Umbilics of surfaces in the Minkowski 3-space

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Abstract

We prove that any closed and convex surface in the Minkowski 3-space of class C^3 has at least two umbilic points. This shows that the Carathéodory conjecture for surfaces in the Euclidean 3-space is true for surfaces in the Minkowski 3-space.

1 Introduction

The Carathéodory conjecture states that any smooth closed and convex surface in the Euclidean 3-space has at least two umbilic points. Various attempts were made to prove this conjecture (see for example [4] for a survey article and [3] for the latest results on the problem using the mean curvature flow on the space of oriented lines in $\mathbb{R}^3$).

We prove in this paper that any closed and convex surface in the Minkowski 3-space of class C^3 has at least two umbilic points (Theorem 3.3). For ovaloids, we can even specify the nature of the umbilic points (Theorem 3.4). We give some preliminaries in section 2 and prove the main results in section 3.

2 Preliminaries

The *Minkowski space* $(\mathbb{R}^3_1, \langle , \rangle)$ is the vector space $\mathbb{R}^3$ endowed with the pseudo-scalar product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -u_0v_0 + u_1v_1 + u_2v_2$, for any $\boldsymbol{u} = (u_0, u_1, u_2)$ and $\boldsymbol{v} = (v_0, v_1, v_2)$

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in $\mathbb{R}^3_1$. We say that a non-zero vector $\boldsymbol{u} \in \mathbb{R}^3_1$ is *spacelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$, *lightlike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and *timelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$. The norm of a vector $\boldsymbol{u} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$. The set of all lightlike vectors form the lightcone

$$LC^* = \{\boldsymbol{u} \in \mathbb{R}^3_1 \setminus \{\underline{0}\} \,|\, \langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0\}.$$

The lightcone can be considered as the cone in $\mathbb{R}^3$ minus the origin given by

$$\{(u_0, u_1, u_2) \in \mathbb{R}^3 \setminus \{(0,0,0)\} \mid -u_0^2 + u_1^2 + u_2^2 = 0\}.$$

A plane $P_c^{\boldsymbol{v}} = \{\boldsymbol{u} \in \mathbb{R}^3_1 \,|\, \langle \boldsymbol{u}, \boldsymbol{v} \rangle = c\}$, for some constant $c \in \mathbb{R}$, is called respectively, spacelike, timelike or lightlike if $\boldsymbol{v}$ is timelike, spacelike or lightlike. Fixing $\boldsymbol{v}$ and varying c gives a family of parallel planes with $P_0^{\boldsymbol{v}}$ passing through the origin (i.e., is a vector space). The vector $\boldsymbol{v}$ is called the *"normal"* vector to $P_c^{\boldsymbol{v}}$. Every non-zero vector in a spacelike plane $P_0^{\boldsymbol{v}}$ is spacelike. There are two linearly independent lightlike vectors in a timelike plane $P_0^{\boldsymbol{v}}$ and a unique lightlike vector in a lightlike plane $P_0^{\boldsymbol{v}}$. The normal vector $\boldsymbol{v}$ is transverse to $P_c^{\boldsymbol{v}}$ if this plane is spacelike or timelike but determines the unique lightlike direction in $P_0^{\boldsymbol{v}}$ if the plane $P_c^{\boldsymbol{v}}$ is lightlike.

Let S be a surface in $\mathbb{R}^3_1$ (of class C^3). The pseudo-scalar product in $\mathbb{R}^3_1$ induces a metric on S. If S is closed, then this metric must be degenerate at some point on S (see for example Lemma 3.1). This happens at points p on S where the tangent space T_pS is a lightlike plane. We call the locus of points where the induced metric on S is degenerate *the Locus of Degeneracy* and denote it by LD.

Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to S$ be a local parametrisation of S and let

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle, \quad G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle$$

denote the coefficients of the first fundamental form of S with respect to $\boldsymbol{x}$. We identify the LD and its pre-image in U by $\boldsymbol{x}$. Then the LD (in U) is the zero set of the C^2-function $\delta(u,v) = (F^2 - EG)(u,v)$. Therefore, the LD is a closed subset of S. We observe that the LD of a generic closed surface is a smooth curve, but we do not make the genericity assumption here. We can have, for instance, a convex surface with an LD that has interior points.

