On pairs of regular foliations in $\mathbb{R}^3$ and singularities of map-germs

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Abstract

We study germs of pairs of codimension one regular foliations in $\mathbb{R}^3$. We show that the discriminant of the pair determines the topological type of the pair. We also consider various classifications of the singularities of the discriminant.

1 Introduction

We study in this paper germs (at the origin) of pairs of codimension one regular foliations in $\mathbb{R}^3$. We can assume that the foliations are the leaves of germs of differential 1-forms ω and η. We seek to obtain local models of the pairs (ω, η) under two natural equivalence relations. We say that two pairs (ω_i, η_i), $i = 1, 2$, are smoothly (resp. topologically) equivalent if there exists a germ of a diffeomorphism (resp. homeomorphism) of $\mathbb{R}^3, 0$ that sends the leaves of ω_1 to those of ω_2 and the leaves of η_1 to those of η_2. The smooth classification is treated in §3 and is related to the classification of certain divergent diagrams. This means that the models obtained in §3 are up to formal equivalence (the diffeomorphism above is replaced by an invertible formal power series).

The topological classification is dealt with in §4. An important feature of the pair (ω, η) is its discriminant $D(\omega, \eta)$ which is the locus of points where the 1-form ω is a multiple of η. Alternatively, $D(\omega, \eta)$ is the locus of points where the 2-form $\omega \wedge \eta = 0$. This is generically a germ of a space curve. We show in §4 (Theorem 4.1) that the discriminant determines the local topological type of the pair (ω, η), and obtain a complete list of discrete topological models. More precisely, if the discriminant is transverse away from the origin to the pair of foliations, then the topological type of the pair is determined by the number of branches of the discriminant in each half region delimited by the leaf of ω (or η) through the origin, provided this number does not exceed two. If there are three or more branches of the discriminant in one of the half regions, then there is no discrete topological model of the pair (Remark 4.2). This is a generalisation of the result in [13] for pairs of germs of regular foliations in the plane.

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Because the discriminant plays a key role in the topological classification, it is of interest to analyse its singularities. For this we require some notation. We can set, in the coordinates system (x, y, z), $w = df$ and $\eta = dz$, where $f : \mathbb{R}^3, 0 \to \mathbb{R}, 0$ is a germ of a smooth function. Then $D(\omega, \eta)$ is the zero fibre of the map-germ $F_{\omega,\eta} : \mathbb{R}^3, 0 \to \mathbb{R}^2$ given by $F_{\omega,\eta}(x, y, z) = (f_x(x, y, z), f_y(x, y, z))$.

Let $\mathcal{E}_n$ be the local ring of germs of functions $\mathbb{R}^n, 0 \to \mathbb{R}$ and m_n its maximal ideal (which is the subset of germs that vanish at the origin). Denote by $\mathcal{E}(n, p)$ the p-tuples of elements in $\mathcal{E}_n$. Let $\mathcal{A} = \mathcal{R} \times \mathcal{L} = Diff(\mathbb{R}^n, 0) \times Diff(\mathbb{R}^p, 0)$ denotes the group of right-left equivalence which acts smoothly on $m_n.\mathcal{E}(n, p)$ by $(h, k).G = k \circ G \circ h^{-1}$. We have another group of interest, namely the contact group $\mathcal{K}$. The group $\mathcal{K}$ is the set of germs of diffeomorphisms $H = (h, H_1) \in Diff(\mathbb{R}^{n+p}, 0)$, with $h \in Diff(\mathbb{R}^n, 0)$. Then H acts on $m_n.\mathcal{E}(n, p)$ as follows: $G = H.F$ if and only if $H(x, F(x)) = (h(x), G(h(x)))$ (this means that the graphs of F and G are diffeomorphic).

It is important to observe here that the action of the group $\mathcal{K}$ is a natural one to use when one seeks to understand the singularities of the zero fibres of germs in $m_n\mathcal{E}(n, p)$. Indeed, if two germs are $\mathcal{K}$-equivalent, then their zero fibres are diffeomorphic. The action of the group $\mathcal{A}$ is finner than that of $\mathcal{K}$. If two germs F and G are $\mathcal{A}$-equivalent, that is $G = k \circ F \circ h^{-1}$ for some $(h, k) \in \mathcal{A}$, then the fibres $G^{-1}(c)$ and $F^{-1}(k^{-1}(c))$ are diffeomorphic, for any c close to $0 \in \mathbb{R}^p$. So the group $\mathcal{A}$ preserves also the smooth structure of nearby fibres to the zero fibre.

We carry out in §5 various classifications of map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$. In §5.1 we give a brief description of the singularities of the discriminant via an action on families of matrices. When at least one foliation is regular, the action reduces to that of the contact group $\mathcal{K}$ on $m_3.\mathcal{E}(3, 2)$. However, when both foliations are singular at the origin, the bifurcations in the discriminant curve in generic families of pairs of foliations are best described using the family of matrices framework. (For related papers see [3, 6], and [15, 16] for applications to implicit differential equations.)

As pointed out above, the key ingredient in the topological classification of the pairs (df, dz) is the number of branches of the discriminant in each semi-space delimited by the plane $z = 0$ (the leaf of dz through the origin). It is therefore natural to seek a classification of the singularities of map-germs $F : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ under an action that preserves the smooth type of the zero fibre of F as well as the leaf $z = 0$ of dz. This action is that of the subgroup $\mathcal{K}_V$ of $\mathcal{K}$, where the changes of coordinates in the source preserve the variety V given by $z = 0$. Clearly, such action preserves the number of branches, in each semi-space $z > 0$ and $z < 0$, of the zero fibre of any map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$. The classification of the $\mathcal{K}_V$-simple map-germs (see Definition 2.3) is carried out in §5.2.

We also classify the simple singularities of the discriminant under an action that preserves the foliation of dz. Let $\mathcal{K}^*$ be the subgroup of $\mathcal{K}$ (and indeed of $\mathcal{K}_V$) where the changes of coordinates in the source preserve the horizontal planes (i.e., the foliation defined by dz). In §5.3 we list the $\mathcal{K}^*$-simple singularities of map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ and show that these coincide with the $\mathcal{K}_V$-simple singularities. The action $\mathcal{K}^*$ does not only preserves the number of branches in each semi-space $z > 0$ and $z < 0$ of the zero fibre of any map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$, but also the contact of the zero fibre with the horizontal planes (see §5.3).

Normally, in classification works, listings of simple orbits or of orbits of certain codimensions are sought. We choose to list here the simple orbits of the actions under consideration as our aim is to provide examples of the singularities of the discriminant $D(\omega, \eta)$ and of topologically determined pairs (ω, η).

The paper is organised as follows. In §2 we give some preliminaries of concepts from singularity theory. In §3 we classify pairs of regular foliations up to formal equivalence and give a complete topological classification in §4. We deal with the various classifications of the singularities of the discriminant in §5.

2 Preliminaries

As highlighted in the introduction, the discriminant of the pair of germs of foliations (ω, η) in $\mathbb{R}^3$ plays a key role in the topological classification of the pair. When not empty, the discriminant is the zero fibre of a map-germ $F_{\omega,\eta} : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$, so we consider various classifications of the singularities of map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$. For this, we need some standard notation from singularity theory (see [2, 17] for survey articles). We present below some concept using the group $\mathcal{A}$, but these concepts are also valid for the other groups used in the paper.

