On the dominance of roots of characteristic equations for neutral functional differential equations

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Abstract

We present a sufficient condition for a zero of a function that arises typically as the characteristic equation of a linear functional differential equations of neutral type, to be simple and dominant. This knowledge is useful in order to derive the asymptotic behaviour of solutions of such equations. A simple characteristic equation, arisen from the study of delay equations with small delay, is analyzed in greater detail.

In the study of the solutions for linear autonomous functional differential equations (FDE), one can derive important information of its asymptotic properties from the spectral properties of the infinitesimal generator of the solution semigroup. Such spectrum is formed by point spectrum only, as the zeros of an entire function, called the *characteristic function*. Verduyn Lunel [12] showed that the solution of linear autonomous FDE of neutral type can be written down as a series expansion of the solution operator restricted to the generalized eigenspaces, images of the so called *spectral projection* into such invariant eigenspaces. In Frasson & Verduyn Lunel [5], the large time behaviour of solutions was derived, based on the knowledge of a dominant eigenvalue, that is, an eigenvalue so that its real part is sufficiently larger, uniformly, than the real part of all other eigenvalues. Analogous results can

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be obtained for a finite set of eigenvalues which dominates the others. We observe that characteristic equations of the same type we study here can arise from other types of FDE, like differential-difference equations with periodic coefficients. (See for instance [5].)

To show dominance of a root of characteristic equation is often not easy. Our main result is the presentation of a sufficient condition, given by conditions (9)–(10), for a zero of a characteristic function to be simple and dominant. This information, combined with results in [5], yields to a precise description of the large time behaviour of solutions of the FDE. These results are a generalization of results in [5], where the characteristic equations where restricted to those with discrete delays, like in equation (7).

The conditions that we derive have been first obtained by Driver *et al.* [3] in their study of delay equations with small delay. Following these results, Arino & Pituk [1] and Faria & Huang [4] provided improvements of results in [3]. Kordonis, Niyianni & Philos [8] and Philos & Purnaras [10, 9] obtained interesting results on the asymptotic behaviour of solutions if a zero of the characteristic equation satisfies the same conditions as we impose. The advantage of our approach is that not all dominant eigenvalues satisfy those restrictive conditions, but still one can obtain information about the large time behaviour of solutions. For instance, all non-simple dominant eigenvalues cannot satisfy the conditions. Theorem 7 provides an example of this situation, where we analyze the dominance of a simple characteristic equation, and show the dominance of a root, further than the limits of conditions (9)–(10).

This paper is organized in the following way. In section 1 we present shortly linear autonomous FDE and the spectral properties associated to the solution semigroup. In section 2 we prove our main result, namely a sufficient condition for a zero of the characteristic equation to be simple and dominant, with an estimate for a spectral gap. We also present an application of the results in the large time behavior of solutions of FDE, obtained in connection with results in [5]. In section 3, inspired by [3] and the forecited subsequent works, we analyze in greater detail a delay equation with one time lag and constant coefficients, improving results in [3, 8, 5].

1 Introduction to FDE

Let $\mathcal{C} = \mathcal{C}\big([-r,0],\mathbb{C}\big)$ denote the Banach space of continuous complex-valued functions from $[-r,0]$ ($r > 0$) endowed with the supremum norm. From the Riesz representation theorem (see for instance Rudin [11]) it follows that every bounded linear mapping $L : \mathcal{C} \to \mathbb{C}$ can be represented by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \tag{1}$$

where η is a complex valued function of bounded variation on $[0,r]$ normalized so that $\eta(0) = 0$ and η is continuous from the right in $(0,r)$. As usual in the theory of delay equations, for a continuous complex-valued function x defined in $[-r,\infty)$, we define $x_t \in \mathcal{C}$ by $x_t(\theta) = x(t+\theta)$, $-r \leqslant \theta \leqslant 0$ and $t \geqslant 0$.