Pei [6] defined an $\mathbb{R}P^2$-valued Gauss map on S. In $\boldsymbol{x}(U)$, this is simply the map $PN : \boldsymbol{x}(U) \to \mathbb{R}P^2$ which associates to a point $p = \boldsymbol{x}(q)$ the projectivisation of the vector $(\boldsymbol{x}_u \times \boldsymbol{x}_v)(q)$, where "$\times$" denotes the wedge product in $\mathbb{R}^3_1$. Away from the LD, the $\mathbb{R}P^2$-valued Gauss map can be identified with the de Sitter Gauss map $\boldsymbol{x}(U_1) \to S^2_1$ on the Lorentzian part of the surface and with the hyperbolic Gauss map $\boldsymbol{x}(U_2) \to H^2(-1)$ on its Riemannian part. (Here U_1 and U_2 are open sets with $U = U_1 \cup U_2 \cup LD$.) Both maps are given by $\boldsymbol{N} = \boldsymbol{x}_u \times \boldsymbol{x}_v / ||\boldsymbol{x}_u \times \boldsymbol{x}_v||$. The shape operator $A_p(\boldsymbol{v}) = -d\boldsymbol{N}_p(\boldsymbol{v})$ is a self-adjoint operator on $\boldsymbol{x}(U) \setminus LD$. We denote by

$$\begin{array}{rcccl} l &=& -\langle \boldsymbol{N}_u, \boldsymbol{x}_u \rangle &=& \langle \boldsymbol{N}, \boldsymbol{x}_{uu} \rangle, \\ m &=& -\langle \boldsymbol{N}_u, \boldsymbol{x}_v \rangle &=& \langle \boldsymbol{N}, \boldsymbol{x}_{uv} \rangle, \\ n &=& -\langle \boldsymbol{N}_v, \boldsymbol{x}_v \rangle &=& \langle \boldsymbol{N}, \boldsymbol{x}_{vv} \rangle \end{array}$$

the coefficients of the second fundamental form on $\boldsymbol{x}(U) \setminus LD$. When A_p has real eigenvalues, we call them the *principal curvatures* and their associated eigenvectors the *principal directions* of S at p. We observe that there are always two principal curvatures at points in the Riemannian part of S but this is not true at points in its Lorentzian part ([5]). The lines of principal curvature, which are the integral curves of the principal directions, are solutions of the binary quadratic differential equation (BDE for short)

$$(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)du^2 = 0. \tag{1}$$

The discriminant of the above equation, which is the set points in $U \setminus LD$ where it determines a unique direction, is denoted the *Lightlike Principal Locus* (LPL) in [5]. It is the zero set of the function $((Gl - En)^2 - 4(Gm - Fn)(Fl - Em))(u, v)$.

A *spacelike umbilic point* (resp. *timelike umbilic point*) is a point in the Riemannian part (resp. Lorentzian part) of the surface where the coefficients of equation (1) vanish simultaneously. (The coefficients of a BDE are its coefficients when viewed as a quadratic form in du and dv.) Spacelike and timelike umbilic points can also be characterised as the points p where the shape operator A_p is a multiple of the identity map.

One can extend the lines of principal curvature across the LD as follows ([5]). As equation (1) is homogeneous in l, m, n, we can multiply these coefficients by $||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ and substitute them by

$$\bar{l} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uu} \rangle, \quad \bar{m} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uv} \rangle, \quad \bar{n} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{vv} \rangle.$$

This substitution does not alter the pair of foliations on $\boldsymbol{x}(U) \setminus LD$. The new equation is defined on the LD and defines the same pair of foliations associated to the de Sitter (resp. hyperbolic) Gauss map on the Lorentzian (resp. Riemannian) part of $\boldsymbol{x}(U)$. The extended lines of principal curvature are the solution curves of the BDE

$$(G\bar{m} - F\bar{n})dv^2 + (G\bar{l} - E\bar{n})dudv + (F\bar{l} - E\bar{m})du^2 = 0. \tag{2}$$

We still call the directions determined by equation (2) at points on the LD principal directions. We do not have a shape operator at points on the LD. For this reason, we define a *lightlike umbilic point* as a point on the LD where the coefficients of equation (2) vanish simultaneously.

