Projections of space curves and duality

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Abstract

We study the flat geometry of orthogonal projections of a generic space curve to planes. For a single projection, we do this by considering submersions on a (singular) plane curve. This is an alternative method to the classification of divergent diagrams carried out in [9]. We redraw the bifurcation diagrams of the orthogonal projections of space curves adding the information about their flat geometry. We also study the duals of the projected curves and the way they bifurcate as the direction of projection varies locally in S^2.

1   Introduction

In [9], J. J. Nuño-Ballesteros and F. Dias obtained a classification of the singularities of orthogonal projections of a generic space curve $\gamma : I \to \mathbb{R}^3$ which takes into account the flat geometry of the projected plane curve α. They did this by classifying germs of divergent diagrams (f, α): $\mathbb{R}, 0 \xleftarrow{f} I, 0 \xrightarrow{\alpha} \mathbb{R}^2, 0$ up to smooth changes of coordinates (see §4). When f is the height function h on α along its normal direction, the equivalence relation between two such plane curve diagrams is denoted by $\mathcal{A}_h$ in [9]. The $\mathcal{A}_h$-singularities of α capture information about its flat geometry.

The classification in [9] is a refinement of that in [8] where David showed that any orthogonal projection of a projection-generic γ can have singularities in only the following ten $\mathcal{A}$-classes: $A_0, A_1, A_2, A_3, A_4, A_5, D_4, D_5, D_6$ and $\tilde{E_7}$ (see [1] for notation). With the additional information on the flat geometry of α, Nuño-Ballesteros and Dias showed that the above classes split into several $\mathcal{A}_h$-classes, totalling seventeen for
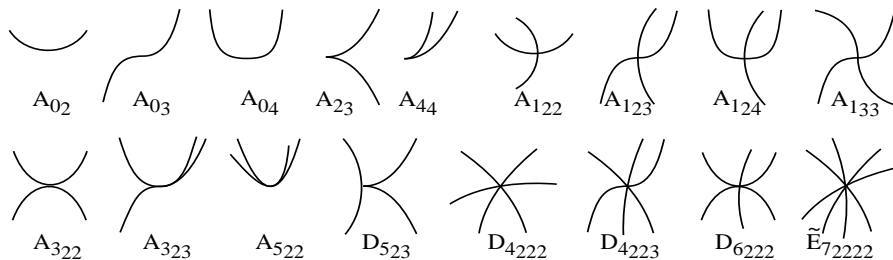
Figure 1: $\mathcal{A}_h$-Singularities of the orthogonal projection of a generic space curve [9].

generic space curves (Figure 1). A geometric characterisation of each of the $\mathcal{A}_h$-classes is given in [9] in terms of the geometry of the space curve γ and of the direction of projection.

Divergent diagrams are hard to deal with as the group acting is not a geometric subgroup ([7]). For plane curve diagrams treated in [9], a classification is obtained by direct computations. We propose here an alternative method that avoids the use of divergent diagrams. We follow the approach in [6] by fixing a model X for the $\mathcal{A}$-singularity of the plane curve α. Then a height function on α can be thought of as a submersion on X (see [6] §3.4 for details). We classify submersions up to smooth changes of coordinates that preserve the model curve X (Theorem 3.1). We have then an action of a geometric subgroup of the contact group $\mathcal{K}$ and the singularity theory classification techniques apply. We show in §4 how to relate our classification in Theorem 3.1 to the results in [9].

The family of orthogonal projections of γ are parametrised by the sphere S^2, and for a generic γ, this family is an $\mathcal{A}_e$-versal deformation of the singularities of α (see for example [11] for the definition of an $\mathcal{A}_e$-versal deformation). We redraw the bifurcation diagrams of α adding the information about its flat geometry (Theorem 5.3).

The flat geometry of the curve α is mainly about its inflections which are obtained by considering the dual curve of α. This dual curve is the discriminant of the family of height functions on α. We give the singularities of the dual curve of α and the way they bifurcate as the projection varies locally in S^2 (Theorem 6.1).

2   Preliminaries

Let $\mathcal{E}_n$ be the local ring of germs of functions $\mathbb{R}^n, 0 \to \mathbb{R}$ and $\mathcal{M}_n$ its maximal ideal (which is the subset of germs that vanish at the origin). Denote by $\mathcal{E}(n, p)$ the p-tuples of elements in $\mathcal{E}_n$. Let $\mathcal{A} = \mathcal{R} \times \mathcal{L} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^p, 0)$ denote the group of right-left equivalence which acts smoothly on $\mathcal{M}_n.\mathcal{E}(n, p)$ by $(h, k).G = k \circ G \circ h^{-1}$. The contact group $\mathcal{K}$ is the set of germs of diffeomorphisms of $\mathbb{R}^n \times \mathbb{R}^p, 0$ which can be

written in the form $H(x, y) = (h(x), H_1(x, y))$, with $h \in Diff(\mathbb{R}^n, 0)$ and $H_1(x, 0) = 0$ for x near 0. This means that $\pi \circ H = h \circ \pi$ where $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the canonical projection. Thus H is a fibred mapping over the diffeomorphism h and preserves the 0-section $\mathbb{R}^n \times \{0\}$. The set of germs of diffeomorphisms of $\mathbb{R}^n \times \mathbb{R}^p, 0$ in the form (I, H), where I is the germ of the identity map of $\mathbb{R}^n, 0$, is denoted by $\mathcal{C}$. The group $\mathcal{K}$ is the semi-direct product of $\mathcal{R}$ and $\mathcal{C}$, and we write $\mathcal{K} = \mathcal{R} \rtimes \mathcal{C}$. The group $\mathcal{K}$ acts on $\mathcal{M}_n.\mathcal{E}(n, p)$ as follows: $G = H.F$ if and only if $(x, G(x)) = H(h^{-1}(x), F(h^{-1}(x)))$; see [12] for details. When $p = 1$, the group $\mathcal{C}$ is just the multiplication by germs of functions in $\mathcal{E}_n$.

It is important to observe that the action of the group $\mathcal{K}$ is a natural one to use when one seeks to understand the singularities of the zero fibres of germs in $\mathcal{M}_n.\mathcal{E}(n, p)$. Indeed, if two germs are $\mathcal{K}$-equivalent, then their zero fibres are diffeomorphic. The action of the group $\mathcal{A}$ is finer than that of $\mathcal{K}$. If two germs F and G are $\mathcal{A}$-equivalent, that is $G = k \circ F \circ h^{-1}$ for some $(h, k) \in \mathcal{A}$, then the fibres $G^{-1}(c)$ and $F^{-1}(k^{-1}(c))$ are diffeomorphic, for any c close to $0 \in \mathbb{R}^p$.

The tangent space to the $\mathcal{A}$-orbit of F at the germ F is given by $L\mathcal{A}\cdot F = \mathcal{M}_n.\{F_{x_1}, \ldots, F_{x_n}\} + F^*(\mathcal{M}_p).\{e_1, \ldots, e_p\}$, where F_{x_i} denotes partial derivatives with respect to x_i $(i = 1, \ldots, n)$, $e_1, \ldots, e_p$ denote the standard basis vectors of $\mathbb{R}^p$ considered as elements of $\mathcal{E}(n, p)$, and $F^*(\mathcal{M}_p)$ is the pull-back of the maximal ideal in $\mathcal{E}_p$. The extended tangent space to the $\mathcal{A}$-orbit of F at the germ F is given by $L_e\mathcal{A}\cdot F = \mathcal{E}_n.\{F_{x_1}, \ldots, F_{x_n}\} + F^*(\mathcal{E}_p).\{e_1, \ldots, e_p\}$, and the codimension of the extended orbit is $d_e(F, \mathcal{A}) = \dim_{\mathbb{R}}(\mathcal{E}(n, p)/L_e\mathcal{A} \cdot F$.

Let $k \geq 1$ be an integer. We denote by $J^k(n, p)$ the space of kth order Taylor expansions without constant terms of elements of $\mathcal{E}(n, p)$ and write j^kF for the k-jet of F. A germ F is said to be $k-\mathcal{A}$-*determined* if any G with $j^kG = j^kF$ is $\mathcal{A}$-equivalent to F (notation: $G \sim F$). The k-jet of F is then called a sufficient jet.

Our goal in §3 is to classify functions on a plane curve C. This means that we require that the diffeomorphisms in $\mathbb{R}^2$ preserve C. We follow the method in [6] and recall some results from there. Let $X, 0 \subset \mathbb{R}^n, 0$ be the germ of a reduced analytic sub-variety of $\mathbb{R}^n$ at 0 defined by a polynomial h in $\mathbb{R}[x_1, \ldots, x_n]$. Following Definition 3.1 in [6], a diffeomorphism $\phi : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ is said to preserve X if $\phi(X), 0 = X, 0$ (i.e., $\phi(X)$ and X are equal as germs at 0). The group of such diffeomorphisms is a subgroup of the group $\mathcal{R}$ and is denoted by $\mathcal{R}(X)$. We define $\mathcal{K}(X)$ to be the subgroup of $\mathcal{K}$ given by $\mathcal{K}(X) = \mathcal{R}(X) \rtimes \mathcal{C}$.

Let $\Theta(X)$ be the $\mathcal{E}_n$-module of vector fields in $\mathbb{R}^n$ tangent to X and let $\Theta_1(X) = \{\xi \in \Theta(X) \,|\, j^1\xi = 0\}$ (i.e., $\Theta_1(X)$ is the set of germs of vector fields in $\Theta(X)$ with no constant or linear terms). Define $\Theta(X)\cdot f = \mathcal{E}_n.\{\xi f \,|\, \xi \in \Theta(X)\}$. If f is a smooth function, then we have the following tangent spaces to the $\mathcal{K}(X)$-orbit of f at the germ f: $L\mathcal{K}_1(X)\cdot f = \Theta_1(X)\cdot f + f^*(\mathcal{M}_1^2).\mathcal{E}_n$ and $L\mathcal{K}(X)\cdot f = \Theta(X)\cdot f + f^*(\mathcal{M}_1).\mathcal{E}_n$.

The $\mathcal{K}(X)$-codimension of f is defined as $d(f, \mathcal{K}(X)) = \dim_{\mathbb{R}}(\mathcal{M}_2/L\mathcal{K}(X) \cdot f)$.

