Caustics of surfaces in the Minkowski 3-space

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Abstract

The caustic of a smooth surface in the Euclidean 3-space is the envelope of the normal rays to the surface. It is also the locus of the centres of curvature (the focal points) of the surface. This is why it is also referred to as the focal set of the surface. It has Lagrangian singularities and its generic models are given in [1] (see Figure 2). The aim of this paper is to define the caustic $C(M)$ of a smooth surface M embedded in the Minkowski 3-space and to study its geometry. We denote by the LD the locus of points on M where the metric is degenerate. If M is a closed surface then its LD is not empty. At a point on the LD the "normal" line to M is lightlike and is tangent to M. Also, the focal set of M is not defined at points on the LD. We define the caustic of M as the bifurcation set of the family of distance squared functions on M. Then $C(M)$ coincides with the focal set of $M \setminus LD$ and provides an extension of the focal set to the LD. We study the local behaviour of the metric on $C(M)$.

1 Introduction

Let M be a smooth and orientable surface in the Euclidean 3-space. To a point $p \in M$ are associated two focal points $p + (1/\kappa_1(p))N(p)$ and $p + (1/\kappa_2(p))N(p)$, where $\kappa_1(p)$ and $\kappa_2(p)$ are the principal curvatures of M at p and N is a unit normal vector at p. The focal set of M is the locus of its focal points. The focal set of M coincides with the caustic generated by its normal lines and is the bifurcation set of the family of distance squared functions on M. Therefore, by a results in [1], for a generic M the caustic $C(M)$ is locally diffeomorphic to one of the models in Figure 2. The affine geometry of the focal set is also of interest and captures some of the extrinsic geometry of the surface M (see for example [2, 3, 9, 13]).

We consider in this paper surfaces M embedded in the Minkowski 3-space $\mathbb{R}^3_1$. The induced metric ρ on M may be degenerate at some points on M (this is indeed the case

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on any closed surface in $\mathbb{R}^3_1$). We label the locus of such points the *Locus of Degeneracy* (LD). At a point $p \in LD$, the tangent plane T_pM to M is lightlike, so the "normal" to the surface at p is the unique lightlike direction in T_pM. Away from the LD, the surface admits locally a unit normal vector which is spacelike if ρ is Lorentzian and timelike if ρ is Riemannian. If $\boldsymbol{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3_1$ is a local parametrisation of M at $p \notin LD$, then the shape operator $-d\boldsymbol{N}_p$, where $\boldsymbol{N}$ is the Gauss map, may have real eigenvalues. When it does we call them the principal curvatures. To these are associated two focal points which trace the focal set of M. However, $\boldsymbol{N}$ is not well defined on the LD and the principal curvatures near the LD tend to infinity as the point tends to the LD. Therefore the focal set is not defined at points on the LD. In contrast, the caustic $C(M)$ of M is defined even on the LD. The caustic is also the bifurcation set of the family of distance squared functions on M and $C(M \setminus LD)$ coincides with the focal set of $M \setminus LD$. Again, by a result in [1], for a generic M the caustic $C(M)$ is locally diffeomorphic to one of the models in Figure 2. We study in this paper the behaviour of the induced metric on $C(M)$.

We give in §2 some preliminaries on the geometry of surfaces in the Minkowski space and a brief review of the lines of principal curvature of a surface as these are related to its caustic. We study in §3 the behaviour of the induced metric on the caustic of a Lorentzian surface patch and deal in §4 with a surface patch where the induced metric has varying signature.

2 Preliminaries

2.1 Surfaces in $\mathbb{R}^3_1$

The *Minkowski space* $(\mathbb{R}^3_1, \langle , \rangle)$ is the vector space $\mathbb{R}^3$ endowed with the pseudo-scalar product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -u_0v_0 + u_1v_1 + u_2v_2$, for any $\boldsymbol{u} = (u_0, u_1, u_2)$ and $\boldsymbol{v} = (v_0, v_1, v_2)$ in $\mathbb{R}^3_1$. We say that a vector $\boldsymbol{u} \in \mathbb{R}^3_1$ is *spacelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$, *lightlike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ and *timelike* if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$. The norm of a vector $\boldsymbol{u} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$.

We have the following pseudo-spheres in $\mathbb{R}^3_1$ with centre $p \in \mathbb{R}^3_1$ and radius $r > 0$,

$$\begin{array}{rcl} H^2(p, -r) & = & \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = -r^2\}, \\ S^2_1(p, r) & = & \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = r^2\}, \\ LC^*(p) & = & \{\boldsymbol{u} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{u} - p, \boldsymbol{u} - p \rangle = 0\}. \end{array}$$

We denote by $H^2(-r)$ and $S^2_1(r)$ the pseudo-spheres centred at the origin in $\mathbb{R}^3_1$.

We consider embeddings $\boldsymbol{i} : M \to \mathbb{R}^3_1$ of a smooth surface M. The set of such embeddings is endowed with the Whitney C^∞-topology. We say that a property is *generic* if it is satisfied by a residual set of embeddings of M in $\mathbb{R}^3_1$. We shall identify $\boldsymbol{i}(M)$ with M.

The pseudo-scalar product in $\mathbb{R}^3_1$ induces a metric on M which can be degenerate at some points on M. We call the locus of such points *the Locus of Degeneracy* and denote it by LD.

Let $\boldsymbol{x} : U \subset \mathbb{R}^2 \to M$ be a local parametrisation of M. To simplify notation, we write $\boldsymbol{x}(U) = M$. Let

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle, \quad G = \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle$$

denote the coefficients of the first fundamental form of M with respect to $\boldsymbol{x}$. The integral curves of the lightlike directions on M are the solution curves of the binary quadratic differential equation (BDE)

$$Edu^2 + 2Fdudv + Gdv^2 = 0. \tag{1}$$

We identify the LD and its pre-image in U by $\boldsymbol{x}$. Then the LD (in U) is given by $F^2 - EG = 0$ and is the *discriminant curve* of the BDE (1) (the discriminant curve of a BDE is the set of points where the directions it determines coincide). We assume in this paper that the LD is either empty or is a smooth curve that splits the surface locally into a Riemannian and a Lorentzian region (i.e., where the induced metric has signature 2 and 1 respectively). The unique lightlike direction on the LD is, in general, transverse to the LD. Then the configuration of the lightlike curves is locally topologically equivalent to Figure 1 left. The unique lightlike direction on the LD can be tangent to the LD at isolated points. Then the BDE (1) has a singularity, and we assume that it is (well) folded (see for example [4] for terminology and [12] for a survey paper on BDEs). The configuration of the lightlike curves at a folded singularity is locally topologically equivalent to one of the last three cases in Figure 1. The assumptions on the LD and on the singularities of the BDE (1) are satisfied for generic embeddings of M in $\mathbb{R}^3_1$.

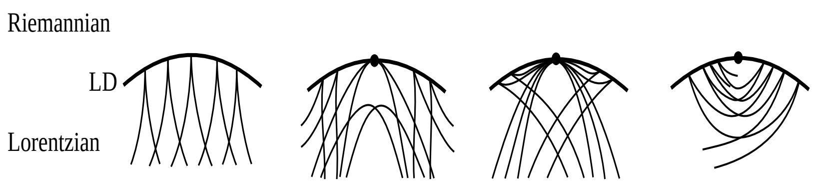

Figure 1: Stable local topological configurations of the lightlike curves at points on the LD.

