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THE FUNDAMENTAL THEOREM OF CALCULUS FOR MULTIDIMENSIONAL BANACH SPACE-VALUED HENSTOCK VECTOR INTEGRALS

Abstract

In the present paper we give the Fundamental Theorem of Calculus for the variational or Henstock vector integrals ${}^K\!\int_{\mathbb{R}} \alpha\, df$ and ${}^K\!\int_{\mathbb{R}} d\alpha\, f$ of multidimensional Banach space-valued functions.

Introduction

In [1, Proposition 3.2], Bongiorno and Di Piazza gave a characterization of the functions which are Kurzweil-Henstock vector integrals of the form

$$h(t) = {}^K\!\int_{[a,t]} f\, dg$$

(${}^K\!\int$ for the Kurzweil integral) considering the one-dimensional real-valued case. They state that:

a) if g and F belong to $ACG^*([a,b])$ and $f : [a,b] \to \mathbb{R}$ is such that $F'(t) = f(t)g'(t)$ for m-almost every $t \in [a,b]$ (m for the Lebesgue measure), then f is Kurzweil-Henstock integrable with respect to g (we write $f \in H_g([a,b])$ and for every $t \in [a,b]$, $F(t) = {}^K\!\int_{[a,t]} f\, dg$.

And reciprocally,

b) if $g \in ACG^*([a,b])$ and $f \in H_g([a,b])$, then $\tilde{f}_g \in ACG^*([a,b])$, where $\tilde{f}_g(t) = {}^K\!\int_{[a,t]} f\, dg$ for each $t \in [a,b]$, and there exists $(\tilde{f}_g)'(t) = f(t)g'(t)$ for m-almost every $t \in [a,b]$.

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In order to prove b), Bongiorno and Di Piazza use the result

$$^{K}\!\!\int_{[a,b]} fg' = ^{K}\!\!\int_{[a,b]} f\,dg \tag{$*$}$$

referring mainly to [5, p. 186]. However, in this reference, or else in [5, p. 186], McLeod affirms that in general

$$^{K}\!\!\int_{[a,b]} f\cdot h = ^{K}\!\!\int_{[a,b]} f\,d\tilde{h}\,, \tag{$**$}$$

where $\tilde{h}(t) = ^{K}\!\!\int_{[a,t]} h$. But $(**)$ does not always hold. In [2, p. 37], a counter-example for Banach space-valued functions is given and we present it in Example 1 below. In the real case it is even possible that $\int_{[a,t]} h \neq \int_{[a,t]} d\tilde{h}$ for m-almost every $t \in [a,b]$. It suffices to take, for instance, $h : [0,1] \to \mathbb{R}$ defined by $h(t) = 1/q$ if $t = p/q$ and $p \in \mathbb{N}$ such that $q \neq 0$, q/p and p/q, and $h(t) = 0$ otherwise. For conditions in which $(**)$ holds, the reader may want to consult [2] or [3].

In the present paper we give the Fundamental Theorem of Calculus for the variational or Henstock vector integrals $^{K}\!\!\int_{\mathbb{R}} \alpha\,df$ and $^{K}\!\!\int_{\mathbb{R}} d\alpha\,f$ of multidimensional Banach space-valued functions.

1 Basic Terminology

For simplicity of proofs and notation, we consider only the two-dimensional case.

Let X and Y be Banach spaces and $f : R \to X$ be a function defined in a compact interval $R \subset \mathbb{R}^2$ (with sides parallel to the coordinate axes). Given $t, s \in \mathbb{R}^2$ with $t \leq s$ (i.e., $t_i \leq s_i$, $i = 1, 2$), we denote by $[t, s]$ the corresponding closed interval and we write $|[t, s]| = m([t, s])$, where m denotes the Lebesgue measure.

Any finite set of closed nonoverlapping intervals J_i of R such that $\cup J_i = R$ is called a division of R and denoted by (J_i). A tagged division of R is a pair $d = (\xi_i, J_i)$, where (J_i) is a division of R and $\xi_i \in J_i$ for every i. Each ξ_i is called the tag of J_i. We denote by TD_R the set of all tagged divisions of R. A tagged partial division d of R is any subset of a tagged division of R and we write $d \in TPD_R$. A gauge of a set $E \subset R$ is a function $\delta : E \to]0, \infty[$ and $d = (\xi_i, J_i) \in TPD_R$ is δ-fine if for each i,

$$J_i \subset B_{\delta(\xi_i)}(\xi_i) = \left\{ t \in R\,;\, |t - \xi_i| < \delta(\xi_i) \right\}.$$

Let J be a closed interval with sides h and k, $h \leq k$. Given $0 < c < 1$, J is said to be c-regular if $h/k \geq c$ and $d = (\xi_i, J_i) \in TPD_R$ is c-regular if each J_i is c-regular.

