$$\begin{cases} \mathcal{D}\big(A^{\mathrm{T}}\big) = \big\{\psi \in \mathcal{C}' \mid \frac{d\psi}{d\xi} \in \mathcal{C}',\ D'\frac{d\psi}{d\xi} = -L'\psi\big\} \\ \quad A^{\mathrm{T}}\psi = -\frac{d\psi}{d\xi}. \end{cases} \tag{2.12}$$

The spectra of A and A^{T} coincide. If we define

$$\mathcal{M}_\lambda(A^{\mathrm{T}}) = \mathcal{N}\big((\lambda I - A^{\mathrm{T}})^{k_\lambda}\big),$$

and if $k_\lambda = 1$, then

$$\mathcal{M}_\lambda(A^{\mathrm{T}}) = \big\{\theta \mapsto e^{-\lambda\theta} v \mid\ 0 \leqslant \theta \leqslant r,\ v \in \mathbb{C}^{n*},\ v\Delta(\lambda) = 0\big\}. \tag{2.13}$$

We denote by Φ_λ the m_λ-vector $\operatorname{row}\{\varphi_1, \ldots, \varphi_{m_\lambda}\}$, where $\varphi_1, \ldots, \varphi_{m_\lambda}$ form a basis of eigenvectors and generalized eigenvectors of A at λ. Let $\psi_1, \ldots, \psi_{m_\lambda}$ be a basis of eigenvectors and generalized eigenvectors of A^{T} at λ. Define the column m_λ-vector Ψ_λ by $\operatorname{col}\{\psi_1, \ldots, \psi_{m_\lambda}\}$ and let $(\Psi_\lambda, \Phi_\lambda) = \big((\psi_i, \varphi_j)\big)$, $i, j = 1, 2, \ldots, m_\lambda$. The matrix $(\Psi_\lambda, \Phi_\lambda)$ is nonsingular and thus can be normalized to be the identity. The decomposition of $\mathcal{C}$ can be written explicitly as

$$\varphi = P_\lambda\varphi + (I - P_\lambda)\varphi,$$

where $P_\lambda \varphi \in \mathcal{M}_\lambda$ and $(I - P_\lambda)\varphi \in \mathcal{Q}_\lambda$ and

$$\begin{aligned} \mathcal{C} &= \mathcal{M}_\lambda \oplus \mathcal{Q}_\lambda \\ \mathcal{M}_\lambda &= \{\varphi \in \mathcal{C} : \varphi = \Phi_\lambda b \text{ for some } m_\lambda\text{-vector } b\}, \\ \mathcal{Q}_\lambda &= \{\varphi \in \mathcal{C} : (\Psi_\lambda, \varphi) = 0\}. \end{aligned}$$

The spaces $\mathcal{M}_\lambda$ and $\mathcal{Q}_\lambda$ are closed subspaces that are invariant under $T(t)$.

We finish this section with exponential estimates on the complementary subspace $\mathcal{Q}_{\lambda_d}$ when λ_d is simple and a *dominant eigenvalue of* A, that is, there exists a $\epsilon > 0$ such that if λ is another eigenvalue of A, then $\operatorname{Re}\lambda < \operatorname{Re}\lambda_d - \epsilon$. The next lemma shows the importance of computing the projections P_λ explicitly.

Lemma 2.1. *Suppose that λ_d is a dominant eigenvalue of A. For $\delta > 0$ sufficiently small there exists a positive constant $K = K(\delta)$ such that*

$$\|T(t)(I - P_\lambda)\varphi\| \leqslant K e^{(\operatorname{Re}\lambda_d - \delta)t} \|\varphi\|, \qquad t \geqslant 0. \tag{2.14}$$

Proof. From the fact that λ_d is dominant, it follows that we can choose $\delta > 0$ sufficiently small such that

$$\sigma(A \mid \mathcal{Q}_{\lambda_d}) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z < \operatorname{Re}\lambda_d - 2\delta\}.$$

Therefore, the lemma follows from the spectral mapping theorem for retarded functional differential equations (see Theorem IV.2.16 of [1]) or from the spectral mapping theorem for neutral equations (see Corollary 9.4.1 of [4]). □

3. Computing the spectral projections

In this section we explicitly compute the spectral projections on the generalized eigenspaces of the infinitesimal generator A associated with equation (2.1) and explicitly given by (2.6). First we recall the standard approach based on duality and the transposed equation. We continue with the computation of the spectral projections using resolvent computations and Dunford calculus. The advantage of the second approach is that the computations not only become much simpler, but also easily generalize to periodic equations which we will pursue in Section 6.

3.1. Spectral projection via duality.

For a simple eigenvalue λ of an operator A, the spectral projection onto the eigenspace $\mathcal{M}_\lambda$ can be explicitly given by

$$P_\lambda \varphi = \langle \varphi_\lambda^*, \varphi \rangle \, \varphi_\lambda,$$

where φ_λ is an eigenvector at λ for A, φ_λ^* is an eigenvector at λ for A^* and $\langle \varphi_\lambda^*, \varphi_\lambda \rangle = 1$ (here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathcal{C}$ and the dual space $\mathcal{C}^*$). Before we can compute the projection explicitly, we need some more definitions in order to avoid the adjoint operator A^* and the duality pairing between $\mathcal{C}$ and its dual space $\mathcal{C}^*$.

It turns out that using a specific bilinear form, it is possible to view the transposed equation (2.5) as the 'adjoint' equation. This approach is developed in [4] and in Appendix A, we present the functional analytic foundation for this approach.

For now, consider the Hale bilinear form $(\cdot,\cdot)$ which is given by

$$(\psi,\varphi)=\psi(0)\varphi(0)-\int_{-r}^{0}d_\theta\Big[\int_{-r}^{\theta}\psi(\theta-\xi)d\mu(\xi)\Big]\varphi(\theta)+\int_{-r}^{0}\int_{-r}^{\theta}\psi(\theta-\xi)d\eta(\xi)\varphi(\theta)d\theta. \tag{3.1}$$

In Appendix A we show that the dual of A using the bilinear form $(\cdot,\cdot)$ is A^{T}. The following lemma provides us with an explicit formula for the spectral projection onto the eigenspace $\mathcal{M}_\lambda$ corresponding to a simple eigenvalue λ_d.

Lemma 3.1. *Let A be given by (2.6). If λ is a simple eigenvalue of A, then the spectral projection P_λ onto $\mathcal{M}_\lambda(A)$ along $\mathcal{R}\big((\lambda I-A)^{k_\lambda}\big)$ can be written explicitly as follows*

$$P_\lambda\varphi=e^{\lambda\,\cdot}\Big[\frac{d}{dz}\det\Delta(\lambda)\Big]^{-1}\operatorname{adj}\Delta(\lambda)\Big(D\varphi+\int_{-r}^{0}d_\tau[\lambda\mu(\tau)+\eta(\tau)]\int_{0}^{-\tau}e^{-\lambda\sigma}\varphi(\sigma+\tau)\,d\sigma\Big), \tag{3.2}$$

where $\operatorname{adj}\Delta(\lambda)$ *denotes the matrix of cofactors of* $\Delta(\lambda)$.

Proof. Let ψ_λ and φ_λ be a basis for $\mathcal{M}_\lambda(A^{\mathrm{T}})$ and $\mathcal{M}_\lambda(A)$, respectively, such that $(\psi_\lambda,\varphi_\lambda)=1$. The spectral projection is given by

$$P_\lambda\varphi=\varphi_\lambda(\psi_\lambda,\varphi). \tag{3.3}$$

Since $z=\lambda$ is a simple zero of $\det\Delta(z)$, we know from (2.10) and (2.13) that

$$\psi_\lambda(s)=e^{-\lambda s}d_\lambda,\qquad 0\leqslant s\leqslant r,\qquad d_\lambda\Delta(\lambda)=0,$$
$$\varphi_\lambda(\theta)=e^{\lambda\theta}c_\lambda,\qquad -r\leqslant\theta\leqslant 0,\qquad \Delta(\lambda)c_\lambda=0.$$

So, using the bilinear form (3.1),

$$(\psi_\lambda,\varphi)=d_\lambda\Big[\varphi(0)-\int_{-r}^{0}d_s\Big(\int_{-r}^{s}e^{-\lambda(s-\theta)}\,d\mu(\theta)\Big)\varphi(s)+\int_{-r}^{0}\int_{-r}^{s}e^{-\lambda(s-\theta)}\,d\eta(\theta)\varphi(s)ds\Big]. \tag{3.4}$$

Therefore

$$\begin{aligned}(\psi_\lambda,\varphi_\lambda)&=d_\lambda\Big[c_\lambda-\int_{-r}^{0}d_s\Big(\int_{-r}^{s}e^{-\lambda(s-\theta)}\,d\mu(\theta)\Big)e^{\lambda s}c_\lambda+\int_{-r}^{0}\int_{-r}^{s}e^{-\lambda(s-\theta)}\,d\eta(\theta)\,e^{\lambda s}c_\lambda\,ds\Big]\\&=d_\lambda\Big[I-\int_{-r}^{0}e^{\lambda\theta}d\mu(\theta)+\lambda\int_{-r}^{0}\int_{-r}^{s}e^{\lambda\theta}\,d\mu(\theta)\,ds+\int_{-r}^{0}\int_{-r}^{s}e^{\lambda\theta}\,d\eta(\theta)\,ds\Big]c_\lambda.\end{aligned}$$

Since

$$\int_{-r}^{0}\int_{-r}^{s}e^{\lambda\theta}\,d\eta(\theta)\,ds=-\int_{-r}^{0}\theta e^{\lambda\theta}\,d\eta(\theta)$$

and

$$\lambda \int_{-r}^{0} \int_{-r}^{s} e^{\lambda\theta}\, d\mu(\theta)\, ds = -\lambda \int_{-r}^{0} \theta e^{\lambda\theta}\, d\mu(\theta),$$

it follows that

$$(\psi_\lambda, \varphi_\lambda) = d_\lambda \Big[I - \int_{-r}^{0} e^{\lambda\theta} d\mu(\theta) - \lambda \int_{-r}^{0} \theta e^{\lambda\theta}\, d\mu(\theta) - \int_{-r}^{0} \theta e^{\lambda\theta}\, d\eta(\theta) \Big] c_\lambda.$$