The following special local parametrisations simplify considerably the calculations and make the algebraic conditions involved easier to interpret geometrically. (The proof is standard and is omitted.)

Theorem 2.1 (1) *At any point p on the Lorentzian part of M there is a local parametrisation $\boldsymbol{x} : U \to V \subset M$ of a neighbourhood V of p, such that for any $p' \in V$, the coordinate curves through p' are tangent to the lightlike directions. Equivalently, there exists a local parametrisation with $E \equiv 0$ and $G \equiv 0$ on U.*

(2) *Let p be a point on the LD of a generic surface M. Then there exists a local parametrisation $\boldsymbol{x} : U \to V \subset M$ of a neighbourhood V of p, such that for any $p' = \boldsymbol{x}(q') \in V \cap LD$, the lightlike directions in $T_{p'}M$ are parallel to $\boldsymbol{x}_u(q')$, i.e., $E = F = 0$ on the LD.*

Pei [10] defined an $\mathbb{R}P^2$-valued Gauss map on M. In $\boldsymbol{x}(U)$, this is simply the map $PN : \boldsymbol{x}(U) \to \mathbb{R}P^2$ which associates to a point $p = \boldsymbol{x}(q)$ the projectivisation of the vector $\boldsymbol{x}_u \times \boldsymbol{x}_v(q)$, where "$\times$" denotes the wedge product in $\mathbb{R}^3_1$. Away from the LD, the $\mathbb{R}P^2$-valued Gauss map can be identified with the de Sitter Gauss map $\boldsymbol{x}(U) \to S^2_1(1)$ on the Lorentzian part of the surface and with the hyperbolic Gauss map $\boldsymbol{x}(U) \to H^2(-1)$ on its Riemannian part. Both maps are given by $\boldsymbol{N} = \boldsymbol{x}_u \times \boldsymbol{x}_v / ||\boldsymbol{x}_u \times \boldsymbol{x}_v||$. The map $A_p(\boldsymbol{v}) = -d\boldsymbol{N}_p(\boldsymbol{v})$ is a self-adjoint operator on $\boldsymbol{x}(U) \setminus LD$. We denote by

$$\begin{array}{rcccl} l & = & -\langle N_u, \boldsymbol{x}_u \rangle & = & \langle N, \boldsymbol{x}_{uu} \rangle, \\ m & = & -\langle N_u, \boldsymbol{x}_v \rangle & = & \langle N, \boldsymbol{x}_{uv} \rangle, \\ n & = & -\langle N_v, \boldsymbol{x}_v \rangle & = & \langle N, \boldsymbol{x}_{vv} \rangle \end{array}$$

the coefficients of the second fundamental form on $\boldsymbol{x}(U) \setminus LD$. When A_p has real eigenvalues, we call them the *principal curvatures* and their associated eigenvectors the *principal directions* of M at p. (There are always two principal curvatures on the Riemannian part of M but this is not true on its Lorentzian part.) The lines of principal curvature, which are the integral curves of the principal directions, are solutions of the BDE

$$(Gm - Fn)dv^2 + (Gl - En)dvdu + (Fl - Em)du^2 = 0. \tag{2}$$

The discriminant of the BDE (2) is denoted the *Lightlike Principal Locus* (LPL) in [5, 6].

On the Riemannian part of a generic surface, the LPL consists of isolated points labelled *spacelike umbilic points* (these are points where A_p is a multiple of the identity map). At none spacelike umbilic points, there are always two orthogonal spacelike principal directions.

On the Lorentzian part of a generic surface, the LPL is either empty or is a smooth curve except at isolated points where it has Morse singularities of type node. Such points are labelled *timelike umbilic points* (these are also points where A_p is a multiple of the identity map). The LPL consists of points where the principal directions coincide and become lightlike. There are two principal directions on one side of the LPL and none on the other. When there are two of them, they are orthogonal and one is spacelike while the other is timelike. The configurations of the lines of principal curvature at points of the LPL are studied in [6].

One can extend the lines of principal curvature across the LD as follows ([6]). As equation (2) is homogeneous in l, m, n, we can multiply these coefficients by $||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ and substitute them by

$$\bar{l} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uu} \rangle, \quad \bar{m} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uv} \rangle, \quad \bar{n} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{vv} \rangle.$$

This substitution does not alter the pair of foliations on $\boldsymbol{x}(U) \setminus LD$. The new equation is defined on the LD and defines the same pair of foliations associated to the de Sitter (resp. hyperbolic) Gauss map on the Lorentzian (resp. Riemannian) part of $\boldsymbol{x}(U)$. The extended lines of principal curvature are the solution curves of the BDE

$$(G\bar{m} - F\bar{n})dv^2 + (G\bar{l} - E\bar{n})dudv + (F\bar{l} - E\bar{m})du^2 = 0. \tag{3}$$

We observe that one of the principal directions on the LD is the unique lightlike direction and the other is spacelike. The LD and LPL can meet tangentially at isolated points. These points are exactly the folded singularities of the BDE (1) of the lightlike foliations ([6]).

2.2 The family of distance squared functions

The family of distance squared functions on M is given by

$$\begin{array}{rccl} d^2: & M \times \mathbb{R}^3_1 & \to & \mathbb{R} \\ & (p, \boldsymbol{v}) & \mapsto & d^2(p, \boldsymbol{v}) = \langle p - \boldsymbol{v}, p - \boldsymbol{v} \rangle. \end{array}$$

We denote by $d^2_{\boldsymbol{v}}$ the function on M given by $d^2_{\boldsymbol{v}}(p) = d^2(p, \boldsymbol{v})$. We take a local parametrisation $\boldsymbol{x} : U \to \mathbb{R}^3_1$ of M at $p = \boldsymbol{x}(q)$ and write $\boldsymbol{x}(U) = M$. We denote by

$$\Sigma(d^2) = \{\xi = ((u, v), \boldsymbol{v}) \in U \times \mathbb{R}^3_1 \,|\, d^2_u(\xi) = d^2_v(\xi) = 0\},$$

and by

$$Bif(d^2) = \{\boldsymbol{v} \in \mathbb{R}^3_1 \,|\, \exists((u, v), \boldsymbol{v}) \in \Sigma(d^2) \text{ such that } rank(Hess(d^2_{\boldsymbol{v}})) < 2 \text{ at } (u, v)\}.$$

