Curves in the Minkowski plane and their contact with pseudo-circles

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Abstract

We study the caustic, evolute, Minkowski symmetry set and parallels of a smooth and regular curve in the Minkowski plane.

1 Introduction

We consider in this paper the geometry of a smooth and regular curve γ in the Minkowski plane $\mathbb{R}^2_1$ which is captured by its contact with pseudo-circles. This contact is studied using the family of distance squared functions on γ.

The points on γ where its tangent direction is lightlike are labelled lightlike points. The evolute of γ (without its inflection points) is well defined away from the lightlike points (§3). However, its caustic $C(\gamma)$ is defined everywhere including at the lightlike points (see §4 and [14] for the caustics of surfaces in the Minkowski 3-space). We determine the generic behaviour of the caustic $C(\gamma)$ at the lightlike points of γ (Proposition 4.1), and show that the caustic of an oval lies in the complement of the interior of γ (Theorem 4.3).

The caustic $C(\gamma)$ is the local stratum of the bifurcation set of the family of distance squared functions on γ. We call the multi-local stratum of the bifurcation set of this family the Minkowski symmetry set (MSS) of γ. We consider in §5 the geometry of the MSS and deal in some details with the MSS of an ellipse.

The family of distance squared functions also gives information about the parallels of γ. These are defined away from the lightlike points of γ. We prove in Theorem 6.2 that the parallels undergo swallowtail transitions at a vertex of γ, which when considered together in $\mathbb{R}^2_1$, give a distinct configuration to that of the parallels of a curve

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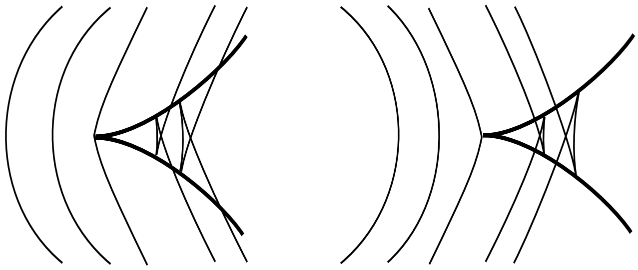

Figure 1: Parallels of a plane curve at a swallowtail transition: left in the Euclidean plane and right in the Minkowski plane.

in the Euclidean plane; see Figure 1. We also prove in Theorem 6.2 that the parallels of curves in the Euclidean plane are as Figure 1, left. (The parallels of curves in the Euclidean plane are always drawn as in Figure 1, left, but to our knowledge, there is no proof that it is the only possible generic configuration for these curves.)

It is worth observing that the concepts and the results in this paper are valid for curves in any Lorentzian plane. In fact, the results can be interpreted in the affine setting (see Remark 5.4).

2 Preliminaries

The *Minkowski plane* $\mathbb{R}^2_1$ is the plane $\mathbb{R}^2$ endowed with the metric induced by the pseudo-scalar product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -u_0v_0 + u_1v_1$, where $\boldsymbol{u} = (u_0, u_1)$ and $\boldsymbol{v} = (v_0, v_1)$ (see for example [10], p55). We say that a non-zero vector $\boldsymbol{u} \in \mathbb{R}^2_1$ is *spacelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$, *lightlike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and *timelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$. We denote by $\boldsymbol{u}^\perp$ the vector given by $\boldsymbol{u}^\perp = (u_1, u_0)$. Thus, $\boldsymbol{u}^\perp$ is "orthogonal" to $\boldsymbol{u}$ (i.e., $\langle \boldsymbol{u}, \boldsymbol{u}^\perp \rangle = 0$). We have $\boldsymbol{u}^\perp = \pm\boldsymbol{u}$ if and only if $\boldsymbol{u}$ is lightlike, and $\boldsymbol{u}^\perp$ is timelike (resp. spacelike) if $\boldsymbol{u}$ is spacelike (resp. timelike).

The norm of a vector $\boldsymbol{u} \in \mathbb{R}^2_1$ is defined by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$. We have the following pseudo-circles in $\mathbb{R}^2_1$ with centre $p \in \mathbb{R}^2_1$ and radius $r > 0$:

$$\begin{array}{rcl} H^1(p, -r) & = & \{q \in \mathbb{R}^2_1 \mid \langle q - p, q - p \rangle = -r^2\}, \\ S^1_1(p, r) & = & \{q \in \mathbb{R}^2_1 \mid \langle q - p, q - p \rangle = r^2\}, \\ LC^*(p) & = & \{q \in \mathbb{R}^2_1 \mid \langle q - p, q - p \rangle = 0\}. \end{array}$$

We denote by $H^1(-r)$, $S^1_1(r)$ and LC^* the pseudo-circles centred at the origin in $\mathbb{R}^2_1$.

We consider embeddings $\gamma : J \to \mathbb{R}^2_1$, where $J = I$ is an open interval of $\mathbb{R}$ or $J = S^1$. The set $Emb(J, \mathbb{R}^2_1)$ of such embeddings is endowed with the Whitney C^∞-topology. We say that a property is *generic* if it is satisfied by curves in a residual

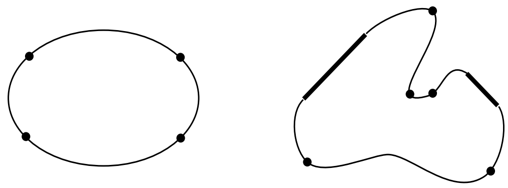

Figure 2: Examples of closed curves with lightlike points (dotes and thick lines). The ellipse on the left has exactly four lightlike points and the curve on the right has two line segments of lightlike points and other isolated lightlike points.

subset of $Emb(J,\mathbb{R}^2_1)$. A curve that satisfies a generic property is called a generic curve.

Let $\gamma \in Emb(I,\mathbb{R}^2_1)$. We say that γ is *spacelike* (resp. *timelike*) if $\gamma'(t)$ is a spacelike (resp. timelike) vector for all $t \in I$. A point $\gamma(t)$ is called a *lightlike point* if $\gamma'(t)$ is a lightlike vector.

Proposition 2.1 *The set of lightlike points of a curve $\gamma \in Emb(S^1,\mathbb{R}^2_1)$ is the union of at least four disjoint non-empty and closed subsets of γ (see Figure 2).*

Proof The lightlike points are those where the tangent line to γ is parallel to $(\pm 1, 1)$. We change the metric in $\mathbb{R}^2$ and consider γ as a curve $\tilde{\gamma}$ in the Euclidean plane $\mathbb{R}^2$. Since $\tilde{\gamma}$ is closed, the image of its Gauss map $N : \tilde{\gamma} \to S^1$ is the whole unit circle S^1. The pre-images of the points $(\pm 1, \pm 1)$ by N have tangent lines parallel to $(\pm 1, 1)$, i.e., they are lightlike points on γ. It follows by the fact that N is a continuous map that the set of lightlike points of γ is the union of at least four disjoint non-empty and closed subsets of γ. □

We apply tools from singularity theory to obtain geometric information about curves in $\mathbb{R}^2_1$. Given a smooth (i.e., C^∞) function $f : J \to \mathbb{R}$ ($J = I$ or S^1), we say that f is singular at $t_0 \in J$ if $f'(t_0) = 0$. We consider the $\mathcal{R}$-singularities of f at $t_0 \in J$, where $\mathcal{R}$ is the group of local changes of parameters in the source that fix t_0. The models for the local $\mathcal{R}$-singularities of functions are $\pm(t-t_0)^{k+1}$, $k \geq 1$, and these are labelled A_k-singularities. The necessary and sufficient conditions for a function f to have an A_k-singularity at t_0 are

$$f'(t_0) = f''(t_0) = \ldots = f^{(k)}(t_0) = 0,\; f^{(k+1)}(t_0) \neq 0.$$

The only stable singularity (ignoring the constant terms) is $\pm(t-t_0)^2$, i.e., the A_1-singularity. (See [5] for more on singularities of functions and their applications to the geometry of curves in the Euclidean plane.)

