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Pairs of foliations on surfaces

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Dedicated to the memory of Professor Carlos Gutierrez

Abstract

We survey in this paper results on a particular set of Implicit Differential Equations (IDEs) on smooth surfaces, called Binary/Quadratic Differential Equations (BDEs). These equations define at most two solution curves at each point on the surface, resulting in a pair of foliations in some region of the surface. BDEs appear naturally in differential geometry and in control theory. The examples we give here are all from differential geometry. They include natural families of BDEs on surfaces. We review the techniques used to obtain local models of BDEs (formal, analytic, smooth and topological). We also discuss some invariants of BDEs and present a framework for studying their bifurcations in generic families.

1. *Introduction*

An implicit differential equation (IDE) is an equation of the form

$$F(x,y,p)=0,\quad p=\frac{dy}{dx} \tag{1.1}$$

where F is a smooth (i.e., C^∞) or real analytic function in some domain in $\mathbb{R}^3$. If $F(q_0)=0$ and $F_p(q_0)\neq 0$ at $q_0=(x_0,y_0,p_0)\in\mathbb{R}^3$ (when not indicated otherwise, subscripts denote partial differentiation), equation (1.1) can be written locally in a neighbourhood of q_0 in the form $p=g(x,y)$. It can then be studied using the methods from the theory of ordinary differential equations.

When $F(q_0)=F_p(q_0)=0$, there may be more then one solution curve of equation (1.1) through points in a neighbourhood U of (x_0,y_0). We deal here mainly with the case when $F_{pp}(q_0)\neq 0$, so there are at most two solution curves through each point in U. In this case, it follows from the division theorem that equation (1.1) can be expressed in a quadratic form

$$a(x,y)dy^2+2b(x,y)dxdy+c(x,y)dx^2=0 \tag{1.2}$$

where a,b,c are smooth or analytic functions in some neighbourhood U of (x_0,y_0) not vanishing simultaneously at any point in U. Equation (1.2) is called a *Binary Differential Equation* (BDE) or *Quadratic Differential Equation*. The functions a,b,c are called the *coefficients of the BDE*.

It is also of interest to study BDEs at points where their coefficients vanish at a given point. We shall label these *Type 2 BDEs* and reserve the label *Type 1 BDEs* to those with coefficients not vanishing simultaneously at any point. There are some crucial differences between the two types of BDEs. For instance, Type 1 BDEs may have finite codimension in the set of all IDEs and can be deformed in this set. However, Type 2 BDEs are of infinite codimension in the set of all IDEs and are deformed in the set of all BDEs. Other differences will be highlighted in the paper.

The *discriminant* of a BDE is the set $\Delta=\{(x,y)\in U:\,(b^2-ac)(x,y)=0\}$. A BDE determines a pair of transverse foliations or no foliations away from the discriminant. Thus, all

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the important features of the equation occur on the discriminant. The discriminant, together with the pair of foliations determined by the BDE is called *the configuration* of the BDE.

BDEs have a long history (see for example [**25**] and [**62**] for historical notes). They appear in, and have application to, control theory, partial differential equations and differential geometry. The examples in this paper are from differential geometry. For applications to control theory see [**25**, **51**]. The paper is organised as follows:

§2: lines of curvature, asymptotic and characteristic curves are classical pairs of foliations on surfaces and are given by BDEs. We consider their configurations on a surface endowed with a Riemannian or a Lorentzian metric. We also discuss the case of surfaces endowed with a metric of mixed type.

§3: the examples in §2 provide a good motivation for seeking models of the configurations of BDEs at points on the discriminant. We review the techniques involved for finding such models and clarify the meaning of the word model (up to formal, analytic, smooth or topological equivalence). We also give a complete list of local singularities of topological codimension ≤ 2.

§4: the discriminant of a BDE is a plane curve. However, the deformation of its singularities cannot always be modelled by the $\mathcal{K}$-deformations of a plane curve singularity. We review some invariants associated to BDEs and review Bruce's symmetric matrices framework ([**6**]) for studying the singularities of the discriminant.

§5: we review briefly a method for studying the bifurcations of a BDE in generic families of BDEs.

§6: we highlight references where work on more general IDEs and homogeneous differential equations of a given degree is carried out. We also give a list of existing local topological models of BDEs of codimension > 2.

An important aspect of BDE which is omitted here is the study of their foliations near a limit cycle. This study is initiated in the pioneering work of Sotomayor and Gutierrez [**64**] where they obtained a formula for the derivative of the Poincaré return map at a limit cycle of the lines of principal curvature on a smooth surface in $\mathbb{R}^3$. The behaviour of the asymptotic and characteristic curves at a limit cycle is also studied in [**32**, **33**].

2. *Examples from differential geometry*

Let S be a smooth surface. We start with the case when S is immersed in the Euclidean space $\mathbb{R}^3$ and denote by "." the scalar product in $\mathbb{R}^3$. Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a local parametrisation of S, and S^2 denotes the unit sphere in $\mathbb{R}^3$. The Gauss map

$$N : \boldsymbol{x}(U) \subset S \to S^2,$$

assigns to each point $p = \boldsymbol{x}(u, v)$ the normal vector $N(p) = (\boldsymbol{x}_u \times \boldsymbol{x}_v/||\boldsymbol{x}_u \times \boldsymbol{x}_v||)(u, v)$ to S at p.

The shape operator $A_p = -d_pN : T_pS \to T_pS$ (or the Weingarten map) has the following properties: it is a self-adjoint operator, i.e., a linear operator with $A_p(w_1).w_2 = w_1.A_p(w_2)$, for any pair of vectors in T_pS; it has always two real eigenvalues κ_1, κ_2 called *the principal curvatures*; it has two orthogonal eigenvectors (when $\kappa_1 \neq \kappa_2$) called the *principal directions*. The integral curves on S of the principal directions line fields are called the *lines of principal curvature*. The points where $\kappa_1 = \kappa_2$ are referred to as *umbilic* points. For generic immersions, the umbilic points are isolated points on S.

Let $E = \boldsymbol{x}_u.\boldsymbol{x}_u$, $F = \boldsymbol{x}_u.\boldsymbol{x}_v$, $G = \boldsymbol{x}_v.\boldsymbol{x}_v$ denote the coefficients of the first fundamental form and $l = N.\boldsymbol{x}_{uu}$, $m = N.\boldsymbol{x}_{uv} = N.\boldsymbol{x}_{vu}$, $n = N.\boldsymbol{x}_{vv}$ those of the second fundamental form on S. Then, the equation of the lines of principal curvature is given by the BDE

$$(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)dv^2 = 0. \tag{2.1}$$

The discriminant of equation (2.1) is the set of umbilic points. Away from such points the lines of curvature form a net of orthogonal curves. The coefficients of equation (2.1) vanish at umbilic points, so we have a BDE of Type 2 there. There are generically three distinct topological configurations of the lines of curvature at umbilic points (Figure 6, top three figures). The configurations were first drawn by Darboux and a rigorous proof was given in [**7**, **64**]. The global behaviour of the lines of principal curvature on closed orientable surfaces in $\mathbb{R}^3$ was first studied in [**64**]. (For historical notes on the study of the lines of curvature see [**62**].)

Two directions $w_1, w_2 \in T_pS$ are *conjugate* if $A_p(w_1).w_2 = 0$. An *asymptotic* direction is a self-conjugate direction, that is $A_p(w).w = 0$. There are two asymptotic directions at a hyperbolic point and none at an elliptic point on the surface. The integral curves of the pair of asymptotic line fields are called the *asymptotic curves*. The equation of the asymptotic curves is given by the BDE

$$ndv^2 + 2mdvdu + ldu^2 = 0. \tag{2.2}$$

The discriminant of equation (2.2) is the parabolic set of the surface. The asymptotic curves form a family of cusps at a generic parabolic point. Their configurations at a cusp of Gauss are given in [**3**, **4**, **50**] (Figure 2, last three figures) and a more general approach for studying the singularities of their equation at such points is given in [**23**, **24**, **51**, **70**]. Generic global properties of these foliations including the study of their limit cycles are given in [**33**].

At elliptic points there is a unique pair of conjugate directions for which the included angle is extremal ([**28**]). These directions are called the *characteristic directions* and their integral curves are called the characteristic curves. Characteristic directions on surfaces in $\mathbb{R}^3$ are studied in [**28**, **55**, **60**] and more recently in [**8**, **15**, **32**, **58**]. In [**32**] they are labelled harmonic mean curvature lines and are defined as curves along which the normal curvature is K/H, where K is the Gaussian curvature and H is the mean curvature of S. The equation of the characteristic curve is given by the BDE

$$\begin{array}{rcl} (2m(Gm-Fn)-n(Gl-En))dv^2 & & \\ +2(m(Gl+En)-2Fln)dvdu & & \\ +(l(Gl-En)-2m(Fl-Em))du^2 & = & 0. \end{array} \tag{2.3}$$

It is shown in [**15**] that the BDEs of the asymptotic, characteristic and principal curves are related. A BDE (1.2) can be viewed as a quadratic form and represented at each point in U by the point $(a(x,y) : 2b(x,y) : c(x,y))$ in the projective plane. Let Γ denote the conic of degenerate quadratic forms. To a point in the projective plane is associated a unique polar line with respect to Γ, and vice-versa. A triple of points is called a self-polar triangle if the polar line of any point of the triple contains the remaining two points. It turns out that, at non parabolic or umbilic points on M, the triple asymptotic, characteristic and principal curves BDEs form a self-polar triangle ([**15**]). In particular, any two of them determine the third one.

In [**29**] is constructed a natural 1-parameter family of BDEs, called *conjugate curve congruence*, that links the asymptotic curves BDE and the principal curves BDE on a smooth surface in $\mathbb{R}^3$. In [**15**], it is constructed a natural 1-parameter family of BDEs, called *reflected conjugate congruence*, linking the characteristic curves BDE and that of the principal curves.