The set $Bif(d^2)$ is the local stratum of the *bifurcation set* of the family d^2, i.e., it is the set of points $\boldsymbol{v} \in \mathbb{R}^3_1$ for which there exists $(u, v) \in U$ such that $d^2_{\boldsymbol{v}}$ has a degenerate local singularity at (u, v). The mapping (d^2_u, d^2_v) is not degenerate at any point in $U \times \mathbb{R}^3_1$. Indeed, its Jacobian matrix, which is a multiple of

$$\begin{pmatrix} \langle \boldsymbol{x}_{uu}, \boldsymbol{x} - \boldsymbol{v} \rangle + \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle & \langle \boldsymbol{x}_{uv}, \boldsymbol{x} - \boldsymbol{v} \rangle + \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle & \boldsymbol{x}^0_u & -\boldsymbol{x}^1_u & -\boldsymbol{x}^2_u \\ \langle \boldsymbol{x}_{uv}, \boldsymbol{x} - \boldsymbol{v} \rangle + \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle & \langle \boldsymbol{x}_{vv}, \boldsymbol{x} - \boldsymbol{v} \rangle + \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle & \boldsymbol{x}^0_v & -\boldsymbol{x}^1_v & -\boldsymbol{x}^2_v \end{pmatrix},$$

has rank 2 at any point in $U\times\mathbb{R}^3_1$ as $\boldsymbol{x}_u = (\boldsymbol{x}^0_u, \boldsymbol{x}^1_u, \boldsymbol{x}^2_u)$ and $\boldsymbol{x}_v = (\boldsymbol{x}^0_v, \boldsymbol{x}^1_v, \boldsymbol{x}^2_v)$ are linearly independent. Therefore the family d^2 is a generating family (see [1] for terminology) and $\Sigma(d^2)$ is a smooth 3-dimensional submanifold of $U \times \mathbb{R}^3_1$. We write $\boldsymbol{v} = (v_0, v_1, v_2)$ and denote by $T^*\mathbb{R}^3_1$ the cotangent bundle of $\mathbb{R}^3_1$ endowed with the canonical symplectic structure (which is metric independent). Then the map $L(d^2) : \Sigma(d^2) \to T^*\mathbb{R}^3_1$, given by

$$L(d^2)((u, v), \boldsymbol{v}) = (\boldsymbol{v}, (\frac{\partial d^2}{\partial v_0}((u, v), \boldsymbol{v}), \frac{\partial d^2}{\partial v_1}((u, v), \boldsymbol{v}), \frac{\partial d^2}{\partial v_2}((u, v), \boldsymbol{v}))),$$

is a Lagrangian immersion, so the map $\pi\circ L(d^2) : \Sigma(d^2) \to \mathbb{R}^3_1$ given by $((u, v), \boldsymbol{v}) \to \boldsymbol{v}$ is a Lagrangian map. The caustic $C(M)$ of M, which is the set of critical values of $\pi \circ L(d^2)$, is precisely $Bif(d^2)$ (see [1] for details). It follows that for a generic surface M, the caustic $C(M)$ is locally diffeomorphic to one of the surfaces in Figure 2.

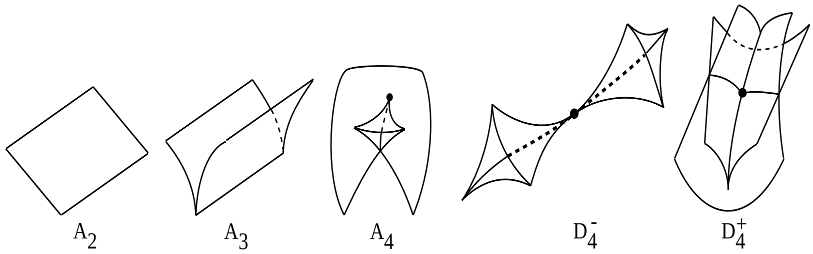

Figure 2: Generic singularities of 2-dimensional caustics.

The local models of the caustic at $\boldsymbol{v}$ corresponding to $p \in M$ on a generic M, depend on the $\mathcal{R}$-singularity type of the distance squared function $d^2_{\boldsymbol{v}}$ at p, where $\mathcal{R}$ is the right group of local diffeomorphisms in the source that fix p. For a generic M, $d^2_{\boldsymbol{v}}$ has local singularities of type $A^{\pm}_1, A_2, A^{\pm}_3, A_4, D^{\pm}_4$ which are modelled by the following functions (together with a multiplication of these functions by -1)

$$(A^{\pm}_1) : u^2 \pm v^2,\ (A_2) : u^2 + v^3,\ (A^{\pm}_3) : u^2 \pm v^4,\ (A_4) : u^2 + v^5,\ (D^{\pm}_4) : u^2v \pm v^3.$$

The caustic is the empty set at an $A^{\pm}_1$-singularity of $d^2_{\boldsymbol{v}}$. At the remaining generic singularities, it is diffeomorphic to a surface in Figure 2 labelled by the singularity type of $d^2_{\boldsymbol{v}}$ at p.

We define *the ridge* on M as the closure of the set of points on M where $d^2_{\boldsymbol{v}}$ (for some $\boldsymbol{v} \in \mathbb{R}^3_1$) has an A_3-singularity. It is the locus of points on M corresponding to the singular points on the caustic. The image of the ridge on the caustic is labelled the *rib curve*. (The notation follows that of Porteous [11] for surfaces in the Euclidean 3-space.)

We can obtain a parametrisation of the caustic, or its defining equations, as follows. We have $d^2_{\boldsymbol{v}}(u, v) = \langle \boldsymbol{x}(u, v) - \boldsymbol{v}, \boldsymbol{x}(u, v) - \boldsymbol{v}\rangle$, so $d^2_{\boldsymbol{v}}$ is singular at $q \in U$ if and only if

$\langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_u\rangle = \langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_v\rangle = 0$ at q, if and only if $\boldsymbol{x}-\boldsymbol{v}$ is parallel to $\boldsymbol{x}_u \times \boldsymbol{x}_v$ at q. That is, $\boldsymbol{x}-\boldsymbol{v} = \mu \boldsymbol{x}_u \times \boldsymbol{x}_v$ for some scalar μ. It is important to observe that this condition includes the case when p is on the LD where $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is parallel to the unique lightlike direction at p.

The singularity of $d^2_{\boldsymbol{v}}$ at q is degenerate if and only if $\boldsymbol{x}(q)-\boldsymbol{v} = \mu \boldsymbol{x}_u \times \boldsymbol{x}_v(q)$ for some scalar μ and $\left((d^2_{\boldsymbol{v}})^2_{uv} - (d^2_{\boldsymbol{v}})_{uu}(d^2_{\boldsymbol{v}})_{vv}\right)(q) = 0$. We have

$$\begin{array}{rcl} \frac{1}{2}(d^2_{\boldsymbol{v}})_{uu} &=& \langle \boldsymbol{x}_u, \boldsymbol{x}_u\rangle + \langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{uu}\rangle = E + \langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{uu}\rangle, \\ \frac{1}{2}(d^2_{\boldsymbol{v}})_{uv} &=& \langle \boldsymbol{x}_u, \boldsymbol{x}_v\rangle + \langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{uv}\rangle = F + \langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{uv}\rangle, \\ \frac{1}{2}(d^2_{\boldsymbol{v}})_{vv} &=& \langle \boldsymbol{x}_v, \boldsymbol{x}_v\rangle + \langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{vv}\rangle = G + \langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{vv}\rangle. \end{array}$$

At q, $\boldsymbol{x}-\boldsymbol{v} = \mu \boldsymbol{x}_u \times \boldsymbol{x}_v$, so $\langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{uu}\rangle = \mu \bar{l}$, $\langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{uv}\rangle = \mu \bar{m}$ and $\langle \boldsymbol{x}-\boldsymbol{v}, \boldsymbol{x}_{vv}\rangle = \mu \bar{n}$, where $\bar{l}, \bar{m}, \bar{n}$ are defined in §2.1. Then

$$\begin{array}{rcl} \frac{1}{2}(d^2_{\boldsymbol{v}})_{uu} &=& E + \mu \bar{l}, \\ \frac{1}{2}(d^2_{\boldsymbol{v}})_{uv} &=& F + \mu \bar{m}, \\ \frac{1}{2}(d^2_{\boldsymbol{v}})_{vv} &=& G + \mu \bar{n}. \end{array}$$