The classification (i.e., the listing of representatives of the orbits) of $\mathcal{K}(X)$-finitely

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determined germs is carried out inductively on the jet level. The method used here is that of the complete transversal adapted in [6] for the $\mathcal{R}(X)$-action. We have the following result which is a version of Theorem 3.11 in [6] for the group $\mathcal{K}(X)$.

Proposition 2.1 *Let* $f : \mathbb{R}^n, 0 \to \mathbb{R}, 0$ *be a germ of a smooth function and let* $h_1, \ldots, h_r$ *be homogeneous polynomials of degree $k + 1$ with the property that*

$$\mathcal{M}_n^{k+1} \subset L\mathcal{K}_1(X){\cdot}f + sp\{h_1, \ldots, h_r\} + \mathcal{M}_n^{k+2}.$$

Then any germ g with $j^k g(0) = j^k f(0)$ is $\mathcal{K}(X)$-equivalent to a germ of the form $f(x) + \sum_{i=1}^l u_i h_i(x) + \phi(x)$, where $\phi(x) \in \mathcal{M}_n^{k+2}$. The vector subspace $sp\{h_1, \ldots, h_r\}$ is called the complete $(k + 1)$-$\mathcal{K}(X)$-transversal of f.

Corollary 2.2 *If $\mathcal{M}_n^{k+1} \subset L\mathcal{K}_1(X){\cdot}f + \mathcal{M}_n^{k+2}$ then f is $k - \mathcal{K}(X)$-determined.*

We also need the following result about trivial families.

Proposition 2.3 ([6]) *Let $F : \mathbb{R}^n \times \mathbb{R}, (0,0) \to \mathbb{R}, 0$ be a smooth family of functions with $F(0,t) = 0$ for t small. Let $\xi_1, \ldots, \xi_p$ be vector fields in $\Theta(X)$ vanishing at $0 \in \mathbb{R}^n$ and let $\partial F/\partial t$ denote the germ $\partial F/\partial t(x,0)$ of the initial speed. Then the family F is $k - \mathcal{R}(X)$-trivial (and hence $k - \mathcal{K}(X)$-trivial) if*

$$\partial F/\partial t \in \langle \xi_1 F, \ldots, \xi_p F \rangle + \mathcal{M}_n^{k+1} \subset \mathcal{E}_{n+1}.$$

For our case $n = 2$, we have the following result adapted from the result in [13], see also Proposition 7.2 in [5].

Proposition 2.4 *Let $X, 0$ be a plane curve with an isolated singularity, defined by a weighted homogeneous map germ $h : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ with weights w_1 and w_2. Then $\Theta(X) = \mathcal{E}_2.\{\xi_1, \xi_2\}$, where $\xi_1 = w_1 x \partial/\partial x + w_2 y \partial/\partial y$ (the Euler vector field) and $\xi_2 = h_y \partial/\partial x - h_x \partial/\partial y$.*

3 Submersions on singular plane curves

Let C be a plane curve with a singularity of $\mathcal{A}_e$-codimension ≤ 2, i.e., it has a defining equation with a singularity of type $A_0, A_1, A_2, A_3, A_4, A_5, D_4, D_5, D_6$ and $\tilde{E}_7$. We take the models of the defining equation of C as in Table 1 second column, and classify germs of submersions of the plane up to $\mathcal{K}(C)$-equivalence. We also use diffeomorphisms in the source that preserve the curve C but do not come from integrating vector fields in $\Theta(C)$.

Table 1: Submersions on singular plane curves.

Name	$\mathcal{A} - $ Model for C	Submersions on C	$\mathcal{K}(C) - codim$
A_0	y	$y + x^k,\ k \geq 1$	$k - 2$
A_1	xy	$y + x^k,\ k \geq 1$	$k - 1$
A_2	$y^2 - x^3$	x y	0 1
A_3	$y(y - x^2)$	x $y + ax^2,\ a \neq 0, -1$ $y + \epsilon x^2 + x^k,\ k \geq 3, \epsilon = 0, -1$	0 2 $k - 1$
A_4	$y^2 - x^5$	x $y + x^2$ $y + x^3$ y	0 1 2 3
A_5	$y(y - x^3)$	x $y + x^2$ $y + ax^3 + x^4,\ a \neq 0, -1$ $y + ax^3,\ a \neq 0, -1$ $y + \epsilon x^3 + x^k,\ k \geq 4, \epsilon = 0, -1$	0 1 3 4 $k - 1$
D_4	$y(y + x)(y - x)$	$y + ax,\ a \neq \pm 1$ $y + \epsilon x + x^k,\ \epsilon = 0, \pm 1, k \geq 2$	1 $k - 1$
D_5	$y(x^2 - y^3)$	$y + x^k,\ k \geq 1$ $x + y^2$ x	$k - 1$ 1 2
D_6	$y(x + y^2)(x - y^2)$	$y + x^k,\ k \geq 1$ $x + ay^2 + y^3$ $x + ay^2,\ a \neq \pm 1$ $x \pm y^2 + y^k,\ k \geq 4$	$k - 1$ 2 3 $k - 1$
$\tilde{E}_7$	$xy(x + y)(x + \lambda y)$	$x + ay + x^2,\ a \neq 0$ $x + ay,\ a \neq 0, 1, \lambda$ $x + \epsilon y + y^k,\ k \geq 2, \epsilon = 0, 1, \lambda$ $y + x^k,\ k \geq 2$	1 2 $k - 1$ $k - 1$

Theorem 3.1 *Any $\mathcal{K}(C)$-finitely determined germ of a submersion $f : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ on the plane curves in column 2 of Table 1 is equivalent, by changes of coordinates that preserve these curves, together with multiplication by germs of functions, to one of the germs in column 3 of Table 1.*

Proof   In all the proof, ξ_1 and ξ_2 are as in Proposition 2.4. To simplify notation we write complete k-transversal for complete $k - \mathcal{K}(C)$-transversal, equivalence for $\mathcal{K}(C)$-equivalence, k-determined for $k - \mathcal{K}(C)$-determined, finitely determined for $\mathcal{K}(C)$-finitely determined, trivial for $\mathcal{K}(C)$-trivial and codimension for $\mathcal{K}(C)$-codimension. We start with the A_k series where we have general results.

A_{2m}-singularity

If $m = 0$, C is regular and the classification reduces to that of functions on surfaces with boundary [2]. Suppose that $m \geq 1$ and take $h(x,y) = y^2 - x^{2m+1}$. Then $\Theta(C) = \langle 2x\frac{\partial}{\partial x} + (2m+1)y\frac{\partial}{\partial y}, 2y\frac{\partial}{\partial x} + (2m+1)x^{2m}\frac{\partial}{\partial y}\rangle$. We integrate the linear parts of the vector fields in $\Theta(C)$ and obtain the following 1-jets of coordinate changes in $\mathcal{R}(C)$: $\eta_1(x,y) = (e^{2\alpha}x, e^{(2m+1)\alpha}y)$ and $\eta_2(x,y) = (x + \beta y, y)$, $\alpha, \beta \in \mathbb{R}$. Let $j^1 f = ax + by$. If $a \neq 0$, then we can eliminate b by η_2 and multiply by $1/a$ to set $j^1 f = x$. It is not difficult to show that f is 1-determined and has codimension 0. If $a = 0$, we can set $j^1 f = y$. Suppose that $j^{k-1} f = y$. Then $\xi_1 f = (2m+1)y$, $\xi_2 f = (2m+1)x^{2m}$ and all the complete k-transversals are empty for $k > 2m$. A complete k-transversal for $2 \leq k \leq 2m-1$ is $y + \lambda x^k$, which is equivalent to $y + x^k$ if $\lambda \neq 0$. For $f = y + x^k$, we have $\xi_1 f = (2m+1)y + 2kx^k$, $\xi_2 f = 2kyx^{k-1} + (2m+1)x^{2m}$. We have $\mathcal{M}_2^{k+1} \subset L\mathcal{K}_1(C)\cdot f + \mathcal{M}_2^{k+2}$, so $y + x^k$ is k-determined. It has codimension $k - 1$ ($x, \ldots, x^{k-1}$ generate a transverse vector space to $L\mathcal{K}(C) \cdot f$ in $\mathcal{M}_2$).

A complete $2m$-transversal to $j^{2m-1} f = y$ is given by $y + \lambda x^{2m}$, $\lambda \in \mathbb{R}$. However, $x^{2m} \in \langle \xi_1(f), \xi_2(f)\rangle$, so by Proposition 2.3, the family $y + \lambda x^{2m}$ is trivial along λ. The germ y is $2m$-determined and has codimension $2m - 1$.

To summarise, a finitely determined germ of a submersion on C is equivalent to one of the following germs: x, $y + x^k$, $2 \leq k \leq 2m - 1$ and y.

A_{2m-1}-singularity

We start with the case $m = 1$ and take $h(x,y) = xy$. Then $\Theta(C) = \langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\rangle$. The 1-jets coordinate changes in $\mathcal{R}(C)$ are $\eta_1(x,y) = (e^\alpha x, e^\alpha y)$ and $\eta_2(x,y) = (e^\alpha x, e^{-\alpha}y)$, $\alpha \in \mathbb{R}$.

Let $j^1 f = ax + by$. Suppose that $a \neq 0$ and $b \neq 0$. Then by changes of coordinates $(x,y) \mapsto (-x, y)$ and $(x,y) \mapsto (x, -y)$ we can suppose that both a and b are positive. These changes of coordinates preserve the curve C but do not come from vector fields in $\Theta(C)$. We can now use the changes of coordinates η_1 and η_2 to set $j^1 f = x + y$. Then $\xi_1 f = x + y$ and $\xi_2 f = x - y$, so $\mathcal{M}_2^2 \subset L\mathcal{K}_1(C)\cdot f + \mathcal{M}_2^3$. Therefore f is 1-determined. It has codimension zero.

Suppose that $a = 0$, then $b \neq 0$ and we can set $j^1 f = y$. For $j^{k-1} f = y$, $\xi_1 f = y$ and $\xi_2 f = -y$. Using Proposition 2.1, any k-jet, $k \geq 2$, with $(k-1)$-jet y is equivalent to $y + \lambda x^k$. This is equivalent to $y + x^k$ if $\lambda \neq 0$. We write $f = y + x^k$. Then $\mathcal{M}_2^{k+1} \subset L\mathcal{K}_1(C)\cdot f + \mathcal{M}_2^{k+2}$, so $y + x^k$ is k-determined. It has codimension $k - 1$.