Consider the scalar linear autonomous functional differential equation

$$\tfrac{d}{dt} Mx_t = Lx_t, \qquad t \geqslant 0, \tag{2}$$

subjected to initial condition $x_0 = \varphi \in \mathcal{C}$, where $L, M : \mathcal{C} \to \mathbb{C}$ are linear continuous, given respectively by

$$L\varphi = \int_0^r d\eta(\theta)\varphi(-\theta), \qquad M\varphi = \varphi(0) - \int_0^r d\mu(\theta)\varphi(-\theta). \tag{3}$$

For more about such differential equations, we refer to [2, 6]. When functions μ and η are step functions, FDE (2) assumes the form

$$\frac{d}{dt}\left[x(t) + \sum_{l=1}^{m} c_l x(t-\sigma_l)\right] = ax(t) + \sum_{j=1}^{k} b_j x(t-\tau_j). \tag{4}$$

In order to ensure existence and uniqueness of solutions of (2) for $t \geqslant 0$, one can impose the additional hypothesis that

$$\lim_{\theta \downarrow 0} \mu(\theta) = 0, \tag{5}$$

which is fulfilled, for instance, when the operator M is a difference operator, or equivalently when μ is a step function. This hypothesis is sufficient, but not necessary. See [5] for more. Our results on the dominance are independent of hypothesis (5).

Let $T(t)$ be the strongly continuous semigroup defined by $T(t)\varphi = x_t$, where $x(\cdot)$ is the solution of (2) subjected to the initial condition $x_0 = \varphi$ and let A be its infinitesimal generator. It is known that there is a close connection between the spectral properties of A and the characteristic function $\Delta(z)$ given by

$$\Delta(z) = z\Big[1 - \int_0^r e^{-z\theta} d\mu(\theta)\Big] - \int_0^r e^{-z\theta} d\eta(\theta). \tag{6}$$

When FDE (2) is in the form (4), the characteristic function takes the form

$$\Delta(z) = z\left(1 + \sum_{l=1}^{n} c_l e^{-z\sigma_l}\right) + a + \sum_{j=1}^{m} b_j e^{-z\tau_j}. \tag{7}$$

One can compute explicitly the resolvent of A and derive that the spectrum $\sigma(A)$ of A is point spectrum only, formed by the zeros of $\Delta(\cdot)$. For each eigenvalue $\lambda \in \sigma(A)$, its ascent k_λ is the order of λ as zero of $\Delta(\cdot)$, and one has the decomposition $\mathcal{C} = \mathcal{M}_\lambda \oplus \mathcal{Q}_\lambda$. The generalized eigenspace $\mathcal{M}_\lambda = \ker(\lambda I - A)^{k_\lambda}$ and $\mathcal{Q}_\lambda$ are invariant under $T(t)$ and $\mathcal{M}_\lambda$ is a finite dimensional subspace of $\mathcal{C}$, of dimension k_λ. In Kaashoek & Verduyn Lunel [7] and in Section IV.3 of Diekmann *et al.* [2] a systematic procedure has been developed to construct a canonical basis for $\mathcal{M}_\lambda$ using Jordan chains. From the spectral theory, it follows that the spectral projection P_λ into $\mathcal{M}_\lambda$ is the residue of $(zI - A)^{-1}$ at $z = \lambda$. See [13] and [5, sec. 3] for more.

An eigenvalue λ_d of A is said to be *dominant* if there is an $\epsilon > 0$ such if $\lambda \neq \lambda_d$ is another eigenvalue of A, then $\Re\lambda < \Re\lambda_d - \epsilon$. Such ϵ is called a *spectral gap* for $\Delta(\cdot)$. From [13] (see also [5, Lemma 2.1]) we obtain a characterization of the large time behaviour of solutions of FDE (2) as follows. Let λ_d be a dominant eigenvalue of A. Then for a spectral gap $\epsilon > 0$, there exists $K > 0$, such that

$$\|(I - P_{\lambda_d})x_t(\varphi)\| \leqslant K e^{(\Re\lambda_d - \epsilon)t} \|(I - P_{\lambda_d})\varphi\|, \quad t \geqslant 0, \ \varphi \in \mathcal{C}. \tag{8}$$

2 Sufficient conditions of dominance of roots of characteristic equations

Consider the characteristic function $\Delta(\cdot)$ given by (6). For such a function, define the auxiliary function $V : \mathbb{C} \to [0, \infty)$ by

$$V(z) = \int_0^r \big(1 + |z|\theta\big) \left|e^{-z\theta}\right| d|\mu|(\theta) + \int_0^r \theta \left|e^{-z\theta}\right| d|\eta|(\theta), \tag{9}$$

where $|\mu|$ and $|\eta|$ denote respectively the total variation functions of μ and η in (6). Our main result is the following theorem.