Given a map-germ $F \in m_n.\mathcal{E}(n, p)$, θ_F denotes the set of germs of vector fields along F (these are sections of the pull-back of the tangent bundle of the target manifold). We set $\theta_n = \theta_{id_{\mathbb{R}^n,0}}$ and $\theta_p = \theta_{id_{\mathbb{R}^p,0}}$, where $id_{\mathbb{R}^n,0}$ and $id_{\mathbb{R}^p,0}$ denote the germs of the identity maps on $\mathbb{R}^n, 0$ and $\mathbb{R}^p, 0$ respectively. One can define the homomorphisms $tF : \theta_n \to \theta_p$, with $tF(\psi) = DF.\psi$, and $wF : \theta_n \to \theta_p$, with $wF(\phi) = \phi \circ F$.

The action of the group $\mathcal{A}$ on $m_n.\mathcal{E}(n, p)$ is described in the introduction. The tangent space to the $\mathcal{A}$-orbit of F at the germ F is given by

$$\begin{array}{rcl} L\mathcal{A}(F) & = & tF(m_n.\theta_n) + wF(m_p.\theta_p) \\ & = & m_n.\{F_{x_1}, \ldots, F_{x_n}\} + F^*(m_p).\{e_1, \ldots, e_p\}, \end{array}$$

where F_{x_i} denotes partial derivatives with respect to x_i $(i = 1, \ldots, n)$, $e_1, \ldots, e_p$ the standard basis vectors of $\mathbb{R}^p$ considered as elements of $\mathcal{E}(n, p)$, and $F^*(m_p)$ the pull-back of the maximal ideal in $\mathcal{E}_p$.

The extended tangent space to the $\mathcal{A}$-orbit of F at the germ F is given by

$$\begin{array}{rcl} L_e\mathcal{A}(F) & = & tF(\theta_n) + wF(\theta_p) \\ & = & \mathcal{E}_n.\{F_{x_1}, \ldots, F_{x_n}\} + F^*(\mathcal{E}_p).\{e_1, \ldots, e_p\}. \end{array}$$

The codimension of the orbit of F is given by

$$d(F, \mathcal{A}) = \dim_{\mathbb{R}}(m_n.\mathcal{E}(n, p)/L\mathcal{A}(F))$$

and the codimension of the extended orbit is

$$d_e(F, \mathcal{A}) = \dim_{\mathbb{R}}(\mathcal{E}(n, p)/L_e\mathcal{A}(F)) .$$

Let $k \geq 1$ be an integer. We denote by $J^k(n, p)$ the space of kth order Taylor expansions without constant terms of elements of $\mathcal{E}(n, p)$ and write j^kF for the k-jet of F. A germ F

is said to be $k-\mathcal{A}$-*determined* if any G with $j^kG = j^kF$ is $\mathcal{A}$-equivalent to F (notation: $G \sim F$). The k-jet of F is then called a sufficient jet. (See for example [2, 5, 17] for finite determinacy criteria.)

The classification (i.e., the listing of representatives of the orbits) of finitely determined germs is carried out inductively on the jet level. The method used here is that of the complete transversal ([4]) together with Mather's Lemma ([12]), given below, where $\mathcal{A}_1$ denotes the normal subgroup of $\mathcal{A}$ whose elements have 1-jets at 0 equal to the identity.

Proposition 2.1 (Complete transversal, [4]) *Let G be a k-jet in $J^k(n,p)$, and let T be a vector subspace of the set $H^{k+1}(n,p)$ of homogeneous jets of degree $k+1$, such that*

$$H^{k+1}(n,p) \subset T + L(J^{k+1}\mathcal{A}_1)(G)\,.$$

Then any $(k+1)$-jet $j^{k+1}F$ with $j^kG = j^kF$ is $J^{k+1}\mathcal{A}_1$-equivalent to $G+t$ for some $t \in T$. (The vector subspace T is called the complete $(k+1)$-transversal of G.)

Lemma 2.2 (Mather's Lemma, [12]) *Let $\mathcal{G}$ be a Lie group acting smoothly on a finite dimensional manifold X. Let V be a connected submanifold of X. Then V is contained in a single orbit of $\mathcal{G}$ if and only if*

1. *for each $x \in V$, $T_xV \subset T_x\mathcal{G}(x) = L\mathcal{G}(x)$;*
2. $\dim T_x\mathcal{G}(x)$ *is constant for all $x \in V$.*

The notion of simple germs is defined in [1] as follows.

Definition 2.3 ([1]) *Let X be a manifold and $\mathcal{G}$ a Lie group acting on X. The modality of a point $x \in X$ under the action of $\mathcal{G}$ on X is the least number m such that a sufficiently small neighbourhood of x may be covered by a finite number of m-parameter families of orbits. The point x is said to be simple if its modality is 0, that is, a sufficiently small neighbourhood intersects only a finite number of orbits. The modality of a finitely determined map-germ is the modality of a sufficient jet in the jet-space under the action of the jet-group.*

The results on finite determinacy and complete transversal are stated above for the group $\mathcal{A}$ and were initially proved for the groups $\mathcal{L}$, $\mathcal{R}$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{A}$ (see [17]). However, Damon showed that these results are also valid for a larger class of subgroups of $\mathcal{K}$ and $\mathcal{A}$, which he called *geometric subgroups* of $\mathcal{K}$ and $\mathcal{A}$ ([7]). These are subgroups that satisfy some algebraic properties that ensure that all the results on finite determinacy and versal unfoldings are valid for the action of such subgroups on $m_n.\mathcal{E}(n,p)$.

We classify here the simple germs in $m_n.\mathcal{E}(n,p)$ under the action of the geometric subgroups of $\mathcal{K}$ given in the introduction (references to proofs that such subgroups are geometric are given in §5). We shall denote by $m(X)$ the maximal ideal in $\mathcal{E}_n$, where X are the coordinates in $\mathbb{R}^n,0$. We consider the case $n=3$ and $p=2$ and denote by (x,y,z) the coordinates in the source and by (u,v) the coordinates in target.

3 Formal classification of pairs (ω, η)

We are interested in this section (and in the next) in classifying the pairs (ω, η). We can attempt to obtain a classification up to smooth changes of coordinates in $\mathbb{R}^3, 0$ and multiplication by non zero functions. We can set, as before, $\eta = dz$ and assume that the foliation of ω is given by the level sets of a smooth function f. The smooth models of pair (df, dz) are then given by the models of germs of functions $f : \mathbb{R}^3, 0 \to \mathbb{R}, 0$ up to diffeomorphisms in the source that preserve the horizontal planes (i.e., the leaves of $\eta = dz$). These diffeomorphisms are in the form $(\phi_1(x, y, z), \phi_2(x, y, z), \phi_3(z))$ and form a subgroup $\mathcal{G}$ of the right group $\mathcal{R}$. We can also allow changes of coordinates in the target as these do not alter the structure of the fibres of f and those of the function z. We can then classify germs of functions in m_3 up to $\mathcal{A}^* = \mathcal{G} \times \mathcal{L}$-equivalence. However, the group $\mathcal{A}^*$ is not a Damon geometric subgroup [7], so the classification is in the formal category. (Basically, we show that there exists an integer k such that $m_3^l \subset L\mathcal{A}^*(f) + m_3^{l+1}$ for any $l > k$. Because the group $\mathcal{A}^*$ is not a geometric subgroup, there is no version of the Preparation Theorem to guarantee that $m_3^{k+1} \subset L\mathcal{A}^*(f)$ and conclude that f is finitely $\mathcal{A}^*$-determined. All we can assert is that there are formal power series $\hat{h}$ and $\hat{k}$ such that $j^k f = \hat{k} \circ f \circ \hat{h}^{-1}$.)