If $a \neq 0$ and $b = 0$, then the change of coordinate $(x,y) \mapsto (y, x)$ which preserves the curve C brings us back to the case $a = 0$ and $b \neq 0$.

When $m \geq 2$, we take $h(x,y) = y(y - x^m)$. Then $\Theta(C) = \langle x\frac{\partial}{\partial x} + my\frac{\partial}{\partial y}, (2y - x^m)\frac{\partial}{\partial x} + myx^{m-1}\frac{\partial}{\partial y}\rangle$ and the 1-jets of coordinate changes in $\mathcal{R}(C)$ are $\eta_1(x,y) = (e^\alpha x, e^{m\alpha}y)$ and $\eta_2(x,y) = (x + \beta y, y)$, $\alpha, \beta \in \mathbb{R}$.

The 1-jets of submersions are equivalent to x or y and the germ x is 1-determined and has codimension 0.

Consider the case $j^{k-1}f = y$. Then $\xi_1 f = my$, $\xi_2 f = myx^{m-1}$. A complete k-transversal, $k \geq 2$, is given by $y + \lambda x^k$. If $k \neq m$, we can use η_1 to set $\lambda = \pm 1$. The germ $y \pm x^k$ is k-determined and has codimension $k - 1$. If $k = m$ then λ is a parameter modulus. We write $f = y + ax^m$, so $\xi_1 f = my + amx^m$ and $\xi_2 f = m(2a + 1)yx^{m-1} - amx^{2m-1}$. Therefore all the complete k-transversals are empty for $k \geq 2m - 1$ and $a \neq 0, -1$. A complete k-transversal for $m + 1 \leq k \leq 2m - 2$ is $y + ax^m + \lambda x^k$, which is equivalent to $y + ax^m \pm x^k$ if $\lambda \neq 0$. The germ $f = y + ax^m \pm x^k$ is k-determined and has codimension $k - 1$.

A complete $2m - 1$-transversal is given by $f = y + ax^m + \lambda x^{2m-1}$, $\lambda \in \mathbb{R}$. However, by Proposition 2.3, this family is trivial along λ for $a \neq 0, -1$. Hence $j^{2m-1}f$ is equivalent to $y + ax^m$. The germ $y + ax^m$ is $(2m - 1)$-determined and has codimension $2m - 2$ for $a \neq 0, -1$.

For $a = 0$, the sufficient jets are equivalent to $y \pm x^k$, $k > m$ and have codimension $k - 1$. If $a = -1$, the sufficient jets are equivalent to $y - x^m \pm x^k$, $k > m$ and have also codimension $k - 1$.

D_4-singularity

We take $h(x, y) = y^3 - yx^2 = y(y - x)(y + x)$ for a D_4-singularity, so $\Theta(C) = \langle x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, (3y^2 - x^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \rangle$ and the 1-jets of the coordinate changes in $\mathcal{R}(C)$ are $\eta_1(x, y) = (e^\alpha x, e^\alpha y), \alpha \in \mathbb{R}$.

Take $j^1 f = ax + by$. Then we have the following orbits in the 1-jet space: $x + a'y$ and y. The parameter a' is a modulus. The germ $f = x + ay$ (we write a for a') is 1-determined and has codimension 1 when $a \neq \pm 1$. When $a = \pm 1$, the germs $x \pm y$ (equivalent to $y \pm x$) yield sufficient jets in the form $y \pm x + x^k$, $k \geq 2$, which are k-determined and have codimension $k - 1$. Similarly for the 1-jet y. It yields sufficient jets in the form $y + x^k$, $k \geq 2$, which is k-determined and has codimension $k - 1$.

D_5-singularity

We take $h(x, y) = yx^2 - y^4 = y(x^2 - y^3)$. Then $\Theta(C) = \langle 3x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}, (x^2 - 4y^3) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} \rangle$ and the 1-jets of coordinate changes in $\mathcal{R}(C)$ are $\eta_1(x, y) = (e^{3\alpha}x, e^{2\alpha}y), \alpha \in \mathbb{R}$. The 1-jets of submersions are equivalent to $x + y$, x or y.

Consider the case $f = x + y$. Then $\xi_1 f = 3x + 2y$, $\xi_2 f = x^2 + 2xy - 4y^3$. It is not hard to show that f is 1-determined and has codimension 0.

If $j^{k-1}f = x$, then $\xi_1 f = 3x$ and $\xi_2 f = x^2 - 4y^3$, so all complete k-transversals are empty for $k \geq 4$. A complete 2-transversal is $x + \lambda y^2$, which is equivalent to $x + y^2$ if $\lambda \neq 0$. The germ $f = x + y^2$ is 2-determined and has codimension 1. If $\lambda = 0$, then the germ x is 3-determined and has codimension 2.

The case $j^{k-1}f = y$ yields sufficient germs in the form $y \pm x^k$, $k \geq 2$, which have codimension $k - 1$.

D_6-singularity

We take $h(x, y) = yx^2 - y^5 = y(x - y^2)(x + y^2)$ so that $\Theta(C) = \langle 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, (x^2 - $

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$5y^4)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}\rangle$ and the 1-jets of the coordinate changes in $\mathcal{R}(C)$ are $\eta_1(x,y) = (e^{2\alpha}x, e^{\alpha}y)$, $\alpha \in \mathbb{R}$. Then the 1-jets of submersions are equivalent to $x+y$, x or y. The germ $x+y$ is 1-determined and has codimension 0. We also obtain a family $y + x^k$ which is k-determined and has codimension $k-1$.

Suppose that $j^{k-1}f = x$. Then $\xi_1 f = 2x$ and $\xi_2 f = x^2 - 5y^4$, so all complete k-transversals are empty for $k \geq 5$. A complete 2-transversal is $x + ay^2$, where a is a modulus. Similar calculations to those above yield the following sufficient jets: $x + ay^2 + y^3$ of codimension 2 and $x + ay^2$ of codimension 3 provided $a \neq \pm 1$. When $a = \pm 1$, we obtain the series $x \pm y^2 + y^k$, $k \geq 4$ which has codimension $k-1$.

$\tilde{E}_7$-*singularity*

We take $h(x,y) = xy(x+y)(x+\lambda y)$ so that $\Theta(C) = \langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, (x^3 + 2(\lambda+1)x^2 + 3\lambda xy^2)\frac{\partial}{\partial x} - (3x^2y + 2(\lambda+1)xy^2 + \lambda y^3)\frac{\partial}{\partial y}\rangle$ and the 1-jets of the coordinate changes in $\mathcal{R}(C)$ are $\eta_1(x,y) = (e^{\alpha}x, e^{\alpha}y)$, $\alpha \in \mathbb{R}$. Then the 1-jets of submersions are equivalent to $x + ay$ or y (a is a modulus).

The complete 2-transversal for $x + ay$ is $x + ay + \mu x^2$ which gives two sufficient jets. One is $x + ay + x^2$, $a \neq 0$, and has codimension 1 and the other is $x + ay$, $a \neq 0, \pm 1, \lambda$, and has codimension 2. If $a = 0, \pm 1, \lambda$, we get the following sufficient jets $x + \epsilon y + y^k$, $k \geq 2$, $\epsilon = 0, \pm 1, \lambda$, which has codimension $k-1$.

The complete k-transversal for y is $y + \mu x^k$, which can be taken to $y + x^k$ with the coordinate change. This is k-determined and has codimension $k-1$. $\qquad\square$

4   Comparison of Theorem 3.1 with results in [9]

Let $\mathbb{R}_r, 0_r$ denote r copies of the germ $\mathbb{R}, 0$ of the real line. Following [9], given a multi-germ of plane curves $\alpha : \mathbb{R}_r, 0_r \to \mathbb{R}^2, 0$ with limiting orthogonal directions v_i at $\alpha_i(0_i)$, the multi-germ $h_\alpha : \mathbb{R}_r, 0_r \to \mathbb{R}_r, 0_r$ of height functions on α along $v = (v_1, ..., v_r)$ is given by $h_i(t) = \alpha_i(t) \cdot v_i$, $i = 1, \ldots, r$. The following definition is given in [9].

Two plane curves singularities $\alpha, \beta : \mathbb{R}_r, 0_r \to \mathbb{R}^2, 0$ are said to be $\mathcal{A}_h$-equivalent if there are germs of diffeomorphisms h, H, K such that the following diagram commutes

$$
\begin{array}{ccccc}
\mathbb{R}_r, 0_r & \xleftarrow{h_\alpha} & \mathbb{R}_r, 0_r & \xrightarrow{\alpha} & \mathbb{R}^2, 0 \\
\downarrow h & & \downarrow H & & \downarrow K \\
\mathbb{R}_r, 0_r & \xleftarrow{h_\beta} & \mathbb{R}_r, 0_r & \xrightarrow{\beta} & \mathbb{R}^2, 0
\end{array}
$$

A classification of general divergent diagrams $\mathbb{R}_r, 0_r \xleftarrow{h} \mathbb{R}_r, 0_r \xrightarrow{\alpha} \mathbb{R}^2, 0$ is given in [9]. This is then used to obtain an $\mathcal{A}_h$-classification of plane curves singularities, and in particular, those of orthogonal projections of space curves.

It is shown in [9] that an orthogonal projection of a generic space curve (see Definition 5.2) has only the $\mathcal{A}_h$-singularities listed in the second column of Table 2. The

Table 2: Submersions which recover the $\mathcal{A}_h$-classes of projections of space curves.

