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Projections of timelike surfaces in the de Sitter space

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Abstract

We study in this paper projections of embedded timelike hypersurfaces M in S^n_1 along geodesics. We deal in more details with the case of surfaces in S^3_1, characterise geometrically the singularities of the projections and prove duality results analogous to those of Shcherbak for central projections of surfaces in $\mathbb{R}P^3$.

1. *Introduction*

We study in this paper the contact of timelike hypersurfaces in the de Sitter space S^n_1 with geodesics. The contact is measured by the singularities of projections along geodesics to transverse sets. There are three types of geodesics in S^n_1, spacelike, timelike and lightlike ([**13**]). In the case of spacelike and timelike geodesics we project, respectively, to orthogonal hyperbolic and elliptic de Sitter hyperquadrics. For a lightlike geodesic, we project to a transverse space as the orthogonal space contains the geodesic. We give in section 3 the expressions for the families of projections along the three types of geodesics.

Given a point p on a timelike hypersurface $M \subset S^n_1$, there is a well defined unit normal vector $e(p) \in S^n_1$ to M at p; see [**4**] and section 2. If M is orientable, then $e(p)$ is globally defined. However, it is always locally defined and our investigation here is local in nature. We have the (de Sitter) Gauss map

$$\begin{array}{cccc} \mathbb{E}: & M & \to & S^n_1 \\ & p & \mapsto & e(p) \end{array}$$

with the property that its differential map (the Weingarten map) $-d\mathbb{E}_p$ is a self-adjoint operator on T_pM ([**4**]). As M is timelike, the restriction of the pseudo-scalar product in the Minkowski space to T_pM is also a pseudo scalar product. Therefore, $-d\mathbb{E}_p$ does not always have real eigenvalues. When these are real, we call the associated eigenvectors the *principal directions* of M at p. For timelike surfaces in S^3_1 there is a curve, labelled the *lightlike principal locus* in [**7**, **10**] (LPL for short), that separates regions on M where there are two distinct principal directions and regions where there are none. On the LPL there is a unique double principal direction. One can also define the concept of an asymptotic direction on a surface M in S^n_1. We say that $v \in T_pM$ is an asymptotic direction at $p \in M$ if $\langle d\mathbb{E}_p(v), v\rangle = 0$, see section 2 for details.

We show in section 4 that the singularities of the projections of surfaces in S^3_1 along the three types of geodesics capture some aspects of the extrinsic geometry of the surface related to the Gauss map $\mathbb{E}$. Indeed, the singularity at $p \in M$ of a given projection is of type cusp or worse if and only if the tangent to the geodesic at p is an asymptotic direction (Theorems 4.2 and 4.3). We characterise geometrically in section 4 all the generic singularities of the projections along geodesics. For instance the LPL is picked up as the locus of points where the projections along the lightlike geodesics have singularities of type cusp. The projections also pick up special

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points on the LPL (Theorem 4.3), namely the singular points of the configuration of the lines of principal curvature.

The first author introduced duality concepts between hypersurfaces in the pseudo spheres in the Minkowski space [**4**, **5**]; see section 6 for details. We use these concepts to prove in section 5 duality results between some surfaces associated to a timelike surface $M \subset S^3_1$ and special curves on the dual surface M^* of M. The results are analogous to those of Shcherbak in [**16**] for central projections of surfaces in $\mathbb{R}P^3$, and to those of Bruce-Romero Fuster in [**2**] for orthogonal projections of surfaces in the Euclidean space $\mathbb{R}^3$.

The work in this paper is part of a project on projections of submanifolds embedded in the pseudo-spheres in the Minkowski space $\mathbb{R}^n_1$ via singularity theory. We dealt in [**8**] with the contact of (hyper)surfaces with geodesics in the hyperbolic space (see also [**11**]) and in [**9**] with their contact with horocycles.

2. *Preliminaries*

We start by recalling some basic concepts in hyperbolic geometry (see for example [**14**] for details). The *Minkowski* $(n+1)$*-space* $(\mathbb{R}^{n+1}_1, \langle , \rangle)$ is the $(n+1)$-dimensional vector space $\mathbb{R}^{n+1}$ endowed by the *pseudo scalar product* $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$, for $\boldsymbol{x} = (x_0, \dots, x_n)$ and $\boldsymbol{y} = (y_0, \dots, y_n)$ in $\mathbb{R}^{n+1}_1$. We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $= 0$ or < 0 respectively. The norm of a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a vector $v \in \mathbb{R}^{n+1}_1$ and a real number c, a hyperplane with pseudo normal v is defined by

$$HP(v, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, v \rangle = c\}.$$

We say that $HP(v, c)$ is a *spacelike*, *timelike* or *lightlike hyperplane* if v is timelike, spacelike or lightlike respectively. We have the following three types of pseudo-spheres in $\mathbb{R}^{n+1}_1$:

$$\begin{array}{rll} \textit{Hyperbolic } n\textit{-space}: & H^n(-1) & = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}, \\ \textit{de Sitter } n\textit{-space}: & S^n_1 & = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}, \\ (\textit{open}) \textit{ lightcone}: & LC^* & = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}. \end{array}$$

We also define the *lightcone* $(n-1)$*-sphere*

$$S^{n-1}_+ = \{\boldsymbol{x} = (x_0, \dots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0, \ x_0 = 1\}.$$

A hypersurface given by the intersection of S^n_1 with a spacelike (resp. timelike) hyperplane is called an *elliptic hyperquadric* (resp. *hyperbolic hyperquadric*).

A smooth embedded hypersurface M in S^n_1 is said to be timelike if its tangent space T_pM at any point $p \in M$ is a timelike vector space. Some aspects of the extrinsic geometry of timelike hypersurfaces in S^n_1 are studied in [**4**, **7**, **10**].

Let M be a timelike hypersurface embedded in S^n_1. Given a local chart $\boldsymbol{i} : U \to M$, where U is an open subset of $\mathbb{R}^{n-1}$, we denote by $\boldsymbol{x} : U \to S^n_1$ such embedding, identify $\boldsymbol{x}(U)$ with U through the embedding $\boldsymbol{x}$ and write $M = \boldsymbol{x}(U)$. Since $\langle \boldsymbol{x}, \boldsymbol{x} \rangle \equiv 1$, we have $\langle \boldsymbol{x}_{u_i}, \boldsymbol{x} \rangle \equiv 0$, for $i = 1, \dots, n-1$, where $u = (u_1, \dots, u_{n-1}) \in U$. We define the spacelike unit normal vector $e(u)$ to M at $\boldsymbol{x}(u)$ by

$$e(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \ldots \wedge \boldsymbol{x}_{u_{n-1}}(u)\|},$$

where $\wedge$ denotes the wedge product of n vectors in $\mathbb{R}^{n+1}_1$ (see for example [**14**]). The de Sitter Gauss map is defined in [**4**] by

$$\begin{array}{rccc} \mathbb{E}: & M & \to & S^n_1 \\ & p & \mapsto & e(p) \end{array}$$

At any $p \in M$ and $v \in T_pM$, one can show that $D_v\mathbb{E} \in T_pM$, where D_v denotes the covariant derivative with respect to the tangent vector v. The linear transformation $A_p = -d\mathbb{E}(p)$ is called the *de Sitter shape operator*. Because the surface M is timelike, the restriction of the pseudo scalar product in $\mathbb{R}^n_1$ to M is a pseudo scalar product. Therefore, the shape operator A_p does not always have real eigenvalues. When these are real, we call them the *principal curvatures* of M at p and the corresponding eigenvectors are called the *principal directions*.