It is not difficult to show that the $\mathcal{A}^*$-classification of the germs $f \in m_3.\mathcal{E}(3, 1)$ is the same as the classification of divergent diagrams (f, z) : $\mathbb{R}, 0 \xleftarrow{f} \mathbb{R}^3, 0 \xrightarrow{z} \mathbb{R}, 0$ carried out by [11]. Recall that a divergent diagram (f_1, f_2) : $\mathbb{R}^p, 0 \xleftarrow{f_1} \mathbb{R}^n, 0 \xrightarrow{f_2} \mathbb{R}^q, 0$ is a pair of map-germs $f_1 : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ and $f_2 : \mathbb{R}^n, 0 \to \mathbb{R}^q, 0$ sharing the same source. Two divergent diagrams $(f_1, f_2), (g_1, g_2)$: $\mathbb{R}^p, 0 \xleftarrow{f_1} \mathbb{R}^n, 0 \xrightarrow{f_2} \mathbb{R}^q, 0$ are said to be equivalent if there exist germs of diffeomorphisms h of $\mathbb{R}^n, 0$, k_1 of $\mathbb{R}^p, 0$ and k_2 of $\mathbb{R}^q, 0$ such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathbb{R}^p, 0 & \xrightarrow{f_1} & \mathbb{R}^n, 0 & \xrightarrow{f_2} & \mathbb{R}^q, 0 \\
{\scriptstyle k_1}\big\downarrow & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle k_2} \\
\mathbb{R}^p, 0 & \xrightarrow[g_1]{} & \mathbb{R}^n, 0 & \xrightarrow{g_2} & \mathbb{R}^q, 0
\end{array}$$

(More on divergent diagrams can be found for example in [8, 9, 11].) We have the following result where we abuse notation and refer to $\mathcal{A}^*$-formally finitely determined germs of pairs (df, dz) when we mean the germ f is $\mathcal{A}^*$-formally finitely determined.

Theorem 3.1 ([11]) The $\mathcal{A}^*$-formally finitely determined germs of pairs of foliations in $\mathbb{R}^3$ are the following:

1. $(d(z + x), dz)$,
2. $(d(z - x^2 \pm y^2), dz)$,
3. $(d(z - x^3 + xz^{2k-1} + y^2), dz)$, $k \geq 1$,
4. $(d(z - x^3 \pm xz^2 + y^2), dz)$,
5. $(d(z - x^3 + x(\pm z^{2k} + \lambda z^{5k-1}) + y^2), dz)$, $k \geq 2$, $\lambda \in \mathbb{R}$.

Remark 3.2 (1) Theorem 3.1 states that a necessary condition for the pair (df, dz) to be $\mathcal{A}^*$-formally finitely determined is that the surface $f^{-1}(0)$ has an A_0, A_1 or an A_2 contact with its tangent plane at the origin. (The tangent plane can be parametrised by a germ of a function $\phi : \mathbb{R}^2, 0 \to \mathbb{R}^3, 0$. The contact of $f^{-1}(0)$ with its tangent plane at the origin is of type A_k if the germ $f \circ \phi : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ has a singularity which is $\mathcal{R}$-equivalent to $\pm x^2 \pm y^{k+1}$.)

(2) The models 1 and 2 in Theorem 3.1 are in fact up to smooth changes of coordinates. Dufour showed in [8] and [9] that model 3 in Theorem 3.1 with $k = 1$ is also up to smooth changes of coordinates. These smooth models are also stable under deformations of the pair (ω, η).

For the other cases, one can reduce by smooth changes of coordinates any N-jet of a pair which is formally finitely determined (N large enough) to one of the pairs in Theorem 3.1 (3) with $k \geq 2$, (4) or (5). This is an importante property and will be used when applying the results in §5.

4 Topological classification of pairs (ω, η)

We consider in this section the topological classification of pairs (ω, η) of regular 1-forms in $\mathbb{R}^3$. We generalise the result in [13] that shows that under a mild hypothesis, the discriminant of the germ of a pair of regular foliations in the plane determines the topological type of the pair. We assume again that $\eta = dz$.

Theorem 4.1 *Let (ω, dz) be a pair of germs, at the origin, of regular 1-forms in $\mathbb{R}^3$. Suppose that the discriminant $D(\omega, dz)$ consists of at most two branches in each semi-space $z > 0$ and $z < 0$ and is transverse, away from the origin, to the pair of foliations. Then the pair (ω, dz) is locally topologically equivalent to one of the following:*

(i) (dx, dz), *if $D(\omega, dz)$ is empty or is an isolated point;*

(ii) $(d(z - x^2 \pm y^2), dz)$, *if $D(\omega, dz)$ has one branch in each semi-space;*

(iii) $(d(z - x^3 + xz + y^2), dz)$, *if $D(\omega, dz)$ has two branches in one semi-space and none in the other;*

(iv) $(d(z - x^3 + xz^2 + y^2), dz)$, *if $D(\omega, dz)$ has two branches in each semi-space.*

Proof Case (i) is not difficult and shall be omitted. We can suppose that $\omega = df$, where f is a germ of a regular function. As the discriminant is transverse, away from the origin, to the pair of foliations, it follows that away from the origin, the foliations have ordinary tangency on the discriminant. (The transversality condition is satisfied if, for example, the discriminant has a $\mathcal{K}^*$-finitely determined singularity at the origin.) So the pair of 1-forms is locally smoothly equivalent to $(d(z - x^2 \pm y^2), dz)$ at each point on the discriminant $D(df, dz)$ that is distinct from the origin. The contact type (elliptic if the pair is equivalent to $(d(z - x^2 - y^2), dz)$ and hyperbolic if it is equivalent to $(d(z - x^2 + y^2), dz)$) is of course constant on each branch of the discriminant. Let (dg, dz) denotes a model pair in Theorem 4.1 and $D(dg, dz)$ its discriminant.

(ii) Let $h : D(\omega, dz) \to D(dg, dz)$ be an increasing homeomorphism with $h(0) = 0$. One can show that the contact type of the foliations is the same on both branches of the discriminant. Furthermore, the leaf L_0 of ω through the origin has the same type of contact with $z = 0$ as the other points on the discriminant, that is, L_0 intersect the plane $z = 0$ at a single point in the elliptic case and in two curves in the hyperbolic case. We deal with the two types of contact separately.

Suppose that the contact is elliptic. Suppose for simplicity that the points on $D(\omega, dz)$ are local minima of f. Let L be a leaf of ω through $q \in D(\omega, dz)$, with $\pi_z(q) < 0$, where $\pi_z(q)$ is the orthogonal projection to the z-axis. Let L' be the leaf of dg through $h(q)$. There exists a homeomorphism $H : L \to L'$, with $H(L \cap \{z = c\}) = L' \cap \{z = h(c)\}$, for all $c \geq \pi_z(q)$. Let U (resp. U') be the interior of the region delimited by L (resp. L') and the plane $z = c_0$ (resp. $z = h(c_0)$), with c_0 a small positive number (so U and U' contain the origin).