Let $\Im_R$ be the set of all closed intervals contained in R. A function $F : \Im_R \to X$ is called additive on intervals (we write $F \in A(\Im_R, X)$) if we have that $F(J) = F(J_1) + F(J_2)$ for any intervals J, J_1 and J_2, with J_1 and J_2 nonoverlapping and $J = J_1 \cup J_2$.

A function $F \in A(\Im_R, X)$ satisfies the regular Strong Lusin Condition on R (we write $F \in {}^r SL(\Im_R, X)$), if for every $\epsilon > 0$, every $0 < c < 1$ and every $E \subset R$ with $m(E) = 0$ there is a gauge δ of E such that for every c-regular δ-fine $d = (\xi_i, J_i) \in TDP_R$ with $\xi_i \in E$ for each i, we have that $\sum \|F(J_i)\| < \epsilon$.

Let $0 < c < 1$. We say that $F \in A(\Im_R, X)$ is c-differentiable at $\xi \in R$ and that $f(\xi)$ is its c-derivative (we write $D^c F(\xi) = f(\xi)$), if for every $\epsilon > 0$, there is a neighborhood V of ξ such that for each c-regular $J \in \Im_R$ with $\xi \in J \subset V$, we have $\big\|F(J) - f(\xi)|J|\big\| < \epsilon|J|$. If $D^c F(\xi) = f(\xi)$ for every $0 < c < 1$, then F is regularly differentiable at $\xi \in R$ with $f(\xi)$ being its regular derivative (we write ${}^r DF(\xi) = f(\xi)$). We say that F is c-differentiable at R when F is c-differentiable at $\xi \in R$ for every $\xi \in R$, and that F is regularly differentiable at R when F is regularly differentiable at $\xi \in R$ for every $\xi \in R$. Let $L(X, Y)$ denote the space of all continuous functions from X to Y. A function $\alpha \in A\big(\Im_R, L(X, Y)\big)$ is weakly c-differentiable at R if, for every $x \in X$, the function

$$\alpha \cdot x : J \in \Im_R \to \alpha(J) \cdot x \in Y$$

is c-differentiable at R and we write $\alpha \in (D^c)^\sigma\big(\Im_R, L(X, Y)\big)$. If there exists $(D^c)^\sigma(\alpha \cdot x)(\xi)$ for every c and every x, we say that α is weakly regularly differentiable at $\xi \in R$. We write $\alpha \in {}^r D^\sigma\big(\Im_R, L(X, Y)\big)$, if α is weakly regularly differentiable at $\xi \in R$, for each $\xi \in R$.

A function $f : R \to X$ is regularly Henstock integrable with respect to $\alpha \in A\big(\Im_R, L(X, Y)\big)$ (we write $f \in {}^r H^\alpha(R, X)$), if there exists a function $F^\alpha \in A(\Im_R, Y)$ such that for every $\epsilon > 0$ and every $0 < c < 1$, there is a gauge δ of R such that for every c-regular δ-fine $d = (\xi_i, J_i) \in TD_R$ we have that

$$\sum_i \big\|F^\alpha(J_i) - \alpha(J_i) f(\xi_i)\big\| < \epsilon\,.$$

If $\alpha(t) = t$, then we write simply ${}^r H(R, X)$ and F instead of ${}^r H^\alpha(R, X)$ and F^α respectively.

In an analogous way, a function $\alpha : R \to L(X, Y)$ is regularly Henstock integrable with respect to $f \in A(\Im_R, X)$ $\big($we write $\alpha \in {}^r H_f\big(R, L(X, Y)\big)\big)$, if there exists a function $A_f \in A(\Im_R, Y)$ such that for every $\epsilon > 0$ and every

$0 < c < 1$, there is a gauge δ of R such that for every c-regular and δ-fine $d = (\xi_i, J_i) \in TD_R$, we have that

$$\sum_i \big\| A_f(J_i) - \alpha(\xi_i) f(J_i) \big\| < \epsilon\,.$$

More generally we have: a function $\alpha : R \to L(X,Y)$ is regularly Kurzweil integrable with respect to $f \in A(\Im_R, X)$ $\big($we write $\alpha \in {}^rK_f\big(R, L(X,Y)\big)\big)$ if there exists $I \in Y$ (we write $I = {}^{rK}\!\int_{[a,t]} \alpha\, df$) such that for every $\epsilon > 0$ and every $0 < c < 1$, there is a gauge δ of R such that for every c-regular and δ-fine $d = (\xi_i, J_i) \in TD_R$, we have that

$$\Big\| I - \sum_i \alpha(\xi_i) f(J_i) \Big\| < \epsilon\,.$$

Analogously, a function $f : R \to X$ is regularly Kurzweil integrable with respect to $\alpha \in A\big(\Im_R, L(X,Y)\big)$ (we write $f \in {}^rK^\alpha(R,X)$), if there exists $I \in Y$ (we write $I = {}^{rK}\!\int_{[a,t]} d\alpha\, f$) such that for every $\epsilon > 0$ and every $0 < c < 1$, there is a gauge δ of R such that for every c-regular and δ-fine $d = (\xi_i, J_i) \in TD_R$, we have that