Therefore the singularity of $d^2_{\boldsymbol{v}}$ is degenerate if and only if $\boldsymbol{x}-\boldsymbol{v} = \mu \boldsymbol{x}_u \times \boldsymbol{x}_v$ and

$$(F^2 - EG) - \mu(\bar{n}E - 2\bar{m}F + \bar{l}G) + \mu^2(\bar{m}^2 - \bar{l}\bar{n}) = 0. \tag{4}$$

Then the caustic is given by

$$C(M) = \{\boldsymbol{x}(u,v) - \mu \boldsymbol{x}_u \times \boldsymbol{x}_v(u,v) \,|\, (u,v) \in U \text{ and } \mu \text{ is a solution of equation (4)}\}.$$

Away from the LD we can write $\boldsymbol{x}-\boldsymbol{v} = \lambda \boldsymbol{N}$, where $\lambda = \mu ||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ and $\boldsymbol{N} = \boldsymbol{x}_u \times \boldsymbol{x}_v / ||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ is a unit normal vector. Then a singularity of $d^2_{\boldsymbol{v}}$ is degenerate if and only if $\boldsymbol{x}-\boldsymbol{v} = \lambda \boldsymbol{N}$ and

$$(F^2 - EG) - \lambda(nE - 2mF + lG) + \lambda^2(m^2 - ln) = 0. \tag{5}$$

The solutions of equation (5) in λ are equal to minus the radii of curvatures. Therefore the caustic $C(M \setminus LD)$ is the focal set of $M \setminus LD$.

We study in the subsequent sections the induced metric on the caustic $C(M)$. When $C(M)$ is a smooth surface, its normal vector is parallel to one of the principal direction of M (Proposition 3.2). Therefore when the metric on M is Riemannian the caustic is Lorentzian. We consider separately the cases where the metric on M is Lorentzian and when it is degenerate.

In the figures, the Riemannian part of $C(M)$ is in white and its Lorentzian part is in grey.

Remark 2.2 *Following* [7]*, the contact between two submanifolds* $\phi(U)$*, where* $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^k$ *is an immersion and* $\psi^{-1}(0)$*, where* $\psi : \mathbb{R}^k \to \mathbb{R}^p$ *is a submersion, at*

$\phi(u_0) \in \psi^{-1}(0)$ *is measured by the singularities of the map-germ* $\psi \circ \phi : \mathbb{R}^n, u_0 \to \mathbb{R}^p, 0$. *In our case, the singularities of* $d^2_{\boldsymbol{v}}$ *at* p *measure the contact of* M *at* p *with the pseudo-sphere centred at* $\boldsymbol{v}$ *and of radius* $||p - \boldsymbol{v}||$. *The family* d^2 *is generic, so the contact makes sense even when the fibre of* $d^2_{\boldsymbol{v}}$ *is singular, i.e., is a lightcone (see* [8]*). When* M *is a Riemannian (resp. Lorentzian) patch, it can have singular contact (i.e.,* $d^2_{\boldsymbol{v}}$ *is singular) only with pseudo-spheres* $H^2(p, -r)$ *(resp.* $S^2(p, r)$*). At points on the LD, the contact can be singular only when the pseudo sphere is a lightcone.*

3 Lorentzian surface patches

We suppose that $\boldsymbol{x}(U)$ is a Lorentzian surface patch and write $M = \boldsymbol{x}(U)$. The vector $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is spacelike at all points in U, so we can use the unit normal vector $\boldsymbol{N} = \boldsymbol{x}_u \times \boldsymbol{x}_v / ||\boldsymbol{x}_u \times \boldsymbol{x}_v||$.

Let $p = \boldsymbol{x}(q)$ and suppose that p is not on the lightlike principal locus (LPL). Then to p are associated either two distinct points or no points on the caustic $C(M)$ (which coincides with the focal set in this case) and this depends on whether there are two or no principal directions at p. When there are two principal directions, one is spacelike and the other is timelike, so one sheet of $C(M)$ is Lorentzian and the other is Riemannian. We analyse the situation on the LPL, and suppose that $p = \boldsymbol{x}(q) \in LPL$ and $q = (0, 0) \in U$. We define the map $\phi : U \times \mathbb{R} \to \mathbb{R}$ by

$$\phi(u, v, \lambda) = (F^2 - EG)(u, v) - \lambda(nE - 2mF + lG)(u, v) + \lambda^2(m^2 - ln)(u, v).$$

In all the paper, we assume that $(m^2 - ln)(q) \neq 0$, that is, p is not a parabolic point.

Theorem 3.1 *Let* M *be a generic Lorentzian surface patch in* $\mathbb{R}^3_1$ *and* p *a point on the* LPL *of* M.

(1) *The distance squared function has an* A_2*-singularity at* p *if and only if* p *is neither a folded singularity of the lines of principal curvature nor a timelike umbilic point. At folded singularities of the lines of principal curvature the singularity is of type* A_3 *and at a timelike umbilic point it is of type* $D_4^{\pm}$.

(2) *The surface* $\phi^{-1}(0)$ *is smooth in a neighbourhood of* (q, λ) *if and only if* $p = \boldsymbol{x}(q)$ *is not a timelike umbilic point. When* p *is a timelike umbilic point,* $\phi^{-1}(0)$ *is diffeomorphic to a cone.*

(3) *The projection* $\pi : \phi^{-1}(0) \to U$ *is a fold map at* (q, λ) *when* $p = \boldsymbol{x}(q) \in LPL$ *is not a timelike umbilic point. The discriminant of* π *(i.e., the image of the critical set of* π*) is the* LPL.

Proof (1) We take a local parametrisation of M as in Theorem 2.1(1). The LPL is then given by $ln = 0$ and the timelike umbilic points occur when $l = n = 0$. If $p \in LPL$ is not a timelike umbilic point we suppose, without loss of generality, that

$l = 0$ and $n \neq 0$ (we also have $m \neq 0$). Then the unique principal direction on the LPL is parallel to $\boldsymbol{x}_u$, and equation (5) has a double root given by $\lambda = -F/m$. We have $(d^2_{\boldsymbol{v}})_{uu}/2 = \lambda l$, $(d^2_{\boldsymbol{v}})_{uv}/2 = F + \lambda m$ and $(d^2_{\boldsymbol{v}})_{vv}/2 = \lambda n$. Therefore on the LPL, $(d^2_{\boldsymbol{v}})_{uu} = (d^2_{\boldsymbol{v}})_{uv} = 0$ and $(d^2_{\boldsymbol{v}})_{vv} \neq 0$, so $d^2_{\boldsymbol{v}}$ has singularities of type A_k. The singularity is of type A_2 at q if and only if $(d^2_{\boldsymbol{v}})_{uuu} = \lambda l_u \neq 0$ at q, if and only if the unique principal direction at q is not tangent to the LPL, if and only if $p = \boldsymbol{x}(q)$ is not a folded singularity of the lines of principal curvature BDE. When $l_u = 0$, the singularity is generically of type A_3.

At a timelike umbilic point, we have $(d^2_{\boldsymbol{v}})_{uu} = (d^2_{\boldsymbol{v}})_{uv} = (d^2_{\boldsymbol{v}})_{vv} = 0$ and the singularity is generically of type $D_4^{\pm}$.