Theorem 1. *Suppose that $z_0 \in \mathbb{C}$ is a zero of $\Delta(\cdot)$ in (6) such that*

$$V(z_0) < 1. \tag{10}$$

Then z_0 is a simple dominant zero of $\Delta(\cdot)$.

Proof. In order to show that z_0 is a simple zero of $\Delta(\cdot)$, we compute

$$\frac{d}{dz}\Delta(z) = 1 + \int_0^r e^{-z\theta}(-1 + z\theta) d\mu(\theta) + \int_0^r \theta e^{-z\theta} d\eta(\theta).$$

Then we can estimate

$$\begin{aligned}\left|\frac{d}{dz}\Delta(z_0)\right| &\geqslant 1 - \left|\int_0^r e^{-z_0\theta}(-1 + z_0\theta) d\mu(\theta) + \int_0^r \theta e^{-z_0\theta} d\eta(\theta)\right| \\ &\geqslant 1 - V(z_0) > 0,\end{aligned}$$

what shows that z_0 is a simple zero of $\Delta(\cdot)$. It remains to prove that z_0 is dominant. In order to do so, let $0 < \delta < 1$ such that

$$V(z_0) < \delta. \tag{11}$$

We can rewrite (11) to obtain

$$\begin{aligned}&1 - \frac{1}{\delta}\int_0^r \left|e^{-z_0\theta}\right| d|\mu|(\theta) \\ &\qquad > \frac{1}{\delta}\left[|z_0| \int_0^r \theta \left|e^{-z_0\theta}\right| d|\mu|(\theta) + \int_0^r \theta \left|e^{-z_0\theta}\right| d|\eta|(\theta)\right].\end{aligned} \tag{12}$$

Let $\epsilon > 0$ such that

$$1 < e^{\epsilon r} \leqslant \frac{1}{\delta}, \tag{13}$$

and let Λ be the right half plane given by

$$\Lambda = \{z \in \mathbb{C} : \Re z > \Re z_0 - \epsilon\}. \tag{14}$$

For $z \in \Lambda$ and $0 \leqslant \theta \leqslant r$ we have that

$$|e^{-z\theta}| \leqslant e^{-\Re z_0 \theta} e^{\epsilon\theta} \leqslant \frac{\left|e^{-z_0\theta}\right|}{\delta}. \tag{15}$$

If $z \in \Lambda$, let γ the line segment that connects z_0 to z, which is contained in Λ. Therefore, for $0 \leqslant \theta \leqslant r$, from (15),

$$\left|e^{-z_0\theta} - e^{-z\theta}\right| = \left|\int_{z_0}^{z} \theta\, e^{-w\theta} dw\right| = \theta \left|\int_{\gamma} e^{-w\theta} dw\right| \leqslant \frac{\left|e^{-z_0\theta}\right|}{\delta} |z - z_0|\theta. \quad (16)$$

Since $\Delta(z_0) = 0$, we obtain that

$$z_0 = z_0 \int_0^r e^{-z_0\theta} d\mu(\theta) + \int_0^r e^{-z_0\theta} d\eta(\theta), \quad (17)$$

which we can use to rewrite $\Delta(z)$ as follows.

$$\begin{aligned}\Delta(z) &= (z - z_0)\Big[1 - \int_0^r e^{-z\theta} d\mu(\theta)\Big] + z_0 - z_0 \int_0^r e^{-z\theta} d\mu(\theta) - \int_0^r e^{-z\theta} d\eta(\theta) \\ &= (z - z_0)\Big[1 - \int_0^r e^{-z\theta} d\mu(\theta)\Big] \\ &\qquad + z_0 \int_0^r \big(e^{-z_0\theta} - e^{-z\theta}\big) d\mu(\theta) + \int_0^r \big(e^{-z_0\theta} - e^{-z\theta}\big) d\eta(\theta).\end{aligned}$$