This together with the definition of $\Delta(z)$ in (2.7) and the normalization condition, $(\psi_\lambda, \varphi_\lambda) = 1$, yields

$$d_\lambda \Big[\frac{d}{dz} \Delta(\lambda) \Big] c_\lambda = 1. \tag{3.5}$$

Returning to representation (3.3) for P_λ and representation (3.4) for (ψ_λ, φ), we claim that, for a given vector v,

$$c_\lambda d_\lambda v = \Big[\frac{d}{dz} \det \Delta(\lambda) \Big]^{-1} \operatorname{adj} \Delta(z)\, v. \tag{3.6}$$

From the relation

$$\Delta(z) \operatorname{adj} \Delta(z) = \det \Delta(z), \tag{3.7}$$

it follows that for a given vector v, the vector c_λ defined by $c_\lambda = \operatorname{adj} \Delta(\lambda) v$, satisfies $\Delta(\lambda) c_\lambda = 0$. Next we choose d_λ such that $d_\lambda \Delta(\lambda) = 0$ and such that

$$d_\lambda \Big[\frac{d}{dz} \Delta(\lambda) \Big] \operatorname{adj} \Delta(\lambda) v = 1,$$

so that (3.5) is also satisfied. Differentiating equation (3.7) with respect to z yields

$$\frac{d}{dz} \Delta(z) \operatorname{adj} \Delta(z) + \Delta(z) \frac{d}{dz} \operatorname{adj} \Delta(z) = \frac{d}{dz} \det \Delta(z). \tag{3.8}$$

If we multiply (3.8) with $z = \lambda$ from the left by d_λ and from the right by v, and use that $d_\lambda \Delta(\lambda) = 0$ and that (3.5) holds, then it follows that

$$d_\lambda v = \Big[\frac{d}{dz} \det \Delta(\lambda) \Big]^{-1}.$$

This proves (3.6) and completes the proof of the lemma. □

An important application of Lemma 3.1 arises in the situation when $\lambda = \lambda_d$ is a dominant eigenvalue of A.

3.2. **Spectral projection via Dunford calculus.** From standard spectral theory [1, 3, 12], it follows that the spectral projection onto $\mathcal{M}_\lambda$ along $\mathcal{R}\big((\lambda I - A)^{k_\lambda}\big)$ can be represented by a Dunford integral

$$P_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (zI - A)^{-1}\, dz, \tag{3.9}$$

where Γ_λ is a small circle such that λ is the only singularity of $(zI - A)^{-1}$ inside Γ_λ. In order to compute the projection explicitly, we need an explicit formula for the resolvent of A.

Lemma 3.2. *If A is defined by (2.6), then the resolvent $(zI-A)^{-1}$ of A is given by*

$$(zI-A)^{-1}\varphi = e^{z\,\cdot}\Big\{\int_{\cdot}^{0} e^{-z\tau}\varphi(\tau)d\tau + \Delta(z)^{-1}\Big[D\varphi + \int_{-r}^{0} d_\theta\big[z\mu(\theta)+\eta(\theta)\big]\int_{0}^{-\theta} e^{-z\tau}\varphi(\tau+\theta)d\tau\Big]\Big\}, \tag{3.10}$$

where $\Delta(z)$ is given by (2.7).

Proof. Let φ be fixed. If we define $\psi=(zI-A)^{-1}\varphi$, then $\psi\in\mathcal{D}(A)$ and $z\psi - A\psi = \varphi$. From the definition of A, it follows that ψ satisfies the ordinary differential equation

$$z\psi - \frac{d\psi}{d\theta} = \varphi \tag{3.11}$$

with boundary condition

$$D\frac{d\psi}{d\theta} = L\psi. \tag{3.12}$$

Equation (3.11) yields

$$\psi(\theta) = e^{z\theta}\Big[\psi(0) + \int_{\theta}^{0} e^{-z\tau}\varphi(\tau)d\tau\Big]. \tag{3.13}$$

Applying D on both sides of (3.11) and using (3.12), we obtain

$$\begin{aligned}
0 &= zD\psi - L\psi - D\varphi \\
&= z\left[\psi(0) - \int_{-r}^{0} d\mu(\theta)\psi(\theta)\right] - \int_{-r}^{0} d\eta(\theta)\psi(\theta) - D\varphi \\
&= z\psi(0) - z\int_{-r}^{0} d\mu(\theta)\Big(e^{z\theta}\Big[\psi(0) + \int_{\theta}^{0} e^{-z\tau}\varphi(\tau)d\tau\Big]\Big) \\
&\qquad - \int_{-r}^{0} d\eta(\theta)\Big(e^{z\theta}\Big[\psi(0) + \int_{\theta}^{0} e^{-z\tau}\varphi(\tau)d\tau\Big]\Big) - D\varphi \\
&= \Big[zI - z\int_{-r}^{0} d\mu(\theta)e^{z\theta} - \int_{-r}^{0} d\eta(\theta)e^{z\theta}\Big]\psi(0) \\
&\qquad - D\varphi - \int_{-r}^{0} d_\theta\big[z\mu(\theta)+\eta(\theta)\big]\int_{0}^{-\theta} e^{-z\tau}\varphi(\tau+\theta)d\tau \\
&= \Delta(z)\psi(0) - D\varphi - \int_{-r}^{0} d_\theta\big[z\mu(\theta)+\eta(\theta)\big]\int_{0}^{-\theta} e^{-z\tau}\varphi(\tau+\theta)d\tau.
\end{aligned}$$

This allows us to solve for $\psi(0)$ and

$$\psi(0) = \Delta(z)^{-1}\Big[D\varphi + \int_{-r}^{0} d_\theta\big[z\mu(\theta)+\eta(\theta)\big]\int_{0}^{-\theta} e^{-z\tau}\varphi(\tau+\theta)d\tau\Big]. \tag{3.14}$$

Substituting (3.14) into (3.13) yields the desired result. □

To illustrate the power of Dunford calculus, we give next a second simpler proof of Lemma 3.1.

Second proof of Lemma 3.1. Since the formula for the projection in (3.9) is precisely the residue of the resolvent of A in $z = \lambda$, the representation for the resolvent (3.10) yields

$$P_\lambda\varphi = \mathrm{Res}_{z=\lambda}\Big\{ e^{z\,\cdot}\,\Delta(z)^{-1}\Big[D\varphi + \int_{-r}^{0} d_\theta\big[\lambda\mu(\theta)+\eta(\theta)\big]\int_0^{-\theta} e^{-z\tau}\varphi(\tau+\theta)d\tau\Big]\Big\}. \tag{3.15}$$

If λ is a simple eigenvalue of A, then λ is a simple zero of $\det\Delta(z)$ and it suffices to compute $\mathrm{Res}_{z=\lambda}\,\Delta(z)^{-1}$ explicitly and

$$\begin{aligned}\mathrm{Res}_{z=\lambda}\,\Delta(z)^{-1} &= \lim_{z\to\lambda}(z-\lambda)[\det\Delta(z)]^{-1}\operatorname{adj}\Delta(z)\\ &= \lim_{z\to\lambda}\Big[\frac{\det\Delta(z)-\det\Delta(\lambda)}{z-\lambda}\Big]^{-1}\operatorname{adj}\Delta(z)\\ &= \Big[\frac{d}{dz}\det\Delta(\lambda)\Big]^{-1}\operatorname{adj}\Delta(\lambda).\end{aligned}$$

Using this together with (3.15), we arrive at (3.2). □

Example 3.1. Consider the retarded equation

$$\dot x(t) = Bx(t-1), \quad t \geqslant 0, \qquad x_0 = \varphi \in \mathcal{C}, \tag{3.16}$$

where $B \neq 0$ is an $n\times n$-matrix. The characteristic equation is given by

$$\Delta(z) = zI - Be^{-z}. \tag{3.17}$$

For every simple root of $\det\Delta(z)$, the spectral projection is given by

$$(P_\lambda\varphi)(\theta) = \Big[\frac{d}{dz}\det\Delta(\lambda)\Big]^{-1}\operatorname{adj}\Delta(\lambda)\Big(\varphi(0) + B\int_0^1 e^{-\lambda\tau}\varphi(\tau-1)d\tau\Big)e^{\lambda\theta}.$$

In the scalar case, a root λ of Δ is not simple if and only if

$$\begin{cases}\lambda - Be^{-\lambda} = 0,\\ 1 + Be^{-\lambda} = 0.\end{cases}$$

Therefore, if $B \neq -1/e$ or equivalently $\lambda = -1$ is not a root of Δ, then all roots of (3.17) are simple. So the spectral projections are given by

$$(P_\lambda\varphi)(\theta) = \frac{1}{1+\lambda}\Big(\varphi(0) + B\int_0^1 e^{-\lambda\tau}\varphi(\tau-1)d\tau\Big)e^{\lambda\theta}, \tag{3.18}$$

where λ satisfies $\lambda - Be^{-\lambda} = 0$. Furthermore, it follows from Corollary 3.12 of [10] that

$$x_t(\varphi) = \sum_{j=0}^{\infty} P_{\lambda_j}T(t)\varphi = \sum_{j=0}^{\infty} T(t)P_{\lambda_j}\varphi, \quad t > 0,$$

where λ_j, $j = 0, 1, \ldots$, denote the roots of $\lambda - Be^{-\lambda} = 0$, ordered according to decreasing real part. Using (3.18) and the fact that $T(t)e^{\lambda_j\,\cdot} = e^{\lambda_j(t+\cdot)}$, we can now explicitly compute the solution of (3.16) with initial condition $x_0 = \varphi$

$$x(t;\varphi) = \sum_{j=0}^{\infty}\frac{1}{1+\lambda_j}\Big(\varphi(0) + B\int_0^1 e^{-\lambda_j\tau}\varphi(\tau-1)d\tau\Big)e^{\lambda_j t}, \qquad t > 0.$$

(Compare Theorem 6 in [11].)

If $B=-1/e$, then all zeros of $\Delta(z)$ are simple except for $\lambda=-1$. For the simple zeros we can again use (3.18). For the double zero $\lambda=-1$, we have to use (3.15) to compute the projection onto the two dimensional space $\mathcal{M}_{-1}$ and P_{-1} is given by

$$(P_{-1}\varphi)(\theta)=\Big(-\frac{2}{3}\varphi(0)+\frac{8}{3}\int_{-1}^{0}e^{\tau}\varphi(\tau)d\tau+2\int_{-1}^{0}\tau e^{\tau}\varphi(\tau)d\tau\Big)e^{-\theta}$$
$$+2\Big(\varphi(0)-\int_{-1}^{0}e^{\tau}\varphi(\tau)d\tau\Big)\theta e^{-\theta}.$$

Since $T(t)\phi=\phi(t+\cdot)$, where $\phi(\theta)=\theta e^{-\theta}$, we can again give the solution explicitly.