We say that a point on S is an *umbilic point* if it is either a spacelike, timelike or lightlike umbilic point. Thus, a point $p = \boldsymbol{x}(q)$ is an umbilic point if and only if all the coefficients of equation (2) vanish at q.

Remark 2.1 The lines of principal curvatures on a generic surface in $\mathbb{R}^3_1$ are studied in [5]. On the Riemannian part of a generic surface, the LPL, when not empty, consists of isolated points which are spacelike umbilic points. Away from these points, there are always two orthogonal spacelike principal directions. On the Lorentzian part of a generic surface, the LPL, when not empty, is a smooth curve except at isolated points where it has Morse singularities of node type. The singular points of the LPL are precisely the timelike umbilic points. The regular points of the LPL consist of points where the principal directions coincide and become lightlike. There are two principal directions on one side of the LPL and none on the other. When there are two of them, they are orthogonal and one is spacelike while the other is timelike.

Equation (2) determines two directions in T_pS at most points p on the LD. One of these directions is the unique lightlike direction in T_pS and the other is a spacelike. The two directions coincide and become the unique lightlike direction in T_pS at isolated points p on the LD. Generic surfaces do not have lightlike umbilic points. The generic local topological configurations of the lines of principal curvature at points on the LPL and on the LD are given in [5].

We consider here closed and convex surfaces in $\mathbb{R}^3_1$. Convexity is an affine property so is independent of the metric (Euclidean or Lorentzian) in $\mathbb{R}^3$.

We also consider some special convex surfaces. An ovaloid in the Euclidean space $\mathbb{R}^3$ is defined as a surface with everywhere strictly positive Gaussian curvature K. We do not have the concept of Gaussian curvature of a surface in the Minkowski space $\mathbb{R}^3_1$ at point on the LD. (In fact, for generic surfaces, the Gaussian curvature tends to infinity as a point on the LD is approached from either the Riemannian or the Lorentzian part of the surface; see [7].) However we can still define the concept of ovaloids using the contact of the surface with planes (which is an affine property of the surface).

Let $P_c^{\boldsymbol{v}} = \{p \in \mathbb{R}^3 \,|\, \langle p, \boldsymbol{v} \rangle = c\}$ be a plane in $\mathbb{R}^3_1$. The contact of a surface S with $P_c^{\boldsymbol{v}}$ is measured by the singularities of the height function $h : S \to \mathbb{R}$, given by

$$h(p) = \langle p, \boldsymbol{v} \rangle.$$

We say that the contact is of type A_1^+ at $p \in S$ if $p \in P_c^{\boldsymbol{v}}$ and the height function h has a Morse singularity of index 0 or 2 at p, i.e., h can be written in some local coordinate system at p in S in the form $\pm(u^2 + v^2)$. For this, it is necessary and sufficient for the Taylor polynomial of degree 2 of h at p to be a strictly positive or a strictly negative quadratic form.

We say that a closed and convex surface S is an *ovaloid* if it has an A_1^+-contact with its tangent plane T_pS at all $p \in S$. An example of an ovaloid in the Minkowski space $\mathbb{R}^3_1$ is (the "Euclidean sphere")

$$S^2 = \{(u_0, u_1, u_2) \in \mathbb{R}^3_1 \,|\, u_0^2 + u_1^2 + u_2^2 = 1\}.$$

The surface $S^2 \subset \mathbb{R}^3_1$ has two umbilic points ([5] section 4.4.), so is not a totally umbilic surface. (See [2] for the study of geodesics on an ellipsoid in $\mathbb{R}^3_1$.)