Hence we can estimate

$$\begin{aligned}|\Delta(z)| \geqslant |z - z_0| &\left|1 - \int_0^r e^{-z\theta} d\mu(\theta)\right| \\ &- \left|z_0 \int_0^r \big(e^{-z_0\theta} - e^{-z\theta}\big) d\mu(\theta) + \int_0^r \big(e^{-z_0\theta} - e^{-z\theta}\big) d\eta(\theta)\right|. \quad (18)\end{aligned}$$

For $z \in \Lambda$, from (15), we get

$$\left|1 - \int_0^r e^{-z\theta} d\mu(\theta)\right| \geqslant 1 - \left|\int_0^r e^{-z\theta} d\mu(\theta)\right| \geqslant 1 - \frac{1}{\delta} \int_0^r \left|e^{-z_0\theta}\right| d|\mu|(\theta). \quad (19)$$

From (12) and (19), we obtain

$$\left|1 - \int_0^r e^{-z\theta} d\mu(\theta)\right| > \frac{1}{\delta}\Big[|z_0| \int_0^r \theta \left|e^{-z_0\theta}\right| d|\mu|(\theta) + \int_0^r \theta \left|e^{-z_0\theta}\right| d|\eta|(\theta)\Big]. \quad (20)$$

Using (16), we estimate

$$\begin{aligned}&\left|z_0 \int_0^r \big(e^{-z_0\theta} - e^{-z\theta}\big) d\mu(\theta) + \int_0^r \big(e^{-z_0\theta} - e^{-z\theta}\big) d\eta(\theta)\right| \\ &\qquad \leqslant \frac{|z - z_0|}{\delta}\Big[|z_0| \int_0^r \theta \left|e^{-z_0\theta}\right| d|\mu|(\theta) + \int_0^r \theta \left|e^{-z_0\theta}\right| d|\eta|(\theta)\Big]. \quad (21)\end{aligned}$$

Finally, if $z \in \Lambda$ and $|z - z_0| > 0$, from (18), (20) and (21) we get

$$|\Delta(z)| > 0. \tag{22}$$

Hence the only zero of $\Delta(\cdot)$ inside the right half plane Λ is z_0. This completes the proof. □

Remark 2. Estimate (8) shows that an estimate for the spectral gap provides an estimate of the order (exponential type) in which the solution approaches the predicted asymptotic behaviour. Under the condition $V(z_0) < 1$, from equations (11), (13), (14) and (22) we obtain that any ϵ such that

$$0 < \epsilon < \frac{1}{r} \ln \left(V(z_0)^{-1} \right) \tag{23}$$

is a spectral gap for the dominant root z_0.

The analogous result for the particular case where FDE has the form (4), with characteristic function (7), was presented in [5]. However, its proof was shown to have mistakes. Theorem 1 presents a correct proof for that result, that we state here as a corollary, without imposing the restriction of z_0 to be real.

Corollary 3. *If z_0 is a zero of* (7) *such that*

$$\sum_{l=1}^{n} |c_l|(1 + |z_0|\sigma_l) \left| e^{-z_0\sigma_l} \right| + \sum_{j=1}^{m} |b_j|\tau_j \left| e^{-z_0\tau_j} \right| < 1, \tag{24}$$

then z_0 is a simple dominant zero of (7).

Proof. The characteristic function $\Delta(\cdot)$ given in (7) can be put in the form (6) if we set the functions η and μ as step functions respectively with "jumps" of size c_l at σ_l and b_j at τ_j. Hence the total variation functions $|\eta|$ and $|\mu|$ are equally step functions with jumps of size $|c_l|$ at σ_l and $|b_j|$ at τ_j respectively. Then condition $V(z_0) < 1$ on Theorem 1 reads as (24) and the result follows from that theorem. □

Once the dominance of a root of the characteristic equation

$$\Delta(z) = 0$$

is obtained, where $\Delta(z)$ is given in(6), we can apply Corollary 4.1 of [5] to obtain the following characterization for the large time behaviour of solutions of FDE (2).