The contact of a curve $\gamma \in Emb(J, \mathbb{R}^2_1)$ ($J = I$ or S^1) with lines is captured by the singularities of the family of height functions on γ. Let $\boldsymbol{v}$ be a non-zero vector in $\mathbb{R}^2_1$ and consider the parallel lines

$$L^{\boldsymbol{v}}_c = \{p \in \mathbb{R}^2_1 \,|\, \langle p, \boldsymbol{v} \rangle = c\},$$

with $c \in \mathbb{R}$, which are (pseudo)-orthogonal to $\boldsymbol{v}$. The contact of γ with the lines $L^{\boldsymbol{v}}_c$ is measured by the singularities of the height function $h_{\boldsymbol{v}} : J \to \mathbb{R}$, given by

$$h_{\boldsymbol{v}}(t) = \langle \gamma(t), \boldsymbol{v} \rangle.$$

An important observation is that the function $h_{\boldsymbol{v}}$ is defined for all non-zero vectors $\boldsymbol{v}$ including lightlike vectors, and at all points on γ including its lightlike points.

We say that the curve γ has an A_k-contact (resp. $A_{\geq k}$-contact) with $L^{\boldsymbol{v}}_c$ at $\gamma(t_0) \in L^{\boldsymbol{v}}_c$ if $h_{\boldsymbol{v}}$ has an A_k (resp. A_l, $l \geq k$)-singularity at t_0. Thus, the contact of the curve γ with $L^{\boldsymbol{v}}_c$ at $\gamma(t_0) \in L^{\boldsymbol{v}}_c$ is of type

A_1 if and only if $\boldsymbol{v} = \lambda \gamma'^{\perp}(t_0)$ ($\lambda \in \mathbb{R} \setminus \{0\}$) and $\langle \gamma''(t_0), \gamma'^{\perp}(t_0) \rangle \neq 0$;

A_2 if and only if $\boldsymbol{v} = \lambda \gamma'^{\perp}(t_0)$, $\langle \gamma''(t_0), \gamma'^{\perp}(t_0) \rangle = 0$ and $\langle \gamma'''(t_0), \gamma'^{\perp}(t_0) \rangle \neq 0$;

$A_{\geq 2}$ if $\boldsymbol{v} = \lambda \gamma'^{\perp}(t_0)$ and $\langle \gamma''(t_0), \gamma'^{\perp}(t_0) \rangle = 0$.

It follows that γ has an $A_{\geq 1}$-contact with $L^{\boldsymbol{v}}_c$ at $\gamma(t_0) \in L^{\boldsymbol{v}}_c$ if and only if $L^{\boldsymbol{v}}_c$ is the tangent line to γ at $\gamma(t_0)$.

We call a point $\gamma(t_0)$ where γ has an A_2-contact with its tangent line an (ordinary) *inflection* point if $\gamma(t_0)$ is not a lightlike point and a *lightlike inflection point* if $\gamma(t_0)$ is a lightlike point. At such points the curve γ lies on both sides of its tangent line.

3 Spacelike and timelike curves

We consider here some properties of curves that have no lightlike points (see also [11] for related results). Let $\gamma : I \to \mathbb{R}^2_1$ be a spacelike or a timelike curve and suppose that it is parametrised by arc length (i.e., $\|\gamma'(s)\| = 1$ for all $s \in I$). We denote by $\boldsymbol{t}$ the unit tangent vector to γ and let $\boldsymbol{n}$ be the unit normal vector to γ such that $\{\boldsymbol{t}, \boldsymbol{n}\}$ is oriented anti-clockwise. The vector $\boldsymbol{n}$ is timelike (resp. spacelike) if γ is spacelike (resp. timelike). We have

$$\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s)$$

where $\kappa(s)$ is defined to be the *curvature* of γ at s. (It follows from the above setting that $\boldsymbol{n}'(s) = \kappa(s)\boldsymbol{t}(s)$.) Thus,

$$\kappa(s) = \frac{\langle \boldsymbol{t}'(s), \boldsymbol{n}(s) \rangle}{\langle \boldsymbol{n}(s), \boldsymbol{n}(s) \rangle} = (-1)^w \langle \boldsymbol{t}'(s), \boldsymbol{n}(s) \rangle$$

where $w = 1$ if γ is spacelike and $w = 2$ if it is timelike. When γ is not parametrised by arc length, and if t denotes the non-arc length parameter, then

$$\begin{aligned} \boldsymbol{t}(t) &= \frac{\gamma'(t)}{\|\gamma'(t)\|}, \\ \boldsymbol{n}(t) &= (-1)^{w+1}\frac{\gamma'(t)^{\perp}}{\|\gamma'(t)\|}. \end{aligned}$$

It follows by differentiation and using the fact that $d/ds = (1/\|\gamma'(t)\|)d/dt$ that

$$\kappa(t) = (-1)^{w+1}\frac{\langle \gamma''(t), \gamma'(t)^{\perp}\rangle}{|\langle \gamma'(t), \gamma'(t)\rangle|^{\frac{3}{2}}}.$$

Remark 3.1 (1) The curvature of a curve γ in the Minkowski plane is not in general well defined at the lightlike points of γ. For instance, if $\gamma(t_0)$ is an isolated lightlike point and $\gamma''(t_0)$ is not parallel to $\gamma'(t_0)$, then $\langle \gamma''(t_0), \gamma'(t_0)^{\perp}\rangle \neq 0$ and the curvature at points on the spacelike and timelike components of γ tends to infinity as t tends to t_0.

(2) Inflection points of a spacelike or a timelike curve are the points where $\kappa(t) = 0$.

A point $\gamma(t_0)$ is called a *vertex* of γ if $\kappa'(t_0) = 0$ and an *ordinary vertex* if $\kappa'(t_0) = 0$ and $\kappa''(t_0) \neq 0$. (See [13] for a 4-vertex thereom for curves in $\mathbb{R}^2_1$.)

The *evolute* of γ, with its inflection points removed, is defined as the curve in $\mathbb{R}^2_1$ given by

$$e(t) = \gamma(t) - \frac{1}{\kappa(t)}\boldsymbol{n}(t).$$

We have the following elementary result.

Proposition 3.2 (i) *The evolute of a spacelike* (*resp. timelike*) *curve is a timelike* (*resp. spacelike*) *curve.*

(ii) *The evolute of γ is singular at precisely the vertices of γ.*

Proof We suppose that γ is parametrised by arc length. Then,

$$e'(t) = \frac{\kappa'(t)}{\kappa^2(t)}\boldsymbol{n}(t)$$

and the proof follows from the fact that the vectors $\boldsymbol{t}$ and $\boldsymbol{n}$ are of different types (one is spacelike while the other is timelike, or vice versa). □

Proposition 3.3 *Let $\gamma : I \to \mathbb{R}^2_1$ be a connected spacelike or timelike curve. Then γ does not intersect its evolute e.*

Proof Suppose that γ intersects its evolute e. Then there exists $t_1, t_2 \in I$ with $t_1 \neq t_2$ (and assume for simplicity that $t_1 < t_2$), such that

$$\gamma(t_1) - \frac{1}{\kappa(t_1)}\boldsymbol{n}(t_1) = \gamma(t_2).$$

It follows that

$$\gamma(t_1) - \gamma(t_2) = \frac{1}{\kappa(t_1)}\boldsymbol{n}(t_1).$$

But there exists $t_3 \in (t_1, t_2)$ such that $\gamma(t_1) - \gamma(t_2)$ is parallel to $\boldsymbol{t}(t_3)$. This is a contradiction as $\boldsymbol{t}(t_3)$ and $\boldsymbol{n}(t_1)$ are of different types. Therefore, γ cannot intersect its evolute. □

4 Caustics of curves in $\mathbb{R}^2_1$

We consider a curve $\gamma \in Emb(S^1, \mathbb{R}^2_1)$. To study the local properties of γ at $\gamma(t_0)$, we consider the germ $\gamma : \mathbb{R}, t_0 \to \mathbb{R}^2_1$ of γ at t_0.