(2) The expression for ϕ in a local parametrisation with $E = G = 0$ is

$$\phi = (F + \lambda m)^2 - \lambda^2 nl,$$

where F, l, m, n are evaluated at (u, v). On the LPL, $F + \lambda m = 0$ and $m \neq 0$. We can make a change of variable and write $\phi = \xi^2 - nl(\xi - F)^2/m^2$. This is a regular function at $(q, 0)$ if and only if $l(q) \neq 0$ or $n(q) = 0$, i.e., if and only if $p = \boldsymbol{x}(q)$ is not a timelike umbilic point. At a timelike umbilic point ϕ has generically a Morse singularity and is equivalent, by smooth changes of coordinates in the source, to $\xi^2 - (\pm u^2 \pm v^2)$.

(3) The result follows from the fact that $\phi_\lambda = 0$ and $\phi_{\lambda\lambda} \neq 0$ at the points in consideration. □

Proposition 3.2 *The normal to the caustic $C(M)$ at a smooth point $\boldsymbol{v}$ is parallel to the principal direction corresponding to the principal curvature which determines $\boldsymbol{v}$.*

Proof We can assume, without loss of generality, that $\phi_v \neq 0$ and parametrise locally the surface $S = \phi^{-1}(0)$ by $(u, v(u, \lambda), \lambda)$. Then the caustic is parametrised locally by

$$\psi(u, \lambda) = \boldsymbol{x}(u, v(u, \lambda)) - \lambda \boldsymbol{N}(u, v(u, \lambda)).$$

We have

$$\begin{array}{rcl} \psi_u &=& \boldsymbol{x}_u + v_u \boldsymbol{x}_v - \lambda(\boldsymbol{N}_u + v_u \boldsymbol{N}_v), \\ \psi_\lambda &=& v_\lambda \boldsymbol{x}_v - \boldsymbol{N} - \lambda v_\lambda \boldsymbol{N}_v. \end{array}$$

As $\boldsymbol{N}_u$ and $\boldsymbol{N}_v$ are in T_pM, it follows that the caustic is singular if and only if $\psi_u = 0$. The matrix of the shape operator $-d\boldsymbol{N}_p$ with respect to the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ is

$$-\frac{1}{F^2 - EG} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix},$$

so

$$\begin{array}{rcl} \boldsymbol{N}_u &=& -\dfrac{Fm - Gl}{F^2 - EG} \boldsymbol{x}_u - \dfrac{Fl - Em}{F^2 - EG} \boldsymbol{x}_v \\ \boldsymbol{N}_v &=& -\dfrac{Fn - Gm}{F^2 - EG} \boldsymbol{x}_u - \dfrac{Fm - En}{F^2 - EG} \boldsymbol{x}_v. \end{array}$$

Therefore

$$\psi_u = (1+\lambda\frac{Fm-Gl}{F^2-EG}+\lambda v_u\frac{Fn-Gm}{F^2-EG})\boldsymbol{x}_u + (\lambda\frac{Fl-Em}{F^2-EG}+v_u(1+\lambda\frac{Fm-En}{F^2-EG}))\boldsymbol{x}_v$$

and

$$\psi_\lambda = \lambda v_\lambda\frac{Fn-Gm}{F^2-EG}\boldsymbol{x}_u + v_\lambda(1+\lambda\frac{Fm-En}{F^2-EG})\boldsymbol{x}_v - \boldsymbol{N}.$$

We write $\psi_u = a\boldsymbol{x}_u + b\boldsymbol{x}_v$ and $\psi_\lambda = c\boldsymbol{x}_u + d\boldsymbol{x}_v - \boldsymbol{N}$ with a, b, c, d as above, then

$$\psi_u \times \psi_\lambda = (ad-bc)\boldsymbol{x}_u \times \boldsymbol{x}_v - a\boldsymbol{x}_u \times \boldsymbol{N} - b\boldsymbol{x}_v \times \boldsymbol{N}.$$

We have, after simplification,

$$ad-bc = \frac{v_\lambda}{F^2-EG}\big((F^2-EG) - \lambda(nE-2mF+lG) + \lambda^2(m^2-ln)\big)$$

and this is zero as λ is a solution of equation (5). Now,

$$\begin{array}{rcl}\boldsymbol{x}_u \times \boldsymbol{N} & = & \frac{1}{||\boldsymbol{x}_u\times\boldsymbol{x}_v||}\boldsymbol{x}_u \times (\boldsymbol{x}_u \times \boldsymbol{x}_v)\\ & = & \frac{1}{||\boldsymbol{x}_u\times\boldsymbol{x}_v||}(\langle\boldsymbol{x}_u,\boldsymbol{x}_u\rangle\boldsymbol{x}_v - \langle\boldsymbol{x}_v,\boldsymbol{x}_u\rangle\boldsymbol{x}_u)\\ & = & \frac{1}{||\boldsymbol{x}_u\times\boldsymbol{x}_v||}(E\boldsymbol{x}_v - F\boldsymbol{x}_u).\end{array}$$

Similarly,

$$\boldsymbol{x}_v \times \boldsymbol{N} = \frac{1}{||\boldsymbol{x}_u \times \boldsymbol{x}_v||}(F\boldsymbol{x}_v - G\boldsymbol{x}_u).$$

It follows that

$$\psi_u \times \psi_\lambda = \frac{1}{||\boldsymbol{x}_u \times \boldsymbol{x}_v||}\left((aF+bG)\boldsymbol{x}_u - (aE+bF)\boldsymbol{x}_v\right).$$

The result follows by showing that equation (2) is satisfied for $du = aF + bG$ and $dv = -(aE + bF)$. □

Theorem 3.3 *Let M be a generic Lorentzian surface patch in $\mathbb{R}^3_1$ and $p \in LPL$ but is not a timelike umbilic point.*

(1) *Suppose that p is not a folded singularity of the lines of principal curvature. Then there is locally one sheet of the caustic $C(M)$ which is a smooth surface at the point corresponding to p. The induced metric on $C(M)$ is degenerate on the image of the LPL on M, which we denote by the LDC (the locus of degeneracy of the metric on the caustic). The LDC is a smooth curve that splits $C(M)$ into a Riemannian and a Lorentzian region (Figure 3, left).*

(2) *Suppose that p is a folded singularity of the lines of principal curvature. The LPL and the ridge curve meet tangentially at p. The caustic is a cuspidal-edge and each of its smooth components contains part of the LDC (Figure 3, right).*

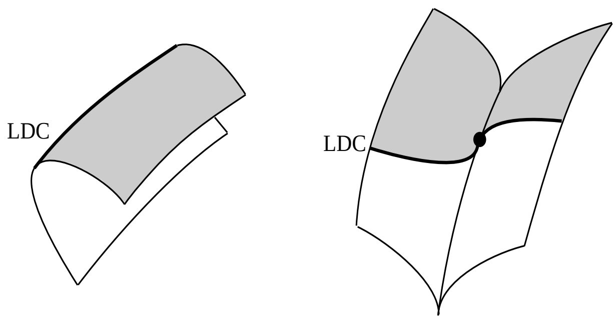

Figure 3: The metric structure on the caustic at general points on the LPL (left) and at a folded singularity of the lines of principal curvature (right).

Proof The results on the differentiable structure of the caustic are deduced from results on 2-dimensional caustics ([1]). We proceed as follows to obtain information about the positions of the relevant curves.

