ASYMPTOTIC CURVES ON SURFACES IN $\mathbb{R}^5$

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Abstract. We study asymptotic curves on generically immersed surfaces in $\mathbb{R}^5$. We characterise asymptotic directions via the contact of the surface with flat objects (k-planes, $k = 1$–4), give the equation of the asymptotic curves in terms of the coefficients of the second fundamental form and study their generic local configurations.

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1. Introduction

Singularity theory made important contributions to the study of extrinsic differential geometry of submanifolds in Euclidean spaces. The idea is to define some natural families of functions or maps on the submanifold and investigate the singularities of such maps. The various types of singularities capture some aspects of the geometry of the submanifold. For example, given a generic smooth surface $M \subset \mathbb{R}^3$, the projection along a tangent direction u at $q \in M$ to a transverse plane is right-left equivalent to a cusp $(x, xy + y^3)$ if u is an asymptotic direction. The singularity of the projection is of type lips/beaks (i.e. right-left equivalent to $(x, x^2y \pm y^3)$) if q is a parabolic point.

For surfaces in $\mathbb{R}^3$, asymptotic directions and parabolic points are characterised in Differential Geometry textbooks in terms of the normal curvature (see for example [12]). However, this approach does not generalise easily to manifolds immersed in higher dimensional spaces. A better approach is to define these concepts in terms of the singularities of maps associated to the contact of the surface with flat objects (k-planes). For 2-dimensional surfaces in $\mathbb{R}^4$ this is done in [17] and [28] in terms of the contact of the surface with 3-dimensional planes and in [7] in terms of its contact with lines. For 2-dimensional surfaces in $\mathbb{R}^n$, $n \geq 5$, this is done in [31] and [34] in terms of the contact of the surface with $(n-1)$-dimensional planes. (See also [26] and [28, 29, 31] for definitions of asymptotic directions using the curvature ellipse.)

We characterise in this paper the asymptotic directions of an immersed 2-dimensional smooth surface M in $\mathbb{R}^5$ in terms of the contact of the surface with k-planes, $k = 1, 2, 3, 4$ (§3). We obtain the differential equation of the asymptotic curves in terms of the coefficients of the second fundamental form (§4) and study the generic local configurations of these curve (§5). Some global consequences are given in §6.

Some aspects of the geometry of surfaces in $\mathbb{R}^5$ is studied in [31] and [29]. The choice of the Euclidean space $\mathbb{R}^5$ is related to the concept of kth-regular immersion of a submanifold M in Euclidean spaces. This is introduced independently by E. A. Feldman [16] and W. Pohl [36]. The cases $n = 3, 4$ and $n \geq 7$ are already studied (see §6 for details). The case $n = 5$ appears to be more complicated and few results are known in this direction so far (see [13] for some partial results). Our study in this paper is part of a project of understanding the geometry of surfaces in $\mathbb{R}^5$.

2. Preliminaries

Let M be a 2-dimensional smooth surface in the Euclidean space $\mathbb{R}^5$ defined locally by an embedding $f : \mathbb{R}^2 \to \mathbb{R}^5$, and denote by TM and NM its tangent and normal bundles. Let $\bar{\nabla}$ denote the Riemannian connection of $\mathbb{R}^5$. Given any vector field Z on M we denote by $\bar{Z}$ its extension to an open set of $\mathbb{R}^5$. Given two tangent vector fields X and Y on M, we define the Riemannian connection on M as $\nabla_X Y = \left(\bar{\nabla}_{\bar{X}}\bar{Y}\right)^{\top}$, which is the orthogonal projection of $\nabla_X Y$ to the tangent plane of M. Let $\mathcal{X}(M)$ (resp. $\mathcal{N}(M)$) denote the spaces of tangent (resp. normal) fields on M. Then the second fundamental form on M is given by

$$\begin{array}{rccc} \alpha : & \mathcal{X}(M) \times \mathcal{X}(M) & \longrightarrow & \mathcal{N}(M) \\ & (X, Y) & \longmapsto & \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y. \end{array}$$

This is a well defined bilinear symmetric map. Given a normal field v on M the map α induces a bilinear symmetric map

$$\begin{array}{rccc} \mathrm{II}_v : & TM \times TM & \longrightarrow & \mathbb{R} \\ & (X, Y) & \longmapsto & \langle \alpha(X, Y), v \rangle. \end{array}$$

The map II_v is also referred to as the second fundamental form along v. The shape operator associated to the normal field v is defined by

$$\begin{array}{rccc} S_v : & TM & \longrightarrow & TM \\ & X & \longmapsto & -\left(\bar{\nabla}_{\bar{X}}\bar{v}\right)^{\top} \end{array}$$

This is a self-adjoint operator and satisfies $\mathrm{II}_v(X, Y) =< S_v(X), Y >$.

Let $q \in M$ and $\{e_1, e_2, e_3, e_4, e_5\}$ be frame in a neighbourhood of q, such that $\{e_1, e_2\}$ is a tangent frame and $\{e_3, e_4, e_5\}$ is a normal orthonormal frame in this neighbourhood. The matrix of the second fundamental form α of f at the point q with respect to this frame is given by

$$\alpha(q) = \begin{bmatrix} a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \end{bmatrix},$$

where $a_i = \langle f_{xx}, e_i \rangle$, $b_i = \langle f_{xy}, e_i \rangle$ and $c_i = \langle f_{yy}, e_i \rangle$, $i = 3, 4, 5$.

The second fundamental form $\alpha(q)$ induces a linear map

$$A_q : N_q M \to \mathcal{Q}_2$$

where $\mathcal{Q}_2$ denotes the space of quadratic forms in two variables, and $A_q(v)$ is the quadratic form associated to II_v at q. We shall write $A_q(v) = \mathrm{II}_v(q)$. If $v \in N_q M$ is represented by its coordinates (v_3, v_4, v_5) with respect to the basis $\{e_3, e_4, e_5\}$, then

$$A_q(v_3, v_4, v_5) = v_3(d^2 f \cdot e_3) + v_4(d^2 f \cdot e_4) + v_5(d^2 f \cdot e_5).$$

We define the following subsets of M:

$$M_i = \{q \in M |\, \mathrm{rank}\alpha_v(q) = i\}.$$

It is shown in [31] that for generically immersed surface in $\mathbb{R}^5$, $M = M_3 \cup M_2$, with M_2 a regular curve on M. Let C denote the cone of degenerate quadratic forms in $\mathcal{Q}_2$. Then we have the following characterisation of points on M.

If $q \in M_3$, then A_q has maximal rank, so $A_q^{-1}(C)$ is a cone in N_qM.

If $q \in M_2$, the image of A_q is a plane through the origin in $\mathcal{Q}_2$. We can classify the points on M_2 according to the relative position of this plane with respect to the cone C. We have the following three cases.

(a) *Hyperbolic type* (denoted by M_2^h): these are the points where $ImA_q \cap C$ consists of two lines. In this case $A_q^{-1}(C)$ is the union of two planes intersecting along the line $\ker \alpha(q)$.

(b) *Elliptic type* (denoted by M_2^e): these are the points where $ImA_q \cap C$ consits of the singular point of C. In this case $A_q^{-1}(C) = \ker \alpha(q)$ is a line.

(c) *Parabolic type* (denoted by M_2^p): these are the points where ImA_q is tangent to C along a line. In this case $A_q^{-1}(C)$ is a plane containing the line $\ker \alpha(q)$.

In all the paper, we assume q to be the origin and take M locally in Monge form

$$(1) \quad \phi(x,y) = (x, y, Q_1(x,y) + f^1(x,y), Q_2(x,y) + f^2(x,y), Q_3(x,y) + f^3(x,y)),$$

where the $f^i, i = 1,2,3$ are germs of smooth functions with zero 2-jets at the origin, and $Q = (Q_1, Q_2, Q_3)$ is a triple of quadratic forms. The flat geometry of submanifolds in $\mathbb{R}^n$ is affine invariant ([5]), so we can make linear changes of coordinates in the source and target and reduce Q to one of the following normal forms:

– (x^2, xy, y^2) if and only if $q \in M_3$,
– $(xy, x^2 \pm y^2, 0)$ if and only if $q \in M_2^h$ (resp. $q \in M_2^e$) for the $+$ (resp $-$) case,
– $(x^2, xy, 0)$ if and only if q is an M_2^p-point.

We shall write

$$\begin{aligned} j^3f^1 &= a_{30}x^3 + a_{31}x^2y + a_{32}xy^2 + a_{33}y^3, \\ j^3f^2 &= b_{30}x^3 + b_{31}x^2y + b_{32}xy^2 + b_{33}y^3, \\ j^3f^3 &= c_{30}x^3 + c_{31}x^2y + c_{32}xy^2 + c_{33}y^3. \end{aligned}$$

A tangent vector $a\phi_x(x,y) + b\phi_y(x,y)$ at the point $\phi(x,y)$ will be identified with the vector (a,b) in $\mathbb{R}^2$.

We need the following notation from singularity theory (see [42]). Let $\mathcal{E}_n$ be the local ring of germs of functions $f : \mathbb{R}^n, 0 \to \mathbb{R}$ and m_n the corresponding maximal ideal. We denote by $\mathcal{E}(n,p)$ the p-tuples of elements in $\mathcal{E}_n$.

Let $\mathcal{A} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^p, 0)$ denote the group of right-left equivalence which acts smoothly on $m_n.\mathcal{E}(n,p)$ by $(h,k).f = k \circ f \circ h^{-1}$.