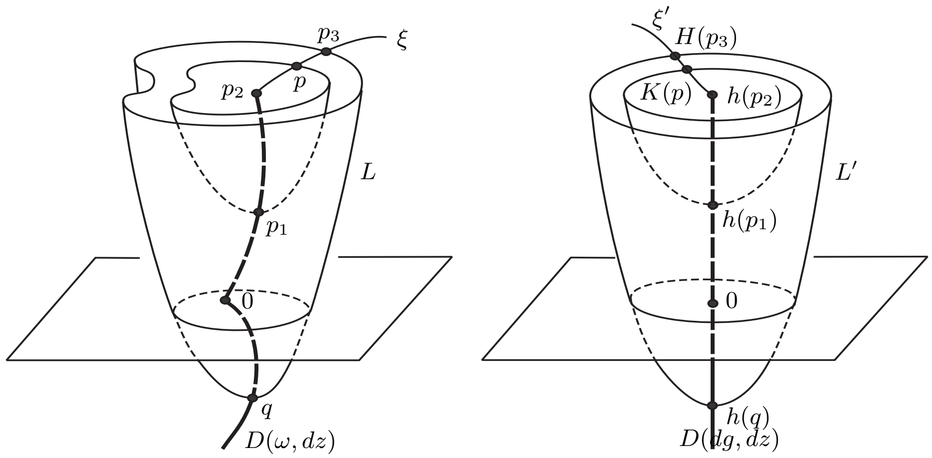

Figure 1: Discriminant with one branch in each half-space (elliptic contact): pair of foliations (left), and the model pair (right).

We consider on each plane $z = c$ the gradient vector field of the restriction of f (resp. g) to this plane and denote it by ξ (resp. ξ'). The fields ξ and ξ' have a unique singularity of type node at the point of intersection of the discriminant with $z = c$, $c \neq 0$. By continuity, the origin is in the closure of very non singular integral curve of ξ (resp. ξ') in $z = 0$. Recall that the integral curves of ξ (resp. ξ') are transverse to the traces of the foliation of $\omega = df$ (resp. dg) on the plane $z = c$.

Given $p \in U$, let L_p be the leaf of ω containing p. This leaf intersects $D(\omega, dz)$ at a unique point p_1. The plane $z = \pi_z(p)$ intersects $D(\omega, dz)$ at a point p_2. This plane intersects the leaf L_p in a closed curve containing p. The flow of ξ provides a homeomorphism between this curve and the curve $L \cap \{z = \pi_z(p)\}$, so the integral curve of ξ through p intersects L at a unique point p_3. The point p determines thus a unique triple $(p_1, p_2, p_3) \in D(\omega, dz) \times D(\omega, dz) \times L$, and vice-versa (Figure 1, left). We define $K(p)$ as the point on the foliation dg determined by the triple $(h(p_1), h(p_2), H(p_3)) \in D(dg, dz) \times D(dg, dz) \times L'$ by reversing the above process (Figure 1, right). The map $K : U \to U'$ is clearly a homeomorphism that sends the pair of foliations (ω, dz) to the model pair.

When the contact is hyperbolic, we consider three leaves of ω. Let L_0 be the leaf through the origin, L_1 an inner leaf and L_2 an outer leaf, i.e., one intersects $D(\omega, dz)$ on one branch and the other on the other branch; Figure 2. The region between the leaves L_1 and L_2 determines a neighbourhood of the origin. We define analogously three leaves L_i', $i = 0, 1, 2$, of dg. We then proceed as in the elliptic case using a homeomorphism $H : L_1 \to L_1'$ that satisfies $H(L_1 \cap \{z = c\}) = L_1' \cap \{z = h(c)\}$ for all c near zero.

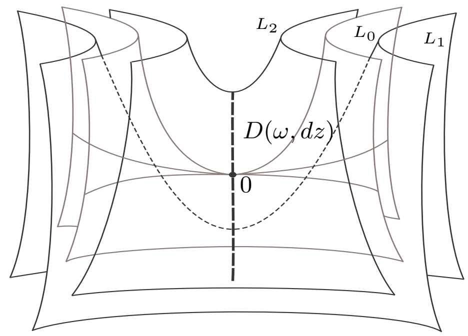

Figure 2: Discriminant with one branch in each half-space, hyperbolic contact.

(iii) Suppose the discriminant $D(\omega, dz)$ has two branches on say, the upper half-space $\mathbb{R}^3_+$, and none in the $\mathbb{R}^3_-$. Then one can show that the contact of the foliations is hyperbolic on one branch, say D_1, and elliptic on the other, say D_2. Furthermore, each leaf that intersects one branch intersects also the other branch. We construct the required homeomorphism in pieces. We start with the upper-half space $\mathbb{R}^3_+$ that contains the discriminant. Let L_0 be the leaf of ω through the origin, L_1 an inner leaf and L_2 an outer leaf of ω; Figure 4, left. We choose a neighbourhood U of L_0 in $\mathbb{R}^3_+$ with boundary L_1, L_2 and the plane $z = 0$; Figure 4, left. We define analogously to leaves L_i', $i = 0, 1, 2$, of dg.

We have a return map on each branch of the discriminant. Given a point q on D_1, the leaf of ω through q intersects D_2 at q'. The leaf of dz through q' intersects D_1 at a point $\alpha_1(q)$. The map $\alpha_1 : D_1 \to D_1$, with $\alpha_1(0) = 0$, is continuous and monotonous. Similarly for the model pair (dg, dz), there exists return map $\alpha_1' : D_1' \to D_1'$. As we are seeking a homeomorphism that preserves foliations, its restriction $h : D_1 \to D_1'$ has to satisfy $h \circ \alpha_1 = \alpha_1' \circ h$. Such an h exists (see for example [14], pp 19-20).

Let $H_1 : L_2 \to L_2'$ be a homeomorphism such that $H_1(L_2 \cap \{z = c\}) = L_2' \cap \{z = h(c)\}$, for all $c \geq 0$. We also need H_1 to send two special curves on L_2 to their analogue on L_2'. These curves are as follows. For $c > 0$ fixed, one separatrix of the saddle singularity of the vector field ξ on $z = c$ intersects the leaf L_2 at two points s_1 and s_2 (ξ is the gradient vector field defined in (ii)); see Figure 3. The closure of the set of such points consists of two curves in $L_2 \cap \{z \geq 0\}$.

We split U into three regions as shown in Figure 3. The leaves through points in region ① do not intersect the discriminant. Therefore we require an extra information to determine the leaf that contains a given point in this region. To do so, we choose a segment of a curve C (resp. C') in the plane $z = 0$ transverse to the traces of the foliation of $\omega = df$ (resp. dg) in region ① (resp. ①′) and joining a regular point of L_0 (resp. L_0') to a point of L_1 (resp.

L_1'). Let $k : C \to C'$ be a homeomorphism (sending the point in L_1 to the point in L_1'). Let p be a point in region 1. The leaf L_p of ω through p cuts C at a unique point p_1 and the horizontal plane through p cuts D_1 at a unique point p_2. The integral curve of the vector field ξ through p determines a unique point p_3 on L_1. The triple $(p_1, p_2, p_3) \in C \times D_1 \times L_2$ determines p uniquely. The image $K(p)$ is set to be the point in region $1'$ determined by the triple $(k(p_1), h(p_2), H_1(p_3)) \in C' \times D_1' \times L_2'$. (If p belongs to the separatrix of the saddle singularity of ξ, we define p_3 to be s_1 in Figure 3.)