Statement (1) follows form Proposition 3.2. The surface $S = \phi^{-1}(0)$ is a double cover of the set of points on M where there are two principal directions. As the normal to $C(M)$ is parallel to a principal direction, one cover of S is mapped to the Riemannian part of $C(M)$ and the other to its Lorentzian part. (Recall that when there are two principal directions one is spacelike and the other is timelike [5, 6].) As the principal direction is lightlike at points on the LPL, it follows that the induced metric on $C(M)$ is degenerate at points on the LDC.

(2) We can assume, without loss of generality, that $\phi_v \neq 0$ and parametrise locally the surface S by $(u, v(u,\lambda), \lambda)$. Using the notation in the proof of Proposition 3.2, the caustic is singular if and only if the coordinates of ψ_u in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ vanish. As λ satisfies equation (5), both coordinates vanish if and only if one of them vanish. Therefore the ridge curve is given by eliminating λ from the following system

$$\left\{ \begin{array}{l} (F^2 - EG) - \lambda(nE - 2mF + lG) + \lambda^2(m^2 - ln) = 0, \\ F^2 - EG + \lambda(Fm - Gl) + \lambda v_u(Fn - Gm) = 0. \end{array} \right. \tag{6}$$

We take a local parametrisation of M as in Theorem 2.1(2) (i.e., with $E = G = 0$) and analyse the lifts of the ridge curve and of the LPL on the surface S. The surface S is parametrised by $(u, v(u,\lambda), \lambda)$ and the lift of the LPL is parametrised by $(u, w(u), -F(u, w(u))/m(u, w(u)))$, for some germ of a smooth function w which satisfies $w'(u) = 0$ at the folded singularity of the lines of principal curvature (i.e., at a point on the LPL which is also points on the ridge curve). Then the tangent direction to the lift of the LPL at such points is parallel to $(1, 0, -(F_u m - F m_u)/m^2)$.

Denote by H the left hand side of the second equation in the system (6), the left hand side of the first equation being ϕ. The tangent to the ridge is orthogonal to the gradients vectors $\nabla\phi$ and ∇H. At the point of interest on the LPL, and with the above setting, $\phi = \phi_\lambda = \phi_u = 0$. Hence the tangent to the lift of the ridge curve to the surface S is parallel to $(\phi_v H_\lambda, 0, -\phi_v H_u)$. The result follows from the fact that the two vectors $(1, 0, -(F_u m - F m_u)/m^2)$ and $(\phi_v H_\lambda, 0, -\phi_v H_u)$ are transverse for generic surfaces and their projections to the (u, v)-parameter space are parallel vectors. $\square$

We deal now with the case when p is a timelike umbilic point. Then the LPL has a Morse singularity of type node at p.

Lemma 3.4 *Let p be a timelike umbilic point of a generic Lorentzian surface patch M in $\mathbb{R}^3_1$. The image of the LPL on the caustic is the union of two smooth curves meeting tangentially at the image of the timelike umbilic point.*

Proof We take a local parametrisation as in Theorem 2.1(2) (i.e., with $E = G = 0$), so the LPL is given by $ln = 0$ and the double root of equation (5) on this set is given by $\lambda = -F/m$. We write $\alpha_1(t) = (u_1(t), v_1(t))$ (resp. $\alpha_2(t) = (u_2(t), v_2(t))$) for a local parametrisation of the curve $l = 0$ (resp. $n = 0$). Then the image of the LPL on the caustic is parametrised by

$$\beta_i(t) = (\boldsymbol{x} + \frac{F}{m}\boldsymbol{N})(\alpha_i(t)), i = 1, 2.$$

It follows that

$$\beta_i' = -\frac{v_i' n}{m}\boldsymbol{x}_u - \frac{u_i' l}{m}\boldsymbol{x}_v + (\frac{F}{m})'\boldsymbol{N},$$

so at the timelike umbilic point $\beta_1' = \beta_2' = (\frac{F}{m})'\boldsymbol{N}$, and this is generically a non-zero vector. $\square$

Theorem 3.5 *Let M be a generic Lorentzian surface patch in $\mathbb{R}^3_1$ and p a timelike umbilic point on M. There are four possible generic configurations for the metric structure on the caustic at a D_4^+ (Figure 4) and two at a D_4^- (Figure 5). These are completely determined by the 3-jet of a parametrisation of M.*

Proof We take here a local parametrisation of the surface in Monge form $(x, y) \mapsto (x, y, f(x, y))$ with the origin mapped to the timelike umbilic point. We write the 2-jet of f in the form $a_0(x^2 - y^2)$. Under the genericity condition, we can suppose that the roots of the cubic part of f are not lightlike. We suppose that one of the roots is spacelike (the case when the root is timelike follows similarly) and make Lorentzian changes of coordinates so that the 3-jet of f is written in the form

$$j^3 f = a_0(x^2 - y^2) + \alpha x(x^2 + axy + by^2).$$

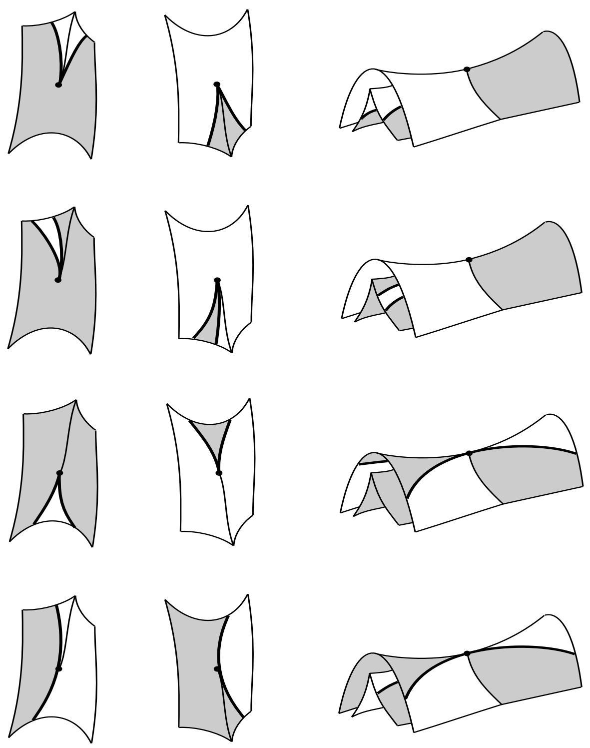

Figure 4: The metric structure on the caustic at a D_4^+-singularity of $d^2_{\boldsymbol{v}}$ (the LDC is the thick curve).

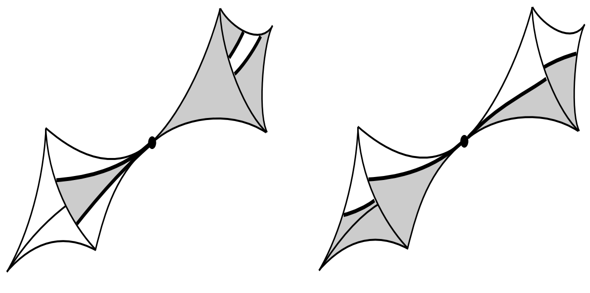

Figure 5: The metric structure on the caustic at a D_4^--singularity of $d^2_{\boldsymbol{v}}$ (the LDC is the thick curve).