$\mathcal{A}$-class	$\mathcal{A}_h$-class	Parametrisation	Submersion
A_0	A_{0_2} A_{0_3} A_{0_4}	(t, t^2) (t, t^3) (t, t^4)	$y + x^2$ $y + x^3$ $y + x^4$
A_1	$A_{1_{22}}$ $A_{1_{23}}$ $A_{1_{24}}$ $A_{1_{33}}$	$(t, t^2); (s^2, s)$ $(t, t^2); (s^3, s)$ $(t, t^2); (s^4, s)$ $(t, t^3); (s^3, s)$	$y + x^2; x + y^2$ $y + x^2; x + y^3$ $y + x^2; x + y^4$ $y + x^3; x + y^3$
A_2	A_{2_3}	(t^2, t^3)	y
A_3	$A_{3_{22}}$ $A_{3_{23}}$	$(t, -t^2); (s, s^2)$ $(t, -t^2); (s, s^3)$	$y + ax^2; y + bx^2,$ $a, b \neq 0, -1, a \neq b$ $y + ax^2; y + x^3$ $a \neq 0, -1$
A_4	A_{4_4}	$(t^2, t^4 + t^5)$	$y + x^2$
A_5	$A_{5_{22}}$	$(t, t^2); (s, s^2 + s^3)$	$y + x^2; y + x^2$
D_4	$D_{4_{222}}$ $D_{4_{223}}$	$(t, t^2); (s^2, s); (u - u^2, u + u^2)$ $(t, t^2); (s^3, s); (u - u^2, u + u^2)$	$y + x^2; y - x + x^2; y + x + x^2$ $y + x^3; y - x + x^2; y + x + x^2$
D_5	$D_{5_{23}}$	$(t^2, t^3 + t^4); (s^2, s)$	$y + x^2; x$
D_6	$D_{6_{222}}$	$(t, -t^2); (s^2, s); (u, u^2 + u^3)$	$y + x^2; x + y^3;$ $x + ay^2, a \neq \pm 1$
$\tilde{E}_7$	$\tilde{E}_{7_{2222}}$	$(t, t^2 + t^3); (s^2 + cs^3, s);$ $(u - u^2, u + u^2); (v - v^2, \lambda v - v^2)$	$y + x^2; x + y^2$ $x - y + x^2; x - \lambda y + y^2$

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symbols for $\mathcal{A}_h$-classes is introduced in [9] and refers to the intersection number of each branch of the plane curve with its tangent line. For example, the notation $A_{1_{23}}$ means that the curve has an A_1-singularity and one of its branches has intersection number equal to 2 with its tangent line while the other branch has intersection number equal to 3 with its tangent line. It is shown in [9] that for generic space curves, the symbol of the $\mathcal{A}_h$-class of its projection determines the $\mathcal{A}_h$-class. This fact allows us to recover all the results in [9] on the $\mathcal{A}_h$-singularities of projections of generic space curves using Theorem 3.1. We give in the fourth column of Table 2 the associated submersions in Theorem 3.1 on the $\mathcal{A}$-model of the singularity of the projected curve. For multi-germs, we associate to each branch a submersion that captures the contact of the branch with its tangent line.

Remark 4.1 For mono-germs, we can recover all the $\mathcal{A}_h$-classification results in [9] using the classification in Theorem 3.1. Consider for example the ramphoid cusp given by $C : y^2 - x^5 = 0$ (A_4-singularity). There are three $\mathcal{A}_h$-classes in this $\mathcal{A}$-class ([9]), namely $(t^2, t^4 + t^5)$, $(t^2, t^5 + t^6)$ and (t^2, t^5). In Theorem 3.1, we have four germs of submersions on C. The fibre $x = 0$ of the germ x is transverse to C, so is not related to the $\mathcal{A}_h$-classification (which deals only with height functions along limiting normal directions). Consider the germ $y + x^2$ and set $Y = y + x^2$. Then $y = Y - x^2$ and substituting in C we obtain $(Y - x^2)^2 - x^5 = Y^2 - 2x^2Y + x^4 - x^5 = 0$. A parametrisation for this curve is precisely $(t^2, t^4 + t^5)$. Similarly, the submersion $y + x^3$ gives the A_h-class $(t^2, t^5 + t^6)$ and the submersion y gives the $\mathcal{A}_h$-class (t^2, t^5).

For multi-germs, some symbols determine the $\mathcal{A}_h$-class. This is the case for example at an A_1 and D_4 singularities. Then we can pair each branch of an $\mathcal{A}_h$-model curve with a submersion from Theorem 3.1. In general, we could have several $\mathcal{A}_h$-classes of a multi-germ singularity which cannot be distinguished by the intersection number of each branch with its tangent line. In those cases, the classification in Theorem 3.1 does not allow us to recover the results in [9].

5 The family of orthogonal projections

The family of orthogonal projections in $\mathbb{R}^3$ is given by

$$
\begin{array}{rccc}
\Pi : & \mathbb{R}^3 \times S^2 & \to & TS^2 \\
& (p, v) & \mapsto & (v, \Pi_v(p))
\end{array}
$$

where $\Pi_v(p) = p - (p \cdot v)v$. Given a space curve $\gamma : I \to \mathbb{R}^3$, where I is an open interval in $\mathbb{R}$, we denote by P the restriction of Π to γ. Thus, the family of orthogonal projections $P : I \times S^2 \to TS^2$ on γ is given by

$$
P(t, v) = (v, P_v(\gamma(t)))
$$

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with $P_v(t) = \Pi_v(\gamma(t)) = \gamma(t) - (\gamma(t) \cdot v)v$. The following is a statement of a result in [8] for the family of orthogonal projections. (In [8] is considered the 3-parameter family of projections from points in $\mathbb{R}^3$ to a plane.)

Theorem 5.1 ([8]) *There is a residual subset $\Omega \subset C^\infty(I, \mathbb{R}^3)$ such that if $\gamma \in \Omega$, then it is an embedding and for any $v \in S^2$, $\Pi_v \circ \gamma$ has only singularities of type $A_0, \ldots, A_5, D_4, D_5, D_6$ and $\tilde{E}_7$.*

The curves in Ω are called *projection-generic* ([8]). The following definition is given in [9]. (The condition (0) in Definition 5.2 is not stated in [9]. We add it here for completion.)

Definition 5.2 ([9], Definition 4.2) A space curve $\gamma \in C^\infty(I, \mathbb{R}^3)$ is said to be *generic* if it is projection-generic and satisfies the following conditions.

(0) If $\tau(t) = 0$ at some $t \in I$, then $\tau'(t) \neq 0$.
(1) Assume that the secant line l to γ at two points $\gamma(t_i), i = 1, 2$, is contained in the osculating planes $O(t_i)$, for any $i = 1, 2$. Then, $\tau(t_i) \neq 0$ for any $i = 1, 2$.
(2) Let l be a cross tangent to γ at $\gamma(t_i)$ for any $i = 1, 2$, tangent in $\gamma(t_1)$. Then $\gamma^{(4)}(t_1) \not\subset O(t_1)$.
(3) Let l be a trisecant line to γ at three points $\gamma(t_i)$, $i = 1, 2, 3$. If $l \subset O(t_1)$, then $\tau(t_1) \neq 0$ and $l \not\subset O(t_i)$, for any $i = 2, 3$.
(4) Let l be a trisecant line to γ at three points $\gamma(t_i)$, $i = 1, 2, 3$. If l is contained in a bitangent plane π to γ at two of these points, then π does not osculate at none of the three points and $\gamma^{(3)}(t_i)$ is not contained in the bitangent plane.
(5) Let l be a quadrisecant line to γ at four points $\gamma(t_i)$, $i = 1, 2, 3, 4$. Then, $l \not\subset O(t_i)$ and $l \parallel \gamma^{(3)}(t_i)$ at most in two points.

It is shown in [9] that the subset of generic curves is residual in $C^\infty(I, \mathbb{R}^3)$. It is also shown in [9] that given a generic space curve γ, for any $v \in S^2$, $P_v = \Pi_v \circ \gamma$ has only the $\mathcal{A}_h$-singularities in column 2 of Table 2. (Condition (0) in Definition 5.2 is added to make it apparent why the A_{0_5}-singularities are excluded. An A_{0_5}-singularity occurs at $t \in I$ if and only if $\tau(t) = \tau'(t) = 0$.)

Let w be a unit vector in $T_v S^2$, so w is given by $w \cdot v = 0$ and $w \cdot w = 1$. We denote by
$$\Delta = \{(v, w) \in S^2 \times S^2 \mid v \cdot w = 0\}.$$

Given $(v, w) \in \Delta$, the height function on the curve $P_v(t)$ along the vector w is given by
$$H_{v,w}(t) = P_v(t) \cdot w = (\gamma(t) - (\gamma(t) \cdot v)v) \cdot w = \gamma(t) \cdot w.$$

This is precisely the height function on the curve γ along the direction w. The family $H : I \times \Delta \to \mathbb{R}$ has parameters in Δ, which is a 3-dimensional manifold.

However, it is trivial along the parameter v. This is why the generic singularities that appear in the members of the family of height functions of a projected space curve are those of $\mathcal{R}_e$-codimension ≤ 2. (For example, this is why we can exclude the A_{0_5}-singularities and why we need the term t^4 in the generic parametrisation $(t^2, t^4 + t^5)$ of the ramphoid cusp.)

Given a generic space curve, the family P is an $\mathcal{A}_e$-versal unfolding of the singularities of the projections along any fixed direction $v_0 \in S^2$ (see [11] for terminology). The $\mathcal{A}$-bifurcation diagrams in $P_v(t)$ as v varies near v_0 are well known ([10, 14]). We redraw these diagrams and include in them the information about the flat geometry of $P_v(t)$.

We consider a modified family of projections which is affine equivalent to P, so the flat geometry of the projected curve is preserved. If the singularity is a local one, we write $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ and project along directions $(\sqrt{1 - v_2^2 - v_3^2}, v_2, v_3)$ near $(1, 0, 0)$ to the fixed plane $(0, v_2, v_3)$. (See the proof of Theorem 5.3 for the multi-local singularities.) The modified family of projections is given by

$$\tilde{P}: \quad I \times \mathbb{R}^2, 0 \quad \to \quad \mathbb{R}^2, 0$$
$$(t, (v_2, v_3)) \quad \mapsto \quad (\gamma_2(t) - \gamma_1(t)v_2, \gamma_3(t) - \gamma_1(t)v_3)$$

The $\mathcal{A}$-bifurcation set $Bif(\tilde{P}, \mathcal{A})$ of $\tilde{P}$ is the set of $(v_2, v_3) \in \mathbb{R}^2, 0$ for which $\tilde{P}_{(v_2, v_3)}(t) = \tilde{P}(t, (v_2, v_3))$ has a singularity of $\mathcal{A}_e$-codimension ≥ 1 at some t. We consider the modified family of height functions $\tilde{H} : \mathbb{R} \times \mathbb{R}^2 \times S^1, (0, (0, 0), w_0) \to \mathbb{R}$, given by $\tilde{H}(t, (v_2, v_3), \omega) = \tilde{P}(t, (v_2, v_3)) \cdot \omega$. We are interested in the inflections of the curve $\tilde{P}_{(v_2, v_3)}(t)$. These are captured by the contact of $\tilde{P}_{(v_2, v_3)}(t)$ with lines and occur when $h(t) = \tilde{H}(t, (v_2, v_3), \omega)$ has an A_2-singularity at some t for some $\omega \in S^1$, or equivalently, when $\kappa(t) = 0$, where κ is the curvature of $\tilde{P}_{(v_2, v_3)}(t)$. We define the $\mathcal{K}$-bifurcation set $Bif(\tilde{H}, \mathcal{K})$ of $\tilde{H}$ as the set of parameters (v_2, v_3) for which there exists $\omega \in S^1$ such that $\kappa(t) = \kappa'(t) = 0$ (i.e., h has an A_3-singularity) or for which there exist t, s, $t \neq s$, with $\tilde{P}_{(v_2, v_3)}(t) = \tilde{P}_{(v_2, v_3)}(s)$ and $\kappa(s) = 0$ (i.e., h has a multi-local singularity of type $A_0 A_2$). Finally, we define the $\mathcal{A}_h$-bifurcation set of $\tilde{P}$ as $Bif(\tilde{P}, \mathcal{A}_h) = Bif(\tilde{P}, \mathcal{A}) \cup Bif(\tilde{H}, \mathcal{K})$.