The family of distance squared functions $f : S^1 \times \mathbb{R}^2_1 \to \mathbb{R}$ on γ is given by

$$f(t, \boldsymbol{v}) = \langle \gamma(t) - \boldsymbol{v}, \gamma(t) - \boldsymbol{v} \rangle.$$

We denote by $f_{\boldsymbol{v}} : S^1 \to \mathbb{R}$ the function given by $f_{\boldsymbol{v}}(t) = f(t, \boldsymbol{v})$. The $\mathcal{R}$-singularity type of $f_{\boldsymbol{v}}$ at t_0 measures the contact of γ at $\gamma(t_0)$ with the pseudo-circle of centre $\boldsymbol{v}$ and radius $||\gamma(t_0) - \boldsymbol{v}||$. The type of the pseudo-circle is determined by the sign of $\langle \gamma(t_0) - \boldsymbol{v}, \gamma(t_0) - \boldsymbol{v} \rangle$.

The catastrophe set of f is defined by

$$\Sigma(f) = \{(t, \boldsymbol{v}) \in S^1 \times \mathbb{R}^2_1 \,|\, f'_{\boldsymbol{v}}(t) = 0\}.$$

We also define

$$Bif(f) = \{\boldsymbol{v} \in \mathbb{R}^2_1 \,|\, \exists (t, \boldsymbol{v}) \in \Sigma(f) \text{ such that } f''_{\boldsymbol{v}}(t) = 0\}.$$

The set $Bif(f)$ is the local stratum of the *bifurcation set* of the family f, i.e., it is the set of points $\boldsymbol{v} \in \mathbb{R}^2_1$ for which there exists $t \in S^1$ such that $f_{\boldsymbol{v}}$ has a degenerate (non-stable) singularity at t, i.e., a singularity of type $A_{\geq 2}$.

The function $g(t, \boldsymbol{v}) = f'_{\boldsymbol{v}}(t) = 2\langle \gamma(t) - \boldsymbol{v}, \gamma'(t) \rangle$ is not singular at any point in $\Sigma(f)$. Indeed, if we write $\gamma(t) = (x(t), y(t))$, then the gradient of g is a multiple of

$$(\langle \gamma(t) - \boldsymbol{v}, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle, x'(t), -y'(t))$$

and is never a zero vector as γ is a regular curve. Therefore, $\Sigma(f)$ is a smooth and regular 2-dimensional submanifold of $S^1 \times \mathbb{R}^2_1$ and the family f is a generating family (see [2] for terminology). We write $\boldsymbol{v} = (v_0, v_1)$ and denote by $T^*\mathbb{R}^2_1$ the cotangent bundle of $\mathbb{R}^2_1$ endowed with the canonical symplectic structure (which is metric independent). We denote by $\pi : T^*\mathbb{R}^2_1 \to \mathbb{R}^2_1$ the canonical projection. Then, the map $L(f) : \Sigma(f) \to T^*\mathbb{R}^2_1$, given by

$$L(f)(t, \boldsymbol{v}) = (\boldsymbol{v}, (\frac{\partial f}{\partial v_0}(t, \boldsymbol{v}), \frac{\partial f}{\partial v_1}(t, \boldsymbol{v}))),$$

is a Lagrangian immersion, so the map $\pi \circ L(f) : \Sigma(f) \to \mathbb{R}^2_1$ given by $(t, \boldsymbol{v}) \to \boldsymbol{v}$ is a Lagrangian map.

The *caustic* $C(\gamma)$ of γ is the set of critical values of the Lagrangian map $\pi \circ L(f)$, and is precisely $Bif(f)$ (see [2] for details). It follows that for a generic curve γ, the caustic $C(\gamma)$ is locally either a regular curve or has a cusp singularity. The local models of the caustic at $\boldsymbol{v}$ corresponding to $t \in S^1$ depend on the $\mathcal{R}$-singularity type of $f_{\boldsymbol{v}}$ at t. For a generic γ, $f_{\boldsymbol{v}}$ has local singularities of type A_1, A_2 or A_3. The caustic is the empty set at an A_1-singularity of $f_{\boldsymbol{v}}$. It is a regular curve at an A_2-singularity of $f_{\boldsymbol{v}}$ and has a cusp singularity at an A_3-singularity of $f_{\boldsymbol{v}}$.

We can obtain a parametrisation of the caustic as follows. We have $f_{\boldsymbol{v}}(t) = \langle \gamma(t) - \boldsymbol{v}, \gamma(t) - \boldsymbol{v} \rangle$, so

$$\frac{1}{2} f'_{\boldsymbol{v}}(t) = \langle \gamma(t) - \boldsymbol{v}, \gamma'(t) \rangle.$$

It follows that $f_{\boldsymbol{v}}$ is singular at t if and only if $\langle \gamma(t) - \boldsymbol{v}, \gamma'(t) \rangle = 0$, equivalently, if and only if $\gamma(t) - \boldsymbol{v} = \mu \gamma'(t)^{\perp}$ for some scalar μ. (This condition includes the lightlike points of γ where $\gamma'(t)^{\perp}$ is parallel to $\gamma'(t)$.)

Differentiating again we get

$$\begin{array}{rcl} \frac{1}{2} f''_{\boldsymbol{v}}(t) & = & \langle \gamma(t) - \boldsymbol{v}, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle \\ & = & \mu \langle \gamma'(t)^{\perp}, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle. \end{array}$$

The singularity of $f_{\boldsymbol{v}}$ at $\gamma(t)$ is degenerate if and only if $f'_{\boldsymbol{v}}(t) = f''_{\boldsymbol{v}}(t) = 0$, equivalently, if and only if $\gamma(t) - \boldsymbol{v} = \mu \gamma'(t)^{\perp}$ and

$$\mu \langle \gamma'(t)^{\perp}, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle = 0. \tag{1}$$

It follows that the caustic of γ is given by

$$C(\gamma) = \{\gamma(t) - \mu \gamma'(t)^{\perp} \,|\, t \in S^1 \text{ and } \mu \text{ is a solution of equation (1)}\}.$$

Away from the lightlike points of γ, we can write $\gamma(t) - \boldsymbol{v} = \lambda \boldsymbol{n}(t)$, where $\lambda = \mu ||\gamma'(t)^{\perp}||$ and $\boldsymbol{n}(t) = (-1)^{w+1} \gamma'(t)^{\perp} / ||\gamma'(t)^{\perp}||$ is the unit normal vector ($w = 1$ if $\gamma(t)$ is spacelike and $w = 2$ if it is timelike). Then a singularity of $f_{\boldsymbol{v}}$ is degenerate if and only if

$$\boldsymbol{v} = \gamma(t) - \frac{1}{\kappa(t)} \boldsymbol{n}(t). \tag{2}$$

This is precisely the evolute of the spacelike and timelike components of γ. As in the case of curves in the Euclidean plane, the evolute of γ (minus its lightlike points) is the locus of its centres of curvature. It is a subset of the caustic, which is the locus of centres of "*osculating*" pseudo-circles (i.e., pseudo-circles that have an $A_{\geq 2}$-contact with γ).