4. Large time behaviour for autonomous FDE's

In this section we shall further investigate the case when $\det\Delta(z)$ has a dominant root $z=\lambda_d$.

Theorem 4.1. *Let A to be given by (2.6). If A has a dominant simple eigenvalue λ_d, then there exist positive numbers ϵ and M such that*

$$\|e^{-\lambda_d t}T(t)\varphi-P_{\lambda_d}\varphi\|\leqslant Me^{-\epsilon t}\tag{4.1}$$

and

$$\lim_{t\to\infty}e^{-\lambda_d t}T(t)\varphi=e^{\lambda_d\cdot}\Big[\frac{d}{dz}\det\Delta(\lambda_d)\Big]^{-1}\operatorname{adj}\Delta(\lambda_d)\Big[D\varphi+\int_{-r}^{0}[\lambda_d d\mu(\tau)+d\eta(\tau)]\int_{0}^{-\tau}e^{-\lambda_d\sigma}\varphi(\sigma+\tau)\,d\sigma\Big].\tag{4.2}$$

Proof. From representation (3.9), it follows that P_λ and A commute, and therefore P_λ and $T(t)$ commute as well. The spectral decomposition with respect to λ_d yields

$$e^{-\lambda_d t}T(t)\varphi=e^{-\lambda_d t}T(t)P_{\lambda_d}\varphi+e^{-\lambda_d t}T(t)(I-P_{\lambda_d})\varphi.$$

From the exponential estimate (2.14), it follows that there exist positive ϵ and M such that

$$\|e^{-\lambda_d t}T(t)(I-P_{\lambda_d})\varphi\|\leqslant Me^{-\epsilon t},\qquad t\geqslant 0.$$

The action of $T(t)$ restricted to a one-dimensional eigenspace $\mathcal{M}_\lambda$ is given by

$$e^{-\lambda_d t}T(t)P_{\lambda_d}=P_{\lambda_d}.$$

This shows (4.1) and using Lemma 3.1, we arrive at (4.2). $\square$

If we evaluate (4.2) at $\theta=0$ we obtain the following corollary.

Corollary 4.1. *Let A be given by (2.6) and suppose that A has a dominant simple eigenvalue λ_d. If $x(t)=x(\,\cdot\,;\varphi)$ denotes the solution of (2.1) with initial data $x_0=\varphi$, then the large time as a function of the initial data φ is given by*

$$\lim_{t\to\infty}e^{-\lambda_d t}x(t)=\Big[\frac{d}{dz}\det\Delta(\lambda_d)\Big]^{-1}\operatorname{adj}\Delta(\lambda_d)\Big[D\varphi+\int_{-r}^{0}[\lambda_d d\mu(\tau)+d\eta(\tau)]\int_{0}^{-\tau}e^{-\lambda_d\sigma}\varphi(\sigma+\tau)\,d\sigma\Big].\tag{4.3}$$

5. Periodic FDE's

A similar approach as presented for autonomous equations yields the asymptotic behaviour of solutions of periodic delay equations. However, in this setting the spectral projection P_λ can only be computed using Dunford calculus. The results in this section provide a new approach and generalize earlier results in [13, 9].

5.1. **Spectral theory for periodic FDE's.** We begin to recall some of the basic theory for linear periodic delay equations that we use in this section. Consider the scalar periodic differential difference equation

$$
\begin{aligned}
\frac{dx}{dt}(t) &= a(t)x(t) + \sum_{j=1}^{m} b_j(t)x(t-r_j), \quad t \geqslant s, \\
x_s &= \varphi, \qquad \varphi \in \mathcal{C},
\end{aligned} \tag{5.1}
$$

where the coefficients a and b_j, $1 \leqslant j \leqslant m$, are real continuous periodic functions with minimal period ω and the delays $r_j = j\omega$ are multiples of the period ω.

To emphasize the dependence of the solution $x(t)$ of (5.1) with respect to the initial condition $x_s = \varphi$, we write $x(t) = x(t; s, \varphi)$. The evolutionary system associated with (5.1) is again given by translation along the solution

$$
T(t,s)\phi = x_t(s,\phi), \tag{5.2}
$$

where $x_t(s,\phi)(\theta) = x(t+\theta; s, \phi)$ for $-m\omega \leqslant \theta \leqslant 0$. The periodicity of the coefficients of (5.1) implies that

$$
T(t+\omega, s+\omega) = T(t,s), \qquad t \geqslant s.
$$

This together with the semigroup property $T(t,\tau)T(\tau,s) = T(t,s)$, $t \geqslant \tau \geqslant s$, yields to

$$
T(t+\omega, s) = T(t,s)T(s+\omega, s) \qquad \text{for } t \geqslant s. \tag{5.3}
$$

The periodicity property (5.3) allows us to define the monodromy map or period map $\Pi(s) : \mathcal{C} \to \mathcal{C}$ associated with (5.1) as follows

$$
\Pi(s)\varphi = T(s+\omega, s)\varphi, \qquad \varphi \in \mathcal{C}. \tag{5.4}
$$

From the general theory for functional differential equations (see [1, 4]), it follows that $\Pi(s)$ is a compact operator, i.e., $\Pi(s)$ is a bounded operator with the property that the closure of the image of the unit ball in $\mathcal{C}$ is compact. Hence, the spectrum $\sigma(\Pi(s))$ of $\Pi(s)$ is at most countable with the only possible accumulation point being zero. If $\mu \neq 0$ belongs to $\sigma(\Pi(s))$, then μ is in the point spectrum of Π, i.e., there exists a $\varphi \in \mathcal{C}$, $\varphi \neq 0$, such that $\Pi(s)\varphi = \mu\varphi$. If μ belongs to the nonzero point spectrum of $\Pi(s)$, then μ is called a *characteristic multiplier* of (5.1) and λ for which $\mu = e^{\lambda\omega}$ (unique up to multiples of $2\pi i$) is called a *characteristic exponent* of (5.1). The characteristic multipliers are in fact independent of s and the *generalized eigenspace* $\mathcal{M}_\mu(s)$ of $\Pi(s)$ at μ is defined to be $\mathcal{N}\big(\mu I - \Pi(s))^{k_\lambda}\big)$, where k_λ is the smallest integer such that

$$
\mathcal{M}_\mu(s) = \mathcal{N}\big((\mu I - \Pi(s))^{k_\lambda}\big) = \mathcal{N}\big((\mu I - \Pi(s))^{k_\lambda + 1}\big).
$$

Since $\Pi(s)$ is compact, there are two closed subspaces $\mathcal{M}_\mu(s)$ and $\mathcal{Q}_\mu(s)$ of $\mathcal{C}$ such that the following properties hold:

(i) $\mathcal{C} = \mathcal{M}_\mu(s) \oplus \mathcal{Q}_\mu(s)$.
(ii) $m_\mu = \dim \mathcal{M}_\mu(s) < \infty$.

(iii) $\mathcal{M}_\mu(s)$ and $\mathcal{Q}_\mu$ are $\Pi(s)$-invariant.
(iv) $\sigma(\Pi(s)|\mathcal{M}_\mu(s)) = \{\mu\}$ and $\sigma(\Pi(s)|\mathcal{Q}_\mu(s)) = \sigma(\Pi(s)) \setminus \{\mu\}$.

Let $\varphi_1(s), \dots, \varphi_{m_\mu}(s)$ be a basis of eigenvectors and generalized eigenvectors of $\Pi(s)$ at μ. Define the row m_λ-vector $\Phi(s) = \{\varphi_1(s), \dots, \varphi_{m_\lambda}(s)\}$. Since $\mathcal{M}_\lambda(s)$ is invariant under $\Pi(s)$, there exists a $m_\lambda \times m_\lambda$ matrix $M(s)$ such that

$$\Pi(s)\Phi(s) = \Phi(s)M(s)$$

and Property (iv) implies that the only eigenvalue of $M(s)$ is $\mu \neq 0$. Therefore, there is a $m_\mu \times m_\mu$ matrix B_s such that $B_s = 1/\omega \log M(s)$.

Define the vector $P(t)$ with elements in $\mathcal{C}$ by

$$P(t) = T(t,0)\Phi e^{-Bt}$$

and let $\Phi = \Phi(0)$, $M = M(0)$ and $B = B_0$. Then, for $t \geqslant 0$,

$$\begin{aligned} P(t+\omega) &= T(t+\omega,0)\Phi e^{-B(t+\omega)} = T(t,0)T(\omega,0)\Phi e^{-B\omega}e^{-Bt} \\ &= T(t,0)\Pi(0)\Phi e^{-B\omega}e^{-Bt} = T(t,0)\Phi M e^{-B\omega}e^{-Bt} \\ &= P(t). \end{aligned}$$

Since $P(t)$ can be extended periodically for $t \in \mathbb{R}$, we conclude that

$$T(t,0)\Phi = P(t)e^{Bt} \qquad \text{for} \quad t \in \mathbb{R}.$$

As for the case $s = 0$, one can define $T(t,s)\Phi(s)$ for all $t \in \mathbb{R}$ and for any real number τ and $\mu \in \sigma(\Pi(s)) \setminus \{0\}$, we have

$$\begin{aligned} \Pi(\tau)T(\tau,s)\Phi(s) &= T(\tau+\omega,\tau)T(\tau,s)\Phi(s) = T(\tau+\omega,s)\Phi(s) \\ &= T(\tau,s)T(s+\omega,s)\Phi(s) = T(\tau,s)\Phi(s)M(s). \end{aligned}$$

Therefore,

$$[\mu I - \Pi(\tau)]T(\tau,s)\Phi(s) = T(\tau,s)\Phi(s)(\mu I - M(s)),$$

and it follows that $\mu \in \sigma(\Pi(\tau))$. Thus the dimension of $\mathcal{M}_\mu(\tau)$ is at least as large as the dimension of $\mathcal{M}_\mu(s)$. Since one can reverse the role of s and τ, we obtain that the characteristic multipliers of (5.1) are independent of the starting time and if $\Phi(s)$ is a basis for $\mathcal{M}_\mu(s)$, then $T(t,s)\Phi(s)$ is a basis for $\mathcal{M}_\mu(t)$ for any $t \in \mathbb{R}$. In particular, the subspaces $\mathcal{M}_\mu(s)$ and $\mathcal{M}_\mu(t)$ are diffeomorphic for all s and t.