Consider the projective space PT_pM of all tangent directions through a point $p \in M$ which is neither an umbilic nor a parabolic point. Conjugation gives an involution on PT_pM, $v \mapsto \overline{v} = C(v)$. There is another involution on PT_pM which is the reflection in either of the principal directions, $v \mapsto R(v)$.

Definition 1. 1. ([**29**]) Let $\Theta : PTM \to [-\pi/2, \pi/2]$ be given by $\Theta(p, v) = \alpha$, where α denotes the oriented angle between a direction v and the corresponding conjugate direction $\overline{v} = C(v)$. The conjugate curve congruence, for a fixed α, is defined to be $\Theta^{-1}(\alpha)$ and is denoted by $\mathcal{C}_\alpha$.

2. ([**15**]) Let $\Phi : PTM \to [-\pi/2, \pi/2]$ be given by $\Phi(p, v) = \alpha$, where α is the signed angle between v and $R(\overline{v}) = R \circ C(v)$. Then, the reflected conjugate curve congruence, for a fixed α, is defined to be $\Phi^{-1}(\alpha)$ and is denoted by $\mathcal{R}_\alpha$.

PROPOSITION 2.1. 1. ([**29**]) *The conjugate curve congruence $\mathcal{C}_\alpha$ of a parametrised surface is given by the BDE*

$$\begin{array}{rcl} (\sin\alpha(mG - nF) - n\cos\alpha\sqrt{EG - F^2})dv^2 & & \\ +(\sin\alpha(lG - nE) - 2m\cos\alpha\sqrt{EG - F^2})dvdu & & \\ +(\sin\alpha(lF - mE) - l\cos\alpha\sqrt{EG - F^2})du^2 & = & 0. \end{array} \tag{2.4}$$

2. ([**15**]) *The reflected conjugate congruence $\mathcal{R}_\alpha$ is given by the BDE*

$$\begin{array}{rcl} (2m(mG - nF) - n(Gl - En))\cos\alpha + (nF - mG)\frac{2mF - lG - nE}{\sqrt{(EG - F^2)}}\sin\alpha dv^2 & & \\ +2(m(lG + nE) - 2lnF)\cos\alpha + (nE - lG)\frac{2mF - lG - nE}{\sqrt{(EG - F^2)}}\sin\alpha dvdu & & \\ +(l(lG - nE) - 2m(lF - mE))\cos\alpha + (mE - lF)\frac{2mF - lG - nE}{\sqrt{(EG - F^2)}}\sin\alpha du^2 & = & 0. \end{array} \tag{2.5}$$

REMARK 1. The concepts of asymptotic, principal and characteristic curves and of conjugate and reflected curve congruences can be associated to any self-adjoint operator on a Riemannian surface ([**65**]).

We turn now to the case of surfaces embedded in a non-Euclidean space. We consider a smooth surface S endowed with a Lorentzian metric, that is, a metric which is locally equivalent to $\lambda(u, v)(dv^2 - du^2)$. We shall refer to such surfaces as *timelike* surfaces. In view of Remark 1, we consider a self-adjoint operator A on S, so $A : TS \to TS$ is a smooth map, where TS is the tangent bundle of S and its restriction $A_p : T_pS \to T_pS$ is a self-adjoint operator. An example of this situation is provided by an immersed timelike surface in the de Sitter space $S^3_1 \subset \mathbb{R}^4_1$, where $\mathbb{R}^4_1$ denotes the Minkowski 4-space. Then, there is a natural Gauss map $\mathbb{E} : S \to S^3_1$ and its derivative is a self-adjoint operator on S ([**48**]).

Because the metric on S is not positive definite, A_p does not always have real eigenvalues. When it does, we label them A-principal curvature and the associated eigenvectors the A-principal directions. The A-lines of principal curvature are given by the BDE (2.1), where E, F, G are the coefficients of the first fundamental form on S and l, m, n are referred to as the coefficients of the A-second fundamental form and are given by the same formulae as those for surfaces in the Euclidean 3-space. The discriminant of the equation is now a curve which is generically either empty or smooth except at isolated points where it has a Morse singularity of type node. We label these singular points *timelike umbilic points* as A_p is a multiple of the identity at such points. The configurations of the A-lines of curvature at timelike umbilic points are those in Figure 6, second and third rows.

The concepts of A-asymptotic and A-characteristic directions and curves can also be defined and their equations are given by the BDEs (2.2) and (2.3) respectively ([**49**]). The local behaviour of these pairs of foliations is distinct from that of their counterpart on a surface in the Euclidean 3-space.

Surfaces that have a mixed type metric give rise to interesting problems. Suppose given a metric $ds^2 = a(u, v)dv^2 + 2b(u, v)dvdu + c(u, v)du^2$ on a smooth surface S, where the set $ac - b^2 = 0$ is a smooth curve on S. Suppose the metric ds^2 is Riemannian in the region $ac - b^2 > 0$, so it is equivalent to $\lambda(u, v)(dv^2 + du^2)$. It is Lorentzian in the region $ac - b^2 < 0$, so it is equivalent to $\lambda(u, v)(dv^2 - du^2)$. Miernowski [**52**] considered the problem of finding an analytic model of the metric at points on the curve $ac - b^2 = 0$. The problem reduces to finding analytic models of BDEs of Type 1 at points on their discriminant (see §3.1).

Given an immersed surface S in the Minkowski space $\mathbb{R}^3_1$, the restriction of the pseudo scalar product in $\mathbb{R}^3_1$ to S gives a metric on S which can be of mixed type. The asymptotic, characteristic and principal curves associated to the Gauss map of S are well defined on the Riemannian and the Lorentzian parts of S. Their extensions to the degenerate locus of the metric on S are studied in [**49**].

3. *Classification*

We denote by $\omega(x, y, dx, dy) = a(x, y)dy^2 + 2b(x, y)dydx + c(x, y)dx^2$ the quadratic form associated to the BDE (1.2) and also use ω to refer to the equation $\omega = 0$. The interest here is local, so we take a, b, c to be germs of functions $\mathbb{R}^2, 0 \to \mathbb{R}$. We consider the origin to be a point on the discriminant. For BDEs of Type 1, we can rotate the coordinate axes in the plane, set $p = dy/dx$ and take p in a neighbourhood of zero. However, for BDEs of Type 2, we take $(dx : dy) \in \mathbb{R}P^1$.

Definition 2. Two germs, at the origin, of BDEs ω_1 and ω_2 are respectively smoothly, analytically or formally equivalent if there exist germs $H = (h_1, h_2) : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ of a smooth, analytic or formal diffeomorphism and $r : \mathbb{R}^2, 0 \to \mathbb{R}$ of a smooth, analytic or formal function not vanishing at 0 such that

$$\omega_2 = r.H^*\omega_1,$$

that is, $\omega_2(x, y, dx, dy) = r(x, y)\omega_1(h_1(x, y), h_2(x, y), dh_1(x, y), dh_2(x, y))$.

Two germs of BDEs are topologically equivalent if there exists a germ of a homeomorphism that takes the configuration of one to the configuration of the other.

The aim is to produce representatives (models, preferably in simple forms) of equivalence classes of the equivalence relations in Definition 2. A more realistic task is to produce models of germs of low codimensions, which we define as follows. We associate to a germ of a BDE $\omega = (a, b, c)$ the jet-extension map

$$\begin{array}{rccc} \Phi : & \mathbb{R}^2, 0 & \to & J^k(2, 3) \\ & (x, y) & \mapsto & j^k(a, b, c)|_{(x,y)} \end{array}$$

where $J^k(2, 3)$ denotes the vector space of polynomial maps of degree $\leq k$ from $\mathbb{R}^2$ to $\mathbb{R}^3$, and $j^k(a, b, c)|_{(x,y)}$ is the k-jet of (a, b, c) at (x, y). (This is simply the Taylor expansion of order k of (a, b, c) at (x, y).)

Definition 3. A singularity of ω is of *codimension* m if the conditions that define it yield a semi-algebraic set V of codimension $m + 2$ in $J^k(2, 3)$, for any $k \geq k_0$.

3.1. *Formal and analytic classifications*

We can reduce, inductively on the k-jet spaces, the coefficients of a BDE to simpler forms by making polynomial changes of coordinates in the plane and multiply by invertible polynomial functions. If this process converges, i.e., the composite of all the changes of coordinates (resp. multiplicative polynomials) converges to an analytic diffeomorphism (resp. non zero analytic function), then the obtained germ of a BDE is an analytic model. Otherwise we have a formal model. The convergence problem is a complicated one, see for example [**1**] for the case of vector fields. However, even if the process is not convergent (which is the case in general), the reduction of the k-jet of a BDE to a simple form is very valuable in practice. The local topological behaviour of the solutions and the relevant invariants of a BDE do, in general, depend only on some initial terms of the coefficients of the BDE. Taking these in simpler forms makes the geometric interpretation of the conditions involved more apparent and the calculations more manageable.

The formal classification of some BDEs of Type 1 is dealt with in [**16**]. We can reduce the constant part of the BDE to one of the following cases

$$dy^2 + dx^2, \quad dy^2 - dx^2, \quad dy^2, \quad 0.$$

(We show below how this is done.) It is shown in [**16**] that a BDE with constant part equivalent to $dy^2 \pm dx^2$ is analytically equivalent to $dy^2 \pm dx^2$. The initial form dy^2 leads to the following representatives of orbits in the space of 1-jets:

$$dy^2 + xdx^2, \quad dy^2 - ydx^2, \quad dy^2.$$

We reproduce from [**16**] the case $dy^2 + xdx^2$ as an example of how the formal reduction is carried out. Suppose a BDE has 1-jet $dy^2 + xdx^2$ and assume that the k-jets of the coefficients of the BDE are $1 + a_k, 2b_k, x + c_k$, with a_k, b_k, c_k belonging to the set of homogeneous polynomials of degree k which we denote by H^k. We make changes of coordinates of the form

$$\begin{aligned} x &= X + p(X,Y) \\ y &= Y + q(X,Y) \end{aligned}$$

with $p \in H^k$ and $q \in H^{k+1}$ and multiply by $1 + r(X,Y), r \in H^k$. Then,

$$dx = (1 + p_X)dX + p_Y dY, \ dy = q_X dX + (1 + q_Y)dY,$$

and the k-jet of the transformed BDE is

$$(1 + a_k + r + 2q_Y, b_k + q_X + Xp_Y, x + c_k + p + 2Xp_X).$$

We are seeking r, p, q so that $a_k + r + 2q_Y = b_k + q_X + Xp_Y = c_k + p + 2Xp_X = 0$, i.e.,

$$\begin{aligned} r + 2q_Y &= -a_k, \\ q_X + Xp_Y &= -b_k, \\ p + 2Xp_X &= -c_k. \end{aligned}$$

This process produces the linear map

$$\begin{array}{rccc} L_k: & H^k \oplus H^{k+1} \oplus H^k & \to & H^k \oplus H^k \oplus H^k \\ & (p,q,r) & \mapsto & (r + 2q_Y, p + 2Xp_X, q_X + Xp_Y) \end{array}$$

The map L_k is surjective and furthermore $\bar{L}_k = L_k|_{q_{k+1}=0}$ is an isomorphism, where q_{k+1} denotes the coefficient of y^{k+1} in q. Therefore, the above linear system has a solution, which means that we can reduce the k-jet of the coefficients of the BDE to $(1, 0, x)$.