Theorem 5.3 *Let γ be a generic space curve γ (as in Definition 5.2). The $\mathcal{A}_h$-bifurcations in $\tilde{P}$ at a singularity with an empty $Bif(\tilde{H}, \mathcal{K})$ are the same as those in ([10, 14]). The $\mathcal{A}_h$-bifurcations in $\tilde{P}$ at a singularity with a non-empty $Bif(\tilde{H}, \mathcal{K})$ are as shown in* Figures 2–5.

Proof We deal with the cases where $Bif(\tilde{H}, \mathcal{K})$ is not empty. The cases in Figure 1 with empty $Bif(\tilde{H}, \mathcal{K})$ are: A_{0_2}, A_{0_3}, $A_{1_{22}}$, $A_{3_{22}}$, $D_{4_{222}}$, $D_{6_{222}}$, $\tilde{E}_{7_{2222}}$ (notation as in Figure 1 and Table 2). Out of the remaining cases, the codimension 1 singularities are the case A_{0_4} with a birth/vanishing of two A_{0_3}-singularities in the bifurcation, the case A_{2_3} with a vanishing of an $A_{1_{22}}$-singularity and a birth of two A_{0_3}-singularities
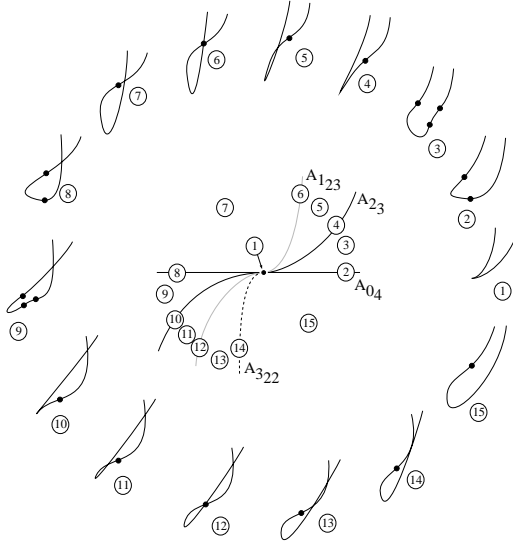
Figure 2: Bifurcations of the ramphoid cusp (the stratification of the parameter space in the middle). The dots represent the points of inflection.

in the bifurcation and the case $A_{1_{23}}$ with an $A_{1_{22}}$-singularity and an A_{0_3}-singularity on both sides of the bifurcation. We treat in some details the cases A_{4_4} and $A_{1_{24}}$ and state the results for the remaining cases.

A_{4_4}-singularity (the ramphoid cusp)

We make changes of coordinates in the source and affine changes of coordinates in $\mathbb{R}^3$ so that $j^5\gamma(t) = (\phi(t), t^2, t^4 + t^5)$, with $j^1\phi = t$. Then $j^5\tilde{P}_{(v_2,v_3)}(t) = (t^2 - \phi(t)v_2, t^4 + t^5 - \phi(t)v_3)$. The family $\tilde{P}$ is an $\mathcal{A}_e$-versal unfolding of the ramphoid cusp and the family $\tilde{H}$ is clearly an $\mathcal{K}_e$-versal unfolding of all the singularities of its members. The components of $Bif(\tilde{P}, \mathcal{A})$ are the following (and are diffeomorphic to their corresponding components of the bifurcation set of the model in [10, 14]):

– The A_{4_4}-stratum, which consists of the origin and where $\tilde{P}_{(0,0)}(t)$ has a ramphoid cusp.

– The A_{2_3} (cusp)-stratum: $\tilde{P}_{(v_2,v_3)}(t)$ has a cusp singularity. Now $\tilde{P}_{(v_2,v_3)}(t)$ is singular if and only if $(v_2, v_3) = (2t + h.o.t., (4t^3 + 5t^4)(1 + h.o.t.))$ (④ and ⑩ in Figure 2).

– The $A_{3_{22}}$-stratum: $\tilde{P}_{(v_2,v_3)}(t)$ has a multi-local singularity where two pieces of the curve have an ordinary tangency at a given point. We know from [10, 14] that this stratum is diffeomorphic to half a line. To find its parametrisation, we use the fact that having an $A_{3_{22}}$-singularity is equivalent to the curve γ having a bitangent plane. This means that there exists $w \in S^2$ such that $h(t, w) = \gamma(t) \cdot w$ has two

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singularities at the same level. That is, $h(t,w) - d = (t - t_1)^2(t - t_2)^2\tilde{h}(t,w)$ for some smooth function $\tilde{h}(t,w)$. The direction of projection is along $(a,b,c) = \gamma(t_2) - \gamma(t_1)$ and the $A_{3_{22}}$-stratum in the (v_2, v_3)-plane is given by $(\frac{b}{a}, \frac{c}{a})$. To obtain the initial terms of the parametrisation of the $A_{3_{22}}$-stratum, we take $w = (w_1, w_2, 1)$ and equate the coefficients of the 5-jets of $h(t,w) - d$ and $(t - t_1)^2(t - t_2)^2\tilde{h}(t,w)$ as functions in t. From that we get $t_2 = -t_1 - t_1^2 + h.o.t$ and this gives the initial terms of the $A_{3_{22}}$-stratum as $(v_2, v_3) = (-t^2 + h.o.t, -t^4 + h.o.t)$ (⑭ in Figure 2).

The components of $Bif(\tilde{H}, \mathcal{K})$ are the following:

– The A_{0_4}-stratum: the height function on $\tilde{P}_{(v_2,v_3)}(t)$ has an A_3-singularity (equivalently, $\tilde{P}_{(v_2,v_3)}(t)$ has a higher order inflection at some point t). This stratum is given by $v_3 = 0$ (② and ⑧ in Figure 2).

– The $A_{1_{23}}$-stratum: the height function on $\tilde{P}_{(v_2,v_3)}(t)$ has a multi-local singularity of type A_0A_2-singularity. This means that there exist s, t, $s \neq t$, such that $\tilde{P}_{(v_2,v_3)}(t) = \tilde{P}_{(v_2,v_3)}(s)$ and $\kappa(s) = 0$. We get thus the following system

$$\gamma_2(t) - \gamma_1(t)v_2 = \gamma_2(s) - \gamma_1(s)v_2, \tag{1}$$
$$\gamma_3(t) - \gamma_1(t)v_3 = \gamma_3(s) - \gamma_1(s)v_3, \tag{2}$$
$$(\gamma_2'(s) - \gamma_1'(s)v_2)(\gamma_3''(s) - \gamma_1''(s)v_3) - (\gamma_2''(s) - \gamma_1''(s)v_2)(\gamma_3'(s) - \gamma_1'(s)v_3) = 0. \tag{3}$$

Equations (1) and (2) give

$$v_2 = \frac{\gamma_2(t) - \gamma_2(s)}{\gamma_1(t) - \gamma_1(s)} \quad \text{and} \quad v_3 = \frac{\gamma_3(t) - \gamma_3(s)}{\gamma_1(t) - \gamma_1(s)} \tag{4}$$

while equation (3) simplifies to

$$\gamma_2'(s)\gamma_3''(s) - \gamma_2''(s)\gamma_3'(s) + (\gamma_1''(s)\gamma_3'(s) - \gamma_1'(s)\gamma_3''(s))v_2 + (\gamma_1'(s)\gamma_2''(s) - \gamma_1''(s)\gamma_2'(s))v_3 = 0. \tag{5}$$

We substitute in equation (5) v_2 and v_3 by their expressions from (4) and write the new equation in the form

$$F(s,t) = \frac{G(s,t)}{\gamma_1(t) - \gamma_1(s)} = 0,$$

where

$$\begin{aligned} G(s,t) &= (\gamma_2'(s)\gamma_3''(s) - \gamma_2''(s)\gamma_3'(s))(\gamma_1(t) - \gamma_1(s)) \\ &\quad - (\gamma_1'(s)\gamma_3''(s) - \gamma_1''(s)\gamma_3'(s))(\gamma_2(t) - \gamma_2(s)) \\ &\quad + (\gamma_1'(s)\gamma_2''(s) - \gamma_1''(s)\gamma_2'(s))(\gamma_3(t) - \gamma_3(s)), \end{aligned}$$

which can be written in a determinant form

$$G(s,t) = \begin{vmatrix} \gamma_1(t) - \gamma_1(s) & \gamma_2(t) - \gamma_2(s) & \gamma_3(t) - \gamma_3(s) \\ \gamma_1'(s) & \gamma_2'(s) & \gamma_3'(s) \\ \gamma_1''(s) & \gamma_2''(s) & \gamma_3''(s) \end{vmatrix}.$$

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(We thank Terry Wall for telling us that G is a determinant and for providing us with a shorter argument, which we reproduce below, for finding a parametrisation of the $A_{1_{23}}$-stratum. Wall also observed that it is no accident that we get a determinant. Its vanishing represents the condition that the vectors $\gamma(s) - \gamma(t), \gamma'(s)$ and $\gamma''(s)$ are coplanar, i.e., there is a plane through $\gamma(t)$ and having intersection number at least 3 with the curve at $\gamma(s)$: projecting from a point on the line joining $\gamma(t)$ and $\gamma(s)$ thus gives an inflexional tangent at the image of $\gamma(s)$ which coincides with the image of $\gamma(t)$.)