Note that if $\mu = e^{\lambda\omega}$ is simple, i.e. $m_\mu = 1$, then $M(s)$ is identical to μ, $\Phi := \Phi(0) = \{\varphi_1\}$ and $P(t)$ assumes the simpler form

$$e^{\lambda t}P(t) = T(t,0)\varphi_1. \tag{5.5}$$

The spectral projection onto $\mathcal{M}_\mu(s)$ along $\mathcal{Q}_\mu(s)$ can again be represented by a Dunford integral (see [3])

$$P_\mu(s) = \frac{1}{2\pi i}\int_{\Gamma_\mu} (zI - \Pi(s))^{-1}\,dz, \tag{5.6}$$

where Γ_μ is a small circle such that μ is the only singularity of $(zI - \Pi(s))^{-1}$ inside Γ_μ.

5.2. **Large time behaviour.** Similarly as for autonomous equations, we can relate the solutions, corresponding to initial data $\varphi \in \mathcal{M}_\mu(s)$, to an arbitrary solution $x_t(s;\varphi)$ by an exponential bound for the remainder.

Theorem 5.1. *Let μ_j, $j = 1, 2, \ldots$, denote the nonzero eigenvalues of $\Pi(s)$ ordered by decreasing modulus and let $\varphi \in \mathcal{C}$. If γ is an arbitrary real number, then there are positive constants ϵ and M such that for $t \geqslant s$*

$$\Big\| x_t(s;\varphi) - \sum_{|\mu_n| \geqslant e^{\gamma\omega}} P_{\mu_n}(s) x_t(s;\varphi) \Big\| \leqslant M e^{(\gamma-\epsilon)(t-s)} \|\varphi\|. \tag{5.7}$$

Proof. Let $k = k(\gamma)$ be the integer such that $|\mu_k| \geqslant e^{\gamma\omega}$ and $|\mu_{k+1}| < e^{\gamma\omega}$. Set $\Sigma = \{\mu_1, \mu_2, \ldots, \mu_k\}$ and define

$$\mathcal{M}_\Sigma(s) = \bigoplus_{\mu \in \Sigma} \mathcal{M}_\mu(s) \quad \text{and} \quad \mathcal{Q}_\Sigma(s) = \bigcap_{\mu \in \Sigma} \mathcal{Q}_\mu(s).$$

Then, one has $\mathcal{C} = \mathcal{M}_\Sigma \oplus \mathcal{Q}_\Sigma$. To prove the exponential estimate, set

$$R_k(s)\varphi = \varphi - \sum_{j=1}^{k} P_{\mu_j}(s)\varphi. \tag{5.8}$$

It follows that $R_k(s)\varphi \in \mathcal{Q}_\Sigma(s)$ and $T(t,s)R_k(s)\varphi \in \mathcal{Q}_\Sigma(t)$ for all $\varphi \in \mathcal{C}$. Furthermore, there exists a constant M_0 such that $\|R_k(s)\varphi\| \leqslant M_0\|\varphi\|$. If ϵ is such that $e^{(\gamma-2\epsilon)\omega} = |\mu_{k+1}|$, then the spectral radius of

$$\widehat{\Pi}(s) \stackrel{\text{def}}{=} \Pi(s)\big|_{\mathcal{Q}_\Sigma(s)}$$

is $e^{(\gamma-2\epsilon)\omega}$. Therefore, $\lim_{n\to\infty} \|\widehat{\Pi}(s)^n\|^{1/n} = e^{(\gamma-2\epsilon)\omega}$ and this implies that for some $m > 0$

$$\|\widehat{\Pi}(s)^m R_k(s)\varphi\| \leqslant e^{(\gamma-\epsilon)m\omega} \|R_k(s)\varphi\|.$$

Since $\widehat{T}(\tau,s) \stackrel{\text{def}}{=} T(\tau,s)|_{\mathcal{Q}_\Sigma(s)}$, $s \leqslant \tau \leqslant s+\omega$ is pointwise bounded, the uniform boundedness principle yields a constant M_1 such that $\|\widehat{T}(\tau,s)\| \leqslant M_1$ for $s \leqslant \tau \leqslant s+\omega$. Set $M_2 = M_1 \max_{j=1,\ldots,m-1} \|\widehat{\Pi}(s)^j\|$ and let $t \geqslant s$ be given. If k_t is the largest integer so that $s + k_t m\omega \leqslant t$, then

$$\|\widehat{T}(t,s)R_k(s)\varphi\| \leqslant M_2 \|\widehat{\Pi}(s)^m\|^{k_t} \|R_k(s)\varphi\| \leqslant M e^{(\gamma-\epsilon)(t-s)}\|\varphi\|,$$

where $M = M_0 M_2$. This proves the exponential estimate for the remainder term. □

Corollary 5.1. *Suppose that $\mu_d = e^{\lambda_d\omega}$ is a simple dominant eigenvalue of $\Pi(s)$. If $P(t)$ is as in (5.5), then there are positive constants M and ϵ so that large time behaviour of the solution $x(t;0,\varphi)$ is given by*

$$\|e^{-\lambda_d t} x_t(0,\varphi) - c(\varphi)P(t)\| \leqslant M e^{-\epsilon t}\|\varphi\|, \qquad t \geqslant 0, \tag{5.9}$$

where $c(\varphi)$ is defined so that $P_{\mu_d}(0)\varphi = c(\varphi)\varphi_1$.

Proof. If $\mu = e^{\lambda\omega}$ is a simple eigenvalue of Π, it follows from Equation (5.5) together with the property that $T(t,s)$ maps $\mathcal{M}_{\mu_d}(s)$ into $\mathcal{M}_{\mu_d}(t)$ diffeomorphically, that

$$\begin{aligned} P_{\mu_d}(t)x_t(0,\varphi) &= P_{\mu_d}(t)T(t,0)\varphi \\ &= T(t,0)P_{\mu_d}(0)\varphi \\ &= c(\varphi)e^{\lambda_d t}P(t). \end{aligned}$$

Therefore the corollary follows directly from Theorem 5.1. □

6. Applications to the large time behaviour

In order to make the analysis of the characteristic equations easier, we assume throughout this section that the coefficients of the functional differential equations are real-valued (the state space remains $\mathcal{C}([-r,0],\mathbb{C}^n)$).

6.1. **Applications for autonomous equations.**

Consider the scalar delay equation

$$\dot{x}(t) = ax(t) + bx(t-\tau), \quad t \geqslant 0, \tag{6.1}$$

with a and b real numbers. The characteristic equation of (6.1) is given by

$$\Delta(z) = z - a - be^{-\tau z}.$$

From Lemma B.3, it follows that if $-e^{-1} < b\tau e^{-a\tau}$, then $\Delta(z)$ has a simple real dominant root and we can use Corollary 4.1 to compute the large time behaviour of the solutions of (6.1). The next theorem presents a new approach for the result in [2], where it was required that $-e^{-1} < b\tau e^{-a\tau} < e$, in order to get the same conclusion.

Theorem 6.1. *If $-e^{-1} < b\tau e^{-a\tau}$, then the long-time behaviour of the solution $x = x(\,\cdot\,;\varphi)$ of (6.1) with initial data $x_0 = \varphi$ is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = \frac{1}{1+b\tau e^{-\lambda_d\tau}}\Big[\varphi(0) + be^{-\lambda_d\tau}\int_{-\tau}^{0} e^{-\lambda_d s}\varphi(s)\,ds\Big], \tag{6.2}$$

where λ_d is the simple real dominant root of $z = a + be^{-\tau z}$.

Proof. Lemma B.3 implies the existence of a simple real dominant zero λ_d of $\Delta(z)$. So if we apply Corollary 4.1 with $r = \tau$, $\mu = 0$ and η a sum of a point-mass at 0 of size a and at $-\tau$ of size b, Equation (6.2) follows from (4.3). □

The second application concerns the main result in [8]. Consider the scalar neutral equation

$$\frac{d}{dt}\Big[x(t) + \sum_{l=1}^{m} c_l x(t-\sigma_l)\Big] = ax(t) + \sum_{j=1}^{k} b_j x(t-h_j). \tag{6.3}$$

The characteristic equation associated with (6.3) is given by

$$\Delta(z) = z\Big(1 + \sum_{l=1}^{m} c_l e^{-z\sigma_l}\Big) - a - \sum_{j=1}^{k} b_j e^{-zh_j}. \tag{6.4}$$

If $\Delta(z)$ in (6.4) has a real zero λ_d that satisfies the condition

$$\sum_{l=1}^{m} |c_l|\,(1+|\lambda_d|\,\sigma_l)e^{-\lambda_d\sigma_l} + \sum_{j=1}^{k} h_j\,|b_j|\,e^{-\lambda_d h_j} < 1, \tag{6.5}$$

then Lemma B.1 implies that λ_d is a simple real dominant zero of (6.4).

Thus, if (6.5) holds, then we can again use Corollary 4.1 to compute the large time behaviour of the solutions of (6.3).

Theorem 6.2. *If λ_d is a real zero of (6.4) such that (6.5) holds, then the large time behaviour of the solution $x = x(\,\cdot\,;\varphi)$ of (6.3) with initial data $x_0 = \varphi$ is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t;\varphi) = \frac{1}{H(\lambda_d)}\Big[\varphi(0) - \sum_{l=1}^{m} c_l\varphi(-\sigma_l) + \sum_{l=1}^{m}\lambda_d c_l\int_0^{-\sigma_l} e^{-\lambda_d\sigma}\varphi(\sigma+\sigma_l)\,d\sigma + \sum_{j=1}^{k} b_j\int_0^{-h_j} e^{-\lambda_d\sigma}\varphi(\sigma+h_j)\,d\sigma\Big], \tag{6.6}$$

where

$$H(\lambda_d) = 1 + \sum_{l=1}^{m} c_l e^{-\lambda_d\sigma_l} - \lambda_d\sum_{l=1}^{m} c_l\sigma_l e^{-\lambda_d\sigma_l} + \sum_{j=1}^{k} b_j h_j e^{-\lambda_d h_j}.$$

Proof. Let $r_1 = \max_{1\leqslant l\leqslant m}\sigma_l$, $r_2 = \max_{1\leqslant j\leqslant k} h_j$ and $r = \max\{r_1, r_2\}$. Let μ be a finite sum of point masses at $-\sigma_l$ of size c_l, $1 \leqslant l \leqslant m$, and let η be a finite sum of point masses at $-h_j$ of size b_j, $1 \leqslant j \leqslant k$. With these definitions we apply Corollary 4.1 in the scalar case. Since $H(\lambda_d)$ is the derivative of $\Delta(z)$ evaluated at $z = \lambda_d$, the theorem follows from (4.3). □

Corollary 6.1. *If $a + \sum_{j=1}^{k} b_j = 0$ and $\sum_{j=1}^{k} h_j\,|b_j| < 1$, then the large time behaviour of the solution $x = x(\,\cdot\,;\varphi)$ of (6.3) with initial data $x_0 = \varphi$ is given by*

$$\lim_{t\to\infty} x(t) = \Big(1 + \sum_{l=1}^{m} c_l + \sum_{j=1}^{k} b_j h_j\Big)^{-1}\Big[\varphi(0) - \sum_{l=1}^{m} c_l\varphi(-\sigma_l) + \sum_{j=1}^{k} b_j\int_0^{-h_j}\varphi(\sigma+h_j)\,d\sigma\Big].$$

The examples in this subsection illustrate that computing the spectral projection on the dominant (finite dimensional) eigenspace yields an easy way to find explicit formulas for the large time behaviour of solutions.