A point p in region 2 (resp. 3) determines a unique triple $(p_1, p_2, p_3) \in D_1 \times D_1 \times L_2$, where p_1 and p_2 are given by the intersection of D_1 with the leaves of ω and dz through p respectively, and p_3 is given by the intersection of the integral curve of ξ through p with L_2 (see Figures 3 and 4). The point $K(p)$ in region $2'$ (resp. $3'$) is defined as the point determined by the triple $(h(p_1), h(p_2), H_1(p_3)) \in D_1' \times D_1' \times L_2'$. It is clear that K is a homeomorphism and extends to points in the plane $z = 0$.

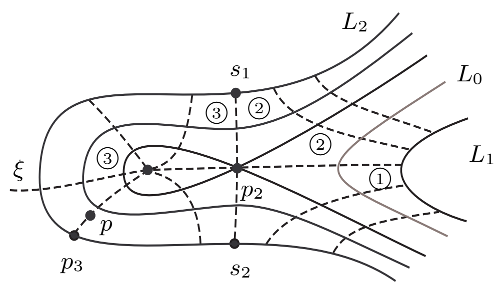

Figure 3: Traces of the foliation ω on $z = c$, $c > 0$.

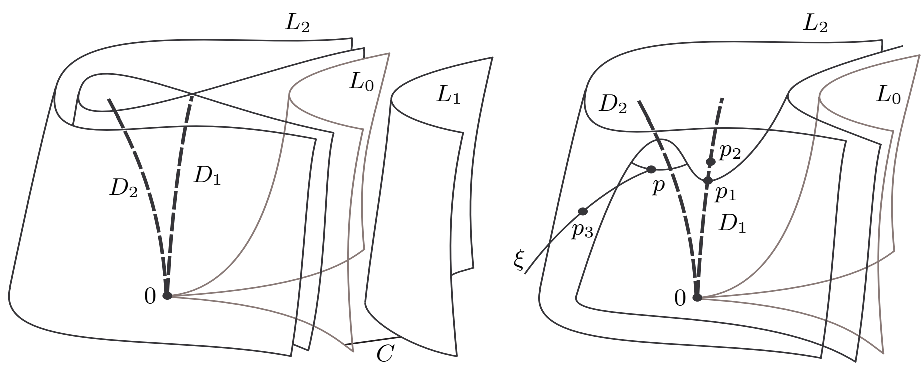

Figure 4: Discriminant with two branches in a half-space: a neighbourhood of L_0 (left), and the triple (p_1, p_2, p_3) associated to a point p in region 3 (right).

In the lower semi-space $\mathbb{R}^3_-$ we consider a neighbourhood V determined by L_0, L_2 and the planes $z = 0$ and $z = c_1$, $c_1 < 0$ close to zero. Let l be a local homeomorphism from the negative part of the z-axis to itself with $l(0) = 0$ and let $H_2 : L_2 \to L_2'$ be a homeomorphism such that $H_2(L_2 \cap \{z = c\}) = L_2' \cap \{z = l(c)\}$, for all $c \leq 0$, c close to zero. The homeomorphism K is defined in the same way as in the upper-half space with h replaced by l. Here we have two regions, one determined by L_1 and L_0 and the other by L_0 and L_2.

The case (iv) is similar to case (iii). □

Remarks 4.2 (1) The observation in [13] (Section 3.2) still holds here for the case when there are more than 2 branches of $D(\omega, dz)$ in one of the semi-spaces. Suppose for simplicity that there are 3 branches D_i, $i = 1, 2, 3$ of the discriminant. Then we obtain two return maps $\alpha_j : D_1 \to D_1$, $j = 1, 2$, one form travelling along the leaves of $\omega = df$ until reaching D_2 and back along the horizontal plane and the other until reaching D_3 and back along the horizontal plane. We have analogously return maps $\alpha_j' : D_1' \to D_1'$ for the possible model pair. A homeomorphism sending the pair of foliations to that of the possible model must induce a homeomorphism $h : D_1 \to D_1$ that conjugates simultaneously the pairs (α_1, α_1') and (α_2, α_2'). Such an h does not exist in general.

We can apply the classification results in Theorem 4.1 to the formal classification in Theorem 3.1.

Corollary 4.3 *Any pair (df, dz) which is formally equivalent to one of the pairs (1)-(5) in Theorem 3.1 is topologically equivalent to, respectively,*

1. (dx, dz),
2. $(d(z - x^2 \pm y^2), dz)$,
3. $(d(z - x^3 + xz + y^2), dz)$,
4. $(d(z - x^3 + xz^2 + y^2), dz)$ *for the (+) case and to* (dx, dz) *for the (−) case,*
5. $(d(z - x^3 + xz^2 + y^2), dz)$ *for the (+) case and to* (dx, dz) *for the (−) case.*

Proof Recall that the first two models and the third for $k = 1$ are smooth models. For the remaining cases, we can reduce the N-jet of (df, dz) for any large N to one of the models in Theorem 3.1.

We need to count the number of branches of the discriminant in each semi-space $z > 0$ and $z < 0$ and show that the discriminant is transverse, away from the origin, to the foliation dz. Take for example the case (5). We know that (df, dz) is smoothly equivalent to $(d(z - x^3 + x(\pm z^{2k} + \lambda z^{5k-1}) + y^2 + g(x, y, z)), dz)$, for some germ of a smooth function g with $j^N g \equiv 0$ and N large enough. The discriminant is the zero fibre of

$$F_{df,dz}(x, y, z) = (-3x^2 + (\pm z^{2k} + \lambda z^{5k-1}) + g_x(x, y, z), 2y + g_y(x, y, z)).$$

By the implicit function theorem, $2y + g_y(x, y, z) = 0$ yields $y = \phi(x, z)$ for some germ of a smooth function ϕ. Now $-3x^2 + (\pm z^{2k} + \lambda z^{5k-1}) + g_x(x, \phi(x, z), z)$ is $\mathcal{K}$-equivalent to $-3x^2 \pm z^{2k}$, so the discriminant is an isolated point in the (−) case and a pair of tangential curves parametrised by z in the (+) case. It is clear that these branches are transverse to the horizontal planes (the foliation of dz). So we can apply Theorem 4.1 to obtain the topological models. These are those given in (5) in the statement of the corollary. □

5 The singularities of the discriminant

We consider here various classifications of the singularities of the discriminant. We start with the action under the contact group $\mathcal{K}$.