Proposition 3.1. ([**16**]) 1. *Suppose that a germ of a BDE has linear part $dy^2 + xdx^2$. Then, for any $k \geq 1$ we can change coordinates and multiply by a non zero function so that the germ of the transformed BDE has k-jet $dy^2 + xdx^2$.*

2. *Suppose the germ of a BDE has linear part $dy^2 + xdx^2$. Then, there exist an analytic change of coordinates which transforms the BDE to $\mu(x,y)(dy^2 + xdx^2)$, where μ is an analytic function not vanishing at the origin.*

The BDEs with 1-jet $dy^2 - ydx^2$ are also considered in [**16**]. The 2-jet can be put in the form $dy^2 - (y + \lambda x^2)dx^2$. It is shown in [**16**] that

PROPOSITION 3.2. ([**16**]) *For almost all value of λ, a BDE with 2-jet $dy^2 - (y + \lambda x^2)dx^2$ is formally equivalent to $dy^2 - (y + \lambda x^2)dx^2$.*

As pointed in the §2, Miernowski [**52**] considered the problem of finding analytic models of a metric $ds^2 = a(u, v)dv^2 + 2b(u, v)dvdu + c(u, v)du^2$ of mixed type at points where $ac - b^2 = 0$. Suppose that the point in consideration is the origin. Miernowski showed that if the 1-jet of ds^2 is equivalent to $dv^2 + udu^2$, then the metric is analytically equivalent to $\mu(u, v)(dv^2 + udu^2)$ (compare Proposition 3.1). However, if the 1-jet of ds^2 is equivalent to $dv^2 - vdu^2$, then Miernowski proved that there is a functional modulus in the classification. As a corollary of his result, a BDE with 2-jet $dy^2 - (y + \lambda x^2)dx^2$ cannot be reduced to the form $\mu(x, y)(dy^2 - (y + \lambda x^2)dx^2)$ by analytic changes of coordinates.

We turn now to BDEs of Type 2. We have the following orbits in the 1-jet space where $\epsilon = \pm 1$:
- $(y, b_1x + b_2y, \epsilon y)$, $b_1 \neq 0$, $2b_1 + \epsilon \neq 0$, $b_1 \neq \frac{1}{2}(b_2^2 - \epsilon)$, and $b_1 \neq \pm b_2 - 1$ when $\epsilon = +1$ ([**17**, **38**])
- $(x + a_2y, 0, y)$, $a_2 > \frac{1}{4}$ ([**66**])
- $(y, \pm x + b_0y, 0)$, $(y, y, 0)$, $(x + y, 0, 0)$, $(x, b_0y, 0)$, $(x + y, -y, 0)$, $(y, 0, 0)$, $(0, 0, 0)$ ([**13**]).

It turns out that there are no discrete orbits in the formal classification. It is shown in [**17**] that a BDE with 1-jet $(y, b_1x + b_2y, \epsilon y)$, $\epsilon = \pm 1$ can be reduced, for almost all values of (b_1, b_2), by a formal diffeomorphism and multiplication by non zero formal power series to $(y, b_1x + b_2y + b(x, y), \epsilon y)$, where $b(x, y)$ is a formal power series with no constant or linear terms. (See also [**42**] for a similar result for the case $\epsilon = -1$.)

3.2. *Smooth and topological classifications*

We deal with BDEs of Type 1 and 2 separately.

BDEs of Type 1

The study of BDEs of Type 1 follows form the general study of IDEs. The IDE (1.1) defines a surface
$$M = \{(x, y, p) \in \mathbb{R}^3 : F(x, y, p) = 0\}$$
in the 3-dimensional space of 1-jets of functions endowed with the contact structure $\alpha = dy - pdx$. Consider the projection $\pi : M \to \mathbb{R}^2$, $\pi(x, y, p) = (x, y)$. Generically, M is a smooth surface (that is 0 is a regular value of F) and the restriction of π to M is either a local diffeomorphism, a fold or cusp map. The set of critical points of the projection is called the *criminant* of the IDE and is given by the equations $F = F_p = 0$. The set of critical values of the projection is called the *discriminant* of the IDE, and is obtained by eliminating p from the equations $F = F_p = 0$.

The multi-valued direction field defined by F in the plane lifts to a single-valued direction field on the surface. This direction field is determined by the vector field
$$\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p}$$
(which is along the intersection of M with the contact planes in $\mathbb{R}^3$). It is of course tangent to M at (x, y, p) and projects to a line through (x, y) with slope p.

If π is a local diffeomorphism at (x, y, p), then the integral curves of ξ around (x, y, p) project to a family of smooth curves around (x, y).

Suppose that $\pi|_M$ has a fold singularity at (x, y, p), i.e., we can choose local coordinates in M and $\mathbb{R}^2$ for which π has the form (u, v^2). This means that $F = F_p = 0$ but $F_{pp} \neq 0$ at (x, y, p). Then, the IDE is locally a BDE at (x, y, p) and the discriminant is a smooth curve. (For BDEs of Type 1, the surface M is smooth if and only if the discriminant is smooth.) Every point in

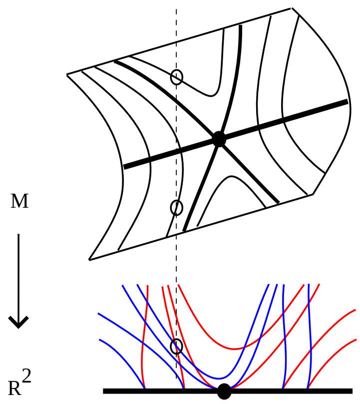

FIGURE 1. *The lifted field ξ and the involution σ on M.*

the plane near (x, y) which is not on the discriminant has two pre-images on M under π. This defines an involution σ on M near (x, y, p), which interchanges pairs of points with the same image under σ (Figure 1). The criminant is the set of fixed points of σ. Thus, locally at (x, y, p), we have a pair (ξ, σ) of a vector field and an involution on M. The classification (smooth or topological) of IDEs is the same as the classification (smooth or topological) of the pairs (ξ, σ). When ξ is regular, the IDE is smoothly equivalent to $dy^2 - xdx^2 = 0$ (i.e., $p^2 - x = 0$) ([**23**]). If ξ has an elementary singularity (saddle/node/focus), then the corresponding point in the plane is called a *folded singularity* of the BDE. At folded singularities, the equation is locally smoothly equivalent to

$$dy^2 + (-y + \lambda x^2)dx^2 = 0, \tag{3.1}$$

with $\lambda \neq 0, \frac{1}{16}$, provided that ξ is linearisable at the singular point; see [**24**, **25**].

Normal forms at folded resonant saddles and nodes are given in [**27**]. At a degenerate elementary singular point of ξ of multiplicity $r \in \mathbb{N}$, $r > 1$, the equation is smoothly equivalent to

$$(\frac{dy}{dx} + \epsilon x^r + Ax^{2r-1})^2 = y$$

where $A \in \mathbb{R}$ and $\epsilon \in \{(\pm 1)^r\}$; see [**26**].

Suppose that $\pi|_M$ has a cusp singularity at (x, y, p), i.e., we can choose local coordinates in M and $\mathbb{R}^2$ for which π has the form $(u, v^3 + uv)$. This means that $F = F_p = F_{pp} = 0$ but $F_{ppp} \neq 0$ at (x, y, p). Then, the IDE is locally a cubic equation in p and the discriminant has a cusp singularity. Bruce [**5**] conjectured that the IDE has a functional modulus for smooth equivalence. Davydov [**24**] proved that the equation has a functional modulus even for the topological equivalence (see also §3.3).

We turn now to the topological equivalence. This can be treated in several ways. It can be done, as in [**24**, **25**], by studying the pair (ξ, σ). Kuzmin [**51**] split the BDE into two ODEs and used the methods of ODE to analyse the behaviour of the solutions. In [**9**, **67**, **69**] we used the method of blowing up (see below). We give below the topological classification of the singularities of codimension ≤ 2.

There are three stable topological models (see [**25**] for references) at folded singularities: a folded saddle if $\lambda < 0$, a folded node if $0 < \lambda < \frac{1}{16}$ and a folded focus if $\frac{1}{16} < \lambda$ in equation (3.1); Figure 2, last three figures respectively. The labelling in the figures refers to the coefficients (a, b, c) of the model BDE. The first two figures in Figure 2 are models away from the discriminant, and the third at points on the discriminant corresponding to regular points of ξ.