Given a map-germ $f \in m_n.\mathcal{E}(n,p)$, θ_f denotes the $\mathcal{E}_n$-module of vector fields along f. We set $\theta_n = \theta_{I_{\mathbb{R}^n}}$ and $\theta_p = \theta_{I_{\mathbb{R}^p}}$. One can define the homomorphisms $tf : \theta_n \to \theta_p$, with $tf(\psi) = Df.\psi$, and $wf : \theta_n \to \theta_p$, with $wf(\phi) = \phi \circ f$.

The extended tangent space to the $\mathcal{A}$-orbit of f at the germ f is given by

$$\begin{aligned} L_e\mathcal{A}.f &= tf(\theta_n) + wf(\theta_p) \\ &= \mathcal{E}_n.\{f_{x_1}, \dots, f_{x_n}\} + f^*(\mathcal{E}_p).\{e_1, \dots, e_p\}, \end{aligned}$$

where subscripts denote partial differentiation, $e_1, \dots, e_p$ the standard basis vectors of $\mathbb{R}^p$ considered as elements of $\mathcal{E}(n,p)$, and $f^*(m_p)$ the pull-back of the maximal ideal in $\mathcal{E}_p$. The $\mathcal{A}_e$-codimension is given by

$$\mathcal{A}_e\text{-}codim(f) = \dim_{\mathbb{R}}(\mathcal{E}(n,p)/L\mathcal{A}_e.f).$$

We denote by $J^k(n,p)$ the space of kth order Taylor expansions without constant terms and write $j^k f$ for the k-jet of f. A germ is said to be $k-\mathcal{A}$-*determined* if any g with $j^k g = j^k f$ is $\mathcal{A}$-equivalent to f (notation: $g \sim f$). The k-jet of f is then called a sufficient jet.

Various classifications (i.e. the listing of representatives of the orbits) of finitely $\mathcal{A}$-determined germs were carried out by various authors for low dimensions n and p. (We shall give references in the appropriate places.)

Let X be a manifold and G a Lie group acting on X. The *modality* of a point $x \in X$ under the action of G on X is the least number m such that a sufficiently small neighbourhood of x may be covered by a finite number of m-parameter families of orbits (see [1]). The point x is said to be *simple* if its modality is 0, that is, a sufficiently small neighbourhood intersects only a finite number of orbits. The modality of a finitely determined map-germ is defined to be the modality of a sufficient jet in the jet-space under the action of the jet-group.

In all the paper, a property is called generic if it is satisfied by a residual subset of immersions $\phi : M \to \mathbb{R}^5$, where the later is endowed with the C^∞-Whitney topology. A given immersion (surface) is called generic if it belongs to a residual subset which is determined by the context in consideration.

3. Characterisations of asymptotic directions

Asymptotic directions on surfaces in $\mathbb{R}^5$ are introduced in [31] in terms of the contact of the surface with 4-dimensional planes. We recall below the definition in [31] of the asymptotic directions and characterise these directions in terms of the contact of the surface with k-planes, $k = 1, 2, 3, 4$.

3.1. **Asymptotic directions and contact with 4-planes.** The contact of the surface with 4-dimensional planes is measured by the singularities of the height function

$$\begin{array}{rccc} H: & M \times S^4 & \longrightarrow & \mathbb{R} \times S^4 \\ & (q,v) & \longmapsto & (h_v(q), v) \end{array}$$

where $h_v(q) = \langle \phi(q), v \rangle$. A height function h_v has a singularity at $q \in M$ if and only if $v \in N_q M$.

It follows from a general result of Montaldi [33] (see also Looijenga's Theorem in [27]) that for a residual set of immersions $\phi : M \to \mathbb{R}^5$, the family H is a generic family of mappings. (The notion of a generic family is defined in terms of transversality to submanifolds of multi-jet spaces, see for example [18].) This means that the singularities of h_v that occur in an irremovable way in the family H are those of $\mathcal{A}_e$-codimension ≤ 4 (the dimension of the parameter space S^4). So h_v has generically a singularity of type $A_{k\leq 5}$, $D_4^{\pm}$ or D_5 (see [1] for notation) and these are versally unfolded by the family H.

We define the *flat ridge of* M as the set of points where the height function, along some normal direction, has a singularity of type A_k with $k \geq 4$. The flat ridge is generically either empty or is a regular curve of A_4-points. The A_5-points form isolated points on this curve. These points are called *higher order flat ridge points*.

It is shown in [31] that for a generic surface, $q \in M_3$ if and only if h_v has only A_k-singularities for any $v \in T_qM$. A point $q \in M_2^h \cup M_2^e$ (resp. $q \in M_2^p$) if and only if there exists $v \in N_qM$ such that h_v has a singularity of type $D_4^{\pm}$ (resp. D_5) at q. This direction v is called *the flat umbilic direction*.

Given $v \in N_qM$, the quadratic forms $\mathrm{II}_v(q)$ and the Hessian $Hess(h_v)(q)$ are equivalent, up to smooth local changes of coordinate in M . So we can identify the quadratic form $A_q(v)$ with $Hess(h_v)(q)$.

A direction $v \in N_qM$ is said to be *degenerate* if q is a non-stable singularity of h_v (i.e. h_v has an $\mathcal{A}_e$-codimension ≥ 1 singularity at q). In this case, the kernel of the Hessian of h_v, $\ker(Hess(h_v)(q))$, contains non zero vectors. Any direction $u \in \ker(Hess(h_v)(q))$ is called a *contact direction associated to v.*

A unit vector $v = (v_3, v_4, v_5) \in N_qM$ is called a *binormal direction* if h_v has a singularity of type A_3 or worse at q. (They are labelled binormal by analogy to the case of curves in $\mathbb{R}^3$.) We have the following result where we assume, without loss of generality, that $v_5 \neq 0$.

Proposition 3.1. ([31]) *Let q be an M_3-point. Then there are at most* 5 *and at least* 1 *binomial directions at q. If M is taken in Monge form (1), then the binomial directions at the origin are along* $(\frac{1}{2}v_4^2, v_4, 1)$ *with*

$$c_{30} + (2b_{30} - c_{31})\frac{v_4}{2} + (a_{30} - 2b_{31} + c_{32})\frac{v_4^2}{4} - (a_{31} - 2b_{32} + c_{33})\frac{v_4^3}{8} + (a_{32} - 2b_{33})\frac{v_4^4}{16} - a_{33}\frac{v_4^5}{32} = 0.$$

Definition 3.2. ([31]) *Let $q \in M$ and $v \in N_qM$ be a binormal direction. An asymptotic direction at q is any contact direction associated to v.*

Remark 3.3. *At an M_3-point q the height function h_v has only singularities of type A_k. So to any binormal direction at q is associated a unique asymptotic direction. It follows from Proposition 3.1 that there are at most* 5 *and at least* 1 *asymptotic directions at any M_3-point. At a point q on the M_2 curve, the height function along the flat umbilic direction has generically a D_4 or a D_5 singularity, so* $\ker(Hess(h_v)(q)) = T_qM$ *and every tangent direction at q could be considered to be asymptotic. However, we shall identify in §3.2 some special directions in T_qM and will reserve the label asymptotic directions at an M_2-point for these special directions.*

Asymptotic directions are also characterised in [31] in terms of normal sections of M. Let v be a degenerate direction at $q \in M_3$ (so $\mathrm{rank}(Hess(h_v)(q)) = 1$), and let θ be a tangent direction in $\ker(Hess(h_v)(q))$. We denote by γ_θ the normal section of the surface M in the tangent direction θ, that is, γ_θ is the curve obtained by the intersection of M with the 4-space $V_\theta = N_qM \oplus \langle\theta\rangle$.

Proposition 3.4. ([31]) *Let $q \in M_3$ and let $v \in N_qM$ be a degenerate direction. Let θ be a tangent direction in* $\ker(Hess(h_v)(q))$. *Then θ is an asymptotic direction if and only if v is the binormal direction at q of the curve γ_θ in the 4-space V_θ.*

The analysis of the contact of the normal sections of M with 3-planes allows us to characterise the flat ridge as follows.

Given a curve $\gamma : \mathbb{R} \to \mathbb{R}^n$, consider its Frenet-Serret frame $\{T, N_1, \cdots, N_{n-1}\}$ and the corresponding curvature functions $\kappa_1, \ldots, \kappa_{n-1}$. We say that a point $q = \gamma(t_0)$ is a *flattening* of γ if $\kappa_{n-1}(t_0) = 0$. The point q is a *degenerate flattening* when $\kappa_{n-1}(t_0) = \kappa'_{n-1}(t_0) = 0$

Proposition 3.5. *Let $q \in M_3$ and let $v \in N_qM$ be a binormal direction. Let θ be its corresponding asymptotic direction and γ_θ the corresponding normal section of M. Then*

(1) *$q = \gamma_\theta(0)$ is a flat ridge point of M if and only if q is a flattening of γ_θ (as a curve in the 4-space V_θ).*

(2) *$q = \gamma_\theta(0)$ is a higher order flat ridge point of M if and only if q is a degenerate flattening of γ_θ.*

Proof. The point q is a singularity of type A_k of the height function h_v on M if and only if it is a singularity of type A_k of $h_v|_{\gamma_\theta}$. Therefore it is a flattening (resp. degenerate flattening) of γ_θ if and only if it is a flat ridge point (resp. higher order flat ridge point) of M. □

3.2. **Asymptotic directions and contact with lines.** If TS^4 denotes the tangent bundle of the 4-sphere S^4, the family of projections to 4-planes is given by

$$\begin{array}{rccc} P: & M \times S^4 & \to & TS^4 \\ & (q, v) & \to & (q, p_v(q)) \end{array}$$

where $p_v(q) = q - < q, v > v$. For a given $v \in S^4$, the map p_v can be considered locally as a germ of a smooth map $\mathbb{R}^2, 0 \to \mathbb{R}^4, 0$. A classification of $\mathcal{A}$-simple singularities of smooth map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^4, 0$ is carried out in [24]; see also [25].