5.1 Families of matrices

Let $\omega = a_1(x,y,z)dx + b_1(x,y,z)dy + c_1(x,y,z)dz$ and $\eta = a_2(x,y,z)dx + b_2(x,y,z)dy + c_2(x,y,z)dz$, where a_i, b_i, c_i, $i = 1,2$, are germs, at the origin, of smooth functions. We associate to the pair (ω, η) the family of matrices

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}(x,y,z),$$

which is a map-germ $A : \mathbb{R}^3, 0 \to M(2,3)$, where $M(n,p)$ is the set of $n \times p$ real matrices. Then the discriminant of (ω, η) is the set of points (x,y,z) where all the 2×2 minors of A vanish. Given the matrix A above and two other square matrices $X \in M(2,2)$ and $Y \in M(3,3)$ whose entries are smooth functions in (x,y,z) and which are invertible at the origin, we can consider the matrix valued function XAY. Clearly its 2×2 minors vanish at precisely the set of points where those of A vanish. Similarly, it is not hard to show that any smooth changes of coordinates in the source, via a diffeomorphism ϕ, takes the zero set of the 2×2-minors of A to that of $A \circ \phi$. All these changes of coordinates form a subgroup $\mathcal{G}$ of the contact group $\mathcal{K}$ acting on the space of families of matrices. (The action of $\mathcal{G}$ can be considered as an action on the entries of the matrices, so we get a subgroup of $\mathcal{K}$ acting on $m_3.\mathcal{E}(3,6)$.) So one can classify the $\mathcal{G}$-singularities of A and obtain the singularities of the discriminant as well as their versal deformations. In the case of regular foliations, it is not hard to show that the matrix A is $\mathcal{G}$ equivalent to

$$\begin{pmatrix} 0 & 0 & 1 \\ g_1(x,y,z) & g_2(x,y,z) & 0 \end{pmatrix},$$

for some germs of smooth functions g_1 and g_2. The action of the group $\mathcal{G}$ reduces to that of the group $\mathcal{K}$ on the space of germs of mappings $(g_1, g_2) : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$. A complete list of the $\mathcal{K}$-simple singularities of map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ can be found, for example, in [10].

The action of the group $\mathcal{G}$ does not preserve the foliations of ω and η. However, as pointed out in the introduction, this approach deals well with the singularities of the discriminant and its bifurcations, especially when both pairs are singular at the origin. Then one can proceed as in [3, 6] and [15, 16].

5.2 $\mathcal{K}_V$-singularities of the discriminant

Here we classify map-germs in $m(x,y,z).\mathcal{E}(3,2)$ under the action of the subgroup $\mathcal{K}_V$ of $\mathcal{K}$ which consists of diffeomorphisms $H = (h, H_1) \in \mathcal{K}$, where h preserves the variety V in the source given by $z = 0$. The diffeomorphism h is in the form $(\phi_1(x,y,z), \phi_2(x,y,z), z\phi_3(x,y,z))$. We obtain below all the $\mathcal{K}_V$-simple germs in $m(x,y,z).\mathcal{E}(3,2)$.

The $\mathcal{K}_V$ tangent space to the orbit of $F \in m(x,y,z).\mathcal{E}(3,2)$ at F is given by

$$L\mathcal{K}_V(F) \quad = \quad m(x,y,z).\{F_x, F_y\} + \mathcal{E}_3.\{zF_z\} + (\mathcal{E}_3.F^*(m(u,v))).\{e_1, e_2\}.$$

The group $\mathcal{K}_V$ is a Damon geometric subgroup (see [7]). A result of Damon [7] states that F is $r-\mathcal{K}_V$-determined if and only if there exists an integer l such that

$$m^{r+1}(x,y,z).\mathcal{E}(3,2) \subset L\mathcal{K}_V(F) + m^{r+l+1}(x,y,z).\mathcal{E}(3,2).$$

The classification is carried out inductively on the jet level.

The 1-jets.

Write $j^1F = (a_1x + a_2y + a_3z, b_1x + b_2y + b_3z)$. If $a_1b_2 - a_2b_1 \neq 0$, then $j^1F \sim (x,y)$. If $a_1b_2 - a_2b_1 = 0$ but one of the coefficients a_i or b_i, $i = 1,2$, is not zero, then $j^1F \sim (x,z)$ or $j^1F \sim (x,0)$. If $a_1 = a_2 = b_1 = b_2 = 0$ then $j^1F \sim (z,0)$ or $j^1F \sim (0,0)$. So the orbits in $J^1(3,2)$ are

$$(x,y), (x,z), (x,0), (z,0)^{(ns)}, (0,0)^{(ns)}.$$

We use above the notation (ns) to indicate that a given jet leads to non-simple orbits and will not be followed. Take for example the 1-jet $(z,0)$. Suppose that $j^1F = (z,0)$. Then any 2-jet over this 1-jet is equivalent to $j^2F = (z + h(x,y), g(x,y))$, where h, g are quadratic forms. These germs form a vector space W of dimension 6. Consider, in the 2-jets space, the subgroup of $\mathcal{K}_V$ that acts on the germs in W. The tangent space to the orbit of j^2F is generated by $x\partial j^2F/\partial x$, $y\partial j^2F/\partial x$, $x\partial j^2F/\partial y$, $y\partial j^2F/\partial y$ and $(h,0)$. This has dimension 5, so the orbit of j^2F is not open in W. Therefore, we must have a modulus at the 2-jet level, which shows that $(z,0)$ leads to non-simple germs. It follows that the 1-jet $(0,0)$ also leads non-simple germs.

It is not hard to show that the germ (x,y) is $1-\mathcal{K}_V$-determined and is stable (i.e., $d_e(F,\mathcal{K}_V) = 0$).

The 2-jets.

• Suppose that $j^1F = (x,z)$. So we can write $F(x,y,z) = (x + h(x,y,z), z + k(x,y,z))$ for some germs of smooth functions h, k with zero 1-jets. We can set $F = (x, z + g(x,y))$ by changes of coordinates in $\mathcal{K}_V$. Writing $g(x,y) = xg_1(x,y) + g_2(y)$ we can eliminate the term $xg_1(x,y)$ by a change of coordinates in $\mathcal{K}_V$. Therefore, any germ with 1-jet (x,z) is $\mathcal{K}_V$-equivalent to one in the form $(x, z + g_2(y))$. The germ $(x, z + g_2(y))$ is $\mathcal{K}_V$-finitely determined if and only if $\mathrm{ord}(g_2(y))$ is finite. In this case it is $\mathcal{K}_V$-equivalent to $(x, z + y^k)$, where $k = \mathrm{ord}(g_2(y))$. It follows from the definition that $d_e(F,\mathcal{K}_V) = k - 2$.

• Suppose that $j^1F = (x,0)$. A complete 2-transversal is given by $(x, a_1y^2 + a_2yz + a_3z^2)$. If $a_1 \neq 0$ then the 2-jet is equivalent to (x,y^2) or $(x, y^2 \pm z^2)$. The germ $(x, y^2 \pm z^2)$ is 2-$\mathcal{K}_V$-determined. If $a_1 = 0$ and $a_2 \neq 0$ then the 2-jet is equivalent to (x, yz). If $a_1 = a_2 = 0$ and $a_3 \neq 0$ then the 2-jet is equivalent to (x, z^2).

The 3-jets.

• Suppose that $j^2F = (x,y^2)$. Then F is $\mathcal{K}_V$-equivalent to $(x, y^2 + g(z))$ for some germ g with a zero 2-jet. The germ $(x, y^2 + g(z))$ is $\mathcal{K}_V$-finitely determined if and only if $\mathrm{ord}(g(z))$

Table 1: Normal forms of $\mathcal{K}_V$-simple map-germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$.