Theorem 4. *Let $x(\cdot)$ be the solution of* (2)*, subjected to the initial condition $x_0 = \varphi \in \mathcal{C}$. If λ_d is a zero of the characteristic equation $\Delta(z)$ given by* (6) *such that $V(\lambda_d) < 1$, where $V(\cdot)$ is given by* (9)*, then the asymptotic behaviour of $x(\cdot)$ is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = \frac{1}{H(\lambda_d)}G(\lambda_d,\varphi), \tag{25}$$

where

$$H(\lambda_d) = 1 + \lambda_d - \int_0^r e^{-\lambda_d\theta}d\mu(\theta) + \int_0^r \theta e^{-\lambda_d\theta}d[\lambda_d\mu(\theta) + d\eta(\theta)] \tag{26}$$

and

$$G(\lambda_d,\varphi) = M\varphi + \int_{-r}^0 [\lambda_d d\mu(\tau) + d\eta(\tau)] \int_0^{-\tau} e^{-\lambda_d\sigma}\varphi(\sigma+\tau)d\sigma. \tag{27}$$

Furthermore, for any ϵ such that

$$0 < \epsilon < \frac{1}{r}\ln\left(V(z_0)^{-1}\right),$$

we have that

$$e^{-\lambda_d t}x(t) - \frac{1}{H(\lambda_d)}G(\lambda_d,\varphi) = o(e^{-\epsilon t}) \tag{28}$$

as $t\to\infty$.

Proof. The result follows from Corollary 4.1 of [5]. Alternatively one can compute the spectral projection on the one dimensional space $\mathcal{M}_{\lambda_d}$ and use estimate (8) to compute the large time behaviour of the solution as described in (28). □

Remark 5. In Theorem 4, if the FDE is the particular form (4), then condition (10) reads as (24), and $H(\lambda_d)$ in (26) and $G(\lambda_d,\varphi)$ in (27) assume the form

$$H(\lambda_d) = 1 + \sum_{l=1}^m c_l e^{\lambda_d\sigma_l} - \lambda_d\sum_{l=1}^m c_l\sigma_l e^{\lambda_d\sigma_l} + \sum_{j=1}^k b_j\tau_j e^{\lambda_d\tau_j}$$

and

$$G(\lambda_d,\varphi) = \varphi(0) - \sum_{l=1}^m c_l\varphi(-\sigma_l) + \sum_{l=1}^m \lambda_d c_l \int_0^{-\sigma_l} e^{-\lambda_d s}\varphi(s+\sigma_l)ds + \sum_{j=1}^k b_j \int_0^{-\tau_j} e^{-\lambda_d s}\varphi(s+\tau_j)ds.$$

3 A simple discrete FDE

Now let us restrict our attention to the FDE

$$\dot{x}(t) = ax(t) + bx(t - \tau), \quad t \geqslant 0 \tag{29}$$

where $a, b \in \mathbb{R}$ are real numbers and $b \neq 0$. The asymptotic behaviour of the solutions of this functional differential equation has being studied by Driver *et al.* [3] and Kordonis, Niyianni & Philos [8]. Its characteristic equation reads as

$$\lambda - a - be^{-\tau\lambda} = 0. \tag{30}$$

Applying Corollary 3 and results in [5], we are able to reproduce the forecited works. However, investigating further the dominance of the real roots of (30), we can extend these results.

For $x \in [-1, \infty)$ we have that $x \mapsto xe^x$ is a strictly increasing function with range $[-e^{-1}, \infty)$. We denote its inverse as $w(\cdot)$[1]. In symbols,

$$y = xe^x,\ x \geqslant -1 \iff x = w(y). \tag{31}$$

For $-e^{-1} < y < 0$, the equation

$$y = xe^x \tag{32}$$

has two real solutions. We observe that $w(y)$ is the larger one. If $y < -e^{-1}$ then (32) admits no real solution x.