$$\Big\| I - \sum_i \alpha(J_i) f(\xi_i) \Big\| < \epsilon\,.$$

Let $R = [a,b]$. If $\alpha \in {}^rK_f\big(R, L(X,Y)\big)$, then we define $\tilde{\alpha}_f(t) = {}^{rK}\!\int_{[a,t]} \alpha\, df$ for each $t \in R$. And, analogously, given $f \in {}^rK^\alpha(R,X)$, we define $\tilde{f}^\alpha(t) = {}^{rK}\!\int_{[a,t]} d\alpha\, f$ for each $t \in R$. If $\alpha(t) = t$, then we simply write ${}^rK(R,X)$ and $\tilde{f}(t) = {}^{rK}\!\int_{[a,t]} f$.

We may associate $F \in A(\Im_R, Y)$ with a function from R to Y which we still denote by F: for $R = [a,b] = [a_1,b_1] \times [a_2,b_2]$ we write

$$F(t) = F\big([a,t]\big) - F\big([a,(a_1,t_2)]\big) - F\big([a,(t_1,a_2)]\big) + F\big([a,a]\big)\,.$$

Reciprocally, we may associate a function $f : R \to Y$ with a function of intervals of R which we also denote by $f : \Im_R \to X$. In this case we write

$$f\big([t,s]\big) = f(s) - f(t_1,s_2) - f(t_2,s_1) + f(t)\,.$$

Thus, when $f \in {}^rH^\alpha\big([a,b],X\big)$, then $F^\alpha\big([a,t]\big) = \tilde{f}^\alpha(t)$ for each $t \in [a,b]$, and analogously, for $\alpha \in {}^rK_f\big([a,b], L(X,Y)\big)$, we have that $A_f\big([a,t]\big) = \tilde{\alpha}_f(t)$ for each $t \in [a,b]$, and therefore we can talk about regular Strong Lusin Condition and regular differentiability of an indefinite integral $\tilde{f}^\alpha$ or $\tilde{\alpha}_f$.

Remark. When X is of finite dimension, then

$$ {}^{r}H^{\alpha}\big([a,b],X\big) = {}^{r}K^{\alpha}\big([a,b],X\big) $$

and

$$ {}^{r}H_f\big([a,b],L(X,Y)\big) = {}^{r}K_f\big([a,b],L(X,Y)\big) $$

and such spaces are called spaces of regularly Kurzweil-Henstock integrable functions or spaces of Mawhin integrable functions.

Example 1. Let $X = l_2([a,b])$ and $Y = \mathbb{R}$. Let $f : [a,b] \to X$ be defined by $f(t) = e_t$ (i.e., $e_t(s) = 1$ if $s = t$, and $e_t(s) = 0$ if $s \neq t$) and let $\alpha : [a,b] \to X' = L(X,\mathbb{R})$ be given by $\alpha(t) = \tilde{e}_t$, where $\tilde{e}_t(x) = \langle e_t, x\rangle$, for every $x \in X$. Then $\alpha(t)f(t) = \langle e_t, e_t\rangle = 1$ and therefore

$$ {}^{K}\!\!\int_{[a,b]} \alpha(t)f(t)\,dt = \int_{[a,b]} dt = b-a\,, $$

where $\int$ denotes the Riemann integral. On the other hand, given $\epsilon > 0$, there exists $\delta > 0$, say

$$ \delta^{\frac{1}{2}} < \frac{\epsilon}{(b-a)^{\frac{1}{2}}}\,, $$

such that for every $d = (\xi_i, t_i) \in TD_{[a,b]}$ with $\max_i\{t_i - t_{i-1}\} < \delta$, we have that

$$ \Big\|\sum_i f(\xi_i)(t_i - t_{i-1})\Big\| = \Big\|\sum_i e_{\xi_i}(t_i - t_{i-1})\Big\| = \Big[\sum_i (t_i - t_{i-1})^2\Big]^{\frac{1}{2}}\,, \tag{1a} $$

where we have used Bessel's equality. But,

$$ \Big[\sum_i (t_i - t_{i-1})^2\Big]^{\frac{1}{2}} < \delta^{\frac{1}{2}} \sum_i (t_i - t_{i-1})^{\frac{1}{2}} = \big[\delta(b-a)\big]^{\frac{1}{2}} < \epsilon\,. \tag{1b} $$

Hence (1a) and (1b) imply that $\tilde{f} = 0$, and so

$$ \int_{[a,b]} \alpha(t)\,d\tilde{f}(t) = 0\,. $$

Now, if $[a,b]$ is a non-degenerate interval, then

$$ 0 < b-a = {}^{K}\!\!\int_{[a,b]} \alpha(t)f(t)\,dt \neq \int_{[a,b]} \alpha(t)\,d\tilde{f}(t) = 0\,. $$