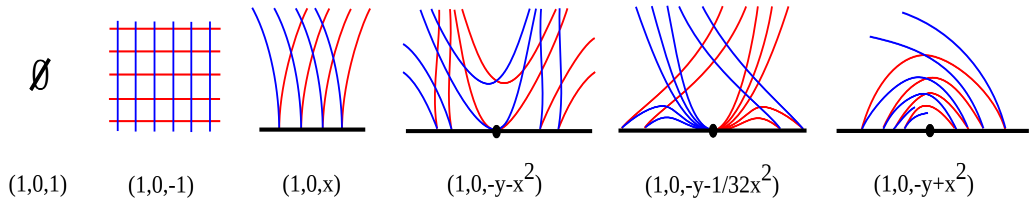

FIGURE 2. *Topological models of stable singularities of BDEs of Type 1.*

Codimension 1 singularities are dealt with in [**9**, **25**, **51**, **67**], see Figure 3. These occur when: the lifted field ξ has a saddle-node singularity (Figure 3, first figure, $\lambda = 0$ in equation (3.1)); the lifted field ξ has equal eigenvalues (Figure 3, second figure, $\lambda = 1/16$ in equation (3.1)); or when the discriminant has a Morse singularity, labelled Morse Type 1 singularities in [**9**] (Figure 3, last 4 figures). The Morse Type 1 singularities are distinguished by the type of the singularity of the discriminant, isolated point or node, and by the type of the folded singularities that appear in a generic deformation (two folded saddles or foci), see [**9**].

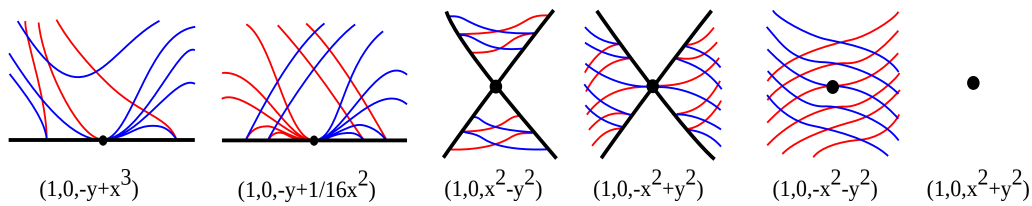

FIGURE 3. *Topological models of codimension* 1 *singularities of BDEs of Type 1.*

Codimension 2 singularities are classified in [**69**]. Degeneracy occurs in three ways: the discriminant is smooth and the lifted field has a degenerate elementary singularity of multiplicity 3 (Figure 4, first two figures); the discriminant has a Morse singularity of type node and the unique direction determined by the IDE at the origin has an ordinary tangency with one of the branches of the discriminant (Figure 4, third figure); the discriminant has a cusp singularity with a limiting tangent transverse to the unique direction determined by the IDE (Figure 4, last two figures).

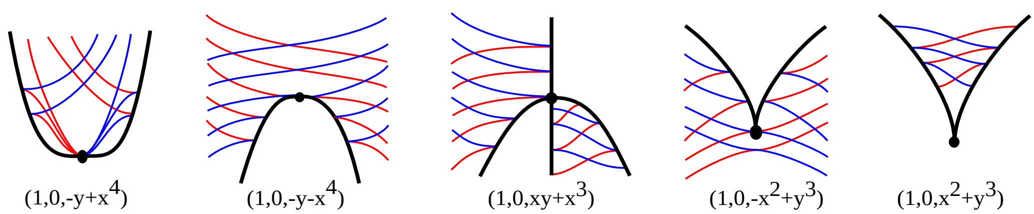

FIGURE 4. *Topological models of codimension* 2 *singularities of BDEs of Type 1.*

We have a general topological result about IDEs with $F = F_p = 0$ and $F_{pp} \neq 0$ at the origin. Such IDEs can be written in the form $\omega = dy^2 + f(x,y)dx^2 = 0$ ([**13**]). We say that ω is finitely topologically determined if there exists $k \in \mathbb{N}$ such that any BDE ω' with $j^k\omega' = j^k\omega$ is topologically equivalent to ω.

THEOREM 3.3. ([**69**]) *A BDE $\omega = dy^2 + f(x,y)dx^2 = 0$ with $f(x,y)$ and $g(x) = f(x,0)$ $\mathcal{K}$-finitely determined is finitely topologically determined.*

The hypotheses in Theorem 3.3 are equivalent to $m(w) < \infty$, where $m(w)$ is the multiplicity of the IDE (§4.1).

BDEs of Type 2

As pointed out in §3.1 there are no discrete local models under formal equivalence for BDEs of Type 2. To my knowledge, smooth equivalence has only been considered in one case in [**42**] (see Remark 2(2)). For topological equivalence there are several ways to proceed. We review here two techniques, another one can be found in [**51**].

One way to proceed when seeking topological models for BDEs of Type 2 is to consider a blowing-up of the singularities of the BDEs. This is first done in [**64**] where topological models of the lines of curvature at umbilic points on a smooth surface in $\mathbb{R}^3$ are sought. Guíñez ([**39**] and elsewhere) used this technique on BDEs whose discriminants are isolated points, labelled there positive quadratic equations. However, Guíñez's technique can be extended to deal with general BDEs ([**57**, **66**, **68**]). We highlight Guíñez's method below for the case when $j^1\omega = (y, b_1x + b_2y, -y)$.

Following the notation in [**39**], let $f_i(\omega)$, $i = 1, 2$ denote the foliation associated to the BDE $\omega = (a, b, c)$, which is tangent to the vector field $a\frac{\partial}{\partial u} + (-b + (-1)^i\sqrt{b^2 - ac}\,)\frac{\partial}{\partial v}$. If ψ is a diffeomorphism and $\lambda(x, y)$ is a non-vanishing real valued function, then ([**39**]) for $k = 1, 2$,

1. $\psi(f_k(\omega)) = f_k(\psi^*(\omega))$, if ψ is orientation preserving;
2. $\psi(f_k(w)) = f_{3-k}(\psi^*(\omega))$, if ψ is orientation reversing;
3. $f_k(\lambda\omega) = f_k(\omega)$, if $\lambda(x, y)$ is positive;
4. $f_k(\lambda\omega) = f_{3-k}(\omega)$, if $\lambda(x, y)$ is negative.

We write $\omega = (y + M_1(x, y), b_1x + b_2y + M_2(x, y), -y + M_3(x, y))$ and consider the directional blowing-up $x = u, y = uv$. (We also need to consider the blowing-up $x = uv, y = v$.) Then, the new BDE $\omega_0 = (u, v)^*\omega$ has coefficients

$$\begin{aligned}
\bar{a} &= u^2(uv + M_1(u, uv)),\\
\bar{b} &= uv(uv + M_1(u, uv)) + u(b_1u + b_2uv + M_2(u, uv)),\\
\bar{c} &= v^2(uv + M_1(u, uv)) + 2v(b_1u + b_2uv + M_2(u, uv)) - uv + M_3(u, uv).
\end{aligned}$$

We can write $(\bar{a}, \bar{b}, \bar{c}) = u(u^2A_1, uB_1, C_1)$ with

$$\begin{aligned}
A_1 &= v + uN_1(u, v),\\
B_1 &= v^2 + b_2v + b_1 + u(N_2(u, v) + vN_1(u, v)),\\
C_1 &= v(v^2 + 2b_2v + 2b_1 + \epsilon) + u(v^2N_1(u, v) + 2vN_2(u, v) + N_3(u, v)),
\end{aligned}$$

and $M_i(u, uv) = u^2N_i(u, v)$, $i = 1, 2, 3$.

The quadratic form $\omega_1 = (u^2A_1, uB_1, C_1)$ can be decomposed into two 1-forms, and to these 1-forms are associated the vector fields

$$X_i = (u^2A_1, -uB_1 + (-1)^i\sqrt{u^2(B_1^2 - A_1C_1)}\,), \quad i = 1, 2.$$

These vector fields are tangent to the foliations defined by ω_1. It is clear that we can factor out the term u in X_i, with an appropriate sign change when $u < 0$. The vector fields

$$Y_i = (uA_1, -B_1 + (-1)^i\sqrt{B_1^2 - A_1C_1}\,), \quad i = 1, 2$$

are then considered. Since the blowing up is orientation preserving if $u > 0$ and orientation reversing if $u < 0$, and we factored out u twice, it follows from the observation above (see [**39**]) that Y_1 corresponds to the foliation $\mathcal{F}_1$ of ω if $u > 0$ and to $\mathcal{F}_2$ if $u < 0$; while Y_2 corresponds to $\mathcal{F}_2$ if $u > 0$ and to $\mathcal{F}_1$ if $u < 0$.

One studies the vector fields Y_i in a neighbourhood of the exceptional fibre $u = 0$, and blows down to obtain the configuration of the integral curves of the original BDE. One can then proceed as in [**9**, **67**, **69**] to show that any two such configurations are homeomorphic.

Another way to proceed when seeking topological models for such BDEs of Type 2 is as follows (see for example [**7**] for the case $c = -a$ and in [**11**] for the general case). Consider the associated surface to the BDE

$$M = \{(x, y, [\alpha : \beta]) \in \mathbb{R}^2, 0 \times \mathbb{R}P^1 : a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\}.$$

As the coefficients of the BDE all vanish at the origin, the exceptional fibre $0 \times \mathbb{R}P^1$ is contained in M. The discriminant function $\delta = b^2 - ac$ plays a key role. When δ has a Morse singularity the surface M is smooth and the projection $\pi : M \to \mathbb{R}^2, 0$ is a double cover of the set $\{(x, y) : \delta(x, y) > 0\}$ ([**11**]; see also [**6**] for a general relation between the singularities of δ and those of M). We label these BDEs *Morse Type 2*. The bi-valued direction field defined by the BDE in the plane lifts to a single direction field ξ on M and extends smoothly to $\pi^{-1}(0)$. Note that the exceptional fibre $0 \times \mathbb{R}P^1 \subset \pi^{-1}(\Delta)$ is an integral curve of ξ. The closure of the set $\pi^{-1}(\Delta) - (0 \times \mathbb{R}P^1)$ is the *criminant* of the equation.

There is an involution σ on $M - (0 \times \mathbb{R}P^1)$ that interchanges points with the same image under the projection to $\mathbb{R}^2, 0$. It is shown in [**11**] that σ extends to M when the coefficients a, b, c are analytic. (In fact the result is true when the coefficients are smooth functions; see Remark 2 in [**66**].) Points on M are identified with their images by σ. A bi-valued field on the quotient space $M' = M/\sigma$ is then studied and models of the configurations of the integral curves of the BDE are obtained by blowing-down.