Theorem 6. *If $-e^{-1} \leqslant b\tau e^{-a\tau}$, then the equation*

$$\Delta(z) = z - a - be^{-\tau z} \tag{33}$$

has a real dominant zero $z = \lambda_d$ given by

$$\lambda_d = a + \frac{1}{\tau} w(b\tau e^{-a\tau}), \tag{34}$$

where the function w is given by (31)*. Furthermore, if $-e^{-1} < b\tau e^{-a\tau}$, then λ_d is a simple root of* (33)*; if $-e^{-1} = b\tau e^{-a\tau}$, then λ_d is a double*

[1]Function $w(\cdot)$ is known as the Lambert W function. In the computer algebra systems Maple and Mathematica, we can access this function under the names `LambertW` and `ProductLog` respectively.

root; if $-e^{-1} > b\tau e^{-a\tau}$ *then there are no real roots of* (33)*. Furthermore, if* $-e^{-1} < b\tau e^{-a\tau} < e$*, then we have that every* ϵ *such that*

$$0 < \epsilon < \frac{1}{\tau} \ln \left(|w(b\tau e^{-a\tau})|^{-1} \right) \tag{35}$$

is a spectral gap for λ_d*.*

Proof. We can rewrite $\Delta(z) = 0$ in the following equivalent way

$$(z-a)\tau e^{(z-a)\tau} = b\tau e^{-a\tau}. \tag{36}$$

Then representation (34) for λ_d and the non existence of real roots of (33) when $-e^{-1} > b\tau e^{-a\tau}$ follow from the discussions about the function w. Differentiating $\Delta(z)$ with respect to z, evaluating it at $z = \lambda_d$ and using the properties of the function $w(\cdot)$, we get

$$\frac{d}{dz}\Delta(\lambda_d) = 1 + w(b\tau e^{-a\tau}).$$

Therefore $\frac{d}{dz}\Delta(\lambda_d) = 0$ if and only if $-e^{-1} = b\tau e^{-a\tau}$. We have $|\frac{d^2}{dz^2}\Delta(z)| > 0$ for all $z \in \mathbb{C}$. Hence we have shown the statements about the order of λ_d as zero of $\Delta(z)$. Notice also that it follows from representation (34) that $b \mapsto \lambda_d$ is a strictly increasing function of b, and $\lambda_d = a$ for $b = 0$.

If $-e^{-1} > b\tau e^{-a\tau}$, the proof that λ_d is a dominant zero of (30) is contained in [5, Lemma B.3]. Recalling (9), we have

$$V(\lambda_d) = |b|\tau e^{-\lambda_d \tau} = |w(b\tau e^{-a\tau})|. \tag{37}$$

When $-e^{-1} < b\tau e^{-a\tau} < e$, we have that $|w(b\tau e^{-a\tau})| < 1$. From (37) and from Remark 2, we obtain estimate (35).

It only remains to show that λ_d is dominant in the case $-e^{-1} = b\tau e^{-a\tau}$. Here we have $\lambda_d = a - 1/\tau$. From the continuity of roots with respect of parameters a and b and the discussion in the previous cases, we have that there is no zero z of $\Delta(z) = 0$ such that $\Re z > a - 1/\tau$. Suppose that $z = (a - 1/\tau) + iy/\tau$ is a root of (36), so that $(z-a)\tau = -1 + iy$. Taking the equations for the real and imaginary part of (36) we obtain respectively

$$\cos y + y \sin y = 1, \qquad -\sin y + y \cos y = 0.$$

Squaring both equations and summing, we obtain that $y = 0$. Therefore all roots $z \neq a - 1/\tau$ of $\Delta(z) = 0$ satisfy $\Re z < a + 1/\tau$. The spectral gap always exists in the case of retarded FDE, since there is a finite number roots z such that $\Re z > \gamma$, for every $\gamma \in \mathbb{R}$. This completes the proof of the theorem. $\square$

We finish applying Theorem 6 to obtain asymptotic behaviour of solutions to the FDE (29).