2 Main Results

For the Henstock vector integral ${}^{rK}\!\!\int_{[a,t]} \alpha\, df$ we have:

Theorem 1. *Let $f \in {}^{r}SL(\Im_R, X)$ and $A \in {}^{r}SL(\Im_R, Y)$ be both regularly differentiable on R, and let $\alpha : [a,b] \to L(X,Y)$ be such that ${}^{r}DA = \alpha \cdot ({}^{r}DF)$ m-almost everywhere on R. Then $\alpha \in {}^{r}H_f\big(R, L(X,Y)\big)$ and $A = \tilde{\alpha}_f$.*

Theorem 2. *Let $f \in {}^{r}SL(\Im_R, X)$ be regularly differentiable on R and $\alpha \in {}^{r}H_g\big(R, L(X,Y)\big)$ be bounded. Then $\tilde{\alpha}_f \in {}^{r}SL(\Im_R, Y)$ and there exists ${}^{r}D(\tilde{\alpha}_f) = \alpha \cdot ({}^{r}Df)$ m-almost everywhere on R.*

And, for the Henstock vector integral ${}^{rK}\!\!\int_{[a,t]} d\alpha\, f$ we have:

Theorem 3. *Let $\alpha \in \big({}^{r}SL\big(\Im_R, L(X,Y)\big) \,\cap\, {}^{r}D^{\sigma}\big(\Im_R, L(X,Y)\big)\big)$, let $F \in {}^{r}SL(\Im_R, Y)$ be regularly differentiable on R and let $f : R \to X$ be such that ${}^{r}DF(t) = {}^{r}D^{\sigma}\big(\alpha \cdot f(t)\big)(t)$ for m-almost every $t \in R$. Then $f \in {}^{r}H^{\alpha}(R,X)$ and $F = \tilde{f}^{\alpha}$.*

And reciprocally:

Theorem 4. *If $\alpha \in \big({}^{r}SL\big(\Im_R, L(X,Y)\big) \,\cap\, {}^{r}D^{\sigma}\big(\Im_R, L(X,Y)\big)\big)$ and $f \in {}^{r}H^{\alpha}(R,X)$ then $\tilde{f}^{\alpha} \in {}^{r}SL(\Im_R, Y)$ and there exists ${}^{r}D\tilde{f}^{\alpha}(t) = {}^{r}D^{\sigma}\big(\alpha \cdot f(t)\big)(t)$ for m-almost every $t \in R$.*

3 Proofs

First we prove the results for the Henstock vector integral ${}^{rK}\!\!\int_{[a,t]} \alpha\, df$.

Theorem 5. *Let $f \in {}^{r}SL(\Im_R, X)$ and $\alpha : R \to L(X,Y)$ such that $\alpha = 0$ m-almost everywhere. Then $\alpha \in {}^{r}H_f\big(R, L(X,Y)\big)$ and $\tilde{\alpha}_f = 0$.*

Proof. Let $E = \{t \in R;\ \alpha(t) \neq 0\}$ and $E_n = \{t \in E;\ n-1 < \|\alpha(t)\| \leq n\}$ for each $n \in \mathbb{N}$. By hypothesis, $m(E) = 0$. Therefore $m(E_n) = 0$ for every n. Since $f \in {}^{r}SL(\Im_R, X)$, then given $n \in \mathbb{N}$, $\epsilon > 0$ and $0 < c < 1$, there is a gauge δ_n of E_n such that for every c-regular δ_n-fine $d_n = (\xi_{n_i}, J_{n_i}) \in TPD_R$ with $\xi_{n_i} \in E_n$ for every i, we have that $\sum_i \big\| f(J_{n_i}) \big\| < \frac{\epsilon}{n \cdot 2^n}$.

Let δ be a gauge of R such that if $\xi \in E_n$ then $\delta(\xi) = \delta_n(\xi)$, and if $\xi \notin E_n$ then $\delta(\xi)$ takes any value in $]0,\infty[$. Hence, for every c-regular δ-fine $d = (\xi_i, J_i) \in TD_R$,

$$\sum_i \big\|\alpha(\xi_i) f(J_i)\big\| = \sum_n \sum_{\xi_i \in E_n} \big\|\alpha(\xi_i) f(J_i)\big\| \leq \sum_n n \sum_{\xi_i \in E_n} \big\| f(J_i)\big\| < \epsilon\,.$$

$\square$

Corollary. *If $f \in {}^rSL(\Im_R, X)$, $\alpha \in {}^rH_f\big(R, L(X,Y)\big)$ and $\beta : [a,b] \to L(X,Y)$ with $\beta = \alpha$ m-almost everywhere, then $\beta \in {}^rH_f\big(R, L(X,Y)\big)$ and $\tilde{\beta}_f = \tilde{\alpha}_f$.*

The next example shows us that the hypothesis of f in Theorem 5 is really needed.