The D_4^+-case
The cubic part of f has one root, so $a^2-4b<0$. If we write $\boldsymbol{v}=(v_0,v_1,v_2)$, then the family of distance squared functions d^2, which we denote here by D, is given by

$$D=-(x-v_0)^2+(y-v_1)^2+(f(x,y)-v_2)^2.$$

The function germ $D_{\boldsymbol{v}}=D(-,\boldsymbol{v})$ has a singularity at the origin if and only if $\boldsymbol{v}=(0,0,-\frac{1}{2a_0})$ and the singularity is of type D_4^+. Consider the set

$$\Omega=\{\xi\in\mathbb{R}^2\times\mathbb{R}^3,((0,0),(0,0,-\frac{1}{2a_0}))\,|\,D_x=D_y=D_{xy}^2-D_{xx}D_{yy}=0 \text{ at } \xi\}.$$

Its projection to $\mathbb{R}^3$ gives the caustic of M. The equation $D_{xy}^2-D_{xx}D_{yy}=0$ involves only x,y and v_2 and determines a cone $\mathcal{C}$ in the (x,y,v_2)-space. A parametrisation of this cone yields a parametrisation of Ω and of the caustic $C(M)$.

For $(x_0,y_0,v_0,v_1,v_2)\in\Omega$, the quadratic part of the Taylor expansion of $D_{\boldsymbol{v}}$ at (x_0,y_0) is a perfect square L^2. The point $(x_0,y_0,f(x_0,y_0))$ is on the ridge curve if and only if the cubic part of $D_{\boldsymbol{v}}$ divides L, if and only if

$$-D_{xxx}D_{xy}^3+3D_{xxy}D_{xy}^2D_{xx}-3D_{xyy}D_{xy}D_{xx}^2+D_{yyy}D_{xx}^3=0$$

at (x_0,y_0,v_0,v_1,v_2). We can now obtain the 1-jets of the parametrisations of the relevant curves. The 1-jet of the ridge curve (in the (x,y)-plane) is given by $y=-\frac{a}{b}x$ and those of the LPL curves by

$$\begin{array}{l} y_1=\frac{2a-b-3}{a-2b}x,\\ y_2=-\frac{2a+b+3}{a+2b}x. \end{array}$$

The 1-jet of the lift of the ridge to the cone $\mathcal{C}$ is given by $(x,-\frac{a}{b}x,-\frac{1}{2a_0}-\frac{\alpha b}{2a_0^2}x)$ and those of the lifts of the LPL curves are, respectively,

$$\begin{array}{l} (x,y_1,-\frac{1}{2a_0}+\frac{\alpha(a^2-ab+b^2-3b)}{2(a-2b)a_0^2}x)\\ (x,y_2,-\frac{1}{2a_0}-\frac{\alpha(a^2+ab+b^2-3b)}{2(a+2b)a_0^2}x). \end{array}$$

The 1-jet of the rib (the image of the ridge on the caustic) is $(0,0,-\frac{1}{2a_0}-\frac{\alpha b}{2a_0^2}x)$, and the 1-jets of the images of the LPL curves on the caustic are, respectively,

$$\begin{array}{l} (0,0,-\frac{1}{2a_0}+\frac{\alpha(a^2-ab+b^2-3b)}{(a-2b)a_0^2}x)\\ (0,0,-\frac{1}{2a_0}-\frac{\alpha(a^2+ab+b^2-3b)}{(a+2b)a_0^2}x). \end{array}$$

The caustic has two sheets at a D_4^+ (Figure 4) (each sheet is the image of a connected component of the cone $\mathcal{C}\setminus\{(0,0,-\frac{1}{2a_0})\}$). We consider the sheet where part of the the rib curve parametrised by $x\geq 0$ lives. The other sheet has symmetrical properties. This part of the rib curve is below the plane $v_2 = -\frac{1}{2a_0}$. We seek the positions of the images of the LPL with respect to the plane $v_2 = -\frac{1}{2a_0}$. These are determined by the signs of $\frac{\alpha(a^2-ab+b^2-3b)}{(a-2b)a_0^2}$ and $-\frac{\alpha(a^2+ab+b^2-3b)}{(a+2b)a_0^2}$. We set $\alpha = 1$ so that the sign is completely determined by (a, b). We obtain a partition of the set $a^2 - 4b > 0$ into regions where the configuration of the caustic is constant. The configurations are as shown in Figure 4.

In order to distinguish between the first two cases in Figure 4 we consider the curve $v_2 = -\frac{1}{2a_0}$ on the caustic. The two cases are distinguished by the relative position of the rib and of the images of the LPL with respect to this curve. This is determined by the relative position of the lifts of the ridge curve and of the LPL curves on $\mathcal{C}$ with respect to the pre-image of the curve $v_2 = -\frac{1}{2a_0}$ on $\mathcal{C}$. A calculation shows that the relative position of the above curves is determined by the sign of $(a-2b)(a+2b)$ (negative for the first configuration in Figure 4 and positive for the second).

The D_4^--case
We follow the same setting as for the D_4^+-case and take $\alpha = 1$. The 1-jets of the LPL curves, of their lift on the cone $\mathcal{C}$ and of their images on the caustic are as for the D_4^+-case. We have three ridge curves in this case and their 1-jets are given by

$$\begin{aligned} y_{r_1} &= -\tfrac{a}{b}x, \\ y_{r_2} &= -\tfrac{1}{2(a-1)b^2}(a^3 - ab^2 + 5ab + (a^2 - 3b - b^2)\sqrt{a^2-4b}\,)x, \\ y_{r_3} &= -\tfrac{1}{2(a-1)b^2}(a^3 + ab^2 - 5ab + (a^2 - 3b - b^2)\sqrt{a^2-4b}\,)x. \end{aligned}$$

The 1-jets of the lifts of the ridge curves to the cone $\mathcal{C}$ are given by

$$\begin{aligned} &(x, y_{r_1}, -\tfrac{1}{2a_0} - \tfrac{b}{2a_0^2}x), \\ &(x, y_{r_2}, -\tfrac{1}{2a_0} - \tfrac{(a^2-4b)}{2a_0^2(b-1)}x), \\ &(x, y_{r_3}, -\tfrac{1}{2a_0} - \tfrac{(a^2-4b)}{2a_0^2(b-1)}x). \end{aligned}$$

There are two possible configurations for the metric structure on the caustic (Figure 5) and these are determined by the relative position of the lifts of the ridge curves and of the LPL on the cone $\mathcal{C}$. The position of these curves on $\mathcal{C}$ is completely determined by a and b. There is a partition of the region $a^2 - 4b > 0$ given by the connected components of the complement of an algebraic curve into regions where each case occur. The algebraic curve has a lengthy expression to reproduce here. If we take for example $(a, b) = (0, -2)$ we get the configuration in Figure 5 left, and for $(a, b) = (-3, -1)$ we get the configuration in Figure 5 right. □

4 Surfaces with a degenerate metric

We start by determining the type of the singularities of $d^2_{\boldsymbol{v}}$ at points on the LD. This will determine the structure, up to diffeomorphism, of the caustic.