We now review some concepts of the extrinsic geometry of embedded timelike surfaces M in S^3_1 (so $n = 3$ above). We denote by (u, v) the coordinates in $U \subset \mathbb{R}^2$. The first fundamental form of the surface M at a point p is the quadratic form $\mathrm{I}_p : T_pM \to \mathbb{R}$ given by $\mathrm{I}_p(v) = \langle v, v\rangle$. If $v = a\boldsymbol{x}_u + b\boldsymbol{x}_v \in T_pM$, then $\mathrm{I}_p(v) = Ea^2 + 2Fab + Gb^2$, where

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u\rangle\,, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v\rangle\,, \quad G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v\rangle\,.$$

are the coefficients of the first fundamental form. Because M is timelike, we have $EG - F^2 < 0$, so at any point $p \in \boldsymbol{x}(U) \subset M$ there are two lightlike directions in T_pM. These are the solutions of $I_p(v) = 0$.

The second fundamental form of the surface M at the point p is the quadratic form $\mathrm{II}_p : T_pM \to \mathbb{R}$ given by $\mathrm{II}_p(v) = \langle A_p(v), v\rangle$, with $A_p = -d\mathbb{E}(p)$. For $v = a\boldsymbol{x}_u + b\boldsymbol{x}_v \in T_pM$, we have $\mathrm{II}_p(v) = la^2 + 2mab + nb^2$, where

$$\begin{array}{rcl} l &=& -\langle e_u, \boldsymbol{x}_u\rangle = \langle e, \boldsymbol{x}_{uu}\rangle \\ m &=& -\langle e_u, \boldsymbol{x}_v\rangle = \langle e, \boldsymbol{x}_{uv}\rangle = \langle e, \boldsymbol{x}_{vu}\rangle = -\langle e_v, \boldsymbol{x}_u\rangle \\ n &=& -\langle e_v, \boldsymbol{x}_v\rangle = \langle e, \boldsymbol{x}_{vv}\rangle \end{array}$$

The shape operator A determines pairs of foliations on M ([**10**]). A *line of principal curvature* is a curve on the surface whose tangent at all points is a principal direction. These form a pair of foliation in some region of M, given by the following binary differential equation

$$(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)du^2 = 0. \tag{2.1}$$

The discriminant function of this equation is

$$\delta(u, v) = \big((Gl - En)^2 - 4(Gm - Fn)(Fl - Em)\big)\,(u, v). \tag{2.2}$$

When $\delta(u, v) > 0$, there are two distinct principal directions at $p = \boldsymbol{x}(u, v)$. These coincide at points where $\delta(u, v) = 0$. There are no principal directions at points where $\delta(u, v) < 0$. We labelled in [**7**, **10**] the locus of points where $\delta(u, v) = 0$ the *Lightlike Principal Locus* (*LPL* for short).

Proposition 2.1. ([**7**, **10**]) (1) *For a generic timelike surface $M \in S^3_1$, the LPL is a curve which is smooth except at isolated points where it has Morse singularities of type node. The singular points are where the shape operator is a multiple of the identity, and are labelled "timelike umbilic points". The LPL is also the set of points on M where the two principal directions coincide and become lightlike.*

(2) *The LPL divides the surfaces into two regions. In one of them there are no principal directions and in the other there are two distinct principal directions at each point. In the latter case, the principal directions are orthogonal and one is spacelike while the other is timelike.*

We also have the concept of asymptotic directions. A direction $v \in T_pM$ is called *asymptotic* if $\mathrm{II}_p(v) = \langle A_p(v), v\rangle = 0$ ([**10**]). An *asymptotic curve* is a curve on the surface whose tangent at all points is an asymptotic direction. The equation of the asymptotic curves is

$$ndv^2 + 2mdudv + ldu^2 = 0. \tag{2.3}$$

The discriminant of equation (2.3) is the locus of points where $m^2 - nl$ vanishes. This is the set of points where the Gauss-Kronecker curvature $K = \det(A_p) = (m^2 - nl)/(F^2 - EG) = 0$ vanishes, and is labelled the (*de Sitter*) *parabolic set*. The parabolic set of a generic surface, when not empty, is a smooth curve. It meets the LPL at isolated points and the two curves are tangential at their points of intersection ([**10**]). In the region $K > 0$ there are two distinct asymptotic directions and there are no asymptotic directions in the region $K < 0$. On the parabolic set $K = 0$ there is a unique asymptotic direction. This direction is tangent to the parabolic set when it is lightlike and this occurs at the point of tangency of the parabolic set with the LPL. These points are the *folded singularities* of the asymptotic curves ([**10**]). On one side of such points, the unique asymptotic direction is spacelike and on the other side it is timelike ([**10**]). (On the LPL one of the asymptotic directions is lightlike and coincides with the unique principal direction there. The generic local topological configurations of the principal and asymptotic curves are studied in [**10**].)

Let $\gamma : I \to M \subset S^3_1$ be a regular curve on a timelike surface M. We can parametrise γ by arc-length and assume that $\gamma(s)$ is unit speed, that is, $\langle \gamma'(s), \gamma'(s)\rangle = \pm 1$. Let $t(s) = \gamma'(s)$ and $w(s) = \gamma(s) \wedge t(s) \wedge e(s)$, where $e(s) = e(\gamma(s))$. The acceleration vector $\gamma''(s)$ is written, in the frame $\{\gamma(s), t(s), e(s), w(s)\}$, in the form

$$\gamma''(s) = \mp\gamma(s) + \kappa_n(s)e(s) + \kappa_g(s)w(s)$$

where $\kappa_g(s)$ is the *geodesic curvature* of γ on M at $\gamma(s)$. When the curve γ is not parametrised by arc length, we re-parametrise by arc-length $l(s) = \int_0^s ||\gamma'(t)||dt$ and the formula for the curvature is

$$\kappa_g(t) = \frac{1}{l'(t)^3} \langle l'(t)\gamma''(t) - l''(t)\gamma'(t), w(t)\rangle \,.$$

A unit speed curve γ is geodesic on M if and only if $\kappa_g \equiv 0$. A point $\gamma(s)$ is called a *geodesic inflection* if $\kappa_g(s) = 0$.

Definition 1. The flecnodal curve of a timelike surface in S^3_1 is the locus of geodesic inflections of the of the asymptotic curves.

3. *The family of projections along geodesics in S^n_1*

We exhibit in this section the expressions for the family of projections along geodesics in S^n_1 for $n \geq 3$ and deal in more details with the case $n = 3$ in the following section. We start with projections along timelike geodesics.