Example 2. Even in the one-dimensional case when the regularity is not used, we may have that if f does not satisfy the Strong Lusin Condition, then α may not be Henstock integrable with respect to f. Take, for instance, an interval $[a,b]$ of the real line with $a \geq 1$, and consider the context of Example 1. Consider arbitrary ξ_i and $[t_{i-1}, t_i] \subset [a,b]$ such that $\xi_i \in [t_{i-1}, t_i]$. Then

$$\big\|f(t_i) - f(t_{i-1})\big\|^2 = \big\|e_{t_i} - e_{t_{i-1}}\big\|^2 = |t_i|^2 + |t_{i-1}|^2 > a^2 + a^2 > 1\,,$$

and hence $f \notin SL([a,b], X)$. We also have that $\alpha(\xi_i)\big[f(t_i) - f(t_{i-1}\big] = 0$ for $\xi_i \in]t_{i-1}, t_i[$ and $\alpha(\xi_i)\big[f(t_i) - f(t_{i-1})\big] \neq 0$ otherwise, and therefore α does not belong to $H_f\big([a,b], L(X,Y)\big)$ (and neither to $K_f\big([a,b], L(X,Y)\big)$).

PROOF OF THEOREM 1.
1) Let $E = \{t \in R$; there exists ${}^rDA(t) = \alpha(t) \cdot {}^rDF(t)\}$. Hence, given $\epsilon > 0$, $0 < c < 1$ and $\xi \in E$, there is a neighborhood V_1 of ξ such that for every closed c-regular interval $J \subset R$, with $\xi \in J \subset V_1$,

$$\big\|A(J) - \alpha(t) \cdot {}^cDf(t)|J|\big\| < \epsilon|J|\,.$$

2) By hypothesis, $f \in {}^rSL(\Im_R, X)$ and $m(R \setminus E) = 0$. Therefore, by the Corollary after Theorem 5, we may suppose that $\alpha(t) = 0$ for every $t \in R \setminus E$.
3) Since $m(R \setminus E) = 0$ and $A \in {}^rSL(\Im_R, Y)$, there is a gauge δ' of $(R \setminus E)$ such that for every c-regular δ'-fine $d = (\xi_i, J_i) \in TPD_R$ with $\xi_i \in R \setminus E$, $\sum \|A(J_i)\| < \epsilon$.
4) Because f is regularly differentiable on R and hence c-differentiable on R, there is a neighborhood V_2 of ξ such that for every c-regular $J \subset R$, with $\xi \in J \subset V_2$, and we have that

$$\big\|\alpha(\xi) \cdot f(J) - \alpha(t) \cdot {}^cDF(t)|J|\big\| < \epsilon|J|\,.$$

5) Finally, let δ be a gauge of R such that $B_{\delta(\xi)}(\xi) \subset (V_1 \cap V_2)$ for each $\xi \in E$, and such that if $\xi \in R \setminus E$, then $\delta(\xi) \leq \delta'(\xi)$ and $B_{\delta(\xi)}(\xi) \subset V_2$. Hence for every c-regular δ-fine $d = (\xi_i, J_i) \in TD_R$, it follows that

$$\sum_i \big\|A(J_i) - \alpha(\xi_i) \cdot f(J_i)\big\| \leq$$

$$\sum_{\xi_i \in E} \big\| A(J_i) - \alpha(\xi_i) \cdot f(J_i) \big\| + \sum_{\xi_i \in R \setminus E} \big\| A(J_i) \big\| + \sum_{\xi_i \in R \setminus E} \big\| \alpha(\xi_i) \cdot \big\| f(J_i) \big\| \,,$$

where the first summand is smaller than $\sum 2\epsilon |J_i| = 2\epsilon |R|$ from 1) and 4). The second summand is smaller than ϵ from 3), and the third summand is equal to zero because we have from 2) that $f(\xi_i) = 0$ for each i. $\square$

Remark. In Theorem 1, we can require that $A \in {}^r SL(\Im_R, Y)$ is regularly differentiable m-almost everywhere on R.

Lemma 6 (Saks-Henstock Lemma)**.**
Given $f \in A(\Im_R, X)$, $\alpha \in {}^r H_f\big(R, L(X,Y)\big)$, let $0 < c < 1$, $\epsilon > 0$, and δ be the gauge of R from the definition of $\alpha \in {}^r H_f\big(R, L(X,Y)\big)$. Then for every c-regular δ-fine $d = (\xi_i, J_i) \in TPD_R$ we have that

$$\sum_i \Big\| \alpha(\xi_i) \cdot f(J_i) - {}^{rK}\!\!\int_{J_i} \alpha \, df \Big\| < \epsilon \,.$$

Proof. The proof follows the standard steps. $\square$

Theorem 7. *If $f \in {}^r SL(\Im_R, X)$ and $\alpha \in {}^r H_f\big(R, L(X,Y)\big)$, then $\tilde{\alpha}_f \in {}^r SL(\Im_R, Y)$.*