Consider the affine chart $p = \beta/\alpha$ (we also need to consider the chart $q = \alpha/\beta$), and set

$$F(x, y, p) = a(x, y)p^2 + 2b(x, y)p + c(x, y).$$

Then, the lifted direction field is parallel to the vector field

$$\xi = F_p\frac{\partial}{\partial x} + pF_p\frac{\partial}{\partial y} - (F_x + pF_y)\frac{\partial}{\partial p}.$$

The singularities of ξ on the exceptional fibre ($F = F_p = 0$) are given by the roots of the cubic

$$\begin{array}{rcl}\phi(p) & = & (F_x + pF_y)(0, 0, p)\\ & = & a_2p^3 + (2b_2 + a_1)p^2 + (2b_1 + c_2)p + c_1,\end{array}$$

where $j^1a = a_1x + a_2y, j^1b = b_1x + b_2y, j^1c = c_1x + c_2y$. The eigenvalues of the linear part of ξ at a singularity are $-\phi'(p)$ and $\alpha_1(p)$, where

$$\alpha_1(p) = 2(a_2p^2 + (b_2 + a_1)p + b_1).$$

Therefore, the cubic ϕ and the quadratic α_1 determine the number and the type of the singularities of ξ (see [**7**, **11**] for details).

The calculations simplify considerably when the 1-jet of the BDE is simplified. For instance, if α_1 and ϕ have no common roots or if ϕ has more than one root ([**11**, **38**]): then one can take

$$j^1(a, b, c) = (y, b_1x + b_2y, \epsilon y), \epsilon = \pm 1.$$

(If α_1 and ϕ have a common root and ϕ has only one root, we can set $j^1(a, b, c) = (x + a_2y, 0, y), a_2 > \frac{1}{4}$, [**66**].)

The topological classification of codimension ≤ 2 singularities of BDEs of Type 2 is obtained using the above methods. We first observe that there are no topologically stable singularities of BDEs of Type 2. The discriminant of such BDEs are always singular and generic deformations within the set of BDEs remove the singularities.

The codimension 1 singularities are classified by the number and type of the singularities of ξ when (b_1, b_2) is away from some special curves in the (b_1, b_2)-plane ([**7**, **11**, **39**], see §3.1 and

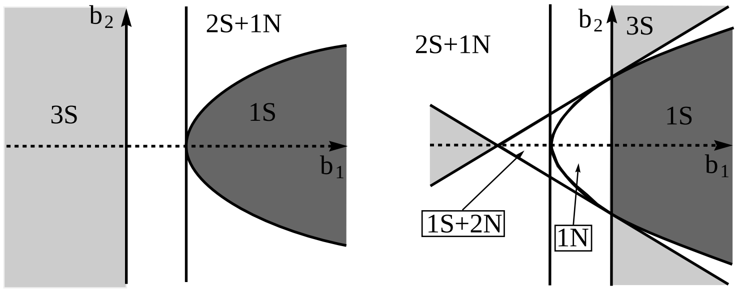

FIGURE 5. *Partition of the (b_1, b_2)-plane, $\epsilon = -1$ left, $\epsilon = +1$ right. The labels refer to the number and type (S for saddle and N for node) of the singularities of ξ.*

Figure 5). There are 3 topological models when the discriminant has an A_1^+ singularity and 5 when it has an A_1^--singularity (Figure 6). The bifurcations of these singularities in generic families are studied in [**12**], see also [**51**] for the case $\epsilon = -1$.

Codimension 2 singularities occur at generic points on the exceptional curves in Figure 5. These are classified in [**66**] using the blowing-up method; see Figure 7 for the models. The models in the first row in Figure 7 correspond to the case where ϕ and α_1 have one common root ($\epsilon = 1$, $b_1 = \pm b_2 - 1$), those in the second row to the case where ϕ has a double root ($2b_1 + \epsilon = 0$ or $b_1 = \frac{1}{2}(b_2^2 - \epsilon)$) and those in the third row to the case where the discriminant has a cusp singularity ($b_1 = 0$). (The first case in the second row in Figure 7 is also classified in [**40**].)

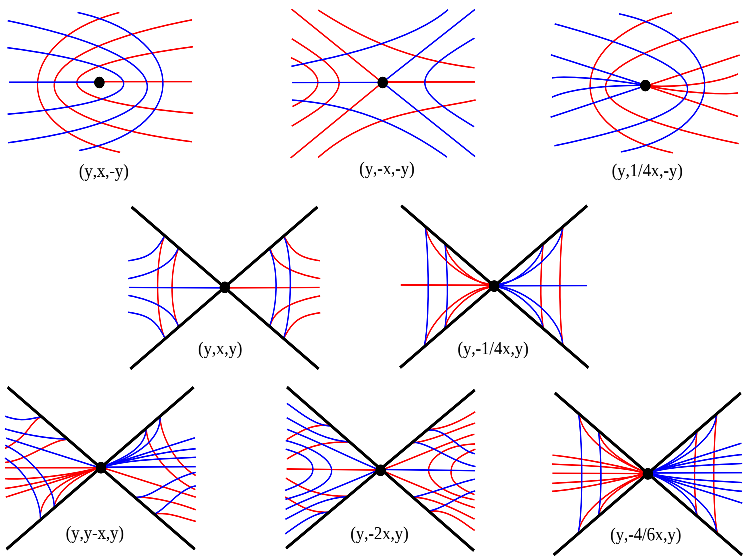

FIGURE 6. *Topological models of codimension 1 singularities of Type 2 BDEs.*

REMARK 2. (1) For BDEs of Type 2, the second method is geometrical and works well when the surface M is smooth ([**7, 11**]). However, when M is singular, the involution σ presents some obstacles. One needs to show that σ extends to the exceptional fibre and this is not trivial.

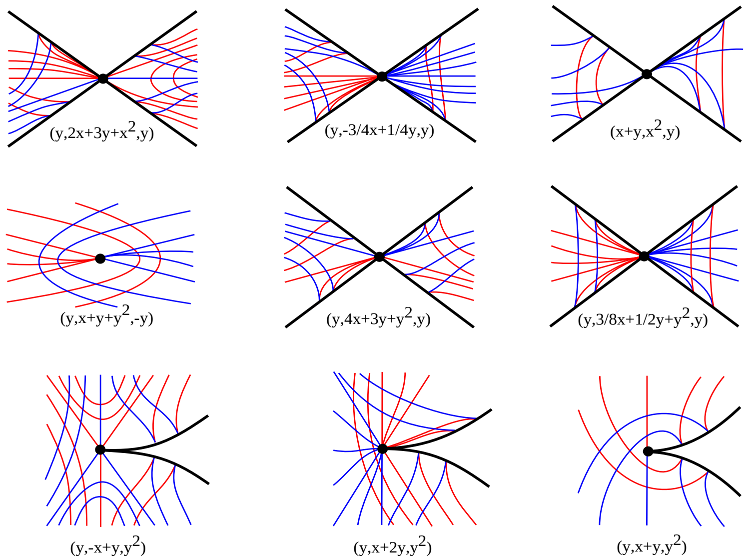

FIGURE 7. *Topological models of codimension 2 singularities of Type 2 BDEs.*

The first method is computational and the calculations are sometimes long and winding. (It is used in [**38**, **39**, **40**, **41**, **42**, **57**, **66**, **68**] to obtain topological models of BDEs with singular associated surface M.)

(2) There are classifications of some more degenerate singularities, motivated by geometric problems. We list the existing cases in the Appendix (§6). It is worth pointing out here that Gutierrez and Guíñez [**42**] showed that BDEs with 1-jet $(y, b_1x + b_2y, -y)$ are topologically 1-determined when the discriminant has a Morse singularity, i.e., when $b_1 \neq 0$. They also proved that any BDE with 1-jet $(y, b_1x + b_2y, -y)$ is smoothly equivalent to $(y, b_1x + b_2y + M_2(x, y), -y + M_3(x, y))$, where M_2, M_3 are smooth functions with zero 1-jets.

3.3. *IDEs with first integrals*

In [**47**], the authors studied germs of IDEs with independent first integral. An IDE is defined to be the surface $M = F^{-1}(0)$ in $PT^*\mathbb{R}^2$ endowed with its canonical contact structure given by the 1-form $\alpha = dy - pdx$. The surface M is supposed to be smooth, so is locally the image of a germ of an immersion $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$. The IDE is then represented by the germ f.

The IDE has a first integral, that is, there exists a germ of a submersion $\mu : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ such that $d\mu \wedge f^*\alpha = 0$ (this means that the integral curves of the lifted field ξ on M are images under f of the level sets of μ). As the solutions of the IDE in the plane are the images under $\pi \circ f$ of the level sets of μ, it is natural to consider the diagram $\mathbb{R}, 0 \xleftarrow{\mu} \mathbb{R}^2, 0 \xrightarrow{\pi\circ f} \mathbb{R}^2, 0$. Consider in general a diagram (g, μ)

$$\mathbb{R}, 0 \xleftarrow{\mu} \mathbb{R}^2, 0 \xrightarrow{g} \mathbb{R}^2, 0 ,$$

where g is a smooth map germ and μ is a germ of a submersion. The diagram (g, μ) is called an integrable diagram if there exists a germ of an immersion $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$ such that $d\mu \wedge f^*\alpha = 0$ and $g = \pi \circ f$. Then (g, μ) is said to be induced by f.

Let $\pi : PT^*\mathbb{R}^2 \to \mathbb{R}^2$ be the natural projection. Two germs of immersions (IDEs) $f : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z$ and $f' : \mathbb{R}^2, 0 \to PT^*\mathbb{R}^2, z'$ are said to be equivalent if there exists germs of diffeomorphisms $\psi : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ and $\phi : \mathbb{R}^2, \pi(z) \to \mathbb{R}^2, \pi(z')$ such that $\hat{\phi} \circ f = f' \circ \pi$, where $\hat{\phi} : PT^*\mathbb{R}^2, z \to PT^*\mathbb{R}^2, z'$ is the lift of ϕ.