We expand $\gamma_i(t) - \gamma_i(s) = (t - s)\gamma_i'(s) + (t - s)^2 \gamma_i''(s)/2 + (t - s)^3 \gamma_i'''(s)/6 + h.o.t.$ Subtracting $(t - s)$ times the second row and $(t - s)^2/2$ times the third from the first, we see that $G(s, t)$ is divisible by $(t - s)^3$. Dividing the first row by this gives a row whose terms of least order are $(\gamma_1'''(0)/6 + (t + 3s)\gamma_1^{(4)}(0)/24, 0, t + 3s)$. The 1-jet of the new determinant is then $2(t + 3s)$. Thus, $G(s, t) = 0$ has a trivial solution $t = s$ and another solution which can be parametrised by $(s, t(s))$. The solution $s = t$ gives the cusp stratum and the solution $(s, t(s))$ yields a parametrisation of the $A_{1_{23}}$-stratum. We have $t(s) = -3s + h.o.t$ and this gives $v_2 = -2s + h.o.t$ and $v_3 = -20s^3 + h.o.t$, so $v_3 = \frac{5}{2}v_2^3 + h.o.t$ (⑥ and ⑫ in Figure 2).

We obtain a stratification of the parameter space (v_2, v_3) determined by $Bif(\tilde{P}, \mathcal{A}_h)$. On each stratum the projections have equivalent $\mathcal{A}_h$-singularities (see the central diagramme in Figure 2). We can now draw the bifurcations in the family of projected curves as the direction of projection varies locally in S^2; see Figure 2.

$A_{1_{24}}$-singularity

We suppose that the two pieces of curves are given by $\gamma_1(t) = (t, \gamma_{12}(t), \gamma_{13}(t))$ with $j^2\gamma_{12} = t^2$, $\gamma_{13}(0) = \gamma_{13}'(0) = 0$, and $\gamma_2(s) = (\gamma_{21}(s), s, \gamma_{23}(s))$ with $j^4\gamma_{21} = s^4$ and $\gamma_{23}(0) = 1$. The modified family of projections along the directions $(v_1, v_2, 1)$ to the (v_1, v_2)-plane is given by the bi-germ $\alpha; \beta$ with

$$\alpha(t) = (\alpha_1(t), \alpha_2(t)) = (t - \gamma_{13}(t)v_1, \gamma_{12}(t) - \gamma_{13}(t)v_2),$$
$$\beta(s) = (\beta_1(s), \beta_2(s)) = (\gamma_{21}(s) - \gamma_{23}(s)v_1, s - \gamma_{23}(s)v_2).$$

The family $\tilde{P} = \alpha; \beta$ is an $\mathcal{A}_e$-versal unfolding of the $A_{1_{24}}$-singularity at $(v_1, v_2) = (0, 0)$, and $H_{\tilde{\alpha}}$ and $H_{\tilde{\beta}}$ are clearly $\mathcal{K}_e$-versal unfoldings of all their singularities. The bifurcation set $Bif(\tilde{P}, \mathcal{A}_h) = Bif(\tilde{H}, \mathcal{K})$ as the component $Bif(\tilde{P}, \mathcal{A})$ is empty. Its 1-dimensional strata are the following.

– The A_{0_4}-stratum: we require the curvature κ_β of β and its derivative to vanish. We consider the map-germ $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ given by $F(s, v_1, v_2) = (\kappa_\beta, \kappa_\beta')(s, v_1, v_2)$. It has maximal rank at the origin if and only if $\gamma_{23}''(0) \neq 0$. This condition is equivalent to $\kappa_{\gamma_2}(0) \neq 0$, so is satisfied for generic space curves. Then it follows by the implicit functions theorem that the A_{0_4}-stratum is a smooth curve. It is not difficult to show that it is given by $v_1 = 0$ (② and ④ in Figure 3 left).
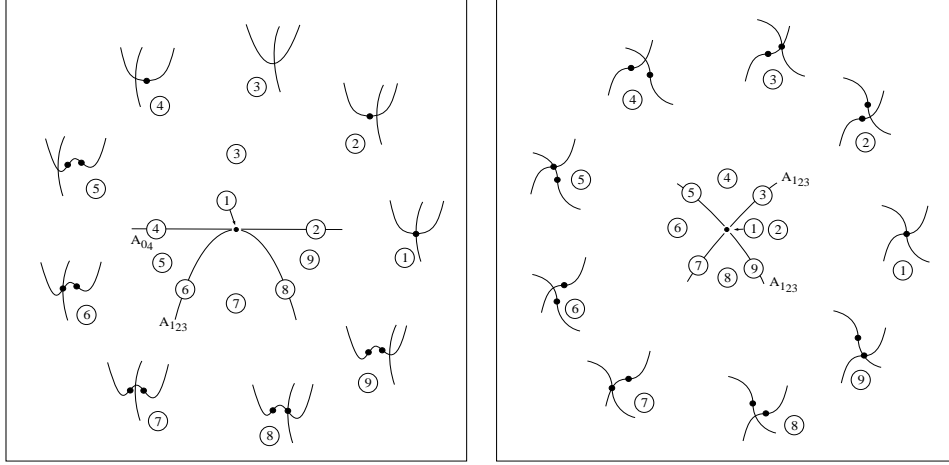
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Figure 3: Bifurcations of an $A_{1_{24}}$-singularity left and of an $A_{1_{33}}$-singularity right.

– The $A_{1_{23}}$-stratum: we require $\alpha_i(t) = \beta_i(s)$, $i = 1, 2$, and $\kappa_\beta(s) = 0$. We consider the map-germ $F : \mathbb{R}^4, 0 \to \mathbb{R}^3, 0$ given by $F(s, t, v_1, v_2) = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \kappa_\beta)(s, t, v_1, v_2)$. It has maximal rank at the origin if and only if $\gamma_{23}''(0) \neq 0$ (which is true for generic space curves; see above). Then it follows by the implicit functions theorem that the $A_{1_{23}}$-stratum is a smooth curve given by $v_1 = \frac{12}{\gamma_{23}''(0)} v_2^2 + h.o.t$ (⑥ and ⑧ in Figure 3 left). The bifurcations in the family of projected curves as the direction of projection varies locally in S^2 are drawn in Figure 3 left.

$A_{1_{33}}$-*singularity*

We use the same setting as for the $A_{1_{24}}$-singularity and take $\gamma_1(t) = (t, \gamma_{12}(t), \gamma_{13}(t))$ with $j^3\gamma_{12} = t^3$, $\gamma_{13}(0) = \gamma_{13}'(0) = 0$ and $\gamma_2(s) = (\gamma_{21}(s), s, \gamma_{23}(s))$ with $j^3\gamma_{21} = s^3$ and $\gamma_{23}(0) = 1$. Then $Bif(\tilde{P}, \mathcal{A}_h) = Bif(\tilde{H}, \mathcal{K})$ and consists of the $A_{1_{23}}$-stratum. For generic space curves, this is given by two smooth and transverse curves $v_2 = \frac{1}{6}\gamma_{23}''(0)v_1 + h.o.t.$ and $v_1 = -\frac{1}{6}\gamma_{23}''(0)v_2 + h.o.t.$; see Figure 3 right for the bifurcations. (As pointed out above, $\gamma_{23}''(0) \neq 0$ means the curve $\kappa_{\gamma_2}(0) \neq 0$.)

$A_{3_{23}}$-*singularity*

We take $\gamma_1(t) = (t, \gamma_{12}(t), \gamma_{13}(t))$ with $j^2\gamma_{12} = t^2$ and $\gamma_{13}(0) = \gamma_{13}'(0) = 0$. We also take $\gamma_2(s) = (s, \gamma_{22}(s), \gamma_{23}(s))$ with $j^3\gamma_{22} = s^3$ and $\gamma_{23}(0) = 1$. Then $Bif(\tilde{P}, \mathcal{A})$ consists of the origin and the $A_{3_{22}}$-stratum which is given by $v_2 = v_1^3 + h.o.t.$ (③ and ⑤ in Figure 4 left). The $Bif(\tilde{H}, \mathcal{K})$ consists of the $A_{1_{23}}$-stratum which is given by $v_2 = -v_1^2 + h.o.t.$ (⑦ and ⑨ in Figure 4 left).

$A_{5_{22}}$-*singularity*

We take $\gamma_1(t) = (t, \gamma_{12}(t), \gamma_{13}(t))$ with $j^2\gamma_{12} = t^2$, $\gamma_{13}(0) = \gamma_{13}'(0) = 0$ and $\gamma_2(s) = (s, \gamma_{22}(s), \gamma_{23}(s))$ with $j^3\gamma_{22} = s^2 + s^3$ and $\gamma_{23}(0) = 1$. Then $Bif(\tilde{P}, \mathcal{A})$ consists of the
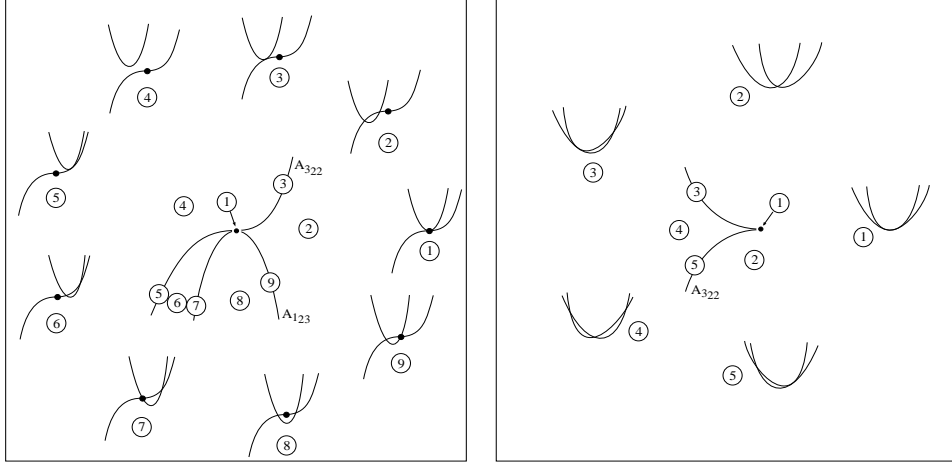
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Figure 4: Bifurcations of an $A_{3_{23}}$-singularity left and $A_{5_{22}}$-singularity right.

origin and the $A_{3_{22}}$-stratum which is given by $(v_1, v_2) = (-\frac{3}{2}s^2 + h.o.t., -2s^3 + h.o.t.)$ (③ and ⑤ in Figure 4 right). As for $Bif(\tilde{H}, \mathcal{K})$, the curvatures of both α and β do not vanish near the origin, so this component of the bifurcation set is empty. See Figure 4 right and compare with that in [10].