It follows by Montaldi's Theorem [33] that for a residual set of immersions $\phi : M \to \mathbb{R}^5$, the family P is a generic family of mappings. So the singularities of p_v that occur in an irremovable way in the family P are those of $\mathcal{A}_e$-codimension ≤ 4, and these are versally unfolded by the family P. For a generic surface, the singularities of p_v are simple and are given in Table 1 (from [24] and [25]).

Table 1: Local singularities of projections of surfaces in $\mathbb{R}^5$ to 4-spaces.

Type	Normal form	$\mathcal{A}_e$-codimension
immersion	$(x, y, 0, 0)$	0
I_k	$(x, xy, y^2, y^{2k+1}),\ k = 1, 2, 3, 4$	k
II_2	$(x, y^2, y^3, x^k y),\ k = 2$	4
$\mathrm{III}_{2,3}$	$(x, y^2, y^3 \pm x^k y, x^l y),\ k = 2,\ l = 3$	4
VII_1	$(x, xy, xy^2 \pm y^{3k+1}, xy^3),\ k = 1$	4

The bifurcation set of the family of projections P (resp. height functions H) is the set of parameter $v \in S^4$ (resp. $u \in S^4$) where p_v (resp. h_u) has a non-stable singularity at some point $q \in M$, i.e. has a singularity of $\mathcal{A}_e$-codimension ≥ 1. We denote by $Bif(P, \mathrm{I}_k)$ (resp. $Bif(H, A_k)$) the stratum of the bifurcation set where p_v (resp. h_u)

has precisely a singularity of type I_k (resp. A_k). We have the following duality result in S^4, analogous to those in [4], [6], [7], [8], [10], [30], [41].

Theorem 3.6. *Suppose that $q \in M_3$. Then to a direction $v \in N_qM$ where h_v has an $A_{k\geq 3}$-singularity at q is associated a unique dual direction $v^* \in T_qM$ where p_{v^*} has an $I_{k\geq 2}$-singularity, and vice-versa. More precisely,*

$$Bif(H, A_3)^* = Bif(P, \mathrm{I}_{k\geq 2}) \quad \textit{and} \quad Bif(P, \mathrm{I}_2)^* = Bif(H, A_{k\geq 3}).$$

Proof. The family of height functions H is a versal unfolding of an A_3-singularity of h_v at a given point $q \in M$. So the closure of $Bif(H, A_2)$ is locally diffeomorphic to the product of a cusp with $\mathbb{R}^2$ (see for example [11]). The singular locus of $Bif(H, A_2)$ is $Bif(H, A_3)$ and is therefore a smooth submanifold of codimension 2 in S^4. Let $v \in Bif(H, A_3)$ be a binormal direction at $q \in M$. We can decompose the limiting tangent space to $Bif(H, A_2)$ at v into a direct sum $T_vBif(H, A_3) \oplus \langle w\rangle$, for some $w \in T_vBif(H, A_2)$.

The 4-dimensional space $T_vBif(H, A_2) \oplus \langle v\rangle$ determines two poles $u_i \in S^4$, $i = 1, 2$. As q varies locally in M, the two poles trace two subspaces of codimension 2 in S^4. These are two copies of the dual of $Bif(H, A_3)$. Indeed a pole determines $T_vBif(H, A_2)$ and this gives $T_vBif(H, A_3)$ by taking the orthogonal complement of w in $T_vBif(H, A_2)$.

We need to show now that projecting along the directions u_i to a transverse 4-space yields a map-germ with an $\mathrm{I}_{k\geq 2}$-singularity at q. We take the surface in Monge form as in (1). We assume, without loss of generality, that $v = (0, 0, 1)$ so the family of height functions can be taken as

$$h(x, y) = v_1x + v_2y + v_3(x^2 + f^1(x, y)) + v_4(xy + f^2(x, y)) + y^2 + f^3(x, y).$$

A point q near the origin is an A_3-singularity of h in the direction $(v_3, v_4, 1)$ if and only if $h_x = h_y = h_{xy}^2 - h_{xx}h_{yy} = 0$ (so $j^2h(q) = L^2$ for some linear term L in x, y) and the cubic part of h at q divides L. Using these equations we can find the limiting tangent space $T_vBif(H, A_2)$ and the poles that it determines. A calculation shows that these poles are the points of intersection of the line through the origin in the direction $(-2, v_4)$ with the unit circle in T_qM. It is not difficult to show that projecting along $(-2, v_4)$ yields a singularity of type $\mathrm{I}_{k\geq 2}$ (generically of type I_k, $2 \leq k \leq 4$). Therefore $Bif(H, A_3)^* = Bif(P, \mathrm{I}_{k\geq 2})$.

Suppose that p_u has an I_2-singularity at q. The family P is a versal unfolding of this singularity so $Bif(P, \mathrm{I}_2)$ is a smooth submanifold of codimension 2 in S^4. In this case, the stratum $Bif(P, \mathrm{I}_1)$ is also a smooth submanifold of codimension 1 in S^4 and one can write $T_uBif(P, \mathrm{I}_1) = T_uBif(P, \mathrm{I}_2) \oplus \langle w\rangle$ for some $w \in T_uBif(P, \mathrm{I}_1)$.

The 4-dimensional space $T_uBif(P, \mathrm{I}_1) \oplus \langle u\rangle$ determines two poles in S^4, and these poles trace two copies of the dual of $T_uBif(P, \mathrm{I}_2)$. Indeed a pole determines $T_uBif(P, \mathrm{I}_1)$ and this gives $T_uBif(P, \mathrm{I}_2)$ by taking the orthogonal complement of w in $T_uBif(P, \mathrm{I}_1)$. A calculation shows that the height function along the direction determined by one of the poles has a singularity of type $A_{k\geq 3}$ (generically of type A_k, $3 \leq k \leq 5$), so the pole is a point on $T_uBif(H, A_{k\geq 3})$. Therefore $Bif(P, \mathrm{I}_2)^* = Bif(H, A_{k\geq 3})$. □

It follows from Theorem 3.6 that to each binormal direction $v \in N_qM$ with $q \in M_3$ is associated a unique (dual) tangent direction $v^* \in T_qM$ where the projection along v^* to a transverse 4-space has a singularity of type I_2 or worse (i.e. of higher $\mathcal{A}_e$-codimension).

Proposition 3.7. *A direction $u \in T_qM$, with $q \in M_3$, is asymptotic if and only if the projection of M along u to a transverse 4-space has an $\mathcal{A}$-singularity of type I_2 or worse.*

Proof. Given a binormal direction $v \in N_qM$, the dual direction $v^* \in T_qM$ generates $\ker(Hess(h_v(q)))$. The result then follows by Theorem 3.6. □

Remark 3.8. *As a consequence of Proposition 3.7, we shall define an asymptotic direction at q as one along which the projection of M at q to a transverse 4-space has an I_2-singularity or worse (compare with Definition 3.2). This definition leads to the existence of at most 5 asymptotic directions at an M_2-point (Proposition 3.9 below; see Remark 3.3).*

We consider now in some details the singularities of the projection to a 4-space. We take M in Monge form as in (1). We assume that the kernel of the projection is along $u \in T_qM$ (otherwise p_u has maximal rank). Then the projection along $u = (u_1, u_2) \in T_qM$ to a transverse 4-space can be written locally in the form

$$p_u(x, y) = (u_2x - u_1y, Q_1(x, y) + f^1(x, y), Q_2(x, y) + f^2(x, y), Q_3(x, y) + f^3(x, y)).$$

We analyse the $\mathcal{A}$-singularities of $p_u(x, y)$. We have the following result, where generic in the M_3-set (resp. M_2-set) means possibly away from some curve (resp. points). The excluded cases are dealt with in Proposition 3.10. (See Table 1 for notation.)

Proposition 3.9. (1) *At generic M_3-points there are at most* 5 *and at least* 1 *tangent directions u where p_u has an $\mathcal{A}$-singularity of type* I_2. *These are the solutions of the following quintic form*

$$c_{30}u_1^5 + (c_{31} - 2b_{30})u_1^4u_2 + (c_{32} - 2b_{31} + a_{30})u_1^3u_2^2 + (c_{33} - 2b_{32} + a_{31})u_1^2u_2^3 + (a_{32} - 2b_{33})u_1u_2^4 + a_{33}u_2^5 = 0.$$

(2) *Suppose that q is a generic M_2-point. Then there at most* 3 *and at least* 1 *tangent directions where p_u has an $\mathcal{A}$-singularity of type* I_2. *These are dual to the flat umbilic direction. There are also two directions (resp. none) where p_u has an $\mathcal{A}$-singularity of type* II_2 *if $q \in M_2^h$ (resp. $q \in M_2^e$), and one direction where p_u has an $\mathcal{A}$-singularity of type* VII_1 *if $q \in M_2^p$. These are dual to the directions giving an A_3-singularity of the height function.*

Proof. The proof follows by making successive changes of coordinates in order to reduce the appropriate jet of p_u to a normal form. The duality results in part (2) also follow by a calculation similar to the one in the proof of Theorem 3.6. However, there is a geometric argument why the dual of the flat umbilic direction consists of at most 3 and at least 1 tangent directions. The bifurcation set of the family of height functions H at a D_4-singularity is the product of the sets in Figure 1 with a line (a D_4-singularity has $\mathcal{A}_e$-codimension 3 and H has 4 parameters and is generically a versal unfolding of this singularity). The limiting tangent spaces of the cuspidal-edges in Figure 1 determine 3 or 1 poles (i.e. dual directions) in S^4. □

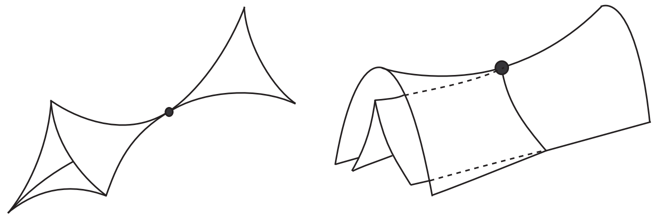

FIGURE 1. The bifurcation set of a D_4-singularity (elliptic left, hyperbolic right).