6.2. **Application for periodic equations.** In this section, we show that also for linear periodic equations, the large time behaviour can be given explicitly by computing a spectral projection onto the dominant eigenvalue of the monodromy operator.

Consider the following linear periodic delay equation

$$\dot{x}(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \quad t \geqslant s, \tag{6.7}$$

where $a(t+\omega) = a(t)$ and $b_j(t+\omega) = b_j(t)$ for $j = 1, 2\ldots, k$. We assume the particular case where $\tau_j = j\omega$ (i.e., the delays are integer multiples of the period ω). The following lemma is clear.

Lemma 6.1. *If $y(t) = e^{-\int_0^t a(s)ds}x(t)$, then y satisfies*

$$\dot{y}(t) = \sum_{j=1}^{k}\hat{b}_j(t)y(t-\tau_j),$$

where $\hat{b}_j(t) = e^{-\int_{t-\tau_j}^{t} a(s)ds} b_j(t)$, $j = 1, 2, \ldots, k$, *are also* ω*-periodic.*

So, it suffices to analyze the following system (recall $\tau_j = j\omega$)

$$\begin{aligned} \dot{x}(t) &= \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \quad t \geqslant s, \\ x_s &= \varphi, \quad \varphi \in \mathcal{C}([-k\omega, 0], \mathbb{C}^n), \end{aligned} \tag{6.8}$$

where b_j are continuous real matrix-valued functions such that $b_j(t+\omega) = b_j(t)$.

Let $\Pi(s) : \mathcal{C} \to \mathcal{C}$ denote the monodromy operator $\Pi(s) = T(s+\omega, s)$, i.e.,

$$(\Pi(s)\varphi)(\theta) = x(\omega + \theta; s, \varphi), \quad -k\omega \leqslant \theta \leqslant 0.$$

Using the differential equation (6.8) and periodicity of b_j, we have the following representation for $\Pi(s)$

$$(\Pi(s)\varphi)(\theta) = \begin{cases} \varphi(0) + \sum_{j=1}^{k} \int_0^{\omega+\theta} b_j(\sigma+s)\varphi\big(\sigma - \tau_j\big)d\sigma, & -\omega \leqslant \theta \leqslant 0, \\ \varphi(\theta + \omega), & -k\omega \leqslant \theta \leqslant -\omega. \end{cases}$$

Since the large time behaviour of the solutions is independent of the starting time s, we can set $s = 0$ and define $\Pi = \Pi(0)$.

Lemma 6.2. *The resolvent* $(zI - \Pi)^{-1}\varphi$ *of the monodromy operator* Π *is given by*

$$(zI - \Pi)^{-1}\varphi(\theta) = \Omega_{-\omega}^{\theta}(1/z)[zI - \Omega_{-\omega}^{0}]^{-1}\big(\varphi(-\omega) + G_{\varphi,z}(0)\big) + G_{\varphi,z}(\theta), \tag{6.9}$$

for $-\omega \leqslant \theta \leqslant 0$ *and*

$$(zI - \Pi)^{-1}\varphi(\theta) = \frac{1}{z^m}(zI - \Pi)^{-1}\varphi(\theta + m\omega) + \sum_{j=0}^{m-1} \frac{\varphi(\theta - j\omega)}{z^{j+1}}, \tag{6.10}$$

for $-\omega \leqslant \theta + m\omega \leqslant 0$, $m = 1, \ldots, k-1$, *where*

$$\begin{aligned} G_{\varphi,z}(\theta) = \frac{1}{z}\Bigg(&\varphi(\theta) - \Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\varphi(-\omega) \\ &+ \sum_{l=1}^{k}\sum_{j=0}^{l-1} \int_{-\omega}^{\theta} \frac{1}{z^{l-j}} \Omega_{\sigma}^{\theta}\big(\tfrac{1}{z}\big) b_l(\sigma)\varphi(\sigma - j\omega)d\sigma\Bigg) \end{aligned} \tag{6.11}$$

and $\Omega_s^t(1/z)$ *denotes the fundamental matrix solution of the differential equation*

$$\frac{d\psi}{dt}(t) = \sum_{l=1}^{k} \frac{1}{z^l} b_l(t)\psi(t).$$

Proof. If $(zI - \Pi)^{-1}\varphi = \psi$ (φ is given), then

$$\varphi = z\psi - \Pi\psi.$$

Suppose at first that φ is differentiable. We can derive the following system of equations

$$\varphi(\theta) = z\psi(\theta) - \psi(\theta + \omega), \qquad -k\omega \leqslant \theta \leqslant -\omega, \tag{6.12}$$

$$\left.\begin{aligned} \tfrac{d}{d\theta}\varphi(\theta) &= z\tfrac{d}{d\theta}\psi(\theta) - \textstyle\sum_{j=1}^{k} b_j(\theta)\psi(\theta - (j-1)\omega), \\ \varphi(-\omega) &= z\psi(-\omega) - \psi(0), \end{aligned}\right\} \quad -\omega \leqslant \theta \leqslant 0. \tag{6.13}$$

Using (6.12) inductively, for $l = 1, 2, \dots, k-1$ and $-\omega \leqslant \theta \leqslant 0$, we obtain

$$\psi(\theta - l\omega) = \frac{1}{z^l}\psi(\theta) + \sum_{j=1}^{l} \frac{1}{z^{l-j+1}}\varphi(\theta - j\omega). \tag{6.14}$$

Using (6.14), we can reduce the equation in (6.13) to a differential equation

$$\frac{d\psi}{d\theta}(\theta) - \sum_{l=1}^{k} b_l(\theta)\frac{1}{z^l}\psi(\theta) = \frac{1}{z}\left[\frac{d\varphi}{d\theta}(\theta) + \sum_{l=2}^{k}\left(b_l(\theta)\sum_{j=1}^{l-1}\frac{1}{z^{l-j}}\varphi(\theta - j\omega)\right)\right]. \tag{6.15}$$

We first solve the homogeneous equation ($\varphi \equiv 0$), i.e.,

$$\frac{d\psi}{d\theta}(\theta) = \sum_{l=1}^{k} b_l(\theta)\frac{1}{z^l}\psi(\theta), \tag{6.16}$$

and let $\Omega_s^t(1/z)$ denote the fundamental matrix solution of (6.16) with $\Omega_s^s(1/z) = I$. Using the variation-of-constants formula, equation (6.15) becomes

$$\psi(\theta) = \Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\psi(-\omega) + G_{\varphi,z}(\theta), \tag{6.17}$$

where

$$G_{\varphi,z}(\theta) = \frac{1}{z}\left(\varphi(\theta) - \Omega_{-\omega}^{\theta}\big(\tfrac{1}{z}\big)\varphi(-\omega) + \sum_{l=1}^{k}\sum_{j=0}^{l-1}\int_{-\omega}^{\theta}\frac{1}{z^{l-j}}\Omega_{\sigma}^{\theta}\big(\tfrac{1}{z}\big)b_l(\sigma)\varphi(\sigma - j\omega)d\sigma\right).$$

We can solve $\psi(-\omega)$ from the boundary condition in (6.13)

$$\varphi(-\omega) = z\psi(-\omega) - \Omega_{-\omega}^{0}\big(\tfrac{1}{z}\big)\psi(-\omega) - G_{\varphi,z}(0). \tag{6.18}$$

Hence

$$\psi(-\omega) = \big[zI - \Omega_{-\omega}^{0}\big(\tfrac{1}{z}\big)\big]^{-1}\big(\varphi(-\omega) + G_{\varphi,z}(0)\big). \tag{6.19}$$

Equations (6.17) and (6.19) yield (6.9). To find ψ on $[-k\omega, -\omega]$, we use again relation (6.12) inductively to obtain (6.10). To finish the proof, note that (6.9) is well defined for $\varphi \in \mathcal{C}$, and we can drop the assumption that φ is differentiable. □

The representation for the resolvent of Π yields important information about the spectral properties of the operator. For example, it follows that the nonzero spectrum of Π, $\sigma(\Pi) \setminus \{0\}$, consists of point spectrum only, given by

$$\sigma(\Pi) \setminus \{0\} = \big\{z \mid \det\big(zI - \Omega_{-\omega}^{0}(1/z)\big) = 0\big\}.$$

Furthermore, questions about completeness of the eigenvectors and generalized eigenvectors of Π (i.e., denseness of the Floquet solutions in $\mathcal{C}([-k\omega, 0], \mathbb{C}^n)$) can be answered using resolvent estimates. Furthermore, using the Dunford representation of the spectral projection P_μ of Π onto a generalized eigenspace $\mathcal{M}_\mu$, $\mu \in \sigma(\Pi)\setminus\{0\}$,

$$P_\mu = \mathrm{Res}_{z=\mu}(zI - \Pi)^{-1},$$

we can explicitly compute the spectral projection of Π using residue calculus. In particular, if μ is a simple eigenvalue of Π, the spectral projection onto the one-dimensional eigenspace is given by

$$P_\mu\varphi = \lim_{z\to\mu}(z - \mu)(zI - \Pi)^{-1}\varphi.$$

Together with Lemma 6.2, this yields a formula for $P_\mu\varphi$ on $[-\omega, 0]$, namely,

$$(P_\mu\varphi)(\theta) = \Omega_{-\omega}^{\theta}(1/\mu)\Big[\lim_{z\to\mu}(z - \mu)\big(zI - \Omega_{-\omega}^{0}(1/z)\big)^{-1}\Big]\big(\varphi(-\omega) + G_{\varphi,\mu}(0)\big), \tag{6.20}$$

where $G_{\varphi,z}$ is given by (6.11). The extension of $P_\mu\varphi$ to $[-k\omega,-\omega]$ can be found using (6.10). So, if we can compute the residue of $[zI-\Omega^\theta_{-\omega}(1/z)]^{-1}$ at μ (which is a question about ordinary differential equations), we can find an explicit formula for P_μ. Like in the autonomous case, the existence of a simple Floquet multiplier μ_d that dominates the others (in the sense that $\mu\neq\mu_d\in\sigma(\Pi(s))$ implies $|\mu|<|\mu_d|$) allows us to give an explicit formula for the large time behaviour of solutions.