Let $HP(v, 0) \cap S^n_1$, $v \in H^n(-1)$, be (a flat) elliptic hyperquadric. Given a point $p \in S^n_1$, there is a unique timelike geodesic in S^n_1 which intersects orthogonally the elliptic hyperquadric at some point $q(v, p)$. We call the point $q(v, p)$ the orthogonal projection of p in the direction v to the elliptic hyperquadric $HP(v, 0) \cap S^n_1$, and consider the fibre bundle

$$\pi_{12} : \Delta_1 = \{(v, q) \in H^n(-1) \times S^n_1 \mid \langle v, q\rangle = 0\} \to S^n_1$$

where π_{12} is the canonical projection (see Appendix). By varying v, we obtain a family of orthogonal projections along timelike geodesics to elliptic hyperquadrics parametrised by vectors in $H^n(-1)$.

THEOREM 3.1. *The family of orthogonal projections in S_1^n along timelike geodesics is given by*

$$\begin{array}{rccc} P_T: & H^n(-1)\times S_1^n & \to & \Delta_1 \\ & (v,p) & \mapsto & (v,q(v,p)) \end{array}$$

where $q(v,p)$ has the following expression

$$q(v,p)=\frac{1}{\sqrt{1+\langle v,p\rangle^2}}\left(p+\langle v,p\rangle v\right).$$

Proof. Let $p\in S_1^n$ and $v\in H^n(-1)$. A timelike geodesic passing through p is parametrised by

$$c(t)=\cosh(t)p+\sinh(t)w, \tag{3.1}$$

for some $w\in H^n(-1)$ tangent to the geodesic at $c(0)=p$ and with $\langle w,p\rangle=0$. At some t_0, we have $c(t_0)=q(v,p)$ and $c'(t_0)=\sinh(t_0)p+\cosh(t_0)w=v$. Thus, $\langle c'(t_0),p\rangle=\langle v,p\rangle$, which gives $\sinh(t_0)=\langle v,p\rangle$. Therefore, $\cosh(t_0)=\sqrt{1+\langle v,p\rangle^2}$. From this we get

$$w=\frac{1}{\sqrt{1+\langle v,p\rangle^2}}\left(v-\langle v,p\rangle p\right).$$

Substituting in (3.1) for $t=t_0$ yields $q(v,p)=\left(p+\langle v,p\rangle v\right)/\sqrt{1+\langle v,p\rangle^2}$. □

We consider next projections along spacelike geodesics. Let $HP(v,0)\cap S_1^n$ be a hyperbolic hyperquadric, so $v\in S_1^n$. Given a point $p\in S_1^n-\{\pm v\}$, there is a unique spacelike geodesic in S_1^n which intersects orthogonally $HP(v,0)\cap S_1^n$ at two points $q^{\pm}(v,p)$. The points $p=\pm v$ are excluded as all the spacelike geodesics orthogonal to $HP(v,0)\cap S_1^n$ pass through these two points. Therefore, their projection is not well defined. We call the points $q^{\pm}(v,p)$ the orthogonal projection of p in the direction v to the hyperbolic hyperquadric $HP(v,0)\cap S_1^n$. We consider the fibre bundle

$$\pi_{52}:\Delta_5=\{(v,q)\in S_1^n\times S_1^n\mid \langle v,q\rangle=0\}\to S_1^n$$

with π_{52} the canonical projection to the second component (see Appendix). By varying v, we obtain a family of orthogonal projections along spacelike geodesics to hyperbolic hyperquadrics parametrised by vectors in S_1^n.

THEOREM 3.2. *The family of orthogonal projections in S_1^n along spacelike geodesics is given by*

$$\begin{array}{rccc} P_S: & S_1^n\times S_1^n-\{(v,\pm v),v\in S_1^n\} & \to & \Delta_5 \\ & (v,p) & \mapsto & (v,q^{\pm}(v,p)) \end{array}$$

where $q^{\pm}(v,p)$ has the following expression

$$q^{\pm}(v,p)=\pm\frac{1}{\sqrt{1-\langle v,p\rangle^2}}\left(p-\langle v,p\rangle v\right).$$

Proof. Let $(v,p)\in S_1^n\times S_1^n$. A spacelike geodesic passing through p is parametrised by

$$c(t)=\cos(t)p+\sin(t)w, \tag{3.2}$$

for some $w \in S_1^n$ tangent to the geodesic at $c(0) = p$ and with $\langle w, p \rangle = 0$. At some t_0, we have $c(t_0) = q(v, p)$ and $c'(t_0) = -\sin(t_0)p + \cos(t_0)w = v$. So $\langle c'(t_0), p \rangle = \langle v, p \rangle$, which gives $\sin(t_0) = -\langle v, p \rangle$. Therefore, $\cos(t_0) = \pm\sqrt{1 - \langle v, p \rangle^2}$. We have $\langle v, p \rangle^2 = 1$ if and only if $p = \pm v$ and this is excluded. Hence,

$$w = \pm \frac{1}{\sqrt{1 - \langle v, p \rangle^2}} (v - \langle v, p \rangle p).$$

Substituting in (3.2) for $t = t_0$ we get $q^{\pm}(v, p) = \pm (p - \langle v, p \rangle v) / \sqrt{1 - \langle v, p \rangle^2}$. The hyperbolic hyperquadric $HP(v, 0) \cap S_1^n$ has two connected components, $q^+(v, p)$ lies on one component and $q^-(v, p)$ on the other. □

We consider now projections along lightlike geodesics, which are lines in S_1^n parallel to lightlike vectors. An orthogonal space to a lightlike geodesic contains the geodesic, so we cannot define projections along lightlike geodesics to orthogonal spaces (which are cylinders). We shall fix instead a space transverse to all lightlike geodesics in S_1^n and project to this space. We denote by $\{e_0, \dots, e_n\}$ the canonical basis of $\mathbb{R}_1^{n+1}$. Any lightlike geodesic intersects transversally the elliptic de Sitter quadric $S^{n-1} = HP(e_0, 0) \cap S_1^n$, so we take S^{n-1} as the space to project to.

We fix a point in S^{n-1}, say $e_1 = (0, 1, 0, \dots, 0)$. A lightlike line through e_1 is parametrised by $e_1 + tv$, $t \in \mathbb{R}$, where $v \in S_+^{n-1} \subset LC^*$. This line lies is S_1^n if and only if $\langle v, e_1 \rangle = 0$. Thus, the lightlike geodesics in S_1^n that pass through e_1 can be parametrised by

$$S_+^{n-2} = \{(1, 0, v_2, \dots, v_n) \in \mathbb{R}^{n+1} : v_2^2 + \dots + v_n^2 = 1\} \subset S_+^{n-1} \subset LC^*.$$

Any lightlike geodesic in S_1^n can be obtained by rotating a lightlike geodesic through e_1. The rotation is in the form $A = \mathrm{Id}_{e_0} \times B$, where B is a rotation in $S^{n-1} = HP(e_0, 0) \cap S_1^n$. Given $v \in S_+^{n-2}$, the geodesics $A(e_1) + tA(v)$, $t \in \mathbb{R}$, obtained by varying $B \in \mathrm{SO}(n-1)$ foliate S_1^n. We can now define the projection in S_1^n along lightlike geodesics as follows.