The idea in [**47**] is to reduce the classification of IDEs with first integrals under the above equivalence to that of germs of integral diagrams. Two germs (g, μ), (g', μ') of integral diagrams are equivalent if the diagram

$$\begin{tikzcd} \mathbb{R}, 0 \arrow[d, "\kappa"'] & \mathbb{R}^2, 0 \arrow[l, "\mu"'] \arrow[r, "g"] \arrow[d, "\psi"'] & \mathbb{R}^2, 0 \arrow[d, "\phi"'] \\ \mathbb{R}, 0 & \mathbb{R}^2, 0 \arrow[l, "\mu'"'] \arrow[r, "g'"] & \mathbb{R}^2, 0 \end{tikzcd}$$

commutes, with κ, ψ, ϕ germs of diffeomorphisms.

Suppose given two germs of IDEs f and f' with first integrals and with the set of critical points of $\pi \circ f$ and $\pi \circ f'$ nowhere dense. Then, ([**47**, Proposition 2.8]), f and f' are equivalent as IDEs if and only if the diagrams $(\pi \circ f, \mu)$ and $(\pi \circ f', \mu')$ are equivalent as integral diagrams.

A weaker equivalence relation of integral diagrams is introduced in [**47**]. Two germs (g, μ), (g', μ') of integral diagrams are weakly equivalent if the diagram

$$\begin{tikzcd} \mathbb{R}^2, 0 \arrow[d, "\phi"'] & \mathbb{R}^2 \times \mathbb{R}, 0 \arrow[l, "(g,\mu)"'] \arrow[r, "\pi_1"] \arrow[d, "\Psi"'] & \mathbb{R}^2, 0 \arrow[d, "\psi"'] \\ \mathbb{R}^2, 0 & \mathbb{R}^2 \times \mathbb{R}, 0 \arrow[l, "(g',\mu')"'] \arrow[r, "\pi_1"] & \mathbb{R}^2, 0 \end{tikzcd}$$

commutes, with ϕ, Ψ, ϕ germs of diffeomorphisms and π_1 is the projection to the first component. Equivalent integral diagrams are weakly equivalent.

The authors in [**47**] used the theory of Legendrian singularities to classified generic integral diagrams under the weakly equivalence. The word generic means the following. The set $Int(U, PT^*\mathbb{R}^2 \times \mathbb{R})$ of integral IDEs with first integral $(f, \mu) : U \subset \mathbb{R}^2 \to PT^*\mathbb{R}^2 \times \mathbb{R}$ is endowed with the Whitney C^∞-topology. A property is generic if the subset of (f, μ) that satisfy it is open and dense in $Int(U, PT^*\mathbb{R}^2 \times \mathbb{R})$.

THEOREM 3.4. ([**47**, Theorem A]) *For almost all differential equation germs with first integral (f, μ), the integral diagram is weakly equivalent to one of the germs in the following finite list:*
(1) $g = (u, v)$, $\mu = v$.
(2) $g = (u^2, v)$, $\mu = v - \frac{1}{3}u^3$.
(3) $g = (u, v^2)$, $\mu = v - \frac{1}{3}u$.
(4) $g = (u^3 + uv, v)$, $\mu = \frac{3}{4}u^4 + \frac{1}{2}u^2 v + v$.
(5) $g = (u, v^3 + uv)$, $\mu = v$.
(6) $g = (u, v^3 + uv^2)$, $\mu = \frac{1}{2}v^2 + v$.

An integral diagram (g, μ) is said to be of generic type if it is weakly equivalent to an integral diagram from the list in Theorem 3.4. Diagrams of generic type are then classified up to the "stronger" equivalence.

THEOREM 3.5. ([**47**, Theorem B]) *An integral diagram of generic type is equivalent to one of the following integral diagrams (g, μ)*
(1) $g = (u, v)$, $\mu = v$.
(2) $g = (u^2, v)$, $\mu = v - \frac{1}{3}u^3$.

(3) $g = (u, v^2)$, $\mu = v - \frac{1}{3}u$.
(4) $g = (u^3 + uv, v)$, $\mu = \frac{3}{4}u^4 + \frac{1}{2}u^2 v + \beta \circ g$,
where $\beta(x, y)$ is a germ of a smooth function with $\beta(0) = 0$ and $\beta_y(0) = \pm 1$.
(5) $g = (u, v^3 + uv)$, $\mu = v + \beta \circ g$,
where $\beta(x, y)$ is a germ of a smooth function with $\beta(0) = 0$.
(6) $g = (u, v^3 + uv^2)$, $\mu = \frac{1}{2}v^2 + \beta \circ g$,
where $\beta(x, y)$ is a germ of a smooth function with $\beta(0) = 0$ and $\beta_x(0) = 1$.

The cases (2) and (3) in Theorem 3.5 are first given in [**22**, **23**]. In (4)-(6) Theorem 3.5, the discriminant is a cusp. We refer to [**47**] for further details and for the relation between Theorem 3.5, the previous classifications and Clairaut type equations.

4. *Invariants*

Consider the two BDEs

$$dy^2 + (x^2 + y^2)dx^2 = 0,$$
$$ydy^2 - 2xdxdy - ydx^2 = 0.$$

The first has a Morse Type 1 singularity and the second a Morse Type 2 singularity. Their discriminant, given by $x^2 + y^2 = 0$, has a Morse singularity of type A_1^+ (isolated point). Consider now the following 1-parameter deformations of these BDEs

$$dy^2 + (x^2 + y^2 + t)dx^2 = 0,$$
$$ydy^2 - 2xdxdy - (y + t)dx^2 = 0.$$

The discriminants in the first family $x^2 + y^2 + t = 0$ undergo the usual Morse transitions (Figure 8, first row). However, the discriminants in the second family $x^2 + y(y + t) = 0$ undergo transitions of type cone sections (Figure 8, second row). One can also show that for the first family, two folded singularities appear on the discriminant for $t < 0$ ([**9**], Figure 8). Three of these singularities appear on the discriminant of the second family for $t \neq 0$ ([**18**]). To explain these phenomena, an invariant of BDEs (multiplicity) is introduced in [**13**] and symmetric matrices are studied in [**6**].

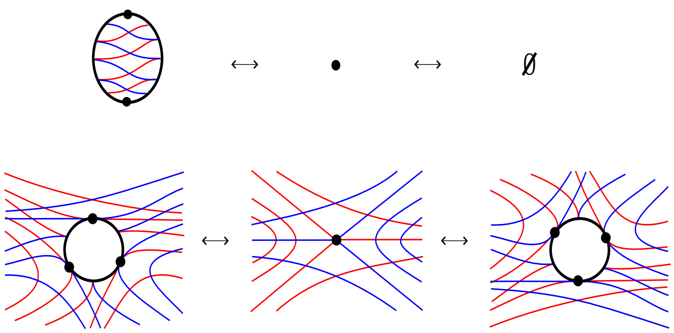

FIGURE 8. *Bifurcations of a Morse Type 1 and a Morse Type 2 singularities.*

4.1. *The multiplicity of a BDE*

We suppose here that the IDE (1.1) is given by an analytic function F and the coefficients of the BDE (1.2) are analytic functions (some of the results are also valid in the smooth category

[**21**]). We can then complexify and denote by $\mathcal{O}(x, y, p)$ the ring of holomorphic function germs $\mathbb{C}^3, 0 \to \mathbb{C}$. We start with IDEs (1.1).

Definition 4. ([**13**]) A *singular point or zero* of the IDE given by $F(x, y, p) = 0$ is a zero of the canonical 1-form $dy - pdx$ on the criminant $F = F_p = 0$. The multiplicity of a singular point is the maximum number of zeros it can split up into under deformations of the equation $F = 0$ (including complex zeros).

Proposition 4.1. ([**13**]) (a) *The multiplicity of a singular point* $((x, y, p) = (0, 0, 0))$ *of the IDE* $F = 0$ *at a fold point of the projection corresponding to a zero of the vector field* ξ *is given by* $\dim_{\mathbb{C}} \mathcal{O}(x, y, p)/\mathcal{O}(x, y, p)\langle F, F_p, F_x + pF_y\rangle$.

(b) *The multiplicity of a non-fold singularity of the projection* $(x, y, p) \to (x, y)$ *is given by* $\dim_{\mathbb{C}} \mathcal{O}(x, y, p)/\mathcal{O}(x, y, p)\langle F, F_p, F_{pp}\rangle$ *provided that the vector field* ξ *is non-zero on the lift.*

(c) *If we have a non-fold singular point of the projection where the vector field* ξ *vanishes, then the multiplicity is the sum of the numbers occurring in* (a) *and* (b).

For BDEs of Type 1, the discriminant is smooth in a generic deformation, so the multiplicity is the number of folded singularities that occur in a generic deformation. If we assume that $a(0, 0) \neq 0$ and $p = 0$, then the multiplicity m of the BDE at $(0, 0, 0)$ is given by

$$m = m(\delta, a\delta_x - b\delta_y),$$

where $m(h, k)$ denotes $\dim_{\mathbb{C}} \mathcal{O}(x, y)/\mathcal{O}(x, y) \langle h, k\rangle$ ([**13**]).

In fact ([**13**]), any BDE of Type 1 can be transformed by changes of coordinates and multiplication by non-zero functions to one in the form $dy^2 + f(x, y)dx^2 = 0$. The multiplicity of the BDE is then given by

$$m = m(f, f_x) = \mu(f) + \mu(f(x, 0)) - 1,$$

where μ denotes the Milnor number of the function germ (which is the multiplicity of its Jacobian ideal).

If we consider the example at the beginning of this section, $f(x, y) = x^2 + y^2$, so $f_x(x, y) = 2x$ and $m = \dim_{\mathbb{C}} \mathcal{O}(x, y)/\mathcal{O}(x, y) \left\langle x^2 + y^2, 2x\right\rangle = 2$. This explains why we have two folded singularities appearing in the deformation (Figure 8, first row)

We turn now to BDEs of Type 2 where we complexify the coefficients.