A surface S is *locally convex* at $p \in S$ if there exists a neighbourhood V of p in S such that V is contained in one of the closed half-spaces determined by the tangent plane T_pS. A convex surface is of course locally convex. Given a local parametrisation $\boldsymbol{x} : U \to S$ of the surface S and $q_0 = (u_0, v_0) \in U$, the height function h along the "normal vector" $\boldsymbol{v} = (\boldsymbol{x}_u \times \boldsymbol{x}_v)(q_0)$ at q_0 can be considered locally as a map $U \to \mathbb{R}$, given by $h(u, v) = \langle \boldsymbol{x}(u, v), \boldsymbol{v}\rangle$. The Taylor polynomial of degree 2 of h at $q_0 = (u_0, v_0) \in U$ is given by

$$\frac{1}{2}\left(h_{uu}(q_0)(u-u_0)^2 + 2h_{uv}(q_0)(u-u_0)(v-v_0) + h_{vv}(q_0)(v-v_0)^2\right)$$

and a necessary condition for S to be locally convex at $p_0 = \boldsymbol{x}(q_0)$ is that

$$(h_{uv}^2 - h_{uu}h_{vv})(q_0) \leq 0.$$

The above condition is true at any point on S including points on the LD.

3 The main results

The proof of the main result relies on the structure of the LD and on the directions determined by equation (2) on this set.

Lemma 3.1 *The LD of a closed surface S in $\mathbb{R}^3_1$ of class C^1 is the union of at least two disjoint non-empty closed subsets of S.*

Proof The LD is the set of points on S where the tangent plane to S is lightlike. Lightlike planes are those tangent to the lightcone LC^* and a key observation is that these planes can be captured by changing the metric on $\mathbb{R}^3$.

We change the metric in $\mathbb{R}^3$ and consider $S \subset \mathbb{R}^3_1$ as a surface $\tilde{S}$ in the Euclidean space $\mathbb{R}^3$. Since $\tilde{S}$ is closed, the image of its Gauss map $N : \tilde{S} \to S^2$ is the whole sphere S^2.

The unit Euclidean normals to the tangent planes to LC^* (viewed as a cone in $\mathbb{R}^3$, see §1) trace the two circles $u_0 = \pm 1/\sqrt{2}$ on S^2. The LD of S is precisely the pre-image of the two circles $u_0 = \pm 1/\sqrt{2}$ by the Gauss map N on $\tilde{S}$. Therefore, the LD consists of at least two disjoint non-empty closed subsets of S. □

Lemma 3.2 *Let S be a closed and convex surface in $\mathbb{R}^3_1$ of class C^3 and $\boldsymbol{x} : U \to S$ a local parametrisation of S.*

(1) *The singular points of $\delta = F^2 - EG$ on the LD are lightlike umbilic points.*

(2) *The unique lightlike principal direction in T_pS at the regular points of δ on the LD is transverse to the LD.*

Proof If $E(q) = 0$ or $G(q) = 0$ at $q \in U$ with $\boldsymbol{x}(q) \in LD$, then $F(q) = 0$. Therefore, we cannot have $E(q) = G(q) = 0$ at points on the LD. We assume, without loss of generality, that $G \neq 0$ on U.

The lightlike directions at points in $\boldsymbol{x}(U)$ are solutions of the equation

$$Gdv^2 + 2Fdudv + Edu^2 = 0,$$

and the unique lightlike direction on the LD is parallel to $G\boldsymbol{x}_u - F\boldsymbol{x}_v$. This is a smooth vector field on $\boldsymbol{x}(U)$, so we can re-parametrise $\boldsymbol{x}(U)$ so that one of the coordinate curves are the integral curves of this vector field. That is, we can choose a local parametrisation of S, that we still denote by $\boldsymbol{x}$, so that the unique lightlike direction on the LD is along $\boldsymbol{x}_u$. With this parametrisation, that we use in the rest of the proof, $E = F = 0$ on the LD.