Given a point $p \in S_1^n$ and $v \in S_+^{n-2}$, there exists a unique $A = \mathrm{Id}_{e_0} \times B$, with $B \in \mathrm{SO}(n-1)$, such that $A(p)$ belongs to the lightlike geodesic $e_1 + tv$. Then $A(p) = e_1 + \langle e_0, p \rangle v$. We define the lightlike projection of p to S^{n-1} along the direction v as the point

$$q(v, p) = A^{-1}(e_1) = A^{-1}(A(p) - \langle e_0, p \rangle v) = p - \langle e_0, p \rangle A^{-1}(v).$$

Definition 2. The family of projections along lightlike geodesics to the de Sitter elliptic hyperquadric S^{n-1} is defined by

$$\begin{array}{rccc} P_L: & S_+^{n-2} \times S_1^n & \to & S^{n-1} \\ & (v, p) & \mapsto & q(v, p) \end{array}$$

where $q(v, p) = p - \langle e_0, p \rangle A^{-1}(v)$, and $A = \mathrm{Id}_{e_0} \times B$, with $B \in \mathrm{SO}(n-1)$, is the unique rotation taking p to a point on the lightlike geodesic $e_1 + tv$, $t \in \mathbb{R}$.

Given an embedded submanifold M in S_1^n, the family of projections of M along geodesics refer to the restriction of the families P_L, P_S and P_T to M. We still denote this restriction by P_L, P_S and P_T respectively. We have the following result where the term generic is defined in terms of transversality to submanifolds of multi-jet spaces (see for example [**3**]).

Theorem 3.3. *For a residual set of embeddings $\boldsymbol{x} : M \to S_1^n$, the families P_L, P_S and P_T are generic families of mappings.*

Table 1: $\mathcal{A}_e$-codimension ≤ 3 local singularities of map-germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ ([**15**]).

Name	Normal form	$\mathcal{A}_e$-codimension
Immersion	(x, y)	0
Fold	(x, y^2)	0
Cusp	$(x, xy + y^3)$	0
4_k ($k = 2$ lips/beaks; $k = 3$ goose)	$(x, y^3 \pm x^k y), k = 2, 3, 4$	$k - 1$
5 (swallowtail)	$(x, xy + y^4)$	1
6 (butterfly)	$(x, xy + y^5 \pm y^7)$	2
7	$(x, xy + y^5)$	3
11_{2k+1} ($k = 2$ gulls)	$(x, xy^2 + y^4 + y^{2k+1}), k = 2, 3$	k
12	$(x, xy^2 + y^5 + y^6)$	3
16	$(x, x^2y + y^4 \pm y^5)$	3

Proof. The theorem follows from Montaldi's result in [**12**] and the fact that $P_L|_{S_0^{n-2}\times M}$, $P_S|_{S_1^n \times M - \{(v,v)\in S_1^n\times M\}}$ and $P_T|_{H^n(-1)\times M}$ are stable maps. □

4. *Projections of timelike surfaces in S_1^3*

A projection along a geodesic is singular at $p \in M$ if and only if the geodesic is tangent to M at p. Therefore, for spacelike surfaces (whose tangent spaces at all points are spacelike) the projections along timelike and lightlike geodesics are always local diffeomorphisms. The study of projections of spacelike surfaces along spacelike geodesics is similar to that of projections of surfaces in $H^3(-1)$ [**9**]. So we deal here with embedded timelike surfaces M in S_1^3. The projection of M at $p_0 \in M$ along a given geodesic can be represented locally by a map-germ from the plane to the plane. These map-germs are extensively studied. We refer to [**15**] for the list of the $\mathcal{A}$-orbits with $\mathcal{A}_e$-codimension ≤ 6, where $\mathcal{A}$ denotes the Mather group of smooth changes of coordinates in the source and target. In Table 1, we reproduce from [**15**] the list of local singularities of $\mathcal{A}_e$-codimension ≤ 3.

We study the local singularities of the projections along the three types of geodesics and characterise them geometrically.

4.1. *Projections along timelike and spacelike geodesics*

It follows from Theorem 3.3 that, for a generic embedding of a timelike surface in S_1^3, only singularities of $\mathcal{A}_e$-codimension ≤ 3 can occur in the members of the family of orthogonal projections of the surface along spacelike and timelike geodesics (3 being the dimension of the parameter spaces S_1^3 and $H^3(-1)$ respectively). We denote by P_S^v (resp. P_T^v) the map $M \to HP(v, 0) \cap S_1^3$ given by $P_S^v(p) = \pi \circ P_S(v, p)$ (resp. $P_T^v(p) = \pi \circ P_T(v, p)$), where π is the projection to the second component. The following result follows from Theorem 3.3.

PROPOSITION 4.1. *For a residual set of embeddings $\boldsymbol{x} : M \to S_1^3$, the projections P_S^v (resp. P_T^v) in the family P_S (resp. P_T) have local singularities $\mathcal{A}$-equivalent to one in Table 1. Moreover, these singularities are versally unfolded by the family P_S (resp. P_T).*

We seek to derive geometric information on M from the local singularities of the projections. We deal with the members of the family P_T and make an observation about those of P_S.

Given $v \in H^3(-1)$ and $p \in S_1^3$, we denote by v^* the parallel transport of v to p along a geodesic orthogonal to $HP(v, 0) \cap S_1^3$. From the proof of Theorem 3.1 we have $v^* =$

$(v - \langle v, p \rangle p) / \sqrt{1 + \langle v, p \rangle^2}$. We observe that the map $H^3(-1) \to T_p S_1^3 \cap H^3(-1)$ given by $v \mapsto v^*/||v^*||$ is a submersion, and the pre-image of a vector w is the curve

$$C_w(t) = \cosh(t)w + \sinh(t)p, t \in \mathbb{R}.$$

(There is a pencil of hyperplanes defining the same elliptic quadric in S_1^3.) We have the following result where the names of the singularities of $\tilde{P}_T^v$ are those in Table 1.