Theorem 7. *Let $x(t)$ be the solution of FDE (29) subjected to the initial condition $x_0 = \varphi \in \mathcal{C}([-\tau, 0], \mathbb{C})$.*

1. *if $-e^{-1} < b\tau e^{-a\tau}$, then the asymptotic behaviour of $x(t)$ is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t} x(t) = \frac{1}{1 + w(b\tau e^{-a\tau})} \left[\varphi(0) + b \int_0^\tau e^{-\lambda_d s} \varphi(s-\tau) ds \right], \quad (38)$$

   where $\lambda_d = a + w(b\tau e^{-a\tau})/\tau$;

2. *if $-e^{-1} = b\tau e^{-a\tau}$, let $\lambda_d = a - 1/\tau$; then the asymptotic behaviour of $x(t)$ as $t \to \infty$ is as follows*

   (a) *if $\varphi(0) + b \int_0^\tau e^{-\lambda_d s} \varphi(s-\tau) ds \neq 0$ then*

$$\lim_{t\to\infty} \frac{e^{-\lambda_d t}}{t} x(t) = \frac{2}{\tau} \left[\varphi(0) + b \int_0^\tau e^{-\lambda_d s} \varphi(s-\tau) ds \right],$$

   (b) *otherwise*

$$\lim_{t\to\infty} e^{-\lambda_d t} x(t) = \frac{2}{3} \left[\varphi(0) + b \int_0^\tau \Big(1 - \frac{3s}{\tau}\Big) e^{-\lambda_d s} \varphi(s-\tau) ds \right].$$

Proof. From Theorem 6, if $-e^{-1} \leqslant b\tau e^{-a\tau}$, then $\lambda_d = a + w(b\tau e^{-a\tau})/\tau$ is a dominant root of the characteristic equation associated to FDE (4), given by (30). Item 1 follows from Theorem 4 and Theorem 6. Furthermore, from Theorem 6, if $-e^{-1} < b\tau e^{-a\tau} < e$, we can estimate the size of the spectral gap and obtain an estimate of type (35).

For item 2, from Theorem 6, it follows that $\lambda_d = a - 1/\tau$ is a dominant double root of the characteristic equation (30). We can compute the spectral projection P_{λ_d}, as explained in [5, Section 3.2], into the two-dimensional generalized eigenspace $\mathcal{M}_{\lambda_d}$. We have that P_{λ_d} is given by

$$\begin{aligned}\big(P_{\lambda_d}\varphi\big)(\theta) = \frac{2}{3} e^{\lambda_d \theta} \left[\varphi(0) + b \int_0^\tau \Big(1 - \frac{3s}{\tau}\Big) e^{-\lambda_d s} \varphi(s-\tau) ds \right] \\ + \frac{2}{\tau} \theta e^{\lambda_d \theta} \left[\varphi(0) + b \int_0^\tau e^{-\lambda_d s} \varphi(s-\tau) ds \right]. \quad (39)\end{aligned}$$

From (39) it follows that $\mathcal{M}_{\lambda_d} = \mathrm{span}\{\phi_0, \phi_1\}$ where ϕ_0, $\phi_1 \in \mathcal{C}([-\tau, 0], \mathbb{C})$ are given respectively by $\phi_0(\theta) = e^{\lambda_d \theta}$ and $\phi_1(\theta) = \theta e^{\lambda_d \theta}$. If $\psi = c_0\phi_0 + c_1\phi_1 \in \mathcal{M}_{\lambda_d}$, then the solution $y(t)$ of (29) subject to initial condition $y_0 = \psi$ is given by

$$y(t) = c_0 e^{\lambda_d t} + c_1 t e^{\lambda_d t}.$$

For $\varphi \in \mathcal{C}$, from (39) we obtain that $P_{\lambda_d}\varphi = c_0(\varphi)\phi_0 + c_1(\varphi)\phi_1$, where

$$\begin{aligned} c_0(\varphi) &= \frac{2}{3}\left[\varphi(0) + b\int_0^\tau \Big(1 - \frac{3s}{\tau}\Big) e^{-\lambda_d s}\varphi(s-\tau)ds\right], \\ c_1(\varphi) &= \frac{2}{\tau}\left[\varphi(0) + b\int_0^\tau e^{-\lambda_d s}\varphi(s-\tau)ds\right]. \end{aligned}$$

The condition in item 2 is equivalent $c_1(\varphi) \neq 0$. From the discussions in the end of Section 1 (see [5, Lemma 2.1]), the asymptotic behaviour of $x(t)$, described in item 2 follows. □

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