We define the subset Ω of $Emb(S^1, \mathbb{R}^2_1)$ such that a curve γ is in Ω if and only if $\langle \gamma''(t), \gamma'(t)\rangle \neq 0$ whenever $\langle \gamma'(t), \gamma'(t)\rangle = 0$ (i.e., the lightlike points of $\gamma \in \Omega$ are not lightlike inflection points). One can show, using Thom's transversality results (see for example [5], Chapter 9 for an analogous proof), that Ω is an open dense subset of $Emb(S^1, \mathbb{R}^2_1)$.

Proposition 4.1 *Let $\gamma \in \Omega$. Then,*

(i) *the lightlike points of γ are isolated points;*

(ii) *the caustic of γ is a regular curve at a lightlike point of γ and has ordinary tangency with γ at such point. Furthermore, γ and its caustic lie locally on opposite sides of their common tangent line at the lightlike point.*

Proof (i) Since the curve γ is in Ω, we have $g'(t) = 2\langle \gamma''(t), \gamma'(t)\rangle \neq 0$ whenever $g(t) = \langle \gamma'(t), \gamma'(t)\rangle = 0$. This implies that the lightlike points, given by $g(t) = 0$, are isolated points.

(ii) For $\gamma \in \Omega$, we can solve equation (1) at a lightlike point $\gamma(t_0)$ to get

$$\mu(t) = -\frac{\langle \gamma'(t), \gamma'(t)\rangle}{\langle \gamma'(t)^{\perp}, \gamma''(t)\rangle}$$

for t near t_0. Then, $\mu(t_0) = 0$ and the caustic $C(\gamma)$ is parametrised locally at t_0 by

$$c(t) = \gamma(t) - \mu(t)\gamma'(t)^{\perp}.$$

We have

$$\mu'(t) = -\frac{2\langle \gamma'(t), \gamma''(t)\rangle}{\langle \gamma'(t)^{\perp}, \gamma''(t)\rangle} - \left(\frac{1}{\langle \gamma'(t)^{\perp}, \gamma''(t)\rangle}\right)'\langle \gamma'(t), \gamma'(t)\rangle$$

and

$$\begin{aligned}\mu''(t) \quad = \quad & -\frac{2\langle \gamma'(t), \gamma'''(t)\rangle}{\langle \gamma'(t)^{\perp}, \gamma''(t)\rangle} - \frac{2\langle \gamma''(t), \gamma''(t)\rangle}{\langle \gamma'(t)^{\perp}, \gamma''(t)\rangle} - 4\left(\frac{1}{\langle \gamma'(t)^{\perp}, \gamma''(t)\rangle}\right)'\langle \gamma'(t), \gamma''(t)\rangle \\ & -\left(\frac{1}{\langle \gamma'(t)^{\perp}, \gamma''(t)\rangle}\right)''\langle \gamma'(t), \gamma'(t)\rangle.\end{aligned}$$

At the lightlike point $\gamma(t_0)$ we have $\gamma'(t_0)^{\perp} = (-1)^{\epsilon}\gamma'(t_0)$, where $\epsilon = 2$ if $\gamma'(t_0) = (\lambda, \lambda)$ and $\epsilon = 1$ if $\gamma'(t_0) = (-\lambda, \lambda)$. Thus,

$$\mu'(t_0) = -\frac{2(-1)^{\epsilon}\langle \gamma'(t_0), \gamma''(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} = -2(-1)^{\epsilon}$$

and

$$\begin{aligned}\mu''(t_0) &= -\frac{2(-1)^\epsilon\langle\gamma'(t_0),\gamma'''(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle} - \frac{2(-1)^\epsilon\langle\gamma''(t_0),\gamma''(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle} \\ &\quad -4\left(\frac{1}{\langle\gamma'(t)^\perp,\gamma''(t)\rangle}\right)'|_{t=t_0}\langle\gamma'(t_0),\gamma''(t_0)\rangle \\ &= \frac{2(-1)^\epsilon\langle\gamma'(t_0),\gamma'''(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle} - \frac{2(-1)^\epsilon\langle\gamma''(t_0),\gamma''(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle}.\end{aligned}$$

It follows now that

$$\begin{aligned}c'(t_0) &= \gamma'(t_0) - \mu'(t_0)\gamma'(t_0)^\perp - \mu(t_0)\gamma''(t_0)^\perp \\ &= \gamma'(t_0) + 2(-1)^\epsilon(-1)^\epsilon\gamma'(t_0) \\ &= 3\gamma'(t_0).\end{aligned}$$

Therefore, γ and $C(\gamma)$ are tangential at $\gamma(t_0)$. Differentiating again, we get

$$\begin{aligned}c''(t_0) &= \gamma''(t_0) - 2\mu'(t_0)\gamma''(t_0)^\perp - \mu''(t_0)\gamma'(t_0)^\perp - \mu(t_0)\gamma'''(t_0)^\perp \\ &= \gamma''(t_0) + 4(-1)^\epsilon\gamma''(t_0)^\perp - 2\left(\frac{\langle\gamma'(t_0),\gamma'''(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle} - \frac{\langle\gamma''(t_0),\gamma''(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle}\right)\gamma'(t_0).\end{aligned}$$

We can take $\{\gamma'(t_0),\gamma''(t_0)\}$ as a system of coordinate of $\mathbb{R}^2_1$ at $\gamma(t_0)$. Then, we can write $c''(t_0) = \alpha\gamma'(t_0) + \beta\gamma''(t_0)$ with

$$\begin{aligned}\beta &= \frac{\langle c''(t_0),\gamma'(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle} \\ &= \frac{\langle\gamma''(t_0) + 4(-1)^\epsilon\gamma''(t_0)^\perp,\gamma'(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle} \\ &= 1 - 4(-1)^\epsilon\frac{\langle\gamma''(t_0),\gamma'(t_0)^\perp\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle} \\ &= 1 - 4(-1)^\epsilon(-1)^\epsilon\frac{\langle\gamma''(t_0),\gamma'(t_0)\rangle}{\langle\gamma'(t_0),\gamma''(t_0)\rangle} \\ &= -3.\end{aligned}$$

We have then

$$\gamma(t) - \gamma(t_0) = ((t-t_0) + h.o.t)\gamma'(t_0) + (\frac{1}{2}(t-t_0)^2 + h.o.t)\gamma''(t_0)$$

and

$$c(t) - c(t_0) = c(t) - \gamma(t_0) = (3(t-t_0) + h.o.t)\gamma'(t_0) + (-\frac{3}{2}(t-t_0)^2 + h.o.t)\gamma''(t_0).$$

This shows that γ and its caustic have an ordinary tangency at $\gamma(t_0)$ and that the two curves lie on opposite sides of their common tangent line at $\gamma(t_0)$. □

Remark 4.2 At a lightlike inflection point $\gamma(t_0)$ of a curve $\gamma \notin \Omega$, the tangent line to γ is always a component of the caustic $C(\gamma)$ (any $\mu \in \mathbb{R}$ is a solution of equation (1) at a lightlike inflection point). The caustic has another component if and only if $\mathrm{ord}(\langle \gamma'(t), \gamma'(t) \rangle) \geq \mathrm{ord}(\langle \gamma'(t)^{\perp}, \gamma''(t) \rangle)$ at $t = t_0$. Then equation (1) can be solved for μ and we obtain a parametrisation of this other component of $C(\gamma)$. This component, which could be singular, passes through $\gamma(t_0)$ if and only if $\mu(t_0) = 0$.