Proposition 3.10. (1) *There may be a curve in M_3 where projecting along one of the directions in Proposition 3.9 (1) yields a singularity of type* I_3 *and isolated points on this curve where the singularity is of type* I_4. *For generic surfaces, this curve is distinct from the flat ridge. That is, the dual of a normal direction along which the height function has an A_4-singularity does not yield in general a projection with an* I_3*-singularity of the projection, and vice-versa.*

(2) *There may be isolated points on M_2 where projecting along one of the directions dual the flat umbilic direction yields a singularity of type* II_3. *There may also be isolated M_2^h-points where projecting along one of the directions not dual the flat umbilic direction yields a singularity of type* $\mathrm{III}_{2,3}$. *The above points are in general distinct from the D_5-points.*

The proof is straightforward and is omitted.

3.3. **Asymptotic directions and contact with 2-planes.** An orthogonal projection from $\mathbb{R}^5$ to a 3-dimensional subspace is determined by its kernel, so we can parametrise all these projections by the Grassmanian $G(2,5)$ of 2-planes in $\mathbb{R}^5$. If w_1, w_2 are two linearly independent vectors in $\mathbb{R}^5$, we denote by $\{w_1, w_2\}$ the plane they generate and by $\pi_{(w_1,w_2)}$ the orthogonal projection from $\mathbb{R}^5$ to the orthogonal complement of $\langle w_1, w_2\rangle$. The restriction of $\pi_{(w_1,w_2)}$ to M, $\pi_{(w_1,w_2)_{|M}}$, can be considered locally at a point $q \in M$ as a map-germ

$$\pi_{(w_1,w_2)_{|M}} : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0.$$

We start with the case where $\langle w_1, w_2\rangle$ is transverse to T_qM, so $\pi_{(w_1,w_2)_{|M}}$ is locally an immersion.

Let $v \in N_qM$ and M_v be the surface patch obtained by projecting M orthogonally to the 3-space $T_qM \oplus \langle v\rangle$ (considered as an affine space through q). We can characterise the asymptotic directions of M at q in terms of the geometry of M_v at q.

Recall that to a smooth surface patch S in an Euclidean 3-space is associated the Gauss map $N : S \to S^2$ which takes a point q on S to a unit normal vector to S at q. The map has generically local singularities of map-germs from the plane to the plane of type fold or cusp (see for example [18]). The fold singularities are precisely the parabolic points of S and form a smooth curve on S. The cusp singularities occur at isolated points on this curve and are called cusps of Gauss.

Proposition 3.11. *Suppose that $q \in M_3$ and let $v \in N_qM$.*

(1) *The direction v is degenerate if and only if q is a parabolic point of M_v. In this case, the unique principal asymptotic direction of M_v at q coincides with the contact direction associated to v.*

(2) *A direction $u \in T_pM$ is asymptotic if and only if there exists $v \in N_pM$ such that q is a cusp of Gauss of M_v and u is its unique asymptotic direction there.*

Proof. (1) We take M in Monge form as in (1) with $Q = (x^2, xy, y^2)$. Given a normal direction $v = (v_3, v_4, v_5)$ at the origin, the surface M_v is parametrised by $\psi(x, y) = (x, y, f(x, y))$ with

$$f(x, y) = v_3(x^2 + f^1(x, y)) + v_4(xy + f^2(x, y)) + v_5(y^2 + f^3(x, y)).$$

The equation of the asymptotic direction of M_v is given by

$$f_{yy}dy^2 + 2f_{xy}dxdy + f_{xx}dx^2 = 0.$$

The discriminant Δ of the above equation is the zero set of the function $\delta = f_{xy}^2 - f_{xx}f_{yy}$ and corresponds to the parabolic set of M_v. We have $j^2f = v_3x^2 + v_4xy + v_5y^2$, so the origin is a parabolic point if and only if $v_4^2 - 4v_3v_5 = 0$, that is, if and only if the height function along v has a degenerate singularity.

(2) The origin is a cusp of Gauss if and only if the unique asymptotic direction of M_v at the origin is tangent to the discriminant Δ (see for example [2]), that is if and only if $\delta = 0$ and $(\delta_x, \delta_y).(-f_{yy}, f_{xx}) = 0$. When $v_5 \neq 0$ (so we can set $v_5 = 1$), this occurs if and only if

$$c_{30} + (2b_{30} - c_{31})\frac{v_4}{2} + (a_{30} - 2b_{31} + c_{32})\frac{v_4^2}{4} - (a_{31} - 2b_{32} + c_{33})\frac{v_4^3}{8} + (a_{32} - 2b_{33})\frac{v_4^4}{16} - a_{33}\frac{v_4^5}{32} = 0.$$

This is exactly the condition for the direction $v = (\frac{v_4^2}{2}, v_4, 1)$ to be binormal (Proposition 3.1). Its dual direction is along $(-2, v_4)$ which is precisely the unique asymptotic direction of M_v at the origin.

If $v_5 = 0$, the origin is a parabolic point of M_v when $v_4 = 0$. We then set $v = (1, 0, 0)$. The origin is a cusp of Gauss of M_v if and only if $a_{33} = 0$. In this case v is also binormal as h_v has a singularity of type $A_{\geq 3}$ at the origin. The dual direction is along $(0, 1)$ which is precisely the unique asymptotic direction of M_v at the origin. □

Proposition 3.12. *Suppose that $q \in M_2$. There are two distinct directions $v \in N_qM$ if $q \in M_2^h$, none if $q \in M_2^e$, and a unique direction if $q \in M_2^p$, where q is a cusp of Gauss of M_v and v^* is the unique asymptotic direction of M_v at q. In addition, there is a unique direction $\bar{v} \in N_qM$ where $M_{\bar{v}}$ has a flat umbilic at q. The asymptotic directions of M at q associated to $\bar{v}$ are the tangent directions to the separatrices of the asymptotic curves of $M_{\bar{v}}$ at q (see Figure 2).*

Proof. The proof is similar to that of Proposition 3.11. If we take the surface in Monge form $(x, y, xy + f^1(x, y), x^2 \pm y^2 + f^2(x, y), f^3(x, y))$, the asymptotic directions $u = (u_1, u_2)$ corresponding to the flat umbilic direction $v = (0, 0, 1)$ are given by $f^3(u_2, u_1) = 0$. The surface M_v is parametrised by $(x, y, f^3(x, y))$ and the tangent to the separatrices of its asymptotic curves are also given by $f^3(u_2, u_1) = 0$ ([9]). □

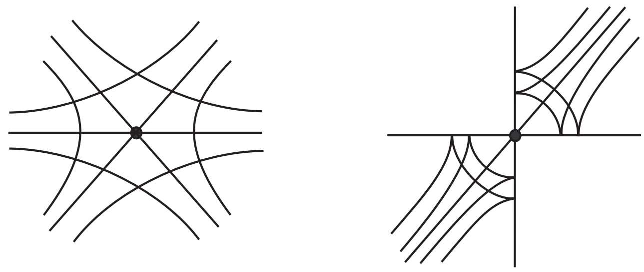

FIGURE 2. Asymptotic curves at a flat umbilic on a surface in $\mathbb{R}^3$ (elliptic left, hyperbolic right).

We deal now with the case when $\pi_{(w_1,w_2)|M}$ is singular. This occurs when the kernel of the projection $\pi_{(w_1,w_2)}$ contains a tangent direction at q. When $\{w_1, w_2\} = T_qM$, the map-germ $\pi_{(w_1,w_2)|M}$ has rank zero at the origin and does not identify the asymptotic directions. We shall assume that $\{w_1, w_2\}$ is distinct from T_qM. Then $\pi_{(w_1,w_2)|M}$ has rank 1 at the origin. It follows by Montaldi's Theorem that for generic surfaces, the irremovable singularities of $\pi_{(w_1,w_2)|M}$ in the family are those of $\mathcal{A}_e$-codimension ≤ 6 (as $\dim G(2,5) = 6$). The $\mathcal{A}$-simple singularities of map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^3, 0$ are classified by Mond [32]; see Table 2. Some non-simple orbits are given in [32] and [38].

Table 2: $\mathcal{A}$-simple singularities of projections of surfaces in $\mathbb{R}^5$ to 3-spaces.