Definition 5. ([**13**]) The multiplicity of a BDE of Type 2 is defined to be the (maximum) number of non-degenerate singular points of the perturbed equations within the set of BDEs, where this is finite.

We observe that if we deform a BDE of Type 2 in the set of all IDEs, then its multiplicity is infinite. Consider, for example, the BDE $yp^2 + 2xp - y = 0$ which has multiplicity 3 using Definition 5 (see below). If we view it as an IDE and consider the deformation $F = tp^n + yp^2 + 2xp - y = 0$, then the equations $F = F_p = F_{pp} = 0$ have a zero at the origin of multiplicity n. Therefore, using Definition 4, the multiplicity of the above BDE is infinite. (The key point here is that one cannot use the division theorem to reduce the BDE to an IDE of a fixed degree.)

PROPOSITION 4.2. ([**13**]) *The multiplicity of a BDE is given by*

$$\begin{aligned} m = & \ \tfrac{1}{2} m(\delta, a\delta_x^2 - 2b\delta_x\delta_y + c\delta_y^2) \\ = & \ m(\delta, a\delta_x - b\delta_y) - m(a,b) \\ = & \ m(\delta, b\delta_x - c\delta_y) - m(b,c). \end{aligned}$$

The formula in Proposition 4.2 is also valid for BDEs of Type 1. For example, when $a(0,0) \neq 0$, $m(a,b) = 0$ and we recover the formula for the multiplicity of a BDE of Type 1 given by $m(\delta, a\delta_x - b\delta_y)$.

If we consider the second example at the beginning of this section, we have $a = y$, $b = -x$, $c = -y$, so the multiplicity $m = m(x^2 + y^2, 4yx) - m(y,x) = 4 - 1 = 3$, which explains why we have three folded singularities appearing in the deformation of the BDE (Figure 8, second row).

4.2. *The singularities of the discriminant*

To a BDE with coefficients (a,b,c) is associated the family of symmetric matrices

$$S(x,y) = \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}.$$

The discriminant of the BDE is precisely the determinant of S. Bruce classified in [**6**] families of symmetric matrices up to an equivalence relation that preserves the singularities of the determinant. Let $S(n,\mathbb{K})$ denote the space of $n \times n$-symmetric matrices with coefficients in the field $\mathbb{K}$ of real or complex numbers. A family of symmetric matrices is a smooth map germ $\mathbb{K}^r, 0 \to S(n,\mathbb{K})$. Denote by $\mathcal{G}$ the group of smooth changes of parameters in the source and parametrised conjugation in the target. Thus, two smooth map-germs A, B are $\mathcal{G}$ equivalent if $B = X^t(A \circ \phi^{-1})X$, where ϕ is a germ of a diffeomorphism $\mathbb{K}^r, 0 \to \mathbb{K}^r, 0$ and $X : \mathbb{K}^r, 0 \to GL(n,\mathbb{K})$. A list of all the $\mathcal{G}$-simple singularities of families of symmetric matrices is obtained in [**6**]. For more on symmetric matrices see [**10**, **36**, **37**].

The 2×2 matrix associated to a BDE of Type 1 is $\mathcal{G}$-equivalent to one in the form

$$\begin{pmatrix} 1 & 0 \\ 0 & f(x,y) \end{pmatrix}.$$

It turns out that, in this case, the $\mathcal{G}$-action reduces to the action of the contact group $\mathcal{K}$ on the ring of function germs $f : \mathbb{K}^2, 0 \to \mathbb{K}, 0$ ([**6**]). This explains, for instance, why we get the usual Morse transitions in the discriminants of the family of BDEs $dy^2 + (x^2 + y^2 + t)dx^2 = 0$ at the beginning of this section (Figure 8, first row).

For BDEs of Type 2, the $\mathcal{G}$-action does not reduce to the action of the contact group $\mathcal{K}$. For the example at the beginning of this section, the matrix $\begin{pmatrix} y & -x \\ -x & -y \end{pmatrix}$ of the BDE is 1-$\mathcal{G}$-determined and a versal $\mathcal{G}$-deformation is given by $\begin{pmatrix} y & -x \\ -x & -y + t \end{pmatrix}$. The zero sets of the determinants of these matrices undergo transitions of type cone sections (Figure 8, second row).

It is worth observing that the action $\mathcal{G}$ models the singularities of the discriminant of a BDE as well as its deformations in generic families of BDEs. The action does not preserve the pair of foliations determined by the BDE. Nevertheless, it provides important information when studying families of BDEs (see §5). All the key local information about the pair of foliations determined by the BDE occurs on the discriminant. It is also worth mentioning that all the $\mathcal{G}$-invariants associated the matrix of a BDE are invariants of the BDE.

4.3. *The index of a BDE*

The index of a BDE with discriminant an isolated point is defined as the index of one direction field determined by the BDE at the singular point. In [**19**, **20**, **21**], Challapa gave the following definition of the index of a BDE at a singular point (with discriminant not necessary an isolated point) when the coefficients are real analytic functions. Consider a family of BDEs $(a(x,y,t), b(x,y,t), c(x,y,t))$. The family is called a good perturbation if the discriminant δ_t is a regular curve for $t \neq 0$ and the BDEs for $t \neq 0$ fixed have only folded singularities. Challapa showed that such good perturbations exist. He defined the index of a folded saddle to be $K(S) = -1/2$ and the index of a folded node and focus to be $K(N) = K(F) = 1/2$ and gave the following definition of the index of an analytic BDE.

Definition 6. Let $\omega = (a, b, c)$ be a germ, at the origin, of an analytic BDE and ω_t be a good perturbation of ω. The index of ω at the origin is defined by

$$I(\omega) = \sum_i K_{\delta_t}(z_i) + \sum_{\delta^t(u_i)<0} \mathrm{index}_{u_i} \nabla\delta_t$$

where $\nabla\delta_t$ denotes the gradient of δ_t , z_i are non-degenerate singular points of ω_t and u_i are the critical points in the negative part of δ_t (i.e., $\nabla\delta_t(u_i) = 0$ and $\delta_t(u_i) < 0$).

Challapa shows that the index I is independent of the choice of a good deformation, it is invariant under analytic changes of coordinates and it satisfies the Poincaré-Hopf formula.

4.4. *Cr-invariant of asymptotic curves at folded singularities*

Consider a surface S immersed in $\mathbb{R}^3$. Uribe-Vargas ([**71**]) produced an invariant of the folded singularities of the asymptotic curves BDEs on S. As asymptotic curves capture the contact of S with lines, one can also consider S immersed in an affine or projective 3-space. Recall that the discriminant of the BDE of the asymptotic curves is the parabolic set P of S. The flecnodal curve F is a curve on which an asymptotic direction has higher contact with the surface. In general the flecnodal curve is a smooth curve on S and is tangent to the parabolic set at the cusp of Gauss/godron point (i.e., at the folded singularities of asymptotic BDE). The flecnodal curve can also be captured using Legendre duality on the BDE of the asymptotic curves, see [**14**]. There is another curve D on S, called the *conodal curve*. It is the closure of the locus of points of contact of S with its bitangent planes. This curve is in general tangent to P and F at a cusp of Gauss.

Let g denotes the cusp of Gauss/godron point. Consider $\pi : PT^*S \to S$ endowed with the canonical contact structure and the Legendrian lifts L_D, L_F, L_P consisting of the contact elements of S tangent to D, F, P around g. Consider also L_g the fibre over g of π. The four Legendrian curves are tangent to the same contact plane Π and their tangent directions determine four lines l_D, l_F, l_P, l_g through the origin in Π.

Definition 7. ([**71**]) The cr-invariant $\rho(g)$ of a godron g is defined as the cross-ratio of the lines l_D, l_F, l_dP and l_g of Π: $\rho(g) = (l_F, l_D, l_P, l_g)$.

Uribe-Vargas used ρ to obtain a classification of the configurations of the curves D, F, P at a cusp of Gauss; see [**71**] for more details.

5. *Bifurcations*

We consider in this section families of germs of BDEs. Two germs of families of BDEs $\tilde{\omega}$ and $\tilde{\tau}$, depending smoothly on the parameters t and s respectively, are said to be *locally fibre topologically equivalent* if, for any of their representatives, there exist neighbourhoods U and W of 0 in respectively the phase space (x, y) and the parameter space t, and a family of homeomorphisms h_t, for $t \in W$, all defined on U such that h_t is a topological equivalence between $\tilde{\omega}_t$ and $\tilde{\tau}_{\psi(t)}$, where ψ is a homeomorphism defined on W. (The map h_t is not required to be continuous in t.)

We associate to a germ of an r-parameter family of BDEs $\tilde{\omega} = (\tilde{a}, \tilde{b}, \tilde{c})$ the jet-extension map

$$\begin{array}{rccc} \Phi : & \mathbb{R}^2 \times \mathbb{R}^r, (0,0) & \to & J^k(2,3) \\ & ((x,y),t) & \mapsto & j^k(\tilde{a}, \tilde{b}, \tilde{c})_t|_{(x,y)} \end{array}$$

where $J^k(2,3)$ denotes the vector space of polynomial maps of degree $\leq k$ from $\mathbb{R}^2$ to $\mathbb{R}^3$, and $j^k(\tilde{a}, \tilde{b}, \tilde{c})_t|_{(x,y)}$ is the k-jet of $(\tilde{a}, \tilde{b}, \tilde{c})$ at (x, y) with t fixed.

The singularity type of the BDE $\tilde{\omega}_0$ determines a semi-algebraic set V in $J^k(2,3)$ of codimension, say, m. The family $\tilde{\omega}$ is said to be a *generic family* if the map Φ is transverse to V in $J^k(2,3)$. A necessary condition for genericity is of course $r \geq m$. It follows from Thom's Transversality Theorem that the set of generic families is residual in the set of smooth map germs $\mathbb{R}^2 \times \mathbb{R}^r, 0 \to \mathbb{R}^3, 0$.