THEOREM 4.2. *Let M be an embedded timelike surface in S_1^3 and $v \in H^3(-1)$.*

(1) *The projection P_T^v is singular at a point $p \in M$ if and only if $v^* \in T_p M$.*

(2) *The singularity of P_T^v at p is of type cusp or worse if and only if v^* is a timelike asymptotic direction at p.*

(3) *The singularity of P_T^v at p is of Type 5 (i.e., swallowtail) if and only if v^* is a timelike asymptotic direction and p is on the flecnodal curve. The singularity is of Type 6 if and only if v^* is tangent to the flecnodal curve at p. At these tangency points, there are generically up to 8 directions on the curve $C_{v^*} \subset H^3(-1)$ where the singularity becomes of Type 7.*

(4) *The singularities of P_T^v at p is of type 4_k, $k = 2, 3, 4$, if and only if p is a parabolic point but not a folded singular point of the asymptotic curves and v^* is the unique timelike asymptotic direction there. There are up to 12 directions on the curve C_{v^*} where the singularity becomes of type 4_3. There are isolated points on the parabolic set where the singularity of the projection along these special directions becomes of Type 4_4.*

(5) *At folded singularity of the asymptotic curves there are up to 12 directions on the curve C_{v^*} where the singularity is of Type 16. Away from these directions the singularity is generically of Type 11_5, and for up to 38 directions on C_{v^*} it becomes of Type 11_7. The singularities of Type 12 do not occur in general.*

Proof. As our study is local in nature, we can make some assumptions about the position of the of surface patch and the choice of the geodesic. We shall assume that the surface patch is at some point p_0 and that this point is taken by a geodesic $\mathcal{C}_1$ to $e_1 = (0, 1, 0, 0) \in HP(e_0, 0) \cap S_1^3$. We also suppose that $e_0 = (1, 0, 0, 0)$ is tangent to $\mathcal{C}_1$ at e_1. The geodesic $\mathcal{C}_1$ can then be parametrised by

$$c_1(t) = \cosh(t)e_1 + \sinh(t)e_0,\ t \in \mathbb{R}.$$

We take, without loss of generality, the point on the surface to be $p_0 = (1, \sqrt{2}, 0, 0)$. The tangent to $\mathcal{C}_1$ at p_0 is parallel to $w = (\sqrt{2}, 1, 0, 0)$. Then the vectors $v \in H^3(-1)$ satisfying $w = v^*/||v^*||$ are in the form $v = (v_0, v_1, 0, 0)$ with $-v_0^2 + v_1^2 = -1$.

The surface patch at p_0 can be taken in Monge form

$$\phi(x, y) = (1 + x, \sqrt{1 + (1 + x)^2 - f^2(x, y) - y^2}, y, f(x, y))$$

where f is a smooth function in some neighbourhood U of the origin in $\mathbb{R}^2$, $(x, y) \in U$ and $f(0, 0) = 0$. (There is nothing special about the above setting, the results are local in nature and are valid for any $v \in H^3(-1)$ and at any point $p_0 \in M$.)

The projection to the elliptic quadric $HP(v, 0) \cap S_1^3 = HP(e_0, 0) \cap S_1^3$ in the direction v (with v as above) is then given by

$$P_T^v(x, y) = q(v, \phi(x, y)) = \frac{1}{\sqrt{1 + \langle v, \phi(x, y) \rangle^2}} \left(\phi(x, y) + \langle v, \phi(x, y) \rangle v \right)$$

(see Theorem 3.1). As we are interested in the $\mathcal{A}$-singularities of the projection, we can simplify the expression of P_T^v by projecting further to the tangent space of the elliptic quadric $HP(e_0, 0) \cap S_1^3$ at e_1. This tangent space is generated by $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ and we take the projection to this space to be the restriction of the canonical projection $\pi : \mathbb{R}^4 \to \mathbb{R}^2$, with $\pi(x_0, x_1, x_2, x_3) = (x_2, x_3)$. Therefore, the modified projection $\tilde{P}_T^{e_0} = \pi \circ P_T^{e_0}$ is a map-germ from the plane to the plane given by

$$\tilde{P}_T^v(x, y) = \frac{1}{\lambda(x, y, v)} \left(y, f(x, y)\right),$$

with $\lambda(x, y, v) = (1 + (-v_0(1 + x) + v_1\sqrt{1 + (1 + x)^2 - f^2(x, y) - y^2}\,)^2)^{1/2}$. The map-germ $\tilde{P}_T^v$ is singular at the origin if and only if $f_x(0, 0) = 0$, if and only if $v^* = w \in T_{p_0}M$.

We can make successive changes of coordinates in the sources and target and write the appropriate k-jet of $\tilde{P}_T^v$ in the form $(y, g(x, y))$. We can then obtain the conditions on the coefficients of the Taylor expansion of f for $\tilde{P}_T^v$ to have a given singularity at the origin. The calculations are carried out using Maple. For example, the 2-jet of $\tilde{P}_T^v$ is $\mathcal{A}$-equivalent to $(y, a_{20}x^2 + a_{21}xy)$. We have a fold singularity if and only if $a_{20} \neq 0$. The condition $a_{20} = 0$ means that $\phi_x(0, 0) = \sqrt{2}w$ is an asymptotic direction at p_0. The remaining calculations are done similarly but are too lengthy to reproduce here. □

For projections along a spacelike geodesic, we choose the geodesic $\mathcal{C}_2$ given by $c_2(t) = \cos(t)e_3 + \sin(t)e_1$ and project to the hyperbolic quadric $HP(e_1, 0) \cap S^3_1$. We take the point $p_0 = (0, \sqrt{2}/2, 0, \sqrt{2}/2)$ on the surface (and on $\mathcal{C}_2$) and project the surface patch around p_0 along geodesics parallel to $\mathcal{C}_2$. We take the surface in Monge form

$$\phi(x, y) = (f(x, y), \sqrt{1 + f^2(x, y) - (\frac{\sqrt{2}}{2} + x)^2 - y^2},\, y,\, \frac{\sqrt{2}}{2} + x)$$

with $f(0, 0) = f_x(0, 0) = 0$. The modified projection is then a map-germ from the plane to the plane and is given by

$$\tilde{P}_S^{e_1}(x, y) = \frac{1}{\sqrt{(\frac{\sqrt{2}}{2} + x)^2 + y^2 - f^2(x, y)}}(f(x, y), y).$$

We can obtain the conditions for $\tilde{P}_S^{e_1}$ to have a given singularity at the origin from the coefficients of the Taylor expansion of f and interpret these geometrically. The results are similar to those in Theorem 4.2. One needs to take $v \in S^3_1$ and replace timelike asymptotic direction by spacelike asymptotic direction in the statement of Theorem 4.2. The numbers of directions in the statements also need changing. In statement (3) we have up to 4 directions on C_{v^*} giving singularities of Type 7; in statement (4), there are up to 2 directions on C_{v^*} giving singularities of Type 4_3; in statement (5), there up to 16 directions on C_{v^*} where the singularity becomes of Type 11_7. There are generically no singularities of type 12 or 16.

4.2. *Projections along lightlike geodesics*

We shall give an explicit expression for the projection P_L (Definition 2) in S^3_1. Consider the sphere $S^2 = \{(0, v_1, v_2, v_3) : v_1^2 + v_2^2 + v_3^2 = 1\}$ (we shall drop the first coordinate of points in S^2). Let

$$T_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T_\phi = \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}$$

and their composite

$$T_{(\theta,\phi)} = T_\theta \circ T_\phi = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta & -\cos\theta\sin\phi \\ \sin\theta\cos\phi & \cos\theta & -\sin\theta\sin\phi \\ \sin\phi & 0 & \cos\phi \end{pmatrix}.$$

Any point on $S^2 - (0,0,\pm 1)$ is the image of the point $e_1 = (1,0,0)$ by a rotation $T(\theta, \phi)$, for some $(\theta, \phi) \in [0, 2\pi] \times] - \frac{\pi}{2}, \frac{\pi}{2}[$. (One can consider other rotations to cover the points $(0,0,\pm 1)$.)