Corollary 6.2. *If $\mu_d\neq 0$ is a simple and dominant eigenvalue of Π, then the large time behaviour of the solution $x(\,\cdot\,;0,\varphi)$ of (6.7) is given by*

$$\lim_{t\to\infty}\Omega^t_{-\omega}\big(\tfrac{1}{\mu}\big)x(t;0,\varphi)=\mathrm{Res}_{z=\mu}\big[zI-\Omega^0_{-\omega}\big(\tfrac{1}{z}\big)\big]^{-1}\big(\varphi(-\omega)+G_{\varphi,\mu}(0)\big)$$

where $\Omega^t_s\big(\frac{1}{z}\big)$ is as in Lemma 6.2.

In the scalar case, we can solve $\Omega^t_s(1/z)$, namely,

$$\Omega^t_s\big(\tfrac{1}{z}\big)=\exp\left(\sum_{j=1}^k z^{-j}\int_s^t b_j(\sigma)d\sigma\right). \tag{6.21}$$

The poles of resolvent $[z-\Omega^0_{-\omega}(1/z)]^{-1}$ are the zeros μ of the equation

$$\mu-\exp\left(\sum_{j=1}^k \mu^{-j}\int_{-\omega}^0 b_j(\sigma)d\sigma\right)=0. \tag{6.22}$$

Define

$$B_j=\frac{1}{\omega}\int_{-\omega}^0 b_j(t)\,dt.$$

For $\mu\in\sigma(\Pi)\setminus\{0\}$, we can write $\mu=e^{\lambda\omega}$ and (6.22) can be reduced to a simpler form

$$\lambda-\sum_{j=1}^k e^{-j\lambda\omega}B_j=\frac{2k(\lambda)\pi i}{\omega}. \tag{6.23}$$

If a Floquet exponent satisfies (6.23) for $k(\lambda)\neq 0$, then $\tilde\lambda=\lambda-2k(\lambda)\pi i/\omega$ is also a Floquet exponent which satisfies (6.23) with $k(\tilde\lambda)=0$. Since we are only interested in the real part of λ, it suffices to consider the equation

$$\Delta(\lambda)\overset{\text{def}}{=}\lambda-\sum_{j=1}^k e^{-j\lambda\omega}B_j=0. \tag{6.24}$$

So, in the scalar case, we can again find the large time behaviour of the solutions of (6.7) when μ_d is a simple dominant eigenvalue of Π. (See Appendix B for conditions for the existence of such a μ_d.)

Corollary 6.3. *Consider the scalar periodic equation given by (6.7). If $\mu_d=e^{\lambda_d\omega}$ is a simple real dominant eigenvalue of Π, then the large time behaviour of the solution $x(t;0,\varphi)$ is given by*

$$\lim_{t\to\infty}\Omega^t_0\big(\tfrac{1}{\mu_d}\big)x(t;0,\varphi)=\Big[1+\sum_{j=1}^k j\mu_d^{-j}\omega B_j\Big]^{-1}\Big(\varphi(0)+\sum_{l=1}^k\int_{-l\omega}^0\frac{1}{\mu_d^l}\Omega^0_\tau\big(\tfrac{1}{\mu_d}\big)b_l(\tau)\varphi(\tau)d\tau\Big),$$

where $\Omega_s^t(1/\mu_d)$ is given by (6.21).

Proof. The fact that $\mu_d = \Omega^0_{-\omega}(1/\mu_d)$ yields

$$\lim_{z\to\mu_d}(z-\mu_d)\big[z-\Omega^0_{-\omega}\big(\tfrac{1}{z}\big)\big]^{-1} = \Big[\frac{d}{dz}\big(z-\Omega^0_{-\omega}\big(\tfrac{1}{z}\big)\big)\Big|_{z=\mu_d}\Big]^{-1}$$
$$= \Big[1+\sum_{j=1}^{k} j\mu_d^{-j}\omega B_j\Big]^{-1}.$$

Since we can also compute

$$\varphi(-\omega)+G_{\varphi,\mu_d}(0) = \frac{1}{\mu_d}\varphi(0)+\sum_{l=1}^{k}\int_{-l\omega}^{0}\frac{1}{\mu_d^{l+1}}\Omega^0_\tau\big(\tfrac{1}{\mu_d}\big)b_l(\tau)\varphi(\tau)d\tau,$$

it follows that P_{μ_d}, defined by (6.20), is given by

$$P_{\mu_d}\varphi(\theta) = \Omega_0^\theta\big(\tfrac{1}{\mu_d}\big)\Big[1+\sum_{j=1}^{k} j\mu_d^{-j}\omega B_j\Big]^{-1}\Big(\varphi(0)+\sum_{l=1}^{k}\int_{-l\omega}^{0}\frac{1}{\mu_d^{l}}\Omega^0_\sigma\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma)d\sigma\Big).$$

Note that from the representation for $P_{\mu_d}\varphi(\theta)$, it follows that $\mathcal{M}_{\mu_d}$ is spanned by

$$\varphi_1(\theta) = \Omega_0^\theta\big(\tfrac{1}{\mu_d}\big), \quad -k\omega \leqslant \theta \leqslant 0.$$

Furthermore $P_\mu\varphi = c(\varphi)\varphi_1$ with $c(\varphi)$ given by

$$c(\varphi) = \Big[1+\sum_{j=1}^{k} j\mu_d^{-j}\omega B_j\Big]^{-1}\Big(\varphi(0)+\sum_{l=1}^{k}\int_{-l\omega}^{0}\frac{1}{\mu_d^{l}}\Omega^0_\sigma\big(\tfrac{1}{\mu_d}\big)b_l(\sigma)\varphi(\sigma)d\sigma\Big).$$

A direct computation shows that

$$\big(T(t,0)\varphi_1\big)(\theta) = \Omega_0^{t+\theta}\big(\tfrac{1}{\mu_d}\big).$$

and $P(t)$, defined in (5.5), is given by

$$P(t)(\theta) = e^{-\lambda_d(t+\cdot)}\Omega_0^{t+\cdot}\big(\tfrac{1}{\mu_d}\big). \tag{6.25}$$

Applying Corollary 5.1, evaluating the functions involved at $\theta = 0$, and using the fact that $P(t)$ in (6.25) is invertible, we arrive at

$$\lim_{t\to\infty}\Omega_0^t\big(\tfrac{1}{\mu_d}\big)x(t;0,\varphi) = c(\varphi).$$

□

Appendix A. Theoretical foundation of the Hale bilinear form

An autonomous neutral equation (2.1) can be translated into a renewal equation by integrating the equation and isolating the part that explicitly depends on the initial condition, the *forcing function*, see [4]. The forcing function belongs to the space $\mathcal{F}$ defined by the set of functions $f:\mathbb{R}_+\to\mathbb{C}^n$ which are of bounded variation, right continuous and satisfying $f(t) = f(h)$ for all $t \geqslant h$. The resulting renewal equation is given by

$$x(t)+\int_0^t\big[d\mu(\theta-t)+\eta(\theta-t)d\theta\big]x(\theta) = F\varphi(t), \tag{A.1}$$

where $F:\mathcal{C}\to\mathcal{F}$, defined by

$$F\varphi(t) = D\varphi+\int_{-r}^{-t}d\mu(\theta)\varphi(t+\theta)+\int_0^t\Big[\int_{-r}^{-s}d\eta(\theta)\varphi(s+\theta)\Big]ds, \quad t\geqslant 0, \tag{A.2}$$

maps the initial condition $\varphi \in \mathcal{C}$ into the correspondent forcing function of (A.1). Equivalently, we can write

$$x(t) = dk * x + F\varphi,$$

where the Borel measure dk is given by

$$k(\theta) = \mu(\theta) + \int_0^\theta \eta(\tau)d\tau.$$

For each $f \in \mathcal{F}$, the equation

$$x + dk * x = f \tag{A.3}$$

has a unique solution $x(\,\cdot\,; f)$ defined on $\mathbb{R}_+$ (see Chapter 9 of [4]). We can define a semigroup $\{S(t)\}_{t \geqslant 0}$ on $\mathcal{F}$ by

$$S(t)f = Fx_t(\,\cdot\,; f), \tag{A.4}$$

that is, $S(t)f$ is the forcing function correspondent to the solution $x_t(\,\cdot\,; f)$ of (A.3). By construction, for $\varphi \in \mathcal{C}$, the solution $x(\,\cdot\,; F\varphi)$ of (A.1) also satisfies (2.1). So the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{F} \\ {\scriptstyle T(t)}\downarrow & & \downarrow{\scriptstyle S(t)} \\ \mathcal{C} & \xrightarrow[F]{} & \mathcal{F} \end{array}$$

Let $\mathcal{C}^*$ denote the dual space of $\mathcal{C}$ and let $\mathbb{C}^{n*}$ denote the real row matrices of dimension n. By the Riesz representation theorem, the elements of $\mathcal{C}^*$ can be represented by functions of bounded variation $g : \mathbb{R}_- \to \mathbb{C}^{n*}$ that are left continuous on $(-r, 0)$ and satisfy $g(0) = 0$ and $g(s) = g(-r)$ for $s \leqslant -r$. The duality pairing between $\mathcal{C}$ and $\mathcal{C}^*$ is given by

$$\langle g, \varphi \rangle \stackrel{\text{def}}{=} \int_{-r}^0 dg(\theta)\varphi(\theta), \qquad \varphi \in \mathcal{C}.$$

If we define $T^*(t) = T(t)^*$ for $t \geqslant 0$, then $T^*(t)$ defines a semigroup (however, generally not strongly continuous; see [1]). One could ask for the differential equation of which $T^*(t)$ is the solution semigroup and it turns out that there is indeed a close connection between $T^*(t)$ and $S^{\mathrm{T}}(t)$, the semigroup defined in a similar manner as $S(t)$, but starting with the transposed equation (2.5).