Normal form F	$d_e(F, \mathcal{K}_V)$	#branches
(x, y)	0	$1, 1$
$(x, z + y^k)$, $k \geq 2$	$k - 2$	$0, 2$ (k even)
		$1, 1$ (k odd)
$(x, y^2 + z^k)$, $k \geq 2$	$k - 1$	$0, 0$ (k even)
		$0, 2$ (k odd)
$(x, y^2 - z^{2k})$, $k \geq 1$	$2k - 1$	$2, 2$
$(x, yz + y^k)$, $k \geq 3$	$k - 1$	$2, 2$ (k even)
		$1, 3$ (k odd)
$(x, z^2 + y^3)$	3	$1, 1$

is finite. In this case it is $\mathcal{K}_V$-equivalent to $(x, y^2 \pm z^k)$ where $k = \mathrm{ord}(g(z))$. Moreover, $d_e(F, \mathcal{K}_V) = k - 1$.

• Suppose that $j^2F = (x, yz)$. A complete k-transversal can be written in the form $(x, yz + g(y))$, where g is a polynomial with a zero 2-jet. We obtain the series $F = (x, yz + y^k)$, $k \geq 3$ of $\mathcal{K}_V$-finitely determined germs, with $d_e(F, \mathcal{K}_V) = k - 1$.

• Suppose that $j^1F = (x, z^2)$. A complete 3-transversal is given by $(x, z^2 + ay^2z + by^3)$. If $b \neq 0$ then we can set $b = 1$ by a change of scale. We can then apply Mather's Lemma to eliminate the term $(0, y^2z)$. The germ $(x, z^2 + y^3)$ is 3-$\mathcal{K}_V$-determined with codimension 3. If $b = 0$, one can consider a 4-transversal and show that we have a modulus at the 4-jet level. Therefore we have no more simple germs.

We summarise the classification below.

Theorem 5.1 *The $\mathcal{K}_V$-simple map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ are given in Table 1.*

In Table 1, #branches indicates the number of branches of the zero-fibre of the normal form F in the semi-spaces $z > 0$ and $z < 0$. This number is calculated as follows. Take, for example, the normal form $F = (x, z^2 + y^3)$. Then $F^{-1}(0, 0)$ is the cusp $z^2 + y^3 = 0$ in the plane $x = 0$. So there is one branch of $F^{-1}(0, 0)$ in each semi-space $z > 0$ and $z < 0$.

Corollary 5.2 *The $\mathcal{K}_V$-simple singularities of the discriminant $D(df, dz)$ of pairs (df, dz) are given in Table 1. Only those pairs whose discriminant is equivalent to a germ in Table 1 which has a number of branches ≤ 2 in each semi-space bounded by $z = 0$ have a discrete topological model. The topological models are given in Theorem 4.1.*

Corollary 5.3 *Any pair (df, dz) which is formally equivalent to one of the pairs (1)–(5) in Theorem 3.1 has a discriminant which is $\mathcal{K}_V$-equivalent to, respectively,*

1. $(1, 0)$,

2. (x, y),

3. $(x^2+z^{2k-1},y)\sim(x,y^2+z^{2k-1})$,
4. $(x^2\mp z^2,y)\sim(x,y^2\mp z^2)$,
5. $(x^2\mp z^{2k},y)\sim(x,y^2\mp z^{2k})$.

All the formally finitely determined pairs have therefore a discrete topological type. The topological models can be deduced from Theorem 4.1 by looking at the number of branches of the germs (2)–(5) above in Table 1.

Proof The proof follows by a straightforward calculation. $\square$

5.3 $\mathcal{K}^*$-singularities of the discriminant

One can study the singularities of the discriminant up to some action that preserves its contact with the leaves of the foliations. The motivation for this is the following.

We first observe that the discriminant is transverse to the leaves of ω and η at a point p if and only if ω and η have an elliptic or hyperbolic contact at p, that is, (ω,η) is equivalent to $(d(z-x^2\pm y^2),dz)$ (the equivalence here is by smooth changes of coordinates and multiplication by non zero functions, see Theorem 3.1 and Remarks 3.2). The discriminant has an ordinary tangency with the leaves of the foliations at p if and only if (ω,η) is equivalent to $(d(z-x^3+xz+y^2),dz)$. (When the discriminant is smooth, it can be parametrised by a germ of some map $\phi:\mathbb{R},0\to\mathbb{R}^3,p$. A leaf at p of one of the foliations is the zero set of a germ of some function $g:\mathbb{R}^3,p\to\mathbb{R},0$. The discriminant has an ordinary tangency at p with the leaf $g^{-1}(0)$ if the function $g\circ\phi$ satisfies $g(0)=g'(0)=0$ and $g''(0)\neq 0$.)

These points of tangency, that we denote as in [13] by 3_1, are isolated and can be counted. We assume, without loss of generality, that $\eta=dz$ and $\omega=df$, where f is a germ of a smooth function. The points of tangency are characterised by the fact that the tangency of the two leaves $z=0$ and $f^{-1}(0)$ is worse than elliptic or hyperbolic. This is the case if and only if $z=0, f_x=f_y=f_{xx}f_{yy}-f_{xy}^2=0$. So the maximum number of such points that can appear in a deformation of the pair is given by

$$\#3_1=\dim_{\mathbb{R}}(\mathcal{E}_3/\left\langle z,f_x,f_y,f_{xx}f_{yy}-f_{xy}^2\right\rangle)=\dim_{\mathbb{R}}(\mathcal{E}_2/\left\langle g_x,g_y,g_{xx}g_{yy}-g_{xy}^2\right\rangle),$$

where $g(x,y)=f(x,y,0)$. This number is a smooth invariant of the pair (df,dz), that is, two smoothly equivalent germs of pairs of 1-forms have the same number $\#3_1$. It is therefore of interest to classify the singularities of the discriminant up to an equivalence that preserve the leaves of the foliation ω or η.

Recall that the discriminant is the zero fibre of map germ $F_{\omega,\eta}:\mathbb{R}^3,0\to\mathbb{R}^2,0$, so we are seeking a classification of the singularities of map-germs $\mathbb{R}^3,0\to\mathbb{R}^2,0$ under the action of the subgroup $\mathcal{K}^*$ of the contact group $\mathcal{K}$, where the changes of coordinates in the source preserve the horizontal planes (i.e., the leaves of η). The changes of coordinates in the source are in the form $(\phi_1(x,y,z),\phi_2(x,y,z),\phi_3(z))$. (We observe that the $\mathcal{K}^*$-action preserves the contact of any germ in $m(x,y,z).\mathcal{E}(3,2)$ with the horizontal planes. Also if $F_{\omega,\eta}$ has a finitely $\mathcal{K}^*$-determined singularity, then the discriminant is transverse, away from the origin, to the foliation of dz which is one of the conditions required in Theorem 4.1.)

The group $\mathcal{K}^*$ is a Damon geometric subgroup (see [7]). We give below a classification of the simple orbits of the action of $\mathcal{K}^*$ on $m(x,y,z).\mathcal{E}(3,2)$. The $\mathcal{K}^*$ tangent space to the orbit of $F \in m(x,y,z).\mathcal{E}(3,2)$ at F is given by

$$L\mathcal{K}^*(F) = m(x,y,z).\{F_x, F_y\} + m(z).\{F_z\} + F^*m(u,v).\mathcal{E}(3,2).$$

A result of Damon [7] states that F is $r-\mathcal{K}^*$-determined if and only if there exists an integer l such that

$$m^{r+1}(x,y,z).\mathcal{E}(3,2) \subset L\mathcal{K}^*(F) + m^{r+l+1}(x,y,z).\mathcal{E}(3,2).$$

Observe that if two germs F and G are $\mathcal{K}^*$-equivalent then they are $\mathcal{K}_V$-equivalent, so two $\mathcal{K}^*$-equivalent germs lie in the same $\mathcal{K}_V$-orbit. In particular, if a germ F is not $\mathcal{K}_V$-simple, then it is not $\mathcal{K}^*$-simple.