We consider in S^3_1 the rotation $A = \mathrm{Id}_{e_0} \times T_{(\theta,\phi)}$. Let $v \in S^1_+ = \{(1,0,v_2,v_3) : v_2^2 + v_3^2 = 1\} \subset LC^*$. A point $p = (p_0, p_1, p_2, p_3)$ is projected to $q(v,p) \in S^2$ along the lightlike geodesic determined by v (see Definition 2). The point $q(v,p)$ is the image of e_1 by a rotation A for some (θ, ϕ). The point p is on the line $q(v,p) + tA(v)$, and we have $p = q(v,p) + p_0 A(v)$, that is,

$$\begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi \end{pmatrix} + p_0 \begin{pmatrix} 1 \\ -v_2\sin\theta - v_3\cos\theta\sin\phi \\ v_2\cos\theta - v_3\sin\theta\sin\phi \\ v_3\cos\phi \end{pmatrix}.$$

We suppose that $q(v,p) \neq (0,0,\pm 1)$, this implies that $p_1^2 + p_2^2 \neq 0$. We can then solve the above system for θ and ϕ and get

$$\begin{array}{rclcrcl} \cos\phi & = & \dfrac{p_0p_3v_3 + \sqrt{1 - p_3^2 + p_0^2v_3^2}}{1 + p_0^2v_3^2} & & \cos\theta & = & \dfrac{p_0p_2v_2 + p_1\sqrt{1 - p_3^2 + p_0^2v_3^2}}{p_1^2 + p_2^2} \\ \sin\phi & = & \dfrac{p_3 - p_0v_3\sqrt{1 - p_3^2 + p_0^2v_3^2}}{1 + p_0^2v_3^2} & & \sin\theta & = & \dfrac{-p_0p_1v_2 + p_2\sqrt{1 - p_3^2 + p_0^2v_3^2}}{p_1^2 + p_2^2} \end{array}$$

To analyse the singularities of P^v_L, we take $v = (1,0,0,1)$, the point $p_0 = e_1 + v$ so that $q(p_0, v) = e_1$. Then, a local parametrisation of the surface can be taken in the form

$$\phi(x,y) = (1 + x, \sqrt{1 + (1+x)^2 - y^2 - f^2(x,y)}, y, f(x,y))$$

with (x,y) is in some neighbourhood of the origin in $\mathbb{R}^2$, $f(0,0) = 1$ and $f_x(0,0) = 1$. We consider the modified projection $\tilde{P^v_L}$ by projecting further to the tangent space of $T_{e_1}S^2 = \{(0,0,x_2,x_3) : x_2, x_3 \in \mathbb{R}\}$, which we identify with $\mathbb{R}^2$. Then, the resulting map-germ from the plane to the plane is given by the last two components of $q(v,p)$, that is $(\sin\theta\cos\phi, \sin\phi)$, with $\sin\theta$, $\cos\phi$, $\sin\phi$ as above. That is,

$$\tilde{P^v_L} = \left(\frac{y\sqrt{1+(1+x)^2-f(x,y)^2}((1+x)f(x,y)+\sqrt{1+(1+x)^2-f(x,y)^2}}{(1+(1+x)^2)(1+(1+x)^2-f^2(x,y))}, \frac{f(x,y)-(1+x)\sqrt{1+(1+x)^2-f(x,y)^2}}{1+(1+x)^2}\right).$$

We take the 4-jet of f in the form

$$j^4 f(x,y) = 1 + x + a_{11}y + a_{20}x^2 + a_{21}xy + a_{22}y^2 + \sum_{i=0}^{3} a_{3i}x^{3-i}y^i + \sum_{i=0}^{4} a_{4i}x^{4-i}y^i.$$

The surface patch parametrised by ϕ is timelike if and only if $a_{11} \neq 0$. A short calculation shows that the 4-jet of the projection is $\mathcal{A}$-equivalent to the map-germ

$$(x,y) \mapsto (y, a_{20}x^2 + a_{21}xy + a_{30}x^3 + a_{31}x^2y - (\frac{1}{2}a_{11}^2 + a_1a_{21} - a_{32})xy^2 + f_4(x,y)),$$

with

$$\begin{array}{rl} f_4(x,y) = & (a_{40} + \frac{1}{2}a_{20}^2)x^4 + a_{41}x^3y - (\frac{1}{2}a_{11}^2a_{20} + a_{11}(a_{31} + a_{21}) + a_{20}a_{22} + \frac{1}{2}a_{21}^2 - a_{42})x^2y^2 \\ & +(\frac{1}{2}a_{43} - a_{21}a_{22} - a_{11}(a_{22} + a_{32}))xy^3. \end{array}$$

We have a fold singularity if and only if $a_{20} \neq 0$; a cusp singularity if and only if $a_{20} = 0$ and $a_{21}a_{30} \neq 0$; a lips/beaks singularity if and only if $a_{20} = a_{21} = 0$ and $a_{30}(3(a_1^2 + 2a_1a_{21} - 2a_{32})a_{30} + 2a_{31}^2) \neq 0$; a swallowtail singularity if and only if $a_{20} = a_{30} = 0$, and $a_{40} \neq 0$.

To interpret these conditions geometrically we consider the LPL, given by $\delta(x,y) = 0$ in expression (2.2) in section 2. We calculate the coefficients of the first and second fundamental forms and find that the point p_0 is on the LPL if and only if

$$a_{20}(a_{20} - 2a_{11}a_{21} + 4a_{20}a_{11}^4 + 4a_{11}^2a_{20} - 4a_{11}^3a_{21} + 4a_{11}^2a_{22}) = 0.$$

The asymptotic directions at p_0 are given by

$$a_{22}dy^2 + a_{21}dxdy + a_{20}dx^2 = 0.$$

In particular, the singularity of P_L^v is worse than fold ($a_{20} = 0$) if and only if $p_0 \in LPL$ and $v = \phi_x(0, 0)$ is a lightlike asymptotic direction. (Then v is also the double principal direction at p_0, see section 2.) The other asymptotic direction is not lightlike unless the LPL is singular.

When $a_{20} = 0$, we assume that $2a_{11}a_{22} - a_{21} - 2a_{11}^2a_{21} \neq 0$, otherwise the point p_0 is a timelike umbilic point so the LPL is singular there. At such points both asymptotic directions are lightlike and the projection P_L^v along these directions has a cusp singularity.

Suppose that $a_{20} = 0$. Then the 1-jet of the equation of the LPL is given by

$$(2a_{11}a_{22} - 2a_{11}^2a_{21} - a_{21})(-3a_{11}a_{30}x - (-a_{21}^2 + a_{11}a_{31})y).$$

As the surface is timelike, $a_{11} \neq 0$. Therefore, the singularity of the projection is of type swallowtail at p_0 or worse (i.e., $a_{30} = 0$) if and only if the lightlike direction v is tangent to the LPL. (This occur at precisely the folded singularities of the configuration of the principal curves [**10**].)