Proof. Let $E \subset R$ be such that $m(E) = 0$ and let $\beta = \alpha \chi_{(R \setminus E)}$. Then by the Corollary after Theorem 5, $\beta \in {}^r H_f\big(R, L(X,Y)\big)$ and $\tilde{\beta}_f = \tilde{\alpha}_f$. Therefore, given $\epsilon > 0$ and $0 < c < 1$, let δ be the gauge of R from the definition of $\beta \in {}^r H_f\big(R, L(X,Y)\big)$. Then, from the Saks-Henstock Lemma (Lemma 6), it follows that for every c-regular δ-fine $d = (\xi_i, J_i) \in TPD_R$, with $\xi_i \in E$ for each i, we have that

$$\sum_i \big\| \tilde{\alpha}_f(J_i) \big\| = \sum_i \big\| \tilde{\alpha}_f(J_i) - \beta(\xi_j) \cdot f(J_i) \big\| < \epsilon \,,$$

since $\tilde{\beta}_f = \tilde{\alpha}_f$ and $\beta(\xi_i) = 0$ for every i. $\square$

Lemma 8. (See [4, Theorem 2.2]). *If $g \in {}^r H(R, X)$, then there exists ${}^r D\tilde{g} = g$ m-almost everywhere on R.*

Theorem 9. *Let $f \in {}^r SL(\Im_R, X)$ be regularly differentiable and let $\alpha \in {}^r H_f\big(R, L(X,Y)\big)$ be bounded. Then the function $t \in R \to \alpha(t) \cdot {}^r Df(t) \in Y$ is regularly Henstock integrable with*

$${}^{rK}\!\!\int_R \alpha \cdot {}^r Df = {}^{rK}\!\!\int_R \alpha \, df \,.$$

Besides, there exists ${}^r D\tilde{\alpha}_f = \alpha \cdot {}^r Df$ m-almost everywhere on R.

PROOF. 1) Given $\epsilon > 0$, let δ_1 be the gauge of R from the definition of $\alpha \in {}^{r}H_f\big(R, L(X,Y)\big)$.
2) Since f is regularly differentiable, then given $0 < c < 1$ and $\xi \in R$, there is a neighborhood V_ξ of ξ such that for each closed c-regular interval $J \subset R$, with $\xi \in J \subset V_\xi$, we have that $\big\|f(J) - {}^{c}Df(\xi)|J|\big\| < \epsilon|J|$.
3) Let δ be a gauge of R such that for each $\xi \in R$, $\delta(\xi) \le \delta_1(\xi)$ and $B_{\delta(\xi)}(\xi) \subset V_\xi$. Hence, for every c-regular δ-fine $d = (\xi_i, J_i) \in TD_R$, we have that

$$\sum_i \big\|\tilde{\alpha}_f(J_i) - \alpha(\xi_i) \cdot {}^{c}Df(\xi_i)|J_i|\big\| \le$$

$$\sum_i \big\|\tilde{\alpha}_f(J_i) - \alpha(\xi_i) \cdot f(J_i)\big\| + \sum_i \big\|\alpha(\xi_i)\big\| \cdot \big\|f(J_i) - {}^{c}Df(\xi_i)|J_i|\big\|,$$

where the first summand is smaller than ϵ by 1), the second summand is smaller than $\epsilon\|\alpha\|_\infty|R|$ ($\|\ \|_\infty$ for the supremum norm) by 2) and the boundedness of α, and the first part of the theorem holds.

The second part comes immediately from Lemma 8. □

PROOF OF THEOREM 2. This comes immediately from Theorems 5, 7 and 9. □

Now we treat the Henstock vector integral ${}^{rK}\!\int_R d\alpha\, f$. In general, the proofs for ${}^{rK}\!\int_R d\alpha\, f$ are analogous to those for ${}^{rK}\!\int_R \alpha\, df$. However, the reader may want to have a look at Theorem 13 which, unlike Theorem 9, does not need the boundedness hypothesis of the Henstock vector integrable function. This fact allows Theorem 4 to be precisely the reciprocal of Theorem 3 (note that Theorems 1 and 2 are not the reciprocal one of another).

Theorem 10. *Let $\alpha \in {}^{r}SL\big(\Im_R, L(X,Y)\big)$ and $f : R \to X$ with $f = 0$ m-almost everywhere. Then $f \in {}^{r}H^{\alpha}(R, X)$ and $\tilde{f}^{\alpha} = 0$.*

PROOF. Let $E = \{t \in R;\ f(t) \ne 0\}$ and $E_n = \{t \in E;\ n-1 < \|f(t)\| \le n\}$ for each $n \in \mathbb{N}$. By hypothesis, $m(E) = 0$. Therefore $m(E_n) = 0$ for every n. Since $\alpha \in {}^{r}SL\big(\Im_R, L(X,Y)\big)$, then for each n, given $\epsilon > 0$ and $0 < c < 1$, there is a gauge δ_n of E_n such that for every c-regular δ-fine $d_n = (\xi_{n_i}, J_{n_i}) \in TPD_R$ with $\xi_{n_i} \in E_n$ for each i, we have that $\sum \|\alpha(J_{n_i})\| < \epsilon n 2^n$.