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	$(x, y, 0)$	0
Cross-cap	(x, y^2, xy)	0
$S_k^{\pm}$	$(x, y^2, y^3 \pm x^{k+1}y), k \geq 1$	k
$B_k^{\pm}$	$(x, y^2, x^2y \pm y^{2k+1}), k \geq 2$	k
$C_k^{\pm}$	$(x, y^2, xy^3 \pm x^k y), k \geq 3$	k
F_4	$(x, y^2, x^3y + y^5)$	4
H_k	$(x, xy + y^{3k-1}, y^3), k \geq 2$	k

We take M in Monge from (1) and for simplicity q an M_3-point (the results hold at any point on M). Suppose, without loss of generality, that the intersection of the kernel of the projection $\pi_{(w_1,w_2)}$ with T_qM is along $u = (1, 0)$. So the kernel is generated by u and some $v = (v_3, v_4, v_5) \in N_qM$ (and $\pi_{(w_1,w_2)} = \pi_{(u,v)}$). Observe that the dual direction to u is $u^* = (0, 0, 1)$.

If $< u^*, v > \neq 0$, then $v_5 \neq 0$ and $\pi_{(u,v)|M}$ is $\mathcal{A}$-equivalent to

$$g(x, y) = (y, x^2 + f^1, xy + f^2).$$

This map-germ has a cross-cap singularity at the origin.

If $< u^*, v >= 0$, then $v = (v_3, v_4, 0)$ and $\pi_{(u,v)|M}$ is $\mathcal{A}$-equivalent to

$$g(x, y) = (y, v_4(x^2 + f^1) - v_3(xy + f^2), y^2 + f^3).$$

When $v_4 \neq 0$, the 2-jet of $\pi_{(u,v)|M}$ is $\mathcal{A}$-equivalent to $(y, x^2, 0)$ (so all the simple singularities of type S_k, B_k, C_k, F_4, with $\mathcal{A}_e$-codimension ≤ 6 occur, as well as some non-simple cases.)

If $v_4 = 0$, $v = (1, 0, 0)$ and $\pi_{(u,v)|M}$ is $\mathcal{A}$-equivalent to

$$g(x, y) = (y, xy + f^2, y^2 + f^3)$$

which has an H_k singularity provided $f^3_{xxx}(0, 0) \neq 0$. The condition $f^3_{xxx}(0, 0) = 0$ is precisely the condition for $u = (1, 0)$ to be an asymptotic direction at the origin. When this happens, the map-germ g has a non-simple singularity with 2-jet equivalent to $(y, xy, 0)$. So one can characterise asymptotic directions using the singularities of projections to 3-spaces. We have thus the following result.

Proposition 3.13. *Let $u \in T_qM$ and v in the unit sphere $S^2 \subset N_qM$.*

(1) *The projection $\pi_{(u,v)|M}$ has a cross-cap singularity for almost all $v \in S^2$.*

(2) *On a circle of directions v in S^2 minus a point, $\pi_{(u,v)|M}$ has a singularity with 2-jet $\mathcal{A}$-equivalent to $(x, y^2, 0)$.*

(3) *There is a unique direction $v \in S^2$ where $\pi_{(u,v)|M}$ has a singularity of type H_k provided u is not an asymptotic direction. If u is asymptotic, then the singularity becomes non-simple with 2-jet $\mathcal{A}$-equivalent to $(x, xy, 0)$.*

3.4. **Asymptotic curves and contact with 3-spaces.** An orthogonal projection from $\mathbb{R}^5$ to a 2-dimensional subspace is also determined by its kernel, so we can parametrise all these projections by the Grassmanian $G(3, 5)$ of 3-planes in $\mathbb{R}^5$. However, $G(3, 5)$ can be identified with $G(2, 5)$, so the projections can be parametrised by $\{w_1, w_2\} \in G(2, 5)$, where $\{w_1, w_2\}$ is the orthogonal complement of the kernel of the projection. We denote the associated projection by $\Pi_{(w_1,w_2)}$. The restriction of $\Pi_{(w_1,w_2)}$ to M, $\Pi_{(w_1,w_2)|M}$, can be considered locally at a point $q \in M$ as a map-germ

$$\Pi_{(w_1,w_2)|M} : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0.$$

As in the previous section, we expect singularities of $\mathcal{A}_e$-codimension ≤ 6 to occur for generic surfaces. The list of corank 1 singularities of map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$, of $\mathcal{A}_e$-codimension ≤ 6 is given by Rieger [39]. The $\mathcal{A}$- simple singularities in these dimensions, including those with corank 2 ([40]), are shown in Table 3.

Table 3: $\mathcal{A}$-simple singularities of projections of surfaces in $\mathbb{R}^5$ to 2-planes.

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	(x,y)	0
Fold	(x,y^2)	0
Cusp	$(x,xy+y^3)$	0
4_k	$(x,y^3\pm x^k y), k\geq 2$	$k-1$
5	$(x,xy+y^4)$	1
6	$(x,xy+y^5\pm y^7)$	2
7	$(x,xy+y^5)$	3
11_{2k+1}	$(x,xy^2+y^4+y^{2k+1}), k\geq 2$	k
12	$(x,xy^2+y^5+y^6)$,	3
13	$(x,xy^2+y^5\pm y^9)$,	4
14	(x,xy^2+y^5),	5
16	$(x,x^2y+y^4\pm y^5)$,	3
17	(x,x^2y+y^4),	4
$I_{2,2}^{l,m}$	$(x^2+y^{2l+1},y^2+x^{2m+1})$, $l\geq m\geq 1$	$l+m$
$II_{2,2}^{l}$	$(x^2-y^2+x^{2l+1},xy)$, $l\geq 1$	$2l$

We start with the corank 1 singularities. Let $u\in T_qM$, $u^\perp$ an orthogonal vector to u in T_qM and $v=(v_1,v_2,v_3)\in N_qM$. We consider the projection $\Pi_{(u^\perp,v)|M}$.

We take M in Monge form (1) at the origin and suppose, without loss of generality, that the intersection of the kernel of $\Pi_{(u^\perp,v)}$ with T_qM is along $u=(1,0)$. Then

$$\Pi_{(u^\perp,v)|M}(x,y)=(y,v_1(x^2+f^1(x,y))+v_2(xy+f^2(x,y))+v_3(cy^2+f^3(x,y))),$$

where c is equal to 0 or 1 according to q being an M_3 or an M_2 point. Observe that the $\mathcal{A}$-type of the singularities of the above map-germ is independent of c. Therefore, the corank 1 singularities of the projections to 2-planes do not distinguish between the M_3 and M_2 points.

If $v_1\neq 0$, then $\Pi_{(u^\perp,v)|M}$ is $\mathcal{A}$-equivalent to a fold map-germ.

If $v_1=0$ and $v_2\neq 0$, then $j^2\Pi_{(u^\perp,v)|M}\sim_{\mathcal{A}}(y,xy)$. The $\mathcal{A}$-singularities of $\Pi_{(u^\perp,v)|M}$ are given by the normal forms 5, 6 and 7 in Table 3. Non-simple singularities of $\mathcal{A}_e$-codimension ≤ 6 may also occur.

If $v_1=v_2=0$, $\Pi_{(u^\perp,v)|M}(x,y)=(y,f^3(x,y))$, and the singularities are of type 4_k (Table 3) unless $f^3_{xxx}(0,0)=0$. In this case, the singularities are of type 11_{2k+1} (Table 3) or more degenerate. The condition $f^3_{xxx}(0,0)=0$ is precisely the condition for $u=(1,0)$ to be an asymptotic direction at the origin. So one can characterise asymptotic directions using corank 1 singularities of projections to 2-planes.

Proposition 3.14. *Let $u\in T_qM$ and v in the unit sphere $S^2\subset N_qM$.*

(1) *The projection $\Pi_{(u^\perp,v)|M}$ has a fold singularity for almost all $v\in S^2$.*

(2) *On a circle of directions v in S^2 minus a point $\Pi_{(u^\perp,v)|M}$ has a singularity with 2-jet $\mathcal{A}$-equivalent to (x,xy) (equivalently, it is not a fold and has a smooth critical set).*

(3) *There is a unique direction* $v \in S^2$ *where* $\Pi_{(u^\perp,v)|M}$ *has a singularity of type* 4_k *provided* u *is not an asymptotic direction. If* u *is asymptotic, then the singularity is* $\mathcal{A}$*-equivalent to* 11_{2k+1} *or is more degenerate.*

We analyse now the corank 2 singularities of the projection. Let $\{w_1, w_2\}$ be a plane in N_qM and denote by $M_{(w_1,w_2)}$ the surface patch obtained by projecting M orthogonally to the 4-space $T_qM \oplus \{w_1, w_2\}$ (considered as an affine space through q). The map-germ $\Pi_{(w_1,w_2)|M}$ has then a corank 2 singularity at the origin, and this singularity can be characterised in terms of the geometry of $M_{(w_1,w_2)}$.

Points on a generic surface immersed in $\mathbb{R}^4$ are classified in [26], and in [28] and [7] in terms of singularities of certain maps on the surface. In [28], a point is called hyperbolic/parabolic/elliptic if there are 2/1/0 directions in the normal plane such that the associated height function has a degenerate singularity (i.e. worse than Morse). The parabolic points form a curve on the surface. This curve may have generically Morse singularities at isolated points. These singularities are called inflection points of real type if the singularity is a crossing and of imaginary type if it is an isolated point. (When the singularity of the parabolic curve is more degenerate, the inflection is called of flat type.) The following result follows directly from this classification and the $\mathcal{A}$-classification of map-germs from the plane to the plane ([39], [40]).