We consider now some special curves in $\mathbb{R}^2_1$. An oval in the Euclidean plane $\mathbb{R}^2$ is defined as a closed and simple curve with everywhere non-vanishing curvature. The curvature of a curve in $\mathbb{R}^2_1$ is not defined at the lightlike points of the curve. However, we can still define the concept of an oval in the Minkowski plane using the contact of the curve with lines. We say that a closed and simple curve in $\mathbb{R}^2_1$ is an *oval* if it has an A_1-contact with all its tangent lines. (This definition includes the lightlike points. An example of an oval is the circle $S^1 = \{(u_0, u_1) \in \mathbb{R}^2_1 \mid u_0^2 + u_1^2 = 1\}$.)

As an oval is a closed and simple curve, it follows by the Jordan curve theorem that its complement $\mathbb{R}^2_1 \setminus \gamma$ consists of two open and connected subsets of $\mathbb{R}^2$. One of them is bounded and is called the interior of γ and the other is unbounded and is called the exterior of γ.

Theorem 4.3 *Let γ be an oval in the Minkowski plane. Then,*

(i) *γ has exactly four lightlike points;*

(ii) *the caustic of γ is a closed curve which lies in the complement of the interior of γ;*

(iii) *the evolute of each spacelike and timelike component of γ has at least one singular point.*

Proof (i) We use the arguments in the proof of Proposition 2.1. The curve $\tilde{\gamma}$ has nowhere vanishing (Euclidean) curvature as its contact with its tangent lines is of type A_1 (the contact of γ with lines is an affine property and is independent of the metric in $\mathbb{R}^2$). Therefore, the Gauss map N is a diffeomorphism and the result follows.

(ii) The curve γ is an oval, so it has neither inflection points nor lightlike inflection points. Therefore, its caustic is defined everywhere and is a closed curve. It follows from Proposition 4.1(ii) and from the fact that γ is an oval that the caustic of γ, minus the lightlike points, lies in the exterior of γ near the lightlike points. By Proposition 3.3, the evolute of a spacelike or a timelike component of γ does not intersect that component. Thus, the evolute of γ remains in the exterior of γ.

(iii) Let $I = (a, b)$ be an interval parametrising a spacelike or timelike component of γ, with $\gamma(a)$ and $\gamma(b)$ lightlike points. As $\langle \gamma''(t), \gamma'(t)^{\perp} \rangle \neq 0$, the curvature goes to infinity as t tends a or b. The curve γ is an oval, so its curvature has constant sign in I. Therefore, $\lim_{t \to a} \kappa(t) = \lim_{t \to b} \kappa(t) = \pm\infty$ (with $t \in I$). It follows that there exists $t \in I$ such that $\kappa'(t) = 0$, so γ has a vertex at t, and this corresponds to a singular point on the evolute (Proposition 3.2(ii)). $\square$

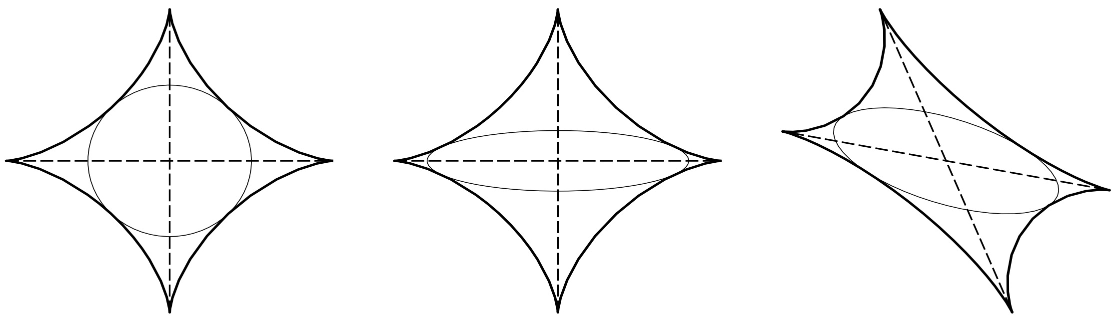

Figure 3: The caustic (thick curve) of a circle (left) and of an ellipse (centre and right) drawn using Maple. The dashed lines are the Minkowski symmetry sets.

Example 4.4 An ellipse $\gamma(t) = (a\cos(t), b\sin(t))$, $t \in \mathbb{R}$ is an oval in $\mathbb{R}^2_1$. Figure 3 shows Maple plots of the caustics of some ellipses. The caustic of a circle ($a = b = 1$) is shown in Figure 3 left. (Recall that the caustic/evolute of a circle in the Euclidean plane is the centre of the circle.) We take $a = 2$ and $b = 1$ in Figure 3 centre. In Figure 3 right, we apply an Euclidean rotation to the ellipse and draw its caustic. We observe that the left figure in Figure 3 can also be found in [9], where the caustic is defined as the envelope of the normal lines to the circle.

5 Minkowski symmetry set

The symmetry set (SS) of a curve in the Euclidean plane is defined as the closure of the locus of centres of bi-tangent circles to the curve ([6, 12]). We define as follows its analogue for a curve in the Minkowski plane.

Definition 5.1 *The Minkowski symmetry set (MSS for short) of a curve γ in the Minkowski plane is the closure of the locus of centres of bi-tangent pseudo-circles to the curve γ.*

The pseudo-circles $H^1(p, -r)$ and $S^1_1(p, r)$ have two connected components and a curve γ can be tangent to either a single component at two distinct points or to each component of these pseudo-circles. If a curve γ is bi-tangent to a "lightcone" $LC^*(p)$, then generically it is tangent to each line of $LC^*(p)$ at a single point. (One can show using Thom's transversality theorem that bi-tangency with a single line of $LC^*(p)$ is not a generic property of curves in $\mathbb{R}^2_1$.)

The contact of γ with pseudo-circles is measured by the family of distance squared functions f (§4). The multi-local stratum of the bifurcation set of f is the set of points $\boldsymbol{v}$ such that $f_{\boldsymbol{v}}$ has singularities at two distinct points t_1 and t_2 with $f_{\boldsymbol{v}}(t_1) = f_{\boldsymbol{v}}(t_2)$.

The MSS has the following properties, some of which are similar to those of the SS.

Theorem 5.2 (i) *The MSS of γ is the closure of the multi-local stratum of the bifurcation set of the family of distance squared functions on γ.*

(ii) *If γ is spacelike or timelike and is parametrised by arc length, then there is a bi-tangent pseudo-circle to γ at $\gamma(t_1)$ and $\gamma(t_2)$ if and only if*

$$\langle \gamma(t_1) - \gamma(t_2), \boldsymbol{t}(t_1) \pm \boldsymbol{t}(t_2) \rangle = 0$$

(*where $+$ or $-$ is determined by the orientation of γ at $\gamma(t_1)$ and $\gamma(t_2)$*).