Let δ be a gauge of R such that if $\xi \in E_n$ then $\delta(\xi) = \delta_n(\xi)$, and if $\xi \notin E_n$ then $\delta(\xi)$ takes any value in $]0, \infty[$. Hence for every c-regular δ-fine $d = (\xi_i, J_i) \in TD_R$,

$$\sum_i \big\|\alpha(J_i) \cdot f(\xi_i)\big\| = \sum_n \sum_{\xi_i \in E_n} \big\|\alpha(J_i) \cdot f(\xi_i)\big\| \le \sum_n n \sum_{\xi_i \in E_n} \big\|\alpha(J_i)\big\| < \epsilon\,.$$

□

Example 3. Even in the one-dimensional case, if α does not satisfy the Strong Lusin Condition, then f may not be Henstock integrable with respect to α. In the context of Example1, we have that

$$\big\|\alpha(t_i) - \alpha(t_{i-1})\big\| = \big\|\tilde{e}_{t_i} - \tilde{e}_{t_{i-1}}\big\| =$$

$$= \sup_{\|x\|\le 1} \Big\{ \big\|\tilde{e}_{t_i}(x) - \tilde{e}_{t_{i-1}}(x)\big\| \Big\} \ge \big\|\tilde{e}_{t_i}(e_{t_i}) - \tilde{e}_{t_{i-1}}(e_{t_i})\big\| = 1\,,$$

for arbitrary ξ_i and $[t_{i-1}, t_i]$ such that $\xi_i \in [t_{i-1}, t_i] \subset [a, b]$. Hence $\alpha \notin SL\big([a,b], L(X,Y)\big)$. We also have that $\big[\alpha(t_i) - \alpha(t_{i-1})\big] f(\xi_i) = 0$ for $\xi_i \in \,]t_{i-1}, t_i[$ and $\big[\alpha(t_i) - \alpha(t_{i-1})\big] f(\xi_i) \ne 0$ otherwise, and so, $f \notin H^\alpha\big([a,b], X\big)$.

Corollary. *Given* $\alpha \in {}^rSL\big(\Im_R, L(X,Y)\big)$, $f \in {}^rH^\alpha(R, X)$ *and a function* $g : R \to X$ *such that* $g = f$ m*-almost everywhere, then* $g \in {}^rH^\alpha(R,X)$ *and* $\tilde{g}^\alpha = \tilde{f}^\alpha$.

PROOF OF THEOREM 3.
1) Let $E = \big\{t \in R;$ there is ${}^rDF(t) = {}^rD^\sigma\big(\alpha \cdot f(t)\big)(t)\big\}$. Hence, given $0 < c < 1$, $\epsilon > 0$ and $\xi \in E$, there exists a neighborhood V_1 of ξ such that for every c-regular $J \in \Im_R$ with $\xi \in J \subset V_1$,

$$\Big\|F(J) - (D^c)^\sigma\big(\alpha \cdot f(\xi)\big)(\xi)|J|\Big\| < \epsilon|J|\,.$$

2) By hypothesis, $\alpha \in {}^rSL\big(\Im_R, L(X,Y)\big)$ and $m(R\setminus E) = 0$, therefore we may suppose that $f(t) = 0$ for every $t \in R \setminus E$ by the Corollary after Theorem 10.
3) Since $m(R \setminus E) = 0$ and $F \in {}^rSL(\Im_R, Y)$, there is a gauge δ' of $R \setminus E$ such that for every c-regular δ-fine $d = (\xi_i, J_i) \in TPD_R$ with $\xi_i \in R \setminus E$ for each i, we have that $\sum \big\|F(J_i)\big\| < \epsilon$.
4) Because $\alpha \in {}^rD^\sigma\big(\Im_R, L(X,Y)\big)$, there is a neighborhood V_2 of ξ such that for every c-regular $J \in \Im_R$ with $\xi \in J \subset V_2$,

$$\Big\|\alpha(J)f(\xi) - (D^c)^\sigma\big(\alpha \cdot f(\xi)\big)(\xi)|J|\Big\| < \epsilon|J|\,.$$

5) Finally, let δ be a gauge of R such that $B_{\delta(\xi)}(\xi) \subset (V_1 \cap V_2)$ for $\xi \in E$, and such that if $\xi \in R \setminus E$ then $\delta(\xi) \le \delta'(\xi)$ and $B_{\delta(\xi)}(\xi) \subset V_2$. Hence, for every c-regular δ-fine $d = (\xi_i, J_i) \in TD_R$, it follows that

$$\sum_i \big\|F(J_i) - \alpha(J_i) f(\xi_i)\big\| \le$$

$$\sum_{\xi_i \in E} \big\|F(J_i) - \alpha(J_i)f(\xi_i)\big\| + \sum_{\xi_i \in R\setminus E} \big\|F(J_i)\big\| + \sum_{\xi_i \in R\setminus E} \big\|\alpha(J_i)\big\| \cdot \big\|f(\xi_i)\big\|\,,$$

where the first summand is smaller than $\sum 2\epsilon|J_i| = 2\epsilon|R|$ from 1) and 4), the second summand is smaller than ϵ from 3), and the third summand is equal to zero because $f(\xi_i) = 0$ from 2). □