Proposition 3.15. *The following hold for a generic immersed surface* M *in* $\mathbb{R}^5$*.*

(1) *The 2-jet of the projection* $\Pi_{(w_1,w_2)|M}$ *is* $\mathcal{A}$*-equivalent to* (x^2, y^2), $(x^2 - y^2, xy)$ *or* (x^2, xy) *if and only if* q *is, respectively, a hyperbolic, elliptic or parabolic point of* $M_{(w_1,w_2)}$*.*

(2) *The 2-jet of the projection* $\Pi_{(w_1,w_2)|M}$ *is* $\mathcal{A}$*-equivalent to* $(x^2 + y^2, 0)$, $(x^2 - y^2, 0)$, *or* $(x^2, 0)$ *if and only if* q *is, respectively, an inflection point of real type, of imaginary type or of flat type of* $M_{(w_1,w_2)}$*.*

Moreover, if $q \in M_3$ *then* $\Pi_{(w_1,w_2)|M}$ *satisfies* (1) *for every plane* $\{w_1, w_2\} \subset N_qM$*. The point* $q \in M_2$ *if and only if there exists a direction* $w_2 \in N_qM$ *such that* q *is an inflection point of* $M_{(w_1,w_2)}$*, for any* $w_1 \in N_qM$*.*

4. Equation of the asymptotic directions

In this section we obtain the equation of the asymptotic directions in terms of the coefficients of the second fundamental form and give another geometric argument why the equation is a quintic form.

We take as in §2 $\phi : U \to \mathbb{R}^5$ to be a local parametrisation of M and choose a frame $e = \{e_1, e_2, e_3, e_4, e_5\}$ depending smoothly on $q \in U$, such that $e_1 = \phi_x(q), e_2 = \phi_y(q)$ and $\{e_3, e_4, e_5\}$ is an orthonormal frame of the normal plane at q.

We consider, without loss of generality, $q \in M_3$. This is not restrictive as M_2-points form a curve on a generic surface M. So the equation obtained at M_3-points is also valid at M_2-points by passing to the limit. We use here the characterisation of an asymptotic direction given in Proposition 3.11.

If we write, in the frame e, $u = (dx, dy) \in T_qM$ and $v = (v_3, v_4, v_5) \in N_qM$, then u is an asymptotic direction of M_v at q if and only if $\mathrm{II}_v(u, u) = 0$, if and only if

(2) $(v_3c_3 + v_4c_4 + v_5c_5)dy^2 + 2(v_3b_3 + v_4b_4 + v_5b_5)dxdy + (v_3a_3 + v_4a_4 + v_5a_5)dx^2 = 0.$

To simplify the notation, we denote by $A/B/C$ the coefficients of $dy^2/2dxdy/dx^2$, respectively, in equation (2). Note that as we are considering $q \in M_3$, at least one of the coefficients A, B, C is not zero at q.

The point q is a parabolic point of M_v if and only if the discriminant function $\delta = B^2 - AC$ of equation (2) is zero at q, that is, if and only if

$$(b_3^2 - a_3c_3)v_3^2 + (2b_4b_3 - a_4c_3 - a_3c_4)v_3v_4 + (2b_5b_3 - a_5c_3 - a_3c_5)v_3v_5 + (b_4^2 - a_4c_4)v_4^2 + (2b_5b_4 - a_5c_4 - a_4c_5)v_4v_5 + (b_5^2 - a_5c_5)v_5^2 = 0. \tag{3}$$

In this case, equation (2) has a unique solution along $(A, -B)$ if $A \neq 0$ or along $(0, 1)$ otherwise.

The point q is a cusp of Gauss of M_v if and only if the unique asymptotic direction u, i.e. the unique solution of equation (2) at q, is tangent to Δ (the zero set of δ). This is the case if $(\delta_x, \delta_y).(A, -B) = 0$ when $A \neq 0$ or $(\delta_x, \delta_y).(0, 1) = 0$ when $A = 0$. When $A = 0$, we have $B = 0$ (and $C \neq 0$) as $\delta = 0$. Therefore $\delta_y = -A_yC$ and the condition becomes $A_y = 0$. So the condition for tangency is

$$\begin{aligned} A\delta_x - B\delta_y = 0 \quad &\text{if} \quad A \neq 0 \\ A_y = 0 \quad &\text{if} \quad A = 0 \end{aligned} \tag{4}$$

By Proposition 3.11, u is an asymptotic direction if and only if equations (2), (3), (4) are satisfied.

Suppose that $A \neq 0$ at q. Equation (3) determines a conic in the projective plane $(v_3 : v_4 : v_5)$ and $A\delta_x - B\delta_y = 0$ a cubic curve. Therefore, by Bézout's theorem, these two curves intersect in at most 6 points. However, if $A = 0$, both equations are satisfied and this gives one of the intersection points of the two curves. This intersection point is of multiplicity 1 unless $A_y = 0$. So the intersection point of multiplicity 1 corresponding to $A = 0$ does not give an asymptotic direction. Hence, the two curves above intersect in at least 1 and at most 5 other points.

If $A = 0$ at q, then $B = 0$ (as $\delta = B^2 - AC$) and these two equations determine a unique direction v in N_qM, given by the point of tangency of the line $A = 0$ with the cone $\delta = 0$ in $\mathbb{R}P^2$. Equations (2)–(4) with $A = 0$ may be satisfied on a curve in M, given by $A_y = 0$, with v the point given by $A = B = 0$. But when $A = A_y = 0$ the cubic $A\delta_x - B\delta_y = 0$ is tangent to the conic at $A = B = 0$. This is a limiting case of when $A \neq 0$ where a binormal direction on the cone approaches the point $A = B = 0$. So here too we have at least 1 and at most 5 asymptotic directions.

The equation of the asymptotic directions can be obtained as follows using Maple. When equations (2)–(4) are satisfied, we can rewrite (2) as $Ady + Bdx = 0$ and equation (4) as $\delta_x dx + \delta_y dy = 0$. We work, without loss of generality, in the chart $v_5 = 1$ and use the resultant to eliminate v_3 and v_4 from $Ady + Bdx$, δ and $\delta_x dx + \delta_y dy$. We then take the relevant component of the resultant. We have the following result, where $a = (a_i), b = (b_i), c = (c_i)$ are the coefficients of the second fundamental form written as vectors and $[\,,\,,\,]$ denotes a 3×3-determinant.

Theorem 4.1. *There is at least one and at most five asymptotic curves passing through any point on a generic immersed surface in $\mathbb{R}^5$. These curves are solutions of the implicit differential equation*

$$A_0 dy^5 + A_1 dx dy^4 + A_2 dx^2 dy^3 + A_3 dx^3 dy^2 + A_4 dx^4 dy + A_5 dx^5 = 0,$$

where the coefficients A_i, $i = 0,1,2,3,4,5$ depend on the coefficients of the second fundamental form and their first order partial derivatives, and are given by

$$A_0 = [\tfrac{\partial c}{\partial y}, b, c],$$

$$A_1 = [\tfrac{\partial c}{\partial x}, b, c] + 2[\tfrac{\partial b}{\partial y}, b, c] + [\tfrac{\partial c}{\partial y}, a, c],$$

$$A_2 = [\tfrac{\partial c}{\partial x}, a, c] + 2[\tfrac{\partial b}{\partial x}, b, c] + [\tfrac{\partial a}{\partial y}, b, c] + 2[\tfrac{\partial b}{\partial y}, a, c] + [\tfrac{\partial c}{\partial y}, a, b],$$

$$A_3 = [\tfrac{\partial a}{\partial x}, b, c] + 2[\tfrac{\partial b}{\partial x}, a, c] + [\tfrac{\partial c}{\partial x}, a, b] + 2[\tfrac{\partial b}{\partial y}, a, b] + [\tfrac{\partial a}{\partial y}, a, c],$$

$$A_4 = [\tfrac{\partial a}{\partial x}, a, c] + 2[\tfrac{\partial b}{\partial x}, a, b] + [\tfrac{\partial a}{\partial y}, a, b],$$

$$A_5 = [\tfrac{\partial a}{\partial x}, a, b].$$

Remark 4.2. *For 2-dimensional surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ the asymptotic curves are given by a quadratic (binary) differential equation in dx, dy. The coefficients of their equations depend only on the coefficients of the second fundamental form (and not on their derivatives).*

If the surface is given in Monge form as in (1) we can obtain the asymptotic directions at the origin using Theorem 4.1. We have the following result where all the partial derivatives are evaluated at the origin.

Corollary 4.3. (1) *Suppose the origin is an M_3-point. Then $u = (u_1, u_2)$ is an asymptotic direction at the origin if and only if*

$$\begin{aligned} u_2^2(f^1_{yyy}u_2^3 + 3f^1_{xyy}u_1u_2^2 + 3f^1_{xxy}u_1^2u_2 + f^1_{xxx}u_1^3) - \\ 2u_1u_2(f^2_{yyy}u_2^3 + 3f^2_{xyy}u_1u_2^2 + 3f^2_{xxy}u_1^2u_2 + f^2_{xxx}u_1^3) + \\ u_1^2(f^3_{yyy}u_2^3 + 3f^3_{xyy}u_1u_2^2 + 3f^3_{xxy}u_1^2u_2 + f^3_{xxx}u_1^3) = 0. \end{aligned}$$

(2) *Suppose the origin is an M_2^h or M_2^e-point. Then $u = (u_1, u_2)$ is an asymptotic direction at the origin if and only if*

$$(u_1^2 \mp u_2^2)(f^3_{yyy}u_2^3 + 3f^3_{xyy}u_1u_2^2 + 3f^3_{xxy}u_1^2u_2 + f^3_{xxx}u_1^3) = 0.$$

(3) *Suppose the origin is an M_2^p-point. Then $u = (u_1, u_2)$ is an asymptotic direction at the origin if and only if*

$$u_1^2(f^3_{yyy}u_2^3 + 3f^3_{xyy}u_1u_2^2 + 3f^3_{xxy}u_1^2u_2 + f^3_{xxx}u_1^3) = 0.$$

5. Generic configurations of the asymptotic curves

For a generic surface, at least one of the coefficients in Theorem 4.1 is not zero at any point $q \in M$. We can assume the point in consideration to be the origin and make linear changes of coordinates in the source so that the coefficient of dy^5 is locally nonzero. We then set $p = \frac{dy}{dx}$ (as $dx = 0$ is not a solution of the equation) so that the equation of the asymptotic curves near the origin is an implicit differential equation (IDE) in the form

$$F(x, y, p) = p^5 + A_1(x, y)p^4 + A_2(x, y)p^3 + A_3(x, y)p^2 + A_4(x, y)p + A_5(x, y) = 0$$

where $A_i(x, y), i = 1, \cdots, 5$ are smooth functions in some neighbourhood U of the origin. We consider F as a multi-germ $U \times \mathbb{R}, (0, 0, p_i) \to \mathbb{R}, 0$, where p_i are the solutions of $F(0, 0, p) = 0$ (there are at most 5 of them).