(iii) *The MSS is a regular curve at p if and only if the bi-tangent pseudo-circle to γ at $\gamma(t_1)$ and $\gamma(t_2)$ is not osculating at $\gamma(t_1)$ or at $\gamma(t_2)$. If this is the case, the tangent line to the MSS at p is the perpendicular bisector to the chord joining $\gamma(t_1)$ and $\gamma(t_2)$.*

(iv) *The MSS is a spacelike curve at a point p in the following cases:* (1) *the curve γ is tangent to each component of a pseudo-circle $H^1(p, -r)$;* (2) *the curve γ is bi-tangent to a single component of a pseudo-circle $S^1_1(p, r)$;* (3) *the curve γ is tangent to one line of $LC^*(p)$ at $\gamma(t_1)$ and to the other line at $\gamma(t_2)$, and $\gamma(t_2) - \gamma(t_1)$ is timelike.*

(v) *The MSS is a timelike curve at a point p in the following cases:* (1) *the curve γ is tangent to each component of a pseudo-circle $S^1_1(p, r)$;* (2) *the curve γ is bi-tangent to a single component of a pseudo-circle $H^1(p, -r)$;* (3) *the curve γ is tangent to one line of $LC^*(p)$ at $\gamma(t_1)$ and to the other line at $\gamma(t_2)$, and $\gamma(t_2) - \gamma(t_1)$ is spacelike.*

(vi) *The MSS has generically no lightlike points.*

Proof The proof of (i) follows from the definition of the MSS and the proof of (ii) is identical to that for the symmetry set of a curve in the Euclidean plane (see [6]).

For (iii), we consider the case where the bi-tangent pseudo-circles are of type $H^1(p, -r)$ and suppose that γ is tangent to both components these pseudo-circles (the other cases follow similarly). We give the pieces of γ at $\gamma(t_1)$ (resp. $\gamma(t_2)$) the orientation of $p + (r\cosh(t), r\sinh(t))$ (resp. $p + (-r\cosh(t), r\sinh(t))$). To simplify notation, we write γ_1 for $\gamma(t_1)$ and γ_2 for $\gamma(t_2)$ and similarly for all information at $\gamma(t_1)$ and $\gamma(t_2)$. The condition for bi-tangency is then given by $g(t_1, t_2) = \langle \gamma_1 - \gamma_2, \boldsymbol{t}_1 + \boldsymbol{t}_2 \rangle = 0$. As $\langle \boldsymbol{t}_2 + \boldsymbol{t}_1, \boldsymbol{n}_2 + \boldsymbol{n}_1 \rangle = 0$, $g(t_1, t_2) = 0$ if and only if $\gamma_1 - \gamma_2 = r(\boldsymbol{n}_1 + \boldsymbol{n}_2)$. The radius r of the bi-tangent pseudo-circle can then be given explicitly in the form

$$r(t_1, t_2) = \frac{\langle \gamma_1 - \gamma_2, \boldsymbol{n}_1 + \boldsymbol{n}_2 \rangle}{2(\boldsymbol{n}_1\boldsymbol{n}_2 - 1)}.$$

(We observe that $\boldsymbol{n}_1\boldsymbol{n}_2 - 1 \neq 0$.) We have

$$\begin{aligned} g_{t_1}(t_1, t_2) &= -(\boldsymbol{n}_1\boldsymbol{n}_2 - 1)(1 - r\kappa_1), \\ g_{t_2}(t_1, t_2) &= (\boldsymbol{n}_1\boldsymbol{n}_2 - 1)(1 + r\kappa_2) \end{aligned}$$

so the MSS is a regular curve at p if and only if $1-r\kappa_1\neq 0$ or $1+r\kappa_2\neq 0$, equivalently, if and only if $H^1(p,-r)$ is not osculating at both $\gamma(t_1)$ and $\gamma(t_2)$.

Suppose that $H^1(p,-r)$ is not osculating at $\gamma(t_1)$. Then we can parametrise locally $g^{-1}(0)$ by $(t_1,t_2(t_1))$ for some smooth function $t_2(t_1)$. The MSS is then parametrised by

$$c(t_1)=\gamma(t_1)-r(t_1,t_2(t_1))\boldsymbol{n}(t_1).$$

We have

$$c'=(1-r\kappa_1)\boldsymbol{t}_1-(r_{t_1}+t_2'r_{t_2})\boldsymbol{n}_1$$

and

$$\begin{aligned} r_{t_1} &= \frac{(1-r\kappa_1)\boldsymbol{t}_1\boldsymbol{n}_2}{2(\boldsymbol{n}_1\boldsymbol{n}_2-1)},\\ r_{t_2} &= -\frac{(1+r\kappa_2)\boldsymbol{t}_2\boldsymbol{n}_1}{2(\boldsymbol{n}_1\boldsymbol{n}_2-1)},\\ t_2' &= \frac{1-r\kappa_1}{1+r\kappa_2}. \end{aligned}$$

Therefore $\langle c',\boldsymbol{n}_1+\boldsymbol{n}_2\rangle=0$, that is $c'(t_1)$ is orthogonal to $\gamma(t_1)-\gamma(t_2)$. To show that the tangent line to the MSS is the perpendicular bisector to the chord joining $\gamma(t_1)$ and $\gamma(t_2)$, it is enough to consider these points on the pseudo-circle $H^1(p,-r)$ and observe the said perpendicular bisector passes through p.

For (iv) and (v), the results are immediate using (iii) for bi-tangency with LC^*. For the other cases, also using (iii), it is enough to choose any two distinct points q_1 and q_2 on $H^1(p,-r)$ (resp. $S^1_1(p,r)$) and consider the vector $\overrightarrow{q_1q_2}$. We can take, without loss of generality, $r=1$ and p to be the origin. Then, a parametrisation of the components of $H^1(-1)$ are given by $(\cosh(s),\sinh(s))$ and $(-\cosh(s),-\sinh(s))$, and those of $S^1_1(1)$ by $(\sinh(s),\cosh(s))$ and $(-\sinh(s),-\cosh(s))$. The result now follows by straightforward calculations.

For (vi), the vector $\overrightarrow{q_1q_2}$ (with q_1 and q_2 as above) is never a lightlike vector, so the only possible case for a point $p\in MSS$ to be lightlike is when one of the lines of LC^* is bi-tangent to the curve γ. However, this does not occur for generic curves in the Minkowski plane. □

We consider now the example of an ellipse in the Minkowski plane.

Proposition 5.3 *The MSS of an ellipse consists of the two segments of lines joining opposite cusps of the caustic of the ellipse (the dashed lines in Figure 3). These segments contain the diagonals of the parallelogram formed by the four tangent lines to the ellipse at its lightlike points (Figure 4, left).*

Proof We make an affine transformation A and take the ellipse to a circle (C) and the rectangle formed by the lightlike tangent lines of the ellipse to a parallelogram (P) (Figure 4, right). The images by A of the families of hyperbole $H^1(p,-r)$ and $S^1_1(p,r)$

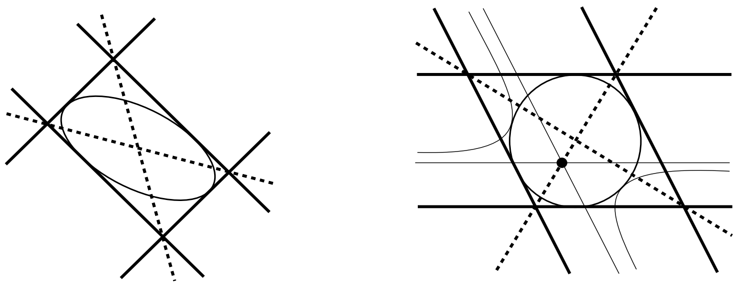

Figure 4: Constructing the MSS of an ellipse (dashed line).

are families of hyperbole $l_1 l_2 = c$ ($c \in \mathbb{R}$) with asymptotes $l_1 = 0$ and $l_2 = 0$ parallel to the sides of the parallelogram (P). The MSS of the ellipse is the pre-image by A of the locus of bi-tangency of the circle (C) with the hyperbole $l_1 l_2 = c$.