(1) The function δ is singular on the LD if and only if $(-E_uG, -E_vG) = (0, 0)$. The coefficients of equation (2) become $(G\bar{m}, G\bar{l}, 0)$ on the LD, and on this set we also have $\boldsymbol{x}_u \times \boldsymbol{x}_v = \lambda \boldsymbol{x}_u$ for some non-zero function λ. Therefore,

$$\bar{l} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uu} \rangle = \lambda \langle \boldsymbol{x}_u, \boldsymbol{x}_{uu} \rangle = \frac{1}{2}\lambda E_u,$$

and similarly,

$$\bar{m} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uv} \rangle = \lambda \langle \boldsymbol{x}_u, \boldsymbol{x}_{uv} \rangle = \frac{1}{2}\lambda E_v.$$

Thus, the coefficients of equation (2) at points on the LD are $(\lambda E_v G, \lambda E_u G, 0)$, and this is $(0, 0, 0)$ at a point $q \in LD$ if and only if the δ is singular at q.

(2) Suppose now that δ is regular on the LD (so the LD is a regular curve). Then we have either $E_u \neq 0$ or $E_v \neq 0$ on this curve. We consider the contact of S with its tangent plane $T_{p_0}S$ at $p_0 = \boldsymbol{x}(q_0) \in LD$. The Taylor polynomial of degree 2 of the height function $h(u, v) = \langle \boldsymbol{x}(u, v), \boldsymbol{x}_u(q_0) \rangle$ along the lightlike "normal vector" $\boldsymbol{x}_u(q_0)$ at q_0 is given by

$$\frac{1}{2}\left(h_{uu}(q_0)(u - u_0)^2 + 2h_{uv}(q_0)(u - u_0)(v - v_0) + h_{vv}(q_0)(v - v_0)^2\right),$$

with

$$\begin{aligned} h_{uu}(q_0) &= \langle \boldsymbol{x}_{uu}(q_0), \boldsymbol{x}_u(q_0) \rangle = \tfrac{1}{2}E_u(q_0), \\ h_{uv}(q_0) &= \langle \boldsymbol{x}_{uv}(q_0), \boldsymbol{x}_u(q_0) \rangle = \tfrac{1}{2}E_v(q_0), \\ h_{vv}(q_0) &= \langle \boldsymbol{x}_{vv}(q_0), \boldsymbol{x}_u(q_0) \rangle = (F_v - \tfrac{1}{2}G_u)(q_0). \end{aligned}$$

The lightlike direction $\boldsymbol{x}_u(q_0)$ is tangent to the LD at $p_0 = \boldsymbol{x}(q_0)$ if and only if $E_u(q_0) = 0$. But as S is convex, $(h_{uv}^2 - h_{uu}h_{vv})(q_0) = \frac{1}{4}(E_v^2 - 2(F_v - \frac{1}{2}G_u)E_u)(q_0) \leq 0$, so $E_u(q_0) = 0$ implies $E_v(q_0) = 0$, and consequently the LD is singular. Therefore, $E_u \neq 0$ at regular points of δ on the LD, that is, the lightlike principal direction is transverse to the LD at the regular points of δ on this set. $\square$

Theorem 3.3 *Let S be a closed and convex surface of class C^3 in $\mathbb{R}^3_1$. Then S has at least two umbilic points.*

Proof Consider the C^3-function $f : S \to \mathbb{R}$ given by $f(p) = p_0$ for any $p = (p_0, p_1, p_2) \in S$. It has a global minimum p_{min} and a global maximum p_{max} (these points need not be unique). The tangent planes to S at p_{min} and p_{max} are spacelike (both are given by $u_0 = 0$). Therefore, p_{min} and p_{max} belong to the Riemannian part of S. Suppose that they belong to the same Riemannian connected component R of S. Let $\gamma : [0, 1] \to R$ be a continuous path in R with $\gamma(0) = p_{min}$ and $\gamma(1) = p_{max}$, and consider the Gauss map $N : \tilde{S} \to S^2$ as in the proof of Lemma 3.1. The continuous curve $N \circ \gamma$ satisfies $N \circ \gamma(0) = (-1, 0, 0)$ and $N \circ \gamma(1) = (1, 0, 0)$, so there exists $t_0 \in (0, 1)$ such that $N \circ \gamma(t_0)$ belongs the equator $u_0 = 0$ on S^2. Therefore, the tangent space to S at $\gamma(t_0)$ is a timelike plane, which is a contradiction as R is supposed to be Riemannian.