If $F(0, 0, p)$ has 5 simple roots then, by the implicit function theorem, the solutions of $F = 0$ consists of a net of 5 transverse smooth curves. Two distinct such nets are not homeomorphic. So discrete topological models do not exist in general for IDEs of degree 5. We shall say here that two IDEs above are equivalent if their solutions are the union of the same number of topologically equivalent foliations.

The surface $F^{-1}(0)$ is generically smooth and the projection $\pi : F^{-1}(0) \to \mathbb{R}^2, 0$ is generically a submersion or has a singularity of type fold, cusp or two transverse folds. The set of singular points of π is called the criminant and its projection to the plane the discriminant of the IDE.

The multi-valued direction field in the plane determined by the IDE lifts to a single direction field on $F^{-1}(0)$. This field is along the vector field

$$\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p}$$

(see for example [3]), where subscript denote partial differentiation at (x, y, p). We analyse ξ around each point $(0, 0, p_i)$ and project down to obtain the configuration of one of the foliations determined by the IDE.

If π is a submersion at $(0, 0, p_i)$ then, by the implicit function theorem, F is equivalent to $p - p_i = g(x, y)$ in a neighbourhood of $(0, 0, p_i)$, where g is a smooth function. So the integral curves are smooth.

If $(0, 0, p_i)$ is a fold singularity of π and is a regular point of ξ, then the configuration of the integral curves in the plane is smoothly equivalent to a family of cusps, Figure 3 ①f (see [14] for references). The field ξ may generically have an elementary singularity (saddle/node/focus) and the configuration of integral curves in the plane is topologically equivalent to a folded-singularity $(p - p_i)^2 - y + \lambda x^2 = 0$, $\lambda \neq 0, \frac{1}{4}$. We have a folded saddle if $\lambda < 0$, a folded node if $0 < \lambda < \frac{1}{4}$ and a folded focus if $\lambda > \frac{1}{4}$ ([14], Figure 3 ①g/h/i).

When π has a cusp singularity at $(0, 0, p_i)$, the equation has the modulus of functions with respect to topological equivalence. There are two types of cusp singularities, the elliptic cusp and the hyperbolic cusp ([14]), Figure 3 ③c/d respectively.

We conclude that the generic configurations of the integral curves of the IDE under consideration are modelled by super-imposing in each quadrant one figure from the left column with one from the right column in Figure 3. We denote this super-imposition by the sign +.

For generic surfaces in $\mathbb{R}^5$, the discriminant of the equation of the asymptotic directions in Theorem 4.1 is smooth. Therefore the cusp singularity (Figure 3 ③) does not occur. The only possible configurations of the asymptotic curves are those obtained from Figure 3 ① and ②. We have the following geometric characterisation of the various possibilities.

Proposition 5.1. (1) *The local configurations of the asymptotic curves of a generically immersed surface in* $\mathbb{R}^5$ *are modeled by super-imposing in each quadrant in* Figure 3 ① *and* ② *one figure from the left column with one from the right column.*

(2) *Let* $q \in M_3$ *be a point on the discriminant* Δ *of the asymptotic IDE and* u *the double asymptotic direction there. Then* q *is a folded-singularity of the asymptotic IDE at* (q, u) *if and only if* q *is an* A_4*-singularity of the height function along* u^* (Figure 3 ① (a or b)+(g, h or i)).

(3) *The discriminant* Δ *intersects transversally the* M_2*-curve at* M_2^p *and* D_5*-points (it may also intersect it at other points). The* D_5*-points are generically not folded singularities, so the configuration of the asymptotic curves at such points is as in* Figure 3 ① (a or b)+(f). *An* M_2^p*-point is (at the appropriate direction) a folded singularity of the IDE of the asymptotic curves and the configurations there are as in Figure* 3 ① (a or b)+(g, h or i).

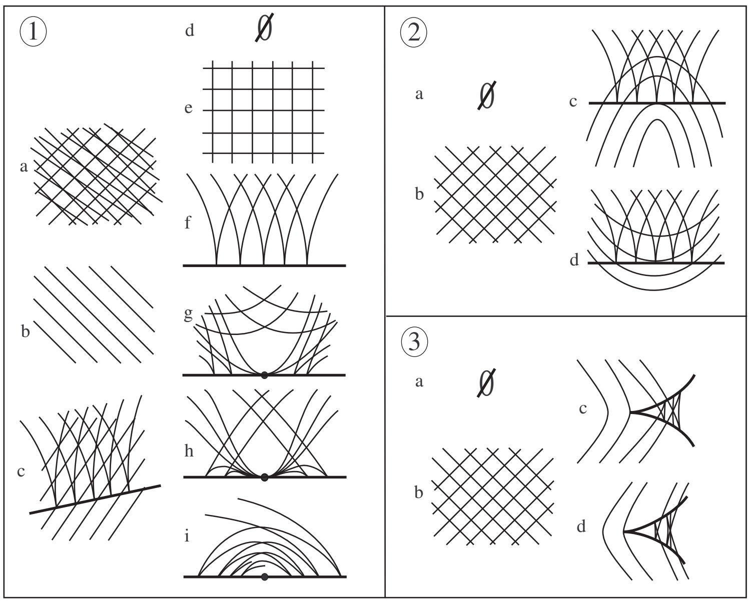

FIGURE 3. Generic configurations of the solutions of an IDE of degree 5 obtained by super-imposing in each quadrant one figure from the left column with one from the right.

Proof. (2) We take the surface in Monge form as in (1) at $q=(0,0)\in M_3$, suppose without loss of generality that $q\in\Delta$ and $u=(1,0)$ is the asymptotic direction there, so $f^3_{xxx}(0,0)=(3f^3_{xxy}-2f^2_{xxx})(0,0)=0$. Then the dual direction of u is $u^*=(0,0,1)$.

We can compute the tangent direction to the discriminant at q using the equation in Theorem 4.1. It is along

$$(3f^3_{xxy}f^3_{xyy}+f^1_{xxx}f^3_{xxy}-2f^3_{xxxy}\,,\,-2f^3_{xxxx}+3(f^3_{xxy})^2),$$

where the partial derivatives are evaluated at the origin. The above direction is parallel to $(1,0)$ if and only if the origin is an A_4-singularity of the height function along the dual direction $(0,0,1)$. (The above expression of the tangent direction can also be used to determine the condition for the configuration of the asymptotic directions to be as in Figure 3 ② (a or b)+(c).)

(3) Suppose that q is not an M_2^p-point. The asymptotic directions at q are either associated to the flat umbilic direction of the height function or to its simple binormal directions (two of them at an M_2^h-point an none at M_2^e). So two of these directions can coincide on Δ in two ways. One way is for two of the asymptotic directions associated to the flat umbilic to coincide at q. Then the point q is a D_5-singularity (this follows for example from Corollary 4.3 (2)). The second way is for one of the asymptotic directions corresponding to the simple binormal directions to coincide with an asymptotic directions corresponding to the flat umbilic direction. If we take the surface in Monge form as in (1) at q, then the condition for this to happen is $j^3f^3(1,\pm1)=0$ (Corollary 4.3 (2)). This can occur generically at isolated points on the M_2-curve and these points are distinct from the M_2^p and D_5-points.

Calculations show the D_5-points are generically not folded singularities (the double asymptotic directions are not tangent to Δ) and that Δ and the M_2-curves are transverse at such points.

Suppose now that q is an M_2^p-point and the surface is in Monge form as in (1) at q. It follows from Corollary 4.3 (3) that q is a point of the discriminant ($(0,1)$ is a double asymptotic direction at q). The tangent direction to the discriminant at q is along the double asymptotic direction $u=(0,1)$, therefore the IDE of the asymptotic curves has generically a folded singularity at q. The tangent direction to the M_2-curve at the origin is along $(f^3_{yyy},\,-f^3_{xyy})$, so it is generically transverse to the discriminant at M_2^p-point. □

Remark 5.2. *The flat ridge of M can be lifted to a regular curve on $N=F^{-1}(0)$ by considering at each of its points the asymptotic direction associated the degenerate binormal direction. (This follows from the following facts. The flat ridge and its lift to the unit normal bundle of the M are generically smooth curves. Therefore the corresponding asymptotic directions form a smooth curve in the tangent bundle of the surface.) It follows from Proposition 5.1(2) that a point $q\in M_3$ is a folded-singularity of the asymptotic IDE if and only if (q,u) is the intersection point on N of the lift of the flat ridge with the criminant. The kernel of the differential of $\pi:N\to M$ at (q,u) is generically transverse to the lift of the flat ridge. Therefore the discriminant and the flat ridge are generically tangential at a folded-singularity of the asymptotic IDEs in the M_3 region.*

6. Global consequences

An immediate consequence of the above local considerations is the following.