Given a bi-tangent hyperbola $l_1 l_2 = c$ to the circle (C), the centre of (C) belongs to the Euclidean symmetry set of $l_1 l_2 = c$. Now, the symmetry set of a hyperbola $l_1 l_2 = c$ consists of the pair of lines which bisect the lines $l_1 = 0$ and $l_2 = 0$. It follows that the point of intersection of $l_1 = 0$ and $l_2 = 0$ is on a diagonal of the parallelogram (P) (Figure 4, right). As the diagonals of the parallelogram are preserved under affine transformations, it follows that the MSS of the ellipse is a subset of the lines containing the diagonals of the parallelogram formed by the four tangent lines to the ellipse at its lightlike points. The result follows now using the fact that the MSS has endpoints at the cusps of the evolute of the ellipse. (See Figure 3 for the MSS of various ellipses in the Minkowski plane. Observe that, in general, the MSS of an ellipse is not along the axes of the ellipse; Figure 3, right.) □

Remark 5.4 (1) The concepts of evolute, caustic and MSS can be associated to a curve in any Lorentzian plane $(\mathbb{R}^2, g)$. We can find a g-orthonormal basis $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ of $\mathbb{R}^2$ so that the expression for g is given, with respect to this basis, by $g(\boldsymbol{u}, \boldsymbol{v}) = -u_0 v_0 + u_1 v_1$, for any $\boldsymbol{u} = (u_0, u_1)$ and $\boldsymbol{v} = (v_0, v_1)$ in $\mathbb{R}^2$ (so we are back to the Minkowski plane).

(2) If we write the g-lightlike lines as $l_i = a_i x + b_i y = 0$, $i = 1, 2$, where (x, y) are the coordinates with respect to the standard basis in $\mathbb{R}^2$, then the g-pseudo-circles centred at the origin are the family of hyperbole (including their asymptotes) $l_1 l_2 = c$, $c \in \mathbb{R}$. Therefore, the results in this paper can be interpreted in the affine setting. They provide information about the contact of a curve in the affine plane $\mathbb{R}^2$ with a given family of hyperbole $l_1 l_2 = c$, translated by any vector in $\mathbb{R}^2$.

6 Parallels

A parallel of a curve γ in the Minkowski plane, with its lightlike points removed, is the curve obtained by moving each point on γ by a fixed distance r along the unit normal $\boldsymbol{n}$ to γ. Thus, a parametrisation of a parallel is given by

$$\eta_r(t) = \gamma(t) + r\boldsymbol{n}(t).$$

It is worth observing that the parallels are not defined at lightlike points (as we require a unit normal vector). Parallels are wave fronts and can be studied following the same approach for curves in the Euclidean plane using the family of distance squared functions (see for example [3]). Consider the map $F : (S^1 \setminus L) \times \mathbb{R}^2_1 \to \mathbb{R} \times \mathbb{R}^2_1$, given by $F(t, \boldsymbol{v}) = (f(t, \boldsymbol{v}), \boldsymbol{v})$, where L denotes the set of the lightlike points of γ and f is the family of distance squared function. The set of critical points $\Sigma(F)$ of F coincides with $\Sigma(f) \cap (S^1 \setminus L) \times \mathbb{R}^2_1$, and is thus a smooth surface (§4). The wave fronts (parallels) associated to γ are the sets $\eta_r = F(\Sigma(F)) \cap \{r\} \times \mathbb{R}^2_1$. Wave fronts have generic Legendrian singularities ([1, 3]) apart from a discrete set of distances r. There are three possible transitions at these values of r ([1]). However, it is shown in [3] that only the A_3-transition occurs (i.e., the swallowtail transitions in Figure 5 right), and this happens at an ordinary vertex of γ.

The A_3-transition in wave fronts is studied by considering (locally) the big front $F(\Sigma(F))$. This big front is a swallowtail surface, that is, $F(\Sigma(F))$ is diffeomorphic to the discriminant of the polynomial $t^4 + \lambda_1 t^2 + \lambda_2 t + \lambda_3$ which is the surface (Figure 5 left)

$$S = \{(\lambda_1, -4t^3 - 2\lambda_1 t, 3t^4 + \lambda_1 t^2),\, t, \lambda_1 \in \mathbb{R}, 0\}.$$

To recover the individual wave fronts, one has to consider generic sections of the surface S. This is done by Arnold [1], where he considered functions $f : \mathbb{R}^3, 0 \to \mathbb{R}, 0$ and allowed changes of coordinates in $\mathbb{R}^3, 0$ that preserve S. Then a generic function is equivalent, under these changes of coordinates, to $f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1$. Therefore, the individual wave fronts undergo the transitions in Figure 5 right.

Our concern here is how the individual fronts are stacked together in $\mathbb{R}^2_1$. For this, one needs to project the sections of S by f to a plane. Then, the problem becomes that of considering the divergent diagramme (f, g)

$$\mathbb{R}^2, 0 \xleftarrow{\;g\;} \mathbb{R}^3, 0 \xrightarrow{\;f\;} \mathbb{R}, 0.$$

Bruce proved in [4] that there are no stable pairs (f, g). (As a consequence, he showed that there are no discrete smooth models for an implicit differential equation (IDE) of cusp type. Davydov [8] showed that there is in fact a functional modulus for an IDE of cusp type even for the topological equivalence. Dara [7] pointed out that there are two possible configurations of the solutions of the IDE of cusp type and these are as in Figure 1.)

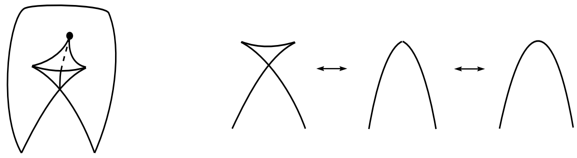

Figure 5: The swallowtail surface left and its generic sections right.

Theorem 6.1 *There are two generic configurations for the family of curves* $g(f^{-1}(c)\cap S)$*,* $c \in \mathbb{R}, 0$*. The two configurations are distinguished by* $g(f^{-1}(0)\cap S)$ *and the image of the singular set of* S *by* g*, which is a cusp. These are as in Figure 1, left, if the cusp and* $g(f^{-1}(0)\cap S)$ *are in the same semi-plane delimited by the limiting tangent line to* $g(f^{-1}(0)\cap S)$ *and as in Figure 1, right, if they are in different semi-planes.*

Proof By Arnold's result ([1]), we can take $f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1$. Then the zero section of f in S is a curve with a singularity of type (t^3, t^4).

We assume that the kernel of dg_0 is transverse to the plane $\lambda_1 = 0$. This insures that the restriction of g to the planes $f^{-1}(c)$ is a local diffeomorphism, so it preserves the structure of the curves $f^{-1}(c)\cap S$. We also assume that the kernel of dg_0 is not parallel to the direction $(1, 0, 0)$. This insures that the image by g of the singular set of S is cusp curve. A map g that satisfy both of the above conditions is a generic map.

Suppose that $f^{-1}(c)\cap S$ has a self-intersection and denote by Δ_c the triangular region whose vertices are the origin and the two cusps of $g((f^{-1}(c)\cap S))$, and whose edges are formed by the image of the singular set of S by g and the segment of $g((f^{-1}(c)\cap S))$ delimited by its singular points (shaded regions in Figure 6).

Then the two configurations of $g(f^{-1}(c)\cap S)$, $c \in \mathbb{R}, 0$, are distinguished by the fact that the self-intersection point of $g(f^{-1}(c)\cap S)$ is inside or outside the triangle Δ_c (Figures 1 and 6). This property depends only on dg_0. To show this, write $g = dg(0)+h$ where h is a smooth map with no linear terms. Let $g_s = dg(0) + sh$, $s \in [0, 1]$. Then $dg_s(0) = dg(0)$ for all $s \in [0, 1]$, so the map $g_s|f^{-1}(c)$ is a local diffeomorphism and maps the singular set of S to a cusp curve. For g_0 and g_1 to give two different configurations, there must exist $s \in [0, 1]$ such that $g_s|f^{-1}(c)$ is not a diffeomorphism (as g_s maps the curve $f^{-1}(c)\cap S$ to one which is not diffeomorphic to it), which is not the case.