The point p_0 is on the parabolic set if and only if $a_{21}^2 - 4a_{20}a_{22} = 0$. Therefore, we have a lips/beaks singularity ($a_{20} = a_{21} = 0$) at the point of tangency of the LPL with the parabolic set. We have thus the following result.

THEOREM 4.3. *The projection P_L^v can have generically the local codimension ≤ 1 singularities in Table 1. The singularity of P_L^v at $p_0 \in M$ is of type*

(1) *fold if and only if v is not a lightlike asymptotic direction at p_0;*

(2) *cusp if and only if v is a lightlike asymptotic direction at p_0 and is transverse to the LPL;*

(3) *swallowtail if and only if v is lightlike asymptotic direction at p_0 and is tangent to the LPL;*

(4) *lips/beaks if and only if v is a lightlike asymptotic direction at p_0 and p_0 is the point of tangency of the LPL with the parabolic curve.*

5. *Duality*

We prove in this section duality result similar to those in [**8**, **9**], and to those in [**16**] for central projections of surfaces in $\mathbb{R}P^3$ and in [**2**] for orthogonal projections of surfaces in $\mathbb{R}^3$.

Let M be an embedded timelike surface in S_1^3. We shall use the duality concepts in [**4**, **5**, **6**], see §6 for details. We denote by A_2^{par} the ruled surface in S_1^3 swept out by the geodesics in S_1^3 passing through a parabolic point of M and with tangent direction there its unique asymptotic direction. We assume that the unique asymptotic direction is not lightlike, so the point is not on the LPL.

Let $p(t)$, $t \in I = (-a, a)$, $a > 0$, be a local parametrisation of the parabolic set of M and $u(t)$ be the unique unit asymptotic direction at $p(t)$. Recall that if $p_0 = p(t_0)$ is on the LPL, $u(t)$ is spacelike on one side of p_0 and timelike on the other side ([**10**] and section 2). The surface A_2^{par} has two connected components given by

$$\begin{aligned} A_2^{S-par} &= \{\cos(s)p(t) + \sin(s)u(t),\ (s,t) \in I \times J,\ \langle u(t), u(t)\rangle = 1\} \\ A_2^{T-par} &= \{\cosh(s)p(t) + \sinh(s)u(t),\ (s,t) \in I \times J,\ \langle u(t), u(t)\rangle = -1\} \end{aligned}$$

with $J = (-b, b)$, for some $b > 0$. We also denote by $A_1||A_1$ the ruled surface swept out by the spacelike or timelike geodesics in S_1^3 that are tangent to M at two points where the normals to M at such points are parallel (i.e., the projection P_S or P_T has a multi-local singularity of type double tangent fold). The surface $A_1||A_1$ has also two component determined by the type of bi-tangent geodesics. We have the following result, whose proof is similar to that of Theorem 3.8 in [**9**].

Theorem 5.1. *Let M^* be the Δ_5-dual of a timelike surface M embedded in S^3_1.*
(1) The Δ_5-dual of the surface A_2^{par} is the cuspidaledge of M^*.
(2) The Δ_5-dual of the surface $A_1||A_1$ is the self-intersection line of M^*.

Proof. (1) We deal with the A_2^{S-par} component (the calculations are similar for the component A_2^{T-par} and are omitted) and parametrise it as above by $y(s,t) = \cos(s)p(t) + \sin(s)u(t)$. The normal to the surface A_2^{S-par} is along

$$y \wedge y_s \wedge y_t = \cos^3(s)p(t) \wedge u(t) \wedge p'(t) + \sin^3(s)p(t) \wedge u(t) \wedge u'(t).$$

At a generic point p on the parabolic set, the asymptotic direction is transverse to the parabolic set, so $p(t) \wedge u(t) \wedge p'(t)$ is along $e(p(t))$. One can prove, following the same arguments in the proof of Lemma 3.11 in [**8**], that $p(t) \wedge u(t) \wedge u'(t)$ is also along $e(p(t))$. Therefore, $y \wedge y_s \wedge y_t$ is along $e(p(t))$, and it follows from this that the normal to the ruled surface A_2^{S-par} is constant along the rulings and is given by the normal vector $e(p(t))$ to M at $p(t)$. This means that A_2^{S-par} is a de Sitter developable surface (i.e., $K \equiv 0$ on A_2^{S-par}). Therefore, the Δ_5-dual of A_2^{S-par} is $\{e(p),\ p \text{ a parabolic point}\}$. This is precisely the singular set (i.e., the cuspidaledge) of M^*, the Δ_5-dual surface of M.

(2) Suppose a multi-local singularity (double tangent fold) occurs at two points p_1 and p_2 on M. The surface $A_1||A_1$ is then a ruled surface generated by spacelike geodesics along a curve C_1 on M through p_1, or a curve C_2 on M through p_2. The normals to the surface at points on C_1 and C_2 that are on the same ruling of $A_1||A_1$ are parallel. Let $q(t)$ be a local parametrisation of the curve C_1 and $u(t)$ be the unit tangent direction to the ruling in $A_1||A_1$ through $q(t)$. A parametrisation of $A_1||A_1$ is given by

$$w(s,t) = \cos(s)q(t) + \sin(s)u(t).$$

The normal to this surface is along $\cos^3(s)V_1(t) + \sin^3(s)V_2(t)$ with $V_1(t) = q(t) \wedge u(t) \wedge q'(t)$ and $V_2(t) = q(t) \wedge u(t) \wedge u'(t)$. These normals are parallel at two points on any ruling, one point being on the curve C_1 and the other on C_2. Therefore, $V_1(t)$ and $V_2(t)$ are parallel, so the normal to the surface $A_1||A_1$ is constant along the rulings of this surface. As these are along the normal to the surface at $q(t)$, it follows that the Δ_5-dual of $A_1||A_1$ is $\{e(p),\ p \in C_1\} = \{e(p),\ p \in C_2\}$. This is precisely the self-intersection line of M^*, the Δ_5-dual surface of M. □

We consider now some components of the bifurcation sets of the families of projections P_S and P_T.

Theorem 5.2. *Let M^* be the Δ_5-dual of a timelike surface M embedded in S^3_1. Then,*
(1) The local stratum $Bif(P_S, lips/beaks)$ of the bifurcation set of P_S, which consists of vectors $v \in S^3_1$ for which the projection P_S^v has a lips/beaks singularity, is a ruled surface. The Δ_5-dual of $Bif(P_S, lips/beaks)$ is the cuspidaledge of M^*.
(2) The multi-local stratum $Bif(P_S, DTF)$ of the bifurcation set of P_S, which consists of vectors $v \in S^3_1$ for which the projection P_S^v has a multi-local singularity of type double tangent fold, is a ruled surface. The Δ_5-dual of this ruled surface is the self-intersection line of M^*.
(3) The local stratum $Bif(P_T, lips/beaks)$ of the bifurcation set of P_T, which consists of vectors $v \in H^3(-1)$ for which the projection P_T^v has a lips/beaks singularity, is a ruled surface. The Δ_1-dual of $Bif(P_T, lips/beaks)$ is the cuspidaledge of M^*.