Let R_1 (resp. R_2) denotes the Riemannian connected component of S which contains p_{min} (resp. p_{max}) and let L_1 (resp. L_2) be its boundary. The sets L_1 and L_2 are part of the LD. It follows from the proof of Lemma 3.1 that L_1 and L_2 are disjoint sets (L_1 is part of the pre-image of the circle $u_0 = -1/\sqrt{2}$ by the Gauss map N, and L_2 is part of the pre-image of the circle $u_0 = 1/\sqrt{2}$ by N).

We consider local parametrisations of S at points on L_1 and L_2. If $\delta = F^2 - EG$ is singular on L_1 and L_2, then the singular points are lightlike umbilic points (Lemma 3.2(1)). As L_1 and L_2 are disjoint, we get at least two umbilic points on S.

Suppose that δ is regular on L_1 (so L_1 is a regular curve; it is also a closed curve). The surface S being closed and convex is homeomorphic to a 2-sphere. Thus R_1 is homeomorphic to a disc. Consider the direction field in R_1 given by equation (2) and which agrees with the unique lightlike direction in T_pS for all $p \in L_1$. This direction field is transverse to L_1 (Lemma 3.2(2)), so by Poincaré-Hopf theorem it must have at least one singularity in R_1. This singularity is a spacelike umbilic point as R_1 is a Riemannian region. We proceed similarly if δ is regular on L_2 to get a second umbilic point of S. If δ is singular at a point on L_2, the singularity is a lightlike umbilic point and gives a second umbilic point of S. $\square$

We showed in [5] section 4.4 that (the Euclidean sphere) S^2 has exactly two umbilic points and both of them are spacelike. We have the following general result.

Theorem 3.4 *The umbilic points of an ovaloid in $\mathbb{R}^3_1$ of class C^3 are all spacelike and there are at least two of them.*

Proof We change the metric in $\mathbb{R}^3$ and consider an ovaloid $S \subset \mathbb{R}^3_1$ as a surface $\tilde{S}$ in the Euclidean space $\mathbb{R}^3$. The fact that the contact of S with its tangent plane (which is independent of the metric) is A_1^+ implies that the Gaussian curvature of $\tilde{S}$ is strictly positive. By Hadamard's theorem, the Gauss map $N : \tilde{S} \to S^2$ is a diffeomorphism

([1]). This implies that the LD of S is the union of two regular (non-empty) disjoint closed curves. These split the surface into three parts, two of them are Riemannian and one is Lorentzian. By Lemma 3.2(2), the unique lightlike principal direction on the LD is transverse to the LD. By Poincaré-Hopf theorem, there is at least one spacelike umbilic point in each Riemannian disc of S.

We now show that there are no timelike umbilic points on S. The timelike umbilic points occur in the Lorentzian part of the surface, so we can take a local parametrisation $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$ where the coordinate curves are lightlike (Theorem 3.1 in [5]). Then $E = G = 0$ in U. The equation of the lines of principal curvature simplifies to $ndv^2 - ldu^2 = 0$, so the timelike umbilic points are the solutions of $l = n = 0$.

Let $q_0 = (u_0, v_0) \in U$ and consider the height function h on $\boldsymbol{x}(U)$ along the unit normal vector $\boldsymbol{N}(q_0) = (\boldsymbol{x}_u \times \boldsymbol{x}_v / ||\boldsymbol{x}_u \times \boldsymbol{x}_v||)(q_0)$. The Taylor polynomial of $h(u,v) = \langle \boldsymbol{x}(u,v), \boldsymbol{N}(q_0)\rangle$ at q_0 is given by

$$\frac{1}{2}\left(l(q_0)(u-u_0)^2 + 2m(q_0)(u-u_0)(v-v_0) + n(q_0)(v-v_0)^2\right),$$

where l, m, n are the coefficients of the second fundamental form. As S is an ovaloid, $(m^2 - nl)(q) < 0$ for any $q \in U$ and consequently $l(q)n(q) \neq 0$ at any $q \in U$. This proves that there are no timelike umbilic points on S. $\square$

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