Similarly as before, we rewrite the transposed equation (2.5) as a renewal equation

$$y(s) + \int_s^0 y(\sigma)\big[d\mu(s-\sigma) + \eta(s-\sigma)d\sigma\big] = F^{\mathrm{T}}\psi(s), \tag{A.5}$$

where

$$F^{\mathrm{T}}\psi(s) = D'\psi + \int_{-r}^s \psi(s-\xi)d\mu(\xi) + \int_s^0 \int_{-r}^\sigma \psi(\sigma-\xi)d\eta(\xi)d\sigma. \tag{A.6}$$

The map F^{T} maps $\psi \in \mathcal{C}'$ onto the space of forcing functions $\mathcal{F}'$ defined by the vector space of functions $g : \mathbb{R}_- \to \mathbb{C}^{n*}$ that are of bounded variation, left continuous and satisfying $g(s) = g(-r)$ for all $s \leqslant -r$.

As before, for each $g \in \mathcal{F}'$, the equation

$$y - y * dk = g, \tag{A.7}$$

has a unique solution $y(\,\cdot\,,g)$ defined in $\mathbb{R}_-$, and we can define a semigroup $S^{\mathrm{T}}(t)$ on $\mathcal{F}'$ by

$$S^{\mathrm{T}}(t)g = F^{\mathrm{T}} y^t(\,\cdot\,,g),$$

that is, $S^{\mathrm{T}}(t)g$ is the forcing function corresponding to the solution $y^t(\,\cdot\,,g)$ of (A.7). By construction, for $\psi \in \mathcal{C}'$, the solution $y(\,\cdot\,,F^{\mathrm{T}}\psi)$ also satisfies (2.5) and therefore the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{F^{\mathrm{T}}} & \mathcal{F}' \\ {\scriptstyle T^{\mathrm{T}}(t)}\big\downarrow & & \big\downarrow{\scriptstyle S^{\mathrm{T}}(t)} \\ \mathcal{C}' & \xrightarrow[F^{\mathrm{T}}]{} & \mathcal{F}' \end{array}$$

One can show that $S^{\mathrm{T}}(t) = T^*(t)$ and therefore

$$F^{\mathrm{T}} T^{\mathrm{T}}(s)\psi = T^*(s) F^{\mathrm{T}}\psi, \qquad \psi \in \mathcal{C}'. \tag{A.8}$$

Relation (A.8) suggests the introduction of a special bilinear form between $\mathcal{C}$ and $\mathcal{C}'$. For $\psi \in \mathcal{C}'$ and $\varphi \in \mathcal{C}$, define

$$\begin{aligned}(\psi,\varphi) &\overset{\text{def}}{=} -\big\langle F^{\mathrm{T}}\psi,\varphi\big\rangle = -\int_{-r}^{0} d[F^{\mathrm{T}}\psi(\theta)]\,\varphi(\theta)\\ &= \psi(0)\psi(0) - \int_{-r}^{0} d_\theta\Big[\int_{-r}^{\theta}\psi(\theta-\xi)d\mu(\xi)\Big]\varphi(\theta)\\ &\qquad\qquad + \int_{-r}^{0}\int_{-r}^{\theta}\psi(\theta-\xi)d\eta(\xi)\varphi(\theta)d\theta.\end{aligned}$$

This is precisely the Hale bilinear form (3.1). Note that with respect to (3.1), the transposed operator A^{T} satisfies

$$(\psi, A\varphi) = (A^{\mathrm{T}}\psi,\varphi), \qquad \psi \in \mathcal{D}\big(A^{\mathrm{T}}\big), \qquad \varphi \in \mathcal{D}\big(A\big).$$

So the bilinear form (3.1) allows us to use the duality between A and A^{T}, instead of, between A and the much more complicated operator A^*.

Appendix B. Properties of roots of characteristic equations

In this appendix, we collect some general properties of roots of characteristic equations. Consider a characteristic equation of the form

$$\Delta(z) = z\Big(1 + \sum_{l=1}^{m} c_l e^{-z\sigma_l}\Big) - a - \sum_{j=1}^{k} b_j e^{-zh_j}, \tag{B.1}$$

where a, b_j $(j = 1,\dots,k)$, c_l $(l = 1,\dots,m)$ are real numbers, and h_j $(j = 1,\dots,k)$, σ_l $(l = 1,\dots,m)$ are positive real numbers.

Given equation (B.1), we introduce a function $V : \mathbb{R} \to \mathbb{R}$, defined by,

$$V(\lambda) = \sum_{l=1}^{m} |c_l|\,(1 + |\lambda|\,\sigma_l)e^{-\lambda\sigma_l} + \sum_{j=1}^{k} |b_j|\,h_j e^{-\lambda h_j}, \qquad \lambda \in \mathbb{R}. \tag{B.2}$$

This function plays a role on estimates of the derivative of $\Delta(z)$ (compare [9, Eq. P(λ_0)]). The next lemma states a sufficient condition for a real root of (B.1) to be simple and dominant.

Lemma B.1. *Suppose that there exists a real zero* λ_0 *of equation (B.1). If*

$$V(\lambda_0) < 1, \tag{B.3}$$

then λ_0 *is a real simple dominant zero of (B.1).*

Proof. Without loss of generality, we can assume that $\lambda_0 = 0$. Therefore, we have

$$\Delta(0) = a + \sum_{j=1}^{k} b_j = 0$$

and condition (B.3) becomes

$$\sum_{l=1}^{m} |c_l| + \sum_{j=1}^{k} h_j \, |b_j| < 1. \tag{B.4}$$

The proof consists of four parts. First we prove that λ_0 is a simple zero of (6.4). Since

$$\frac{d}{dz}\Delta(0) = 1 + \sum_{l=1}^{m} c_l + \sum_{j=1}^{k} h_j b_j,$$

it follows from (B.4) that

$$\left|\frac{d}{dz}\Delta(0)\right| \geqslant 1 - \left|\sum_{l=1}^{m} c_l + \sum_{j=1}^{k} h_j b_j\right| > 0.$$

Thus $\lambda_0 = 0$ is a simple zero of $\Delta(z)$.

Now we prove that there are no other zeros on the imaginary axis. Suppose that for some $\nu \neq 0$, $z = i\nu$ is a zero of $\Delta(z)$. From the equations for the real and imaginary part of $\Delta(i\nu)$, it follows that

$$\begin{cases} \nu \sum_{l=1}^{m} c_l \sin(\nu\sigma_l) - a - \sum_{j=1}^{k} b_j \cos(\nu h_j) = 0, \\ \nu(1 + \sum_{l=1}^{m} c_l \cos(\nu\sigma_l)) + \sum_{j=1}^{k} b_j \sin(\nu h_j) = 0. \end{cases}$$

Since $\nu \neq 0$, we obtain from the equation for the imaginary part that

$$\Big(1 + \sum_{l=1}^{m} c_l \cos(\nu\sigma_l)\Big) + \sum_{j=1}^{k} b_j \frac{\sin(\nu h_j)}{\nu} = 0. \tag{B.5}$$

We can estimate

$$\left|1 + \sum_{l=1}^{m} c_l \cos(\nu\sigma_l)\right| \geqslant 1 - \left|\sum_{l=1}^{m} c_l \cos(\nu\sigma_l)\right| \geqslant 1 - \sum_{l=1}^{m} |c_l| > \sum_{j=1}^{k} h_j \, |b_j| \,. \tag{B.6}$$

On the other hand, using $|\sin(x)| \leqslant |x|$, we have

$$\left|\sum_{j=1}^{k} b_j \frac{\sin(\nu h_j)}{\nu}\right| \leqslant \sum_{j=1}^{k} h_j \, |b_j| \,.$$

Thus we obtain from (B.5) and (B.6) that

$$\sum_{j=1}^{k} h_j \, |b_j| < \left|1 + \sum_{l=1}^{m} c_l \cos(\nu\sigma_l)\right| = \left|\sum_{j=1}^{k} b_j \frac{\sin(\nu h_j)}{\nu}\right| \leqslant \sum_{j=1}^{k} h_j \, |b_j| \,.$$

A contradiction to the assumption that $z = i\nu$, $\nu \neq 0$, is a zero of $\Delta(z)$. This proves that $\lambda = 0$ is the only zero on the imaginary axis.

Next we show that there are no zeros of $\Delta(z)$ with $\operatorname{Re} z > 0$. Suppose that $z = \alpha + i\nu$ satisfies $\Delta(\alpha + i\nu) = 0$. The equations for the real and imaginary part read

$$
\text{(B.7)} \quad \begin{cases} \alpha\big(1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l)\big) + \nu \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \sin(\nu\sigma_l) \\ \qquad\qquad -a - \sum_{j=1}^{k} b_j e^{-\alpha h_j} \cos(\nu h_j) = 0, \\ -\alpha \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \sin(\nu\sigma_l) + \nu\big(1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l)\big) \\ \qquad\qquad + \sum_{j=1}^{k} b_j e^{-\alpha h_j} \sin(\nu h_j) = 0. \end{cases}
$$

If $\nu = 0$ and $\alpha > 0$, then the first equation of (B.7) yields

$$
\alpha\Big(1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\Big) = a + \sum_{j=1}^{m} b_j e^{-\alpha h_j} = \sum_{j=1}^{m} b_j (e^{-\alpha h_j} - 1).
$$

So

$$
\left|1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\right| = \left|\sum_{j=1}^{m} b_j \frac{e^{-\alpha h_j} - 1}{\alpha}\right| \leqslant \sum_{j=1}^{m} |b_j|\, h_j.
$$

On the other hand, using (B.4), we find

$$
\left|1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\right| > \sum_{j=1}^{m} |b_j|\, h_j.
$$

A contradiction to the assumption that $\nu = 0$. Thus we can assume that $\alpha > 0$ and $\nu > 0$. We claim that

$$
\text{(B.8)} \quad \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \sin(\nu\sigma_l) < 0.
$$

Suppose first that the claim holds. Then it follows from the second equation of (B.7) that

$$
1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l) < -\sum_{j=1}^{k} b_j e^{-\alpha h_j} \frac{\sin(\nu h_j)}{\alpha} \leqslant \sum_{j=1}^{k} h_j |b_j|.
$$

On the other hand, using (B.4), we have

$$
1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l) \geqslant 1 - \left|\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l)\right| \geqslant 1 - \sum_{l=1}^{m} |c_l| > \sum_{j=1}^{k} h_j |b_j|.
$$

A contradiction. Thus, if (B.8) holds, there are no zeros of $\Delta(z)$ with $\operatorname{Re} z > 0$.