$D_{5_{23}}$-singularity

We take $\gamma_1(t) = (\gamma_{11}(t), \gamma_{12}(t), \gamma_{13}(t))$ with $j^2\gamma_{11} = t^2$, $j^4\gamma_{12} = t^3 + t^4$, $\gamma_{13}(0) = \gamma'_{13}(0) = 1$ and $\gamma_2(s) = (\gamma_{21}(s), s, \gamma_{23}(s))$ with $j^2\gamma_{21} = s^2$, $\gamma_{23}(0) = \gamma'_{23}(0) = 0$. Then we have a $D_{5_{23}}$-singularity at $t = s = 0$ if and only if $\gamma'''_{11}(0) \neq 4$. The $Bif(\tilde{P}, \mathcal{A})$ consists of the origin, the A_{2_3}-stratum which is given by $v_2 = \frac{3}{4}v_1^2 + h.o.t$ (② and ⑧ in Figure 5 left), the $A_{3_{22}}$-stratum which is given by $v_1 = -v_2^2 + h.o.t$ (⑥ and ⑩ in Figure 5 left), and the the $D_{4_{222}}$-stratum which is given by $v_1 = v_2 + h.o.t$ with $v_2 > 0$ (④ in Figure 5 left). The $Bif(\tilde{H}, \mathcal{K})$ consists of the $A_{1_{23}}$-stratum given by $(v_1, v_2) = (t^2 + (\frac{1}{6}\gamma'''_{11}(0) - 1)t^3 + h.o.t., -3t^2 + (\frac{3}{2}\gamma'''_{11}(0) - 5)t^3 + h.o.t.)$ which is a cusp as $\gamma'''_{11}(0) \neq 4$ (⑫ and ⑭ in Figure 5 left).

$D_{4_{223}}$-singularity

Here we have a tri-germ and we take the three pieces of curves given by $\gamma_1(t) = (t, \gamma_{12}(t), \gamma_{13}(t))$, $\gamma_2(s) = (\gamma_{21}(s), s, \gamma_{23}(s))$ and $\gamma_3(u) = (\gamma_{31}(u), \gamma_{32}(u), \gamma_{33}(u))$ with the following setting: $j^2\gamma_{12} = t^2$, $\gamma_{13}(0) = \gamma'_{13}(0) = 0$, $j^3\gamma_{21} = s^3$, $\gamma_{23}(0) = 1$, $\gamma'_{23}(0) = 0$, $j^2\gamma_{31} = u - u^2$, $j^2\gamma_{32} = u + u^2$, $\gamma_{33}(0) = -1$, $\gamma'_{33}(0) = 0$. The modified family of projections along the directions $(v_1, v_2, 1)$ to the (v_1, v_2)-plane is given by $\alpha; \beta; \delta$ with

$$\alpha(t) = (\alpha_1(t), \alpha_2(t)) = (t - \gamma_{13}(t)v_1, \gamma_{12}(t) - \gamma_{13}(t)v_2),$$
$$\beta(s) = (\beta_1(s), \beta_2(s)) = (\gamma_{21}(s) - \gamma_{23}(s)v_1, s - \gamma_{23}(s)v_2),$$
$$\delta(u) = (\delta_1(u), \delta_2(u)) = (\gamma_{31}(u) - \gamma_{33}(u)v_1, \gamma_{32}(u) - \gamma_{33}(u)v_2).$$
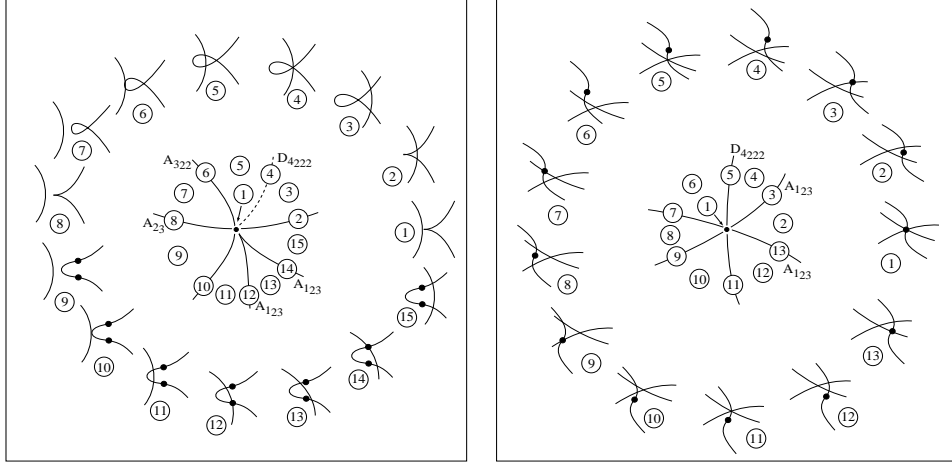
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Figure 5: Bifurcations of a $D_{5_{23}}$-singularity left and of a $D_{4_{223}}$-singularity right.

The $A_{1_{23}}$-stratum consists of two branches. It follows by the implicit functions theorem that one is given by $v_2 = \frac{1}{6}\gamma_{23}''(0)v_1 + h.o.t$ (③ and ⑨ in Figure 5 right) and the other is given by $v_2 = (1 + \frac{1}{12}\gamma_{23}''(0))v_1 + h.o.t$ (⑦ and ⑬ in Figure 5 right). The $D_{4_{222}}$-stratum is also a smooth curve given by $v_2 = 2v_1 + h.o.t$ (⑤ and ⑪ in Figure 5 right). The three strata meet transversally at the origin if and only if $\gamma_{23}''(0) \neq 12$ (i.e., if and only if $\kappa_{\gamma_2}(0) \neq 6$). This is an additional condition on the space curve to those in Definition 5.2. It can be shown using the standard transversality techniques that the set of space curves in Definition 5.2 with this extra condition at a $D_{4_{223}}$-singularity of a projection is also residual in $C^\infty(I, \mathbb{R}^3)$. □

Remark 5.4 The results in Theorem 5.3 (and Theorem 6.1) are specific to the family of projections of space curves. Ideally, we would like to have a theory of versal deformations that captures both the $\mathcal{A}$ and $\mathcal{A}_h$-singularities of plane curves and state that the family of projections of space curves is versal in general. This is an open question.

6 Bifurcations in the dual of the projected curve

Let α be a plane curve and consider the family of height functions $h : I \times S^1 \to \mathbb{R}$, given by $h(t, u) = \alpha(t) \cdot u$. The discriminant of the family h is

$$D := \{(u, h(t, u)) \in S^1 \times \mathbb{R} \mid \frac{\partial h}{\partial t}(t, u) = 0\}$$

and can be viewed as the dual of the curve α, see for example [4]. We consider the dual of the (modified) projection $\tilde{P}_v(t)$ of the space curve γ and the way it bifurcates

as the direction of projection varies locally in S^2. We consider the modified family of height functions given by $\tilde{H}(t, (v, \omega)) = \tilde{P}_v(t) \cdot \omega$.

For v fixed, the discriminant of the family $\tilde{H}_v(t, \omega) = \tilde{P}_v(t) \cdot \omega$ is the dual of the projected curve $\tilde{P}_v(t)$. It is also the fibre $\pi^{-1}(v)$ of the projection $\pi : D(\tilde{H}) \subset S^1 \times \mathbb{R} \times S^2 \to S^2$ of the discriminant of the family $\tilde{H}$. The family $\tilde{H}$ is a versal unfolding of the singularities of $\mathcal{K}_e$-codimension ≤ 2 which appear in its members. We thus have models, up-to diffeomorphisms, of the discriminant of $\tilde{H}$. A natural attempt to obtain models of the dual of the curve $\tilde{P}_v(t)$ is to study sections of $D(\tilde{H})$. However, the map π is degenerate (of infinite codimension; more details are given below). We proceed by analysing separately each case in Figure 1.

The $\mathcal{A}_h$-class of the dual of $\tilde{P}_v(t)$ with v fixed can be obtained using the results in [15]. It follows from [15] that A_{0_2}, $A_{3_{22}}$, A_{4_4}, and $A_{5_{22}}$ are self-dual while we have dual pairs A_{0_3} and A_{2_3}, A_{0_4} and E_{6_4} (a swallowtail), $A_{3_{23}}$ and $E_{7_{23}}$ (a tangency between a cusp and a regular branch). Thus, the dual of an A_{0_2} is a smooth curve, the dual of an A_{0_3} is a cusp, the dual of an A_{0_4} is a swallowtail and undergoes swallowtail transitions. The multi-germs with transverse branches dualise to disjoint curves (this includes all the $A_{1_{ij}}$-singularities, the $D_{4_{222}}$, $D_{4_{223}}$ and $\tilde{E}_{7_{2222}}$ singularities in Figure 1). For the $A_{3_{22}}$ and $A_{5_{22}}$-singularities, the dual curves have the same contact as the curves themselves ([3]). It is not hard to show that we obtain the same bifurcations as those of the original bi-germ (for the $A_{3_{22}}$ see for Figure 4 left around the strata ③ and ⑤, and for the $A_{5_{22}}$-singularity see Figure 4 right). The $D_{6_{222}}$-singularity dualises to a disjoint union of a smooth branch and two tangential branches (i.e., an $A_{3_{22}}$-singularity). Therefore, we need to consider only the bifurcations in the dual of $\tilde{P}_v(t)$ when $\tilde{P}_v(t)$ is in one of the following $\mathcal{A}_h$-class in Figure 1: A_{2_3}, A_{4_4} and $A_{3_{23}}$.