Lemma 11 (Saks-Henstock Lemma)**.** *Given $\alpha \in A\big(\Im_R, L(X,Y)\big)$, $f \in {}^rH^\alpha(R,X)$, let $0 < c < 1$, $\epsilon > 0$, and δ be a gauge of R from the definition of $f \in {}^rH^\alpha(R,X)$. Then for every c-regular δ-fine $d = (\xi_i, J_i) \in TPD_R$, we have that*

$$\sum_i \Big\| \alpha(J_i) \cdot f(\xi_i) - {}^{rK}\!\!\int_{J_i} d\alpha\, f \Big\| < \epsilon\,.$$

PROOF. The proof follows the standard steps. □

Theorem 12. *If $\alpha \in {}^rSL\big(\Im_R, L(X,Y)\big)$ and $f \in {}^rH^\alpha(R,X)$, then $\tilde{f}^\alpha \in {}^rSL(\Im_R, Y)$.*

PROOF. Let $E \subset R$ be such that $m(E) = 0$ and let $g = f\chi_{(R\setminus E)}$. Then, by the Corollary after Theorem 10, $g \in {}^rH^\alpha(R,X)$ and for each $t \in R$, we have that $\tilde{g}^\alpha = \tilde{f}^\alpha$. Given $0 < c < 1$ and $\epsilon > 0$, let δ be the gauge of R from the definition of $g \in {}^rH^\alpha(R,X)$. Then, from the Saks-Henstock Lemma (see Lemma 11), it follows that for every c-regular δ-fine $d = (\xi_i, J_i) \in TPD_R$ with $\xi_i \in E$ for each i, we have that

$$\sum_i \big\|\tilde{f}^\alpha(J_i)\big\| = \sum_i \big\|\tilde{f}^\alpha(J_i) - \alpha(J_i)\cdot g(\xi_i)\big\| < \epsilon\,,$$

since $\tilde{g}^\alpha = \tilde{f}^\alpha$ and $g(\xi_i) = 0$ for every i. □

Theorem 13. *If $\alpha \in {}^rD^\sigma\big(\Im_R, L(X,Y)\big)$ and $f \in {}^rH^\alpha(R,X)$, then the function $g : R \to Y$ defined by*

$$g(t) = {}^rD^\sigma\big(\alpha \cdot f(t)\big)(t)$$

is regularly Henstock integrable with ${}^{rK}\!\int_R g = {}^{rK}\!\int_R d\alpha\, f$. Besides, there exists ${}^rD\tilde{f}^\alpha$ m-almost everywhere on R and, in this case,

$${}^rD\tilde{f}^\alpha(t) = {}^rD^\sigma\big(\alpha \cdot f(t)\big)(t)\,.$$

PROOF. 1) Given $\epsilon > 0$ and $0 < c < 1$, let δ_1 be the gauge of R from the definition of $f \in {}^rH^\alpha(R,X)$.
2) Since $\alpha \in {}^rD^\sigma\big(\Im_R, L(X,Y)\big)$, then for each $\xi \in R$ there is a neighborhood V_ξ of ξ such that for each c-regular $J \in \Im_R$ with $\xi \in J \subset V_\xi$,

$$\big\|\alpha(J)\cdot x - (D^c)^\sigma(\alpha \cdot x)(\xi)|J|\big\| < \epsilon|J|\,.$$

3) Let δ be the gauge of R such that for each $\xi \in R$, $\delta(\xi) \le \delta_1(\xi)$ and $B_{\delta(\xi)}(\xi) \subset V_\xi$. Hence, for every c-regular δ-fine $d = (\xi_i, J_i) \in TD_R$,

$$\sum_i \Big\| \tilde{f}^\alpha(J) - {}^c\!D^\sigma\big(\alpha \cdot f(\xi)\big)(\xi)|J_i| \Big\| \le$$

$$\sum_i \big\| \tilde{f}^\alpha(J_i) - \alpha(J_i) \cdot f(\xi_i) \big\| + \sum_i \Big\| \alpha(J_i) \cdot f(\xi_i) - {}^c\!D^\sigma\big(\alpha \cdot f(\xi_i)\big)(\xi_i)|J_i| \Big\|,$$

where the first summand is smaller than ϵ from 1), and the second summand is smaller than $\epsilon|R|$ from 2), and the first part of the theorem follows.

The second part comes immediately from Lemma 8. □

Proof of Theorem 4. The proof follows directly from Theorems 10, 12 and 13. □

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