Theorem 6.1. *Suppose that M is a closed surface immersed in $\mathbb{R}^5$ with $\chi(M) \neq 0$. Then the discriminant Δ of the asymptotic curves is not empty.*

Proof. If Δ is empty then there is a globally defined asymptotic line field on M (recall that there is at least one asymptotic direction at each point on M). It follows from the Poincaré-Hopf formula ([21]) and the hypothesis on M that this line field has critical points on M. This is a contradiction as the critical points occur on the discriminant. □

We consider now the map $\pi : F^{-1}(0) \to M$ in §5 and denote by $\Sigma\pi$ its singular set (i.e. the criminant).

Theorem 6.2. *Let M be a closed orientable surface generically immersed in $\mathbb{R}^5$ with non zero Euler characteristic $\chi(M)$. If the map π has non vanishing degree then the IDE of the asymptotic curves has a folded singularities.*

Proof. We can choose an orientation on both M and N. The map π determines a decomposition of N as a union $N = N^+ \cup N^-$ of closed surfaces such that $N^+ \cap N^- = \partial N^+ = \partial N^- = \Sigma\pi$, $\pi|_{N^+}$ being an orientation preserving immersion and $\pi|_{N^-}$ an orientation reversing immersion. We have $\chi(N) = \chi(N^+) + \chi(N^-)$, for $\chi(N^+ \cap N^-) = \chi(\Sigma\pi) = 0$. Moreover, since π is a stable map without cusps, the following relation holds (see [37])

$$\chi(N) - 2\chi(N^-) = \chi(M)deg(\pi).$$

That is, $\chi(N^+) - \chi(N^-) = \chi(M)deg(\pi)$.

When $\chi(N) \neq 0$, it follows from Poincaré-Hopf formula ([21]) that there is a critical point of the direction field determined by the IDE of the asymptotic curves on N. Then the result follows from Proposition 5.1.

Suppose that $\chi(N) = 0$. In this case, $\chi(N^+) = -\chi(N^-)$ and thus $2\chi(N^+) = \chi(M)deg(\pi)$. It now follows from the hypothesis that $\chi(N^+) \neq 0$ and the extended Poincaré-Hopf-Morse formula ([19]) implies that the restriction of the direction field to the closed surface N^+ must have singularities (that lie on the criminant curve $\partial N^+ = \Sigma\pi$). Therefore the IDE of the asymptotic curves has a folded singularity. □

Corollary 6.3. *Let M be a closed orientable surface generically immersed in $\mathbb{R}^5$ with non zero Euler characteristic $\chi(M)$. If the map π has a non vanishing degree then M has either M_2 points or flat ridges.*

Proof. The result follows from the geometrical interpretation of folded singularities of the IDE of the asymptotic curves in Proposition 5.1. □

Remark 6.4. *With the hypothesis of Corollary 6.3, one can assert that there exist either parabolic M_2 points (i.e. intersections of the discriminant with the M_2 curve) or tangency points of the flat ridge curve and the discriminant.*

We relate next the global existence of binormal/asymptotic fields with the $2nd$-order regularity problem. Let $f : M \to \mathbb{R}^n$ be an immersion of a surface M in n-space. A point $q \in M$ is said to be 2-*regular* if and only if there exists some local coordinate system $\{x, y\}$ at q such that the subspace generated by the vectors $\{\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}\}$ at q has maximum rank. If this is not the case then q is said to be 2-*singular*. The immersion f is said to be *regular of order 2* if all the points of M are 2-regular. Feldman ([16]) proved that the set of 2-regular immersions of any closed surface M in $\mathbb{R}^n$ is open and dense when $n = 3$ and $n \geq 7$. When $n = 6$, the 2-singular points are generically isolated. Moreover, 2-regular immersions satisfy the h-principle, that is, any immersion of a surface into $\mathbb{R}^6$ can be deformed through a regular homotopy into a 2-regular immersion ([15, 20]). When $n = 4$, the 2-singular points coincide with the inflection points defined by Little in [26]. The existence of inflection points on generic closed surfaces immersed in $\mathbb{R}^4$ was explored in [17] by analysing the behaviour of the asymptotic curves on such surfaces. It is shown in [17] that generic closed locally convex surfaces in $\mathbb{R}^4$ with non vanishing Euler number have inflection points. The case $n = 5$ appears to be more complicated and not many results are known in this direction. Costa obtained in [13] an example of a 2-regular immersion of the 2-sphere into $\mathbb{R}^5$ consisting in a double cover of the Veronese surface (projective plane) immersed in S^4. This is done as follows. Consider the map

$$\begin{array}{rccl} V : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^6 \\ & (x, y, z) & \longmapsto & (x^2, y^2, z^2, xy, xz, yz). \end{array}$$

The restriction of V to the unit sphere S^2 defines a 2-regular immersion of the real projective plane into $\mathbb{R}^6$, known as the *Veronese surface*. It is not difficult to show that $V(S^2)$ is contained in both a hyperplane (of equation $X + Y + Z = 1$, where (X, Y, Z, U, W) are the coordinates in $\mathbb{R}^6$) and a 5-sphere of $\mathbb{R}^6$, and hence in a 4-sphere. By choosing appropriate coordinates on S^2 and on the hyperplane of equation $X + Y + Z = 1$ (identified with $\mathbb{R}^5$), we can locally define $V(S^2)$ by means of the chart $\tilde{V} : \mathbb{R}^2 \to \mathbb{R}^5$, given by

$$\tilde{V}(x, y) = \Big(\frac{y\sqrt{4 - x^2 - y^2}}{2}, \frac{x\sqrt{4 - x^2 - y^2}}{2}, \frac{xy}{2}, \frac{x^2 - y^2}{4}, \frac{3x^2 + 3y^2 - 8}{4\sqrt{3}}\Big).$$

The $2nd$-order regularity of a generic surface in $\mathbb{R}^5$ is related to the generic behaviour of the family of height functions on M. In fact, the 2-singular points coincide with the points of type $M_i, i < 3$, and these are the corank 2 singularities of the height functions ([31]). We can then reinterpret the result in Corollary 6.3 as follows.

Corollary 6.5. *Let M be a generic 2-regular closed orientable surface in $\mathbb{R}^5$ with non zero Euler characteristic. If the map π has non vanishing degree, then M has flat ridge curves and some of them must be tangent to the discriminant curve at some point.*

In the particular case of the Veronese surface $V(S^2)$, it can be shown that all the points on the surface are flat ridges. Indeed, this surface in not generic from the viewpoint of its contacts with hyperplanes.

The $2nd$-order regularity of a surface in $\mathbb{R}^5$ is also related to the global existence of certain degenerate normal fields (called essential) on the surface ([34]). We analyse next the geometrical dynamics associated to such fields.

Given any normal field v on M, we can consider its associated shape operator S_v (see §2). For each $q \in M$ there is an orthonormal basis in T_qM formed by the eigenvectors of S_v (*v-principal directions*). The corresponding eigenvalues k_1 and k_2 are called *maximal* and *minimal v-principal curvature*, respectively. A point q is said to be *v-umbilic* if both v-principal curvatures coincide at q. Denote by $\mathcal{U}_v$ the subset of all the v-umbilic points of M. The v-principal directions define two, mutually orthogonal tangent fields in the region $M - \mathcal{U}_v$, whose critical points are the v-umbilics. The corresponding integral curves are called v-curvature lines. The two foliations, together with their critical points form the *v-principal configuration* of M. The differential equation of the v-curvature lines is given by $S_v\big(X(q)\big) = \lambda(q)X(q)$.

Suppose that v is a binormal field defined locally in some open region S of a surface M immersed in $\mathbb{R}^5$. The matrices of S_v and II_v coincide on S. Therefore, in some appropriate coordinate system, the matrix of S_v coincides with that of $Hess(f_{v(q)})(q)$, $\forall q \in S$. But this implies that one of the eigenvectors of S_v vanishes at every point of S. Therefore, one of the principal foliations of S_v coincides with the asymptotic foliation associated to v. So the v-curvature lines with associated vanishing curvature are also solutions of the implicit differential equation of Theorem 4.1.

The critical points of the v-principal configurations associated to binormal fields on surfaces immersed into $\mathbb{R}^5$ are points of type M_2 and thus 2-singular ([34]). Therefore the only 2-regular surfaces that may admit globally defined binormal fields are tori or Klein bottles.

6.1. **Final Remarks.** The definition of an asymptotic direction at a point on a 2-dimensional surface in the Euclidean space $\mathbb{R}^5$ in terms of singularities of projections to k-planes, $k = 1, 2, 3, 4$, admits a natural generalisation to m-manifolds immersed in $\mathbb{R}^n$. The case of submanifolds of codimension 2 was first treated in [29] and more recently in [35], where the existence and behaviour of asymptotic curves was studied in connection with some global geometrical properties such as the vanishing of the normal curvature, convexity and semi-umbilicity. A problem under investigation is the determination of the configuration of the asymptotic curves on a 2-dimensional surface immersed in $\mathbb{R}^n$.

The study of contact of surfaces immersed in other spaces with special submanifolds which are invariant with respect to the corresponding transformation groups is also of interest. There is currently an extensive programme initiated by S. Izumiya on the study of contact of submanifolds in Minkowski spaces with degenerate objects, such as horospheres (see for example [22, 23] and these papers for more references). For example, horo-asymptotic directions on surfaces in the Hyperbolic 4-space are introduced and studied in [22].

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