The *bifurcation set* of a generic family is the set of parameters t where the associated BDE has a singularity of codimension ≥ 1 at some point $p \in U$. This gives a stratification $\mathcal{S}$ of the parameter space consisting of following strata: the origin (if the singularity at $t = 0$ is isolated), and local and semi-local singularities of codimension s, $1 \leq s \leq m - 1$. The singularity of $\tilde{\omega}_0$ is local, but semi-local singularities can appear in $\tilde{\omega}_t$, for $t \neq 0$. The semi-local singularities are very hard to deal with, and there is so far no general approach to deal with them. Each case is dealt with separately. There is a result in [**69**] which is worth mentioning here.

LEMMA 5.1. ([**69**]) *There are no Poincaré-Andronov (Hopf) bifurcations on the lifted field ξ of an IDE* (1.1) *at a regular point on the criminant.*

When studying bifurcations of a BDE $\tilde{\omega}_0$, the aim is to show that any two generic families of $\tilde{\omega}_0$ are (fibre) topologically equivalent. The strategy we adopted in [**66**, **69**] is the following.

- Obtain a model for the BDE at $t = 0$ (using the methods in §3.2).
- Reduce the N-jet of the family to a normal form (using the formal reduction technique in §3.1).
- Obtain a condition for the family to be generic.
- Show that the bifurcation sets of generic families are homeomorphic.
- Obtain the configuration of the discriminant in each stratum of $\mathcal{S}$ (using the symmetric matrices framework §4.2).
- Show that the number of singularities, their type and their position on the discriminant are constant in each stratum of $\mathcal{S}$. (The results in [**56**] are of use here.)
- Show that the configurations of the integral curves have a constant topological type in each stratum of $\mathcal{S}$.

Models of generic families of BDEs with local codimension 2 singularities and their bifurcations in the families are given in [**66**, **69**]. Singularities of codimension 1 are dealt with in [**9**, **18**, **51**, **67**]. Some degenerate cases are studied in [**54**].

6. *Appendix*

In §2 the pairs of foliations asymptotic/characteristic/principal curves are defined on an immersed surface. These pairs (or some of the them) are also considered on algebraic surfaces or surfaces with a cross-cap singularity. They are given by BDEs whose singularities are of higher codimension ([**34**, **57**, **63**, **68**]).

When $F = F_p = F_{pp} = 0$ at the origin, the solution curves of the IDE form a web (see §3.3). A classification of hexagonal analytic 3-webs $p^3 + ap^2 + bp + c = 0$ is given in [**2**]. Other types of n-web occur in differential geometry. For example, the asymptotic curves on surfaces in $\mathbb{R}^5$ are given by a quintic differential equation $p^5 + a_1p^4 + a_2p^3 + a_3p^2 + a_4p + a_5 = 0$, where a_i, $i = 1 \dots 5$, are smooth functions in (x, y) ([**53**, **59**]).

Systems of IDEs $F_1(t, x, p) = \dots = F_n(t, x, p) = 0$ with $x = (x_1, \dots, x_n)$, $p = (p_1, ..., p_n)$, $p_i = dx_i/dt$ are considered by in [**61**].

Homogeneous differential equations of degree greater than 2 are also considered. In [**44**] are defined lines of curvature on surface in $\mathbb{R}^4$ that are given by a quartic differential equation $a_0dy^4 + a_1dy^3dx + a_2dy^2dx^2 + a_3dydx^3 + a_4dx^4 = 0$, where a_i, $i = 1 \dots 4$, are smooth functions in (x, y). The coefficients all vanish at some special points on the surface. The configuration of the solution curves at such points is given in [**44**].

In [**31**] Fukui and Nuño-Ballesteros studied equations of degree n which have n real solutions away from some isolated (singular) points where all the coefficients vanish. They defined the index of such BDEs at a singular point and proved a Poincaré-Hopf type theorem (see also [**30**]). They also gave a classification of the configuration of the n-web around generic singular points.

Below are some topological models of singularities of BDEs of Type 2 with codimension higher than 2 (see Figure 9). The models are of BDEs with discriminant having a given $\mathcal{K}$-singularity type. However, they do not form an exhaustive list of the topological types of BDEs with discriminant having that $\mathcal{K}$-singularity type. The models are topologically determined by the k-jet of the BDE, where k is the highest degree of the coefficients of the equation. These are as follows.

– The discriminant has an A_1^--singularity, α_1 and ϕ have two common roots, [**68**]:

$$(y, -x + y^2, y),\ (y, -x + xy, y).$$

– The discriminant has an A_2-singularity, $j^1w \sim (x, by, 0)$, [**57**]:

$$(x, -y, x^2),\ (x, y, x^2).$$

– The discriminant has an A_3-singularity, $j^1w \sim (0, x + y, 0)$, [**41**] and [**43**] respectively:

$$(y, x^2, -y),\ (y^2, x + y, -y^2).$$

– The discriminant has an A_3-singularity, $j^1w \sim (0, b_0x, y)$, [**68**]:

$$(\pm y^3, b_0x + b_2y^2, y).$$

The topological type is constant in open regions determined by some exceptional curves in the (b_0, b_2)-plane (see [**68**]).

– The discriminant has an A_3-singularity, $j^1w \sim (\alpha x + y, \pm x, 0)$, [**49**]:

$$(y, x, \pm y^3),\ (y, -x, \pm y^3).$$

– The discriminant has an $X_{1,2}$-singularity [**68**]:

$$(x^2 + y^4, -xy, -x^2 + 2y^2 + y^3),\ (x^2 + y^4, -xy, -x^2 + 2y^2 + xy^2).$$

– The discriminant has an $Y_{5,6}$-singularity [**57**]:

$$(x^2, -xy, 2y^2 - x^3).$$

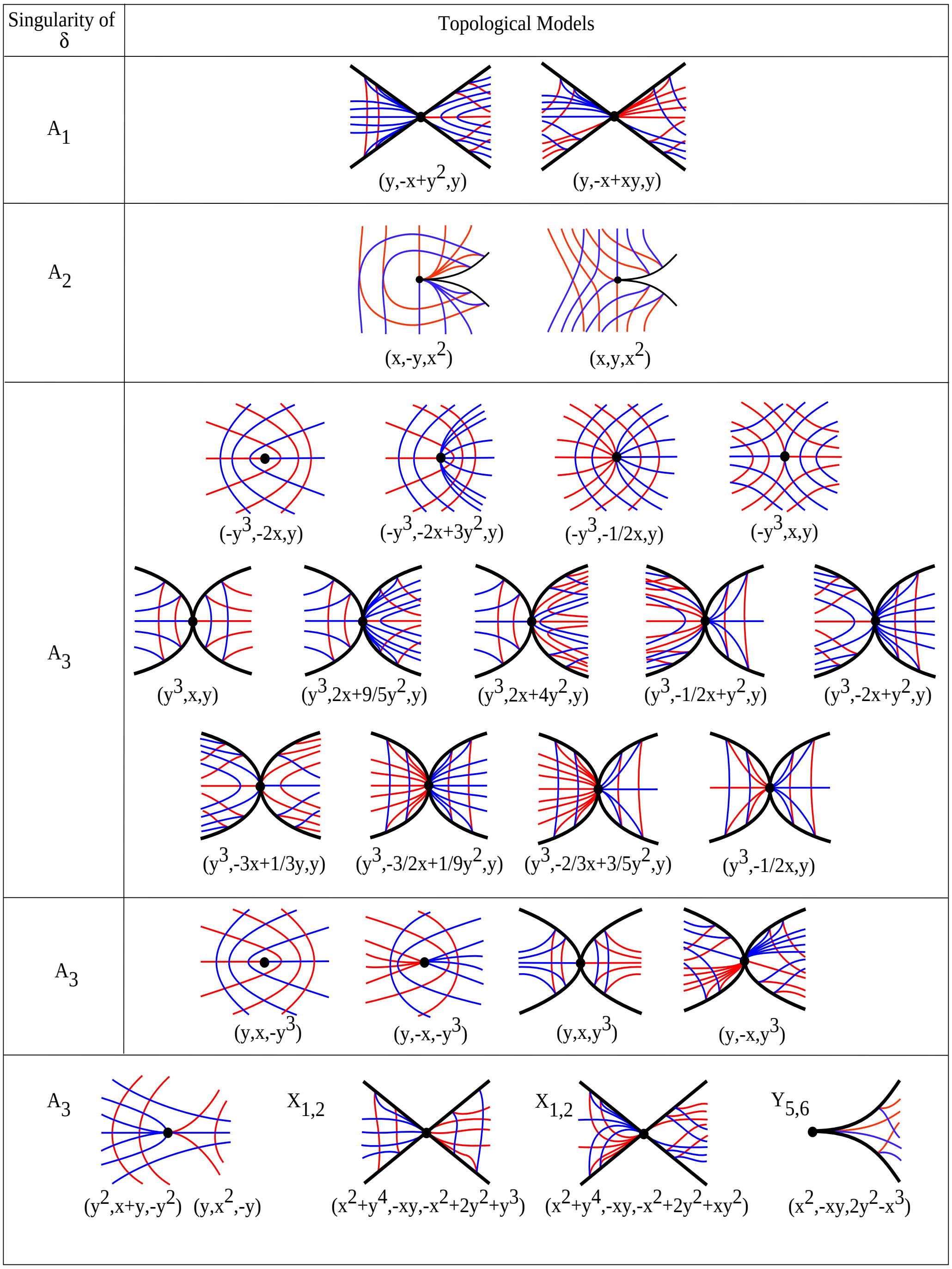

FIGURE 9. *Some topological models of Type 2 BDEs with singularities of codimension > 2.*

– Under some conditions, a BDE with discriminant an isolated point is topologically equivalent to its principal part defined by Newton polyhedra [**45**].

Remark 3. Some of the configurations in Figure 9 are topologically equivalent to those of less degenerate singularities. For instance, those in the second row are equivalent to a folded node and saddle respectively (Figure 2). However, their topological codimensions are distinct and so are their bifurcations.

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