(4) *The multi-local stratum $Bif(P_T, DTF)$ of the bifurcation set of P_T, which consists of vectors $v \in H^3(-1)$ for which the projection P_T^v has a multi-local singularity of type double tangent fold, is a ruled surface. The Δ_5-dual of this ruled surface is the self-intersection line of M^*.*

Proof. We prove (1) as the proof of (2) is similar. It follows from Theorem 4.2(5) that the lips/beaks stratum $Bif(P_S, lips/beaks)$ of the family P_S is given by the set of $v \in S_1^3$ such that v^* is an asymptotic direction at a parabolic point p, where v^* denotes the parallel transport of v to p. Thus, $v^* = u(t)$ when $v \in Bif(P_S, lips/beaks)$, where $u(t)$ is the unique asymptotic direction at $p(t)$.

We have then

$$u(t) = v^* = \frac{1}{\sqrt{1 - \langle v, p(t)\rangle^2}} \left(v - \langle v, p(t)\rangle\, p(t)\right)$$

and hence

$$v = \sqrt{1 - \langle v, p(t)\rangle^2}\, u(t) + \langle v, p(t)\rangle\, p(t).$$

If we set $\sin(s) = \langle v, p(t)\rangle$ we get

$$Bif(P_P, lips/beaks) = \{\cos(s)u(t) + \sin(s)p(t), t \in I, s \in \mathbb{R}\},$$

which shows that $Bif(P_P, lips/beaks)$ is a ruled surface. For the duality result, following Remark 1, we need to find the unit normal vector to $Bif(P_S, lips/beaks)$. Following the same argument in the proof of Theorem 5.1, we find that the normal vector is constant along the rulings of the surface $Bif(P_S, lips/beaks)$ and is along $e(t)$, and the result follows.

□

REMARK 1. It is shown in [**16**] that other strata of the bifurcation set of the family of central projections of a surface in $\mathbb{R}P^3$ are also self-dual. For instance, the strata A_3, A_1^3 and $A_1 \times A_2$ are all self-dual. These results do not hold in our context. If we define the A_3 set as the surface formed by geodesics through points on the flecnodal curve and with tangent at these points along the associated asymptotic direction, then this surface is not in general a ruled surface. This means that the dual of the A_3-set is not the flecnodal curve on the Δ_5-dual surface of M. The situation is similar for the other strata.

6. *Appendix*

We require some properties of contact manifolds and Legendrian submanifolds for the duality results in this paper (for more details see for example [**1**]). Let N be a $(2n+1)$-dimensional smooth manifold and K be a field of tangent hyperplanes on N. Such a field is locally defined by a 1-form α. The tangent hyperplane field K is said to be *non-degenerate* if $\alpha \wedge (d\alpha)^n \neq 0$ at any point on N. The pair (N, K) is a *contact manifold* if K is a non-degenerate hyperplane field. In this case K is called a *contact structure* and α a *contact form*.

A submanifold $\boldsymbol{i} : L \subset N$ of a contact manifold (N, K) is said to be *Legendrian* if dim $L = n$ and $d\boldsymbol{i}_x(T_xL) \subset K_{\boldsymbol{i}(x)}$ at any $x \in L$. A smooth fibre bundle $\pi : E \to M$ is called a *Legendrian fibration* if its total space E is furnished with a contact structure and the fibres of π are Legendrian submanifolds. Let $\pi : E \to M$ be a Legendrian fibration. For a Legendrian submanifold $\boldsymbol{i} : L \subset E$, $\pi \circ \boldsymbol{i} : L \to M$ is called a *Legendrian map*. The image of the Legendrian map $\pi \circ \boldsymbol{i}$ is called a *wavefront set* of $\boldsymbol{i}$ and is denoted by $W(\boldsymbol{i})$.

The duality concepts we use in this paper is one of those introduced in [**4**, **5**, **6**], where five Legendrian double fibrations are considered on subsets of the product of two of the pseudo spheres $H^n(-1)$, S^n_1 and LC^*. We recall here only those that are needed in this paper:
(1) (a) $H^n(-1) \times S^n_1 \supset \Delta_1 = \{(v, w) \mid \langle v, w\rangle = 0\}$,
(b) $\pi_{11} : \Delta_1 \to H^n(-1)$, $\pi_{12} : \Delta_1 \to S^n_1$,
(c) $\theta_{11} = \langle dv, w\rangle|\Delta_1$, $\theta_{12} = \langle v, dw\rangle|\Delta_1$.

(5) (a) $S^n_1 \times S^n_1 \supset \Delta_5 = \{(v, w) \mid \langle v, w\rangle = 0\}$,
(b) $\pi_{51} : \Delta_5 \to S^n_1$, $\pi_{52} : \Delta_5 \to S^n_1$,
(c) $\theta_{51} = \langle dv, w\rangle|\Delta_5$, $\theta_{52} = \langle v, dw\rangle|\Delta_5$.

Here, $\pi_{i1}(v, w) = v$ and $\pi_{i2}(v, w) = w$ for $i = 1, 5$, $\langle dv, w\rangle = -w_0dv_0 + \sum_{i=1}^n w_idv_i$ and $\langle v, dw\rangle = -v_0dw_0 + \sum_{i=1}^n v_idw_i$. The 1-forms θ_{i1} and θ_{i2}, $i = 1, 5$, define the same tangent hyperplane field over Δ_i which is denoted by K_i.

Theorem 6.1. ([**4**, **5**, **6**]) *The pairs (Δ_i, K_i), $i = 1, 5$, are contact manifolds and π_{i1} and π_{i2} are Legendrian fibrations.*

Remarks 1.
(1) Given a Legendrian submanifold $\boldsymbol{i} : L \to \Delta_i$, $i = 1, 5$, Theorem 6.1 states that $\pi_{i1}(\boldsymbol{i}(L))$ is dual to $\pi_{i2}(\boldsymbol{i}(L))$ and vice-versa. We shall call this duality Δ_i-duality.
(2) If $\pi_{11}(\boldsymbol{i}(L))$ is smooth at a point $\pi_{11}(\boldsymbol{i}(u))$, then $\pi_{12}(\boldsymbol{i}(u))$ is the normal vector to the hypersurface $\pi_{11}(\boldsymbol{i}(L)) \subset H^n_+(-1)$ at $\pi_{11}(\boldsymbol{i}(u))$. Conversely, if $\pi_{12}(\boldsymbol{i}(L))$ is smooth at a point $\pi_{12}(\boldsymbol{i}(u))$, then $\pi_{11}(\boldsymbol{i}(u))$ is the normal vector to the hypersurface $\pi_{12}(\boldsymbol{i}(L)) \subset S^n_1$. The same properties hold for the Δ_5-duality.

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