Theorem 6.1 *The bifurcations in the duals of the curves $\tilde{P}_v(t)$ as v varies locally in S^2 are as shown in Figure 6 at an A_{2_3}-singularity (cusp), Figure 7 left at an A_{4_4}-singularity (ramphoid cusp), and Figure 7 right at an $A_{3_{23}}$-singularity.*

Proof   *A_{2_3}-singularity (cusp)*

We take $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, with $j^1\gamma_1 = t$, $j^2\gamma_2 = t^2$ and $j^2\gamma_3 = t^3$. Then

$$\tilde{H}(t, v_2, v_3, \omega) = (\gamma_2(t) - \gamma_1(t)v_2)\omega_1 + (\gamma_3(t) - \gamma_1(t)v_3)\omega_2$$

where $\omega = (\omega_1, \omega_2)$. The function $\tilde{H}_{(0,0)}(t, \omega)$ is singular for any $\omega \in S^1$. Therefore $S^1 \times \{0\}$ is part of the discriminant $\tilde{H}_{(0,0)}(t, \omega)$. The singularity is of type A_1 if $\omega_1 \neq 0$ and of type A_2 otherwise. Near $\omega = (0, 1)$, we take a chart $(w, 1)$ of S^1 and consider $\tilde{H}(t, w)_{(v_2, v_3)} = (\gamma_2(t) - \gamma_1(t)v_2)w + (\gamma_3(t) - \gamma_1(t)v_3)$. At $(v_2, v_3) = 0$,

$$\tilde{H}'(t, w)_{(0,0)} = \gamma_2'(t)w + \gamma_3'(t) = t(2w + 3t + h.o.t.).$$

The solution $t = 0$ of $\tilde{H}'(t, w)_{(0,0)} = 0$ gives the component $S^1 \times \{0\}$ of the dual of $\tilde{P}_v(t)$ and the solution $2w + 3t + h.o.t. = 0$ gives a smooth curve parametrised by
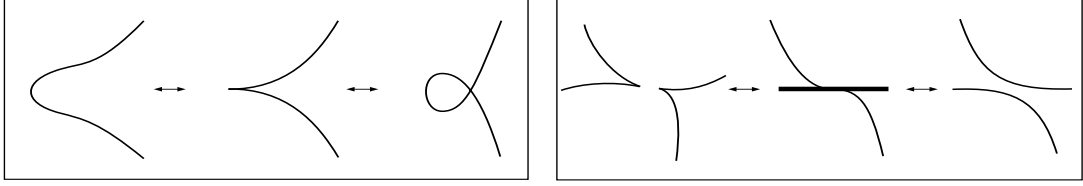
Figure 6: Bifurcation of the cusp singularity (left) and of its dual (right).

$(-3/2t + h.o.t., -1/2t^3 + h.o.t.)$, where we identify locally S^1 with the real line (see Figure 6 right, central figure).

The A_2-singularity of $\tilde{H}_{(0,0)}(t,\omega)$ at $\omega = (0,1)$ is of codimension 1 and the family $\tilde{H}(t,0,v_3,w)$ is a versal deformation of this singularity, so its discriminant is a cuspidal-edge. However the sections of this cuspidal-edge with planes $v_3 = constant$ are not generic as the plane $v_3 = 0$ contains the singular set of the cuspidal-edge. We have

$$\tilde{H}(t,0,v_3,w) = \gamma_3(t) + \gamma_2(t)w - \gamma_1(t)v_3,$$

so $\tilde{H}'(t,0,v_3,w) = \gamma_3'(t) + \gamma_2'(t)w - \gamma_1'(t)v_3 = 0$ gives $w = w(t) = -\frac{1}{\gamma_2'(t)}(\gamma_3'(t) - \gamma_1'(t)v_3)$. We then get a parametrisation of the dual curve in the chart $(w,1)$ of S^1 in the form

$$\left(-\frac{1}{\gamma_2'(t)}(\gamma_3'(t) - \gamma_1'(t)v_3), \gamma_3(t) - \frac{\gamma_2(t)}{\gamma_2'(t)}(\gamma_3'(t) - \gamma_1'(t)v_3) - \gamma_1(t)v_3\right).$$

This is singular if and only if $w'(t) = 0$. This equation has two solutions if $v_3 < 0$ and none if $v_3 > 0$. The singularities of the dual are of cusp type, i.e., are $\mathcal{A}$-equivalent to (t^2, t^3). We observe that the dual curve in the above chart goes to infinity as t tends to 0. We sketch in Figure 6 (right) the bifurcations in the dual curve in an affine chart of $S^1 \times \mathbb{R}$ and represent S^1 in a thick curve.

A_{4_4}-singularity (ramphoid cusp)

We take, as in the proof of Theorem 5.3, $j^5\gamma(t) = (\phi(t), t^2, t^4 + t^5)$ with $j^1\phi = t$. At $\omega = (0,1)$, the height function on the projected curve has an A_3-singularity. The dual of $\tilde{P}_{(0,0)}$ consists of $S^1 \times \{0\}$ (drawn locally in thick in ① Figure 7) together with another curve (its "proper" dual) given by $(-2t^2 - \frac{5}{2}t^3 + h.o.t, -t^4 - \frac{3}{4}t^5 + h.o.t)$. This has a singularity $\mathcal{A}$-equivalent to (t^2, t^5). (We observe that the proper dual curve of a ramphoid cusp which is $\mathcal{A}_h$-equivalent to (t^2, t^5) has a singularity $\mathcal{A}$-equivalent to (t^3, t^5), i.e., an E_8-singularity, [15].) We obtain a stratification of the (v_2, v_3)-plane determined by the following codimension ≥ 1 strata which are the same curves as for their corresponding singularities of the projections: the origin (the A_{4_4}-stratum for $\tilde{P}$); the A_3-stratum (A_{0_4}-stratum for $\tilde{P}$, ② and ⑥ in Figure 7 left) where the height function has an A_3-singularity; the A_1A_1-stratum ($A_{3_{22}}$-stratum for $\tilde{P}$, ⑩ in Figure 7
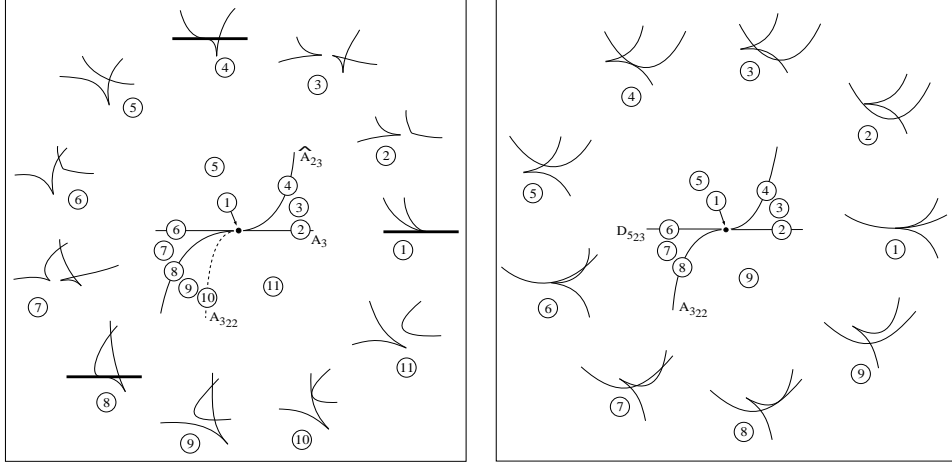
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Figure 7: Bifurcation of the dual of an A_{4_4} left and of an $A_{3_{23}}$ right.

left); the dual of an A_{2_3} which we denote by $\hat{A}_{2_3}$ (④ and ⑧ in Figure 7 left). We can prove that the family of height functions is a versal deformation of the singularities on the codimension 1 strata, so we have the standard pictures of the bifurcations of the discriminant as we cross these strata. For instance, we have the swallowtail transitions as we cross the A_3-stratum (② and ⑥ in Figure 7 left). We can also show that we have another singularity (cusp) on the dual curve on the A_3-stratum, so we have a maximum of three cusps that can appear in the bifurcation of the dual of a ramphoid cusp (Figure 7 left).

$A_{3_{23}}$-singularity

With the same setting as in the proof of Theorem 5.3 the families of height functions on these curves are given by $h_1(t, v_2, v_3, w) = w\alpha_1(t) + \alpha_2(t)$ and $h_2(t, v_2, v_3, w) = w\beta_1(s) + \beta_2(s)$. Now $\partial h_1/\partial t = 0$ gives $w = -\alpha_2'(t)/\alpha_1'(t) = (\gamma_{12}'(t) - \gamma_{13}'(t)v_2)/(1 - \gamma_{13}'(t)v_1)$, and similarly $\partial h_2/\partial t = 0$ gives $w = -\beta_2'(t)/\beta_1'(t) = (\gamma_{22}'(s) - \gamma_{23}'(s)v_2)/(1 - \gamma_{23}'(s)v_1)$, so the dual curves are parametrised by

$$\hat{\alpha}(t) = (w_\alpha(t), w_\alpha(t)\alpha_1(t) + \alpha_2(t)), \text{ with } w_\alpha(t) = -\frac{\alpha_2'(t)}{\alpha_1'(t)},$$

$$\hat{\beta}(s) = (w_\beta(s), w_\beta(s)\beta_1(s) + \beta_2(s)), \text{ with } w_\beta(s) = -\frac{\beta_2'(s)}{\beta_1'(s)}.$$

At $v_1 = v_2 = 0$, we get a bi-germ $(-2t+h.o.t., -t^2+h.o.t.); (-3s^2+h.o.t., -2s^3+h.o.t.)$, which is a cusp with a smooth curve tangent to the cusp (i.e., an E_7-singularity). We expect a transverse intersection of the two curves on some stratum in $S^1 \times \mathbb{R}$ (i.e., a $D_{5_{23}}$-singularity). This happens when $\hat{\alpha}(t) = \hat{\beta}(s) = 0$ and $\hat{\beta}'(s) = 0$. Now $\hat{\beta}'(s) = 0$ implies $w_\beta'(s) = \beta_2'(s) = 0$, which in turn implies that $v_2 = 0$. We can check that the

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two curves $\hat{\alpha}$ and $\hat{\beta}$ do indeed intersect at the cusp of $\hat{\beta}$ when $v_2 = 0$. The curves α and β have ordinary tangency along the $A_{3_{22}}$-stratum. As the dual curves have the same contact ([3]), the $A_{3_{22}}$-stratum is the same for both the curves and their duals. The bifurcations in the dual curves as (v_2, v_3) varies near the origin are given in Figure 7 right. $\square$

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