To prove (B.8) (still assuming $\alpha > 0$), we first take the following combinations of (B.7). The combination α times the first plus ν times the second equation of (B.7) yields

$$
\text{(B.9)} \quad (\alpha^2 + \nu^2)\Big(1 + \sum_{l=1}^{m} c_l e^{-\alpha\sigma_l} \cos(\nu\sigma_l)\Big) \\ + \alpha \sum_{j=1}^{k} b_j \big(e^{-\alpha h_j} \cos(\nu h_j) - 1\big) + \nu \sum_{j=1}^{m} b_j e^{-\alpha h_j} \sin(\nu h_j) = 0,
$$

and the combination ν times the first minus α times the second equation of (B.7) yields

$$(\alpha^2+\nu^2)\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) + \nu\sum_{j=1}^{k} b_j(e^{-\alpha h_j}\cos(\nu h_j)-1) - \alpha\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j) = 0. \tag{B.10}$$

From the first equation of (B.7) and the fact that $1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\cos(\nu\sigma_l)>0$, it follows that

$$\nu\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) \leqslant \sum_{j=1}^{k} b_j(e^{-\alpha h_j}\cos(\nu h_j)-1). \tag{B.11}$$

From the second equation of (B.7) and the fact that $1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\cos(\nu\sigma_l)>0$, it follows that

$$\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j) < \alpha\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l). \tag{B.12}$$

From (B.9) and the fact that $1+\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\cos(\nu\sigma_l)>0$,

$$\alpha\sum_{j=1}^{k} b_j(e^{-\alpha h_j}\cos(\nu h_j)-1) < -\nu\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j). \tag{B.13}$$

From (B.11) and (B.13), it follows that

$$\alpha\nu\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) < -\nu\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j)$$

and hence

$$\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j) < -\alpha\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l). \tag{B.14}$$

From (B.12) and (B.14), it now follows

$$\sum_{j=1}^{k} b_j e^{-\alpha h_j}\sin(\nu h_j) < 0$$

and hence, from (B.10),

$$(\alpha^2+\nu^2)\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) < -\nu\sum_{j=1}^{k} b_j(e^{-\alpha h_j}\cos(\nu h_j)-1). \tag{B.15}$$

Finally, from (B.11) and (B.15), it follows that

$$\sum_{l=1}^{m} c_l e^{-\alpha\sigma_l}\sin(\nu\sigma_l) < 0$$

and this proves the claim (B.8).

Thus we have proved that $\lambda_0=0$ is the only zero of $\Delta(z)$ with $\operatorname{Re} z \geqslant 0$. To complete the proof of the lemma, it remains to prove that $\lambda_0=0$ is dominant.

Suppose to the contrary that, for every $\delta > 0$, there exists a zero of $\Delta(z)$ in the strip $-\delta < \operatorname{Re} z < 0$. Since $\Delta(z)$ is an entire function, its zeros cannot have a finite accumulation point. So it follows that $\Delta(z)$ has a sequence of zero's λ_n such that $\operatorname{Re}\lambda_n$ tends to zero and $|\operatorname{Im}\lambda_n|$ tends to infinity. From (B.4) it follows that for $\operatorname{Re}\lambda_n$ sufficiently small, there exists an $\epsilon > 0$ such that

$$\left|1 + \sum_{l=1}^{m} c_l e^{-\lambda_n \sigma_l}\right| \geqslant 1 - \sum_{l=1}^{m} |c_l|\, e^{-\operatorname{Re}\lambda_n \sigma_l} > \epsilon.$$

Thus

$$\epsilon\, |\lambda_n| < a + \sum_{j=1}^{k} |b_j|\, e^{-\operatorname{Re}\lambda_n h_j}. \tag{B.16}$$

As n tends to infinity the left hand side of (B.16) tends to infinity while the right hand side of (B.16) remains bounded. A contradiction to the assumption that, every $\delta > 0$, there exists a zero of $\Delta(z)$ in the strip $-\delta < \operatorname{Re} z < 0$. This completes the proof that $\lambda_0 = 0$ is dominant simple zero of $\Delta(z)$. □

Remark B.1. Actually, condition (B.3) is sharp. This can be seen by considering the following example

$$\frac{d}{dt}\Big[x(t) - \frac{1}{2}x(t-1)\Big] = \frac{1}{2}x(t) - \frac{1}{2}x(t-1). \tag{B.17}$$

The characteristic equation associated with (B.17) is given by

$$z\Big(1 - \frac{1}{2}e^{-z}\Big) = \frac{1}{2} - \frac{1}{2}e^{-z}.$$

So $c_1 = -1/2$, $a = 1/2$, $b_1 = -1/2$, $\sigma_1 = 1$ and $h_1 = 1$. Thus $a + b_1 = 0$ and $\lambda_0 = 0$ is a real zero. Since $|c_1| + h_1|b_1| = 1$, condition (B.3) fails, and if we differentiate the characteristic equation, it follows that $\lambda_0 = 0$ is not a simple zero.

The next lemma gives sufficient conditions for the existence of a simple and dominant real root of (B.1).

Lemma B.2. *If there exists γ such that*

$$\Delta(\gamma) < 0 \qquad \text{and} \qquad V(\gamma) \leqslant 1,$$

then there exists a unique real root λ_0 of Δ in (γ, ∞) such that $V(\lambda_0) < 1$. Therefore λ_0 is a simple real dominant root of (B.1).

Proof. First of all, we observe that V is positive and strictly decreasing. Indeed, since the functions $h_1(x) = e^{-x}$ and $h_2(x) = (1 + |x|)e^{-x}$ are positive and strictly decreasing for real x, we obtain that functions defined by linear combinations with positive coefficients of h_1 and h_2, with possible positive rescales of the variable x, are positive and strictly decreasing.

Computing the derivative of Δ for $\lambda > \gamma$, we obtain that

$$\begin{aligned}
\frac{d}{d\lambda}\Delta(\lambda) &= 1 + \sum_{l=1}^{m} c_l e^{-\lambda\sigma_l} - \lambda \sum_{l=1}^{m} c_l \sigma_l e^{-\lambda\sigma_l} + \sum_{j=1}^{k} b_j h_j e^{-\lambda h_j} \\
&\geqslant 1 - \Big(\sum_{l=1}^{m} |c_l|\,(1 + |\lambda|\,\sigma_l)e^{-\lambda\sigma_l} + \sum_{j=1}^{k} |b_j|\, h_j e^{-\lambda h_j}\Big) \\
&= 1 - V(\lambda) > 0,
\end{aligned}$$

since $V(\lambda) < V(\gamma) \leqslant 1$. Therefore $\Delta(\lambda)$ is strictly increasing for $\lambda > \gamma$. It is easy to see that

$$\lim_{\lambda\to\infty} \Delta(\lambda) = \infty.$$

This together with the fact that $\Delta(\gamma) < 0$ implies the existence of a unique real root λ_0 of Δ with $\lambda_0 > \gamma$. Since V is strictly decreasing, it follows that $V(\lambda_0) < 1$. The last assertion now follows from Lemma B.1. □

In a simpler case, we can improve the conclusions of Lemma B.2 even further. The next lemma is used in Theorem 6.1 and improves the results in [2] and [9].

Lemma B.3. *If $-e^{-1} < b\tau e^{-a\tau}$, then the equation*

$$\Delta(z) = z - a - be^{-\tau z} \tag{B.18}$$

has a simple real dominant zero $z = \lambda_d$.

Proof. Suppose first that $-e^{-1} < b\tau e^{-a\tau} < 0$. This implies that $b < 0$. If we define $\gamma = a - 1/\tau$, then

$$\Delta(\gamma) = \frac{1}{\tau} - be^{-a\tau-1} < 0,$$

and (recall equation (B.2))

$$V(\gamma) = |b|\,\tau e^{-a\tau+1} = -b\tau e^{-a\tau+1} < 1.$$

So the hypotheses of Lemma B.2 are satisfied and the existence of the simple real dominant zero λ_d of Δ follows.

Before we continue with the case $b > 0$, we need to study how the location of the real roots of Δ depends on b. If λ is a real zero of Δ, then

$$\lambda = a + be^{-\tau\lambda}.$$

Observe that, if $b < 0$ then $\lambda < a$ and if $b > 0$, then $\lambda > a$.

Moreover, for $b > 0$, such real λ always exists, is simple and unique, because $\Delta(a) < 0$,

$$\lim_{z\to\infty} \Delta(z) = \infty$$

and for all z real

$$\frac{d}{d\lambda}\Delta(z) = 1 + b\tau e^{-\tau z} > 0.$$

Denote this root by λ_d. Fix a and consider λ_d as a function of $b > 0$. If we differentiate $\Delta\big(\lambda_d(b)\big)$ with respect to b, we obtain

$$\frac{d\lambda_d}{db} = \frac{e^{-\tau\lambda_d}}{1 + b\tau e^{-\tau\lambda_d}} > 0.$$

Therefore $b \mapsto \lambda_d(b)$ is strictly increasing on $[0,\infty)$.

With these preliminaries, we can now show that in case $b > 0$, the root λ_d is also dominant. Suppose that $z = x + iy$ is another zero of Δ with x, y real numbers and $y > 0$. The equations for the real and imaginary parts for z are given by

$$x - a - b\cos(\tau y)e^{-\tau x} = 0, \tag{B.19}$$

$$y + be^{-\tau x}\sin(\tau y) = 0. \tag{B.20}$$

Define $\bar{b} = b\cos(\tau y)$. Equation (B.19) becomes

$$x - a - \bar{b}e^{-\tau x} = 0. \tag{B.21}$$

If $\bar{b} \leqslant 0$, then (B.21) and the arguments just given imply that $x \leqslant a < \lambda_d$. If $\bar{b} > 0$ then equation (B.20) and $y > 0$ imply that $\tau y \neq k\pi$ for all k integer and therefore $\bar{b} < b$, but this again implies that $x < \lambda_d$.

In order to show that there exists an $\epsilon > 0$ such that actually all roots with $z \neq \lambda_d \in \mathbb{C}$ satisfies $\operatorname{Re} z < \lambda_d - \epsilon$, we argue by contradiction and suppose that such ϵ does not exist. So there is a sequence z_n of zeros of Δ such that $\operatorname{Re} z_n \to \lambda_d$. Since Δ is an analytic function, its zeros are isolated, and therefore the only possibility is that $|\operatorname{Im} z_n| \to \infty$, but (B.20) shows that $\operatorname{Im} z_n$ is in fact bounded. A contradiction. Therefore also in the case $b > 0$, λ_d is a simple real dominant zero of Δ. This completes the proof of the lemma. □

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