We can therefore assume that g is a linear projection along a direction $\boldsymbol{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$, with $u_1^2 + u_2^2 + u_3^2 = 1$, to a transverse plane. As we assume that the kernel of g is not parallel to $(1, 0, 0)$ we can take, for simplicity, $u_3 \neq 0$ and project to the (u_1, u_2)-plane.

The projection of $f^{-1}(0)\cap S$ is the curve

$$l_0(t) = (4u_1u_2t^3 - 3u_1u_3t^4, 4(u_2^2 - 1)t^3 - 3u_2u_3t^4)$$

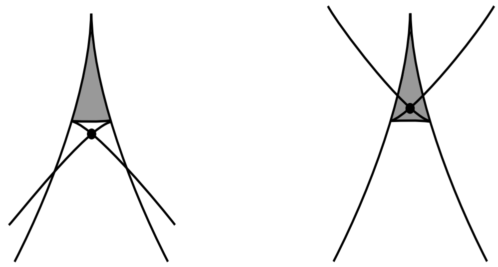

Figure 6: The two generic positions of the intersection point of $g(f^{-1}(c)\cap S)$, outside the shaded region left and inside it right.

and the projection of the singular set of S is the cusp curve

$$c_0(t) = (6(u_1^2-1)t^2 - 8u_1u_2t^3 + 3u_1u_3t^4, 6u_1u_2t^2 - 8(u_2^2-1)t^3 + 3u_2u_3t^4).$$

The limiting tangent directions of the two curves are transverse as $u_1^2+u_2^2-1 \neq 0$. Then the position of the two curves with respect to the limiting tangent line L_0 to l_0 at $t=0$ is determined by the sign of u_1u_3 (positive for the two curves to be in the same semi-plane determined by L_0 and negative if they lie in different semi-planes).

The fibre $f^{-1}(c)\cap S$ is singular if $c<0$. The singular points are given by $6t^2+c=0$ and the self-intersection point is given by $2t^2+c=0$. We project these points along $\boldsymbol{u}$ to the (u_1,u_2)-plane. It is not difficult to show that the projection of the self-intersection point is inside the triangle Δ_c if and only if $u_1u_3<0$ and outside if and only if $u_1u_3>0$. Thus, the configuration of the curves $g(f^{-1}(c)\cap S)$ is determined by the positions of the curves $l_0(t)$ and $c_0(t)$ with respect to the limiting tangent line L_0 to l_0 at $t=0$. □

Theorem 6.2 (a) *The parallels of a curve γ in the Euclidean plane are as in Figure 1, left, at an ordinary vertex of γ.*

(b) *The parallels of a curve γ in the Minkowski plane are as in Figure 1, right, at an ordinary vertex of γ.*

Proof We suppose that γ is parametrised by arc length and apply Theorem 6.1. The projection of the singular set of the big front is the evolute of γ.

(a) The evolute of a curve γ in the Euclidean plane is given by $e(t) = \gamma(t) + 1/\kappa(t)\boldsymbol{n}(t)$. Suppose that $t=0$ is an ordinary vertex of γ, that is $\kappa'(0)=0$ and $\kappa''(0)\neq 0$. Then $e'(0)=0$ and $e''(0) = -\kappa''(0)/\kappa^2(0)\boldsymbol{n}(0)$.

We take $\{\boldsymbol{t}(0),\boldsymbol{n}(0)\}$ as a coordinate system at $e(0)$. Then the evolute is above the axis parallel to $\boldsymbol{t}(0)$ if $\kappa''(0)<0$ and below it if $\kappa''(0)>0$.

The parallel of interest is $\eta_{r_0}(t) = \gamma(t) + r_0\boldsymbol{n}(t)$, with $r_0 = 1/\kappa(0)$. We have $\eta'_{r_0}(0) = \eta''_{r_0}(0) = 0$ and

$$\begin{array}{rcl} \eta'''_{r_0}(0) & = & -\frac{\kappa''(0)}{\kappa(0)}\boldsymbol{t}(0), \\ \eta^{(4)}_{r_0}(0) & = & -\frac{\kappa'''(0)}{\kappa(0)}\boldsymbol{t}(0) - 3\kappa''(0)\boldsymbol{n}(0). \end{array}$$

Then

$$\eta_{r_0}(t) = (-\frac{\kappa''(0)}{3!\kappa(0)}t^3 + h.o.t)\boldsymbol{t}(0) + (-\frac{3}{4!}\kappa''(0)t^4 + h.o.t)\boldsymbol{n}(0),$$

so the parallel η_{r_0} is above the horizontal axis if $\kappa''(0) < 0$ and below it if $\kappa''(0) > 0$. That is, the parallel η_{r_0} and the evolute are always on the same side of the limiting tangent direction to the parallel. It follows by Theorem 6.1 that the parallels of γ have the configuration in Figure 1, left.

(b) The evolute of a curve γ in the Minkowski plane is given by $e(t) = \gamma(s) - \frac{1}{\kappa(t)}\boldsymbol{n}(t)$. At an ordinary vertex $t = 0$, we have $e'(0) = 0$ and $e''(0) = \kappa''(0)/\kappa^2(0)\boldsymbol{n}(0)$.

The parallel of interest is $\eta_{r_0}(t) = \gamma(t) + r_0\boldsymbol{n}(t)$, with $r_0 = -1/\kappa(0)$. Here we have $\eta'_{r_0}(0) = \eta''_{r_0}(0) = 0$ and

$$\begin{array}{rcl} \eta'''_{r_0}(0) & = & -\frac{\kappa''(0)}{\kappa(0)}\boldsymbol{t}(0), \\ \eta^{(4)}_{r_0}(0) & = & -\frac{\kappa'''(0)}{\kappa(0)}\boldsymbol{t}(0) - 3\kappa''(0)\boldsymbol{n}(0). \end{array}$$

Following the same argument above, we conclude that the parallel η_{r_0} and the evolute are always on opposite sides of the limiting tangent direction to the parallel. It follows by Theorem 6.1 that the parallels of γ have the configuration in Figure 1, right. □

Figure 7 shows a Maple plot of the parallels of an ellipse with its lightlike points removed. Observe that the tangent lines to the ellipse at the lightlike points are asymptotes of its parallels (this is also the case at an isolated lightlike point of any curve in the Minkowski plane).

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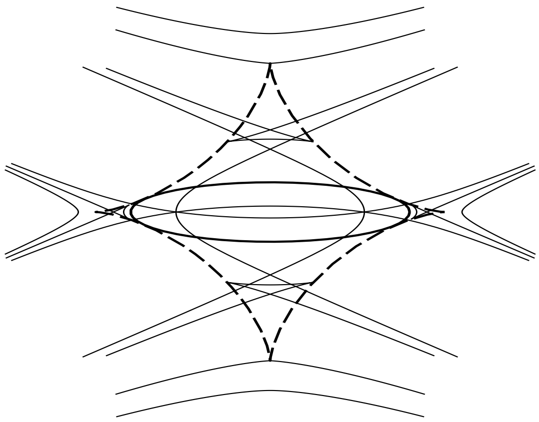

Figure 7: The parallels to an ellipse with its lightlike points removed. (The dashed curve is the caustic of the ellipse.)

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