The classification is carried out inductively on the jet level.

The 1-jets.

The same calculation in §5.2 shows that the $\mathcal{K}^*$-orbits at the 1-jet level coincide with the $\mathcal{K}_V$-orbits, i.e., are given by

$$(x,y), (x,z), (x,0), (z,0)^{(ns)}, (0,0)^{(ns)}.$$

The 1-jet (x,y) is also 1- $\mathcal{K}^*$-determined and is stable.

The 2-jets.

• Suppose that $j^1F = (x,0)$. (The 1-jet (x,z) is a subcase of this one.) We can write $F = (x + h(x,y,z), k(x,y,z))$ for some germs of functions h, k with zero 1-jets. A change of coordinates in the source reduces F to $(x, k_1(x,y,z))$. We can write $k_1(x,y,z) = xk_2(x,y,z) + g(y,z)$ and eliminate the term $xk_2(x,y,z)$ by changes of coordinates in $\mathcal{K}^*$. Therefore, any germ with 1-jet $(x,0)$ is $\mathcal{K}^*$-equivalent to one in the form $(x, g(y,z))$. We still denote by $\mathcal{K}^*$ the subgroup of $\mathcal{K}$ acting on the set of germs of functions $g(y,z) \in m(y,z).\mathcal{E}(2,1)$, with the diffeomorphisms in the source preserving the horizontal lines $z = c$. We have the following result.

Proposition 5.4 *A map-germ $F(x,y,z) = (x, g(y,z))$ is $r-\mathcal{K}^*$-determined (resp. simple) if and only if the function germ $g(y,z)$ is $r-\mathcal{K}^*$-determined (resp. simple). For such finitely determined germs we have $d_e(F,\mathcal{K}^*) = d_e(g,\mathcal{K}^*)$.*

Proof We observe that

$$L\mathcal{K}^*(F) = m(x,y,z).\{(1,0)\} + (m(x).\mathcal{E}_3).\{(0,1)\} + \{(0,\xi) : \xi \in L\mathcal{K}^*(g)\}.$$

The result then follows by applying Damon's determinacy result in [7]. □

A $\mathcal{K}^*$-classification of germs $g(y,z) \in m(y,z).\mathcal{E}(2,1)$ is given in [13]. The simple germs of this action are those in Table 2, see [13]. (We include in the last column in Table 2 the invariant $\#3_1$ mentioned at the beginning of this section. It coincides with the number $\#3_1$ associated to $F(x,y,z) = (x, g(y,z))$.) We have therefore the following result.

Table 2: $\mathcal{K}^*$-simple function-germs $\mathbb{R}^2, 0 \to \mathbb{R}, 0$.

Normal form g	$d_e(g, \mathcal{K}^*)$	$\#3_1$
y	0	0
$z + y^k$, $k \geq 2$	$k-2$	$k-1$
$y^2 + z^k$, $k \geq 2$	$k-1$	k
$yz + y^k$, $k \geq 3$	$k-1$	k
$z^2 + y^3$	3	4

Theorem 5.5 *A map-germ $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ is $\mathcal{K}^*$-simple if and only if it is $\mathcal{K}_V$-simple. So the $\mathcal{K}^*$-simple germs are also those given in Table 1.*

Remark 5.6 (Families of pairs of 1-forms). Let (df, dz) be a pair of germs of regular foliations in $\mathbb{R}^3, 0$ and suppose that the discriminant $D(df, dz)$ has a finitely $\mathcal{K}^*$-determined singularity. Suppose further that a generic deformation of the pair yields a versal deformation of the discriminant. So only $\mathcal{K}^*$-stable singularities of the discriminant will appear in the deformation. This means that the discriminant is either transverse to the leaves of dz or has ordinary contact with one of the leaves. Therefore, the local models that are present in a generic deformation of the pair are $(d(z + x), dz)$, $(d(z - x^2 \pm y^2), dz)$, and $(d(z - x^3 + xz + y^2), dz)$, and these are all smooth local models.

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References

[1] V. I. Arnol'd, S. M. Guseĭn-Zade and A. N. Varchenko, *Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts.* Monographs in Mathematics, 82. Birkhuser, 1985.

[2] J. W. Bruce, Classifications in singularity theory and their applications. New developments in singularity theory (Cambridge, 2000), 3–33, *NATO Sci. Ser. II Math. Phys. Chem.*, 21, Kluwer Acad. Publ., 2001.

[3] J. W. Bruce, On families of symmetric matrices. *Mosc. Math. J.* 3 (2003), 335–360.

[4] J. W. Bruce, N. P. Kirk and A. A. du Plessis, Complete transversals and the classification of singularities. *Nonlinearity* 10 (1997), 253-275.

[5] J. W. Bruce, A. A. du Plessis and C. T. C. Wall, Determinacy and unipotency. *Invent. Math.* 88 (1987), 521–554.

[6] J. W. Bruce and F. Tari, On families of square matrices. *Proc. London Math. Soc.* (3) 89 (2004), 738–762.

[7] J. N. Damon, The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$. *Mem. Amer. Math. Soc.* 50 (1984), no. 306.

[8] J. -P. Dufour, Sur la stabilité des diagrams d'applications différentiables. *Ann. Sci. École Norm. Sup.* (4) 10 (1977), 153–174

[9] J. -P. Dufour, Bi-stabilité des fronces. *C. R. Acad. Sci. Paris Sér. A-B* 285 (1977), A445–A448.

[10] A. A. du Plessis and C. T. C. Wall, *The geometry of topological stability.* London Mathematical Society Monographs. New Series, 9. Oxford Science Publications. Oxford University Press, 1995.

[11] S. Mancini, M. A. S. Ruas and M. A. Teixeira, On divergent diagrams of finite codimension. *Port. Math. (N.S.)* 59 (2002), 179–194.

[12] J. N. Mather, Stability of C^∞ mappings, IV: Classification of stable germs by $\mathcal{R}$-algebras. *Publ. Math., IHES* 37 (1969), 223–248.

[13] R. D. S. Oliveira and F. Tari, Topological classification of pairs of regular foliations in the plane. *Hokkaido Mathematical Journal* 31 (2002), 523–537.

[14] W. Szlenk, *An introduction to the theory of smooth dynamical systems.* John Wiley & Sons, PWIV- Polish Scientific Publishers, Warzawa 1984.

[15] F. Tari, Two-parameter families of binary differential equations. *Preprint, available from http://maths.dur.ac.uk/˜dma0ft.*

[16] F. Tari, Two-parameter families of implicit differential equations. *Discrete Contin. Dynam. Systems* 13 (2005), 139–162.

[17] C. T. C. Wall, Finite determinacy of smooth map-germs. *Bull. London Math. Soc.* 13 (1981), 481–539.

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