Theorem 4.1 *Suppose that p is a point on the locus of degeneracy (the LD) of the induced metric on M. If p is not a folded singular point of the lightlike curves, there are two distinct light cones that have degenerate contact with the surface at p. One of these is centred at p and it has an A_2-contact with M at p. The other is centred away from p and has an A_2-contact with M at p, except maybe at isolated points on the LD where the contact becomes of type A_3. At folded singularities of the lightlike curves, the light cone centred at p is the unique pseudo sphere that has a degenerate contact with M at p and the contact is generically of type A_3.*

Proof Let $\boldsymbol{x}$ be a local parametrisation of M with $p = \boldsymbol{x}(0,0) \in LD$. The origin is a degenerate singularities of $d^2_{\boldsymbol{v}}$ if and only of equation (4) is satisfied. As $F^2 - EG = 0$ at p, equation (4) has two solutions

$$\mu_1 = 0,\ \mu_2 = \frac{\bar{n}E - 2\bar{m}F + \bar{l}G}{\bar{m}^2 - \bar{l}\bar{n}}.$$

At $\mu = \mu_1$, $\boldsymbol{v} = p$ and $(d^2_{\boldsymbol{v}})_{uu} = 2E$, $(d^2_{\boldsymbol{v}})_{uv} = 2F$, $(d^2_{\boldsymbol{v}})_{vv} = 2G$. Generically, one of the coefficients of the first fundamental form is not zero at p, so the singularity of $d^2_{\boldsymbol{v}}$ at the origin is of type $A_{k\geq 2}$. As $\boldsymbol{v} = p$, the pseudo sphere that has this degenerate contact with M is the set of points q with $\langle q - \boldsymbol{v}, q - \boldsymbol{v}\rangle = \langle p - \boldsymbol{v}, p - \boldsymbol{v}\rangle = 0$. This is the light cone $LC^*(p)$ centred at p.

We have, at the origin,

$$\begin{array}{rcl} (d^2_{\boldsymbol{v}})_{uuu} &=& 3E_u, \\ (d^2_{\boldsymbol{v}})_{uuv} &=& E_v + 2F_u, \\ (d^2_{\boldsymbol{v}})_{uvv} &=& G_u + 2F_v, \\ (d^2_{\boldsymbol{v}})_{vvv} &=& 3G_v. \end{array}$$

The singularity of $d^2_{\boldsymbol{v}}$ is of type A_2 unless the cubic part of $d^2_{\boldsymbol{v}}$ divides L, where $L^2 = (Eu^2 + 2Fuv + Gv^2)/2$ is the quadratic part of $d^2_{\boldsymbol{v}}$. We can take $L = -Gu + Fv$. Let

$$C(u,v) = \frac{1}{6}\left((d^2_{\boldsymbol{v}})_{uuu}u^3 + 3(d^2_{\boldsymbol{v}})_{uuv}u^2v + 3(d^2_{\boldsymbol{v}})_{uvv}uv^2 + (d^2_{\boldsymbol{v}})_{vvv}v^3\right)$$

denote the cubic part of $d^2_{\boldsymbol{v}}$ at the origin. Then the singularity is of type A_2 if and only if $C(-G, F) \neq 0$, if and only if

$$E_uG^2 - FG(E_v + 2F_u) + EG(G_u + 2F_v) - EFG_v \neq 0.$$

The above inequality is equivalent to

$$-G(F^2 - EG)_u + F(F^2 - EG)_v \neq 0,$$

which means that the unique lightlike direction $-G\boldsymbol{x}_u + F\boldsymbol{x}_v$ on the LD is not tangent to the LD, or equivalently, p is not a folded singularity of the lightlike BDE. At folded singularities of the lightlike BDE we have generically a singularity of type A_3.

The condition $C(-G, F) = 0$ is equivalent to $\mu_2 = \mu_1 = 0$. (This can be seen by setting $E = F = 0$ on the LD. Then $C(-G, F) = 0$ if and only if $\bar{l} = E_u = 0$, if and only if $\mu_2 = 0 = \mu_1 = 0$.)

When p is not a folded singularity of the lightlike BDE, $\mu_2 \neq 0$, and $d^2_{\boldsymbol{v}}$ with $\boldsymbol{v} = p + \mu_2 \boldsymbol{x}_u \times \boldsymbol{x}_v$ has a singularity of type A_2 at general points on the LD. The pseudo sphere that has this degenerate contact with M at p is the set of points q with $\langle q - \boldsymbol{v}, q - \boldsymbol{v} \rangle = \langle p - \boldsymbol{v}, p - \boldsymbol{v} \rangle = \mu_2^2 \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_u \times \boldsymbol{x}_v \rangle = 0$, which is the light cone $LC^*(\boldsymbol{v})$ centred at $\boldsymbol{v}$. The singularity of $d^2_{\boldsymbol{v}}$ at p can become of type A_3 (when the cubic part of $d^2_{\boldsymbol{v}}$ divides L, L^2 being its quadratic part) at isolated points on the LD. (At such points, the ridge intersects transversally the LD and is transverse to both principal directions.) □

We consider now the behaviour of the metric on the caustic at points on the LD and deal separately with the cases where p is or is not a folded singular point of the lightlike curves.

Theorem 4.2 *Suppose that p is on the locus of degeneracy of the induced metric on M but is not a folded singular point of the lightlike curves. Then the caustic has locally two sheets $C_1(M)$ and $C_2(M)$.*

(1) *The sheet $C_1(M)$ is a smooth surface tangent to M along the LD. The LD on M is also the locus of degeneracy of the induced metric on $C_1(M)$ and splits $C_1(M)$ into a Riemannian part and a Lorentzian part.*

(2) *The sheet $C_2(M)$ is either a smooth surface or a cuspidal-edge, and is Lorentzian at its regular points. The image of the LD on $C_2(M)$ is a lightlike curve.*

Proof (1) We follow the notation in the proof of Theorem 4.1. Denote by $C_1(M)$ the caustic associated to the value $\mu_1 = 0$ at p. The roots of the quadratic equation (4) in μ are distinct, so its discriminant function is strictly positive. Therefore there exists a smooth function $\mu(u, v)$, with $\mu(u, v) = 0$ on the LD near p, which solves equation (4) .

A parametrisation of $C_1(M)$ is given by $\boldsymbol{v}(u, v) = \boldsymbol{x}(u, v) - \mu(u, v)\boldsymbol{x}_u \times \boldsymbol{x}_v(u, v)$. It follows that the LD is also a curve on $C_1(M)$. We have

$$\begin{array}{rcl} \boldsymbol{v}_u &=& \boldsymbol{x}_u - \mu_u \boldsymbol{x}_u \times \boldsymbol{x}_v - \mu(\boldsymbol{x}_u \times \boldsymbol{x}_v)_u \\ \boldsymbol{v}_v &=& \boldsymbol{x}_v - \mu_v \boldsymbol{x}_u \times \boldsymbol{x}_v - \mu(\boldsymbol{x}_u \times \boldsymbol{x}_v)_v \end{array}$$

At $p = \boldsymbol{x}(u, v) \in LD$, $\boldsymbol{x}_u \times \boldsymbol{x}_v \in T_pM$ which implies that $\boldsymbol{v}_u$ and $\boldsymbol{v}_v$ are also in T_pM. That means that $C_1(M)$ and M are tangential along the LD and that the LD is also the locus of degeneracy of the induced metric on $C_1(M)$.

We take a local parametrisation of M with $E = F = 0$ on the LD. The generic assumption on the LD (given by $F^2 - EG = 0$) to be a smooth curve implies that $E_u \neq 0$ or $E_v \neq 0$ on the LD. We assume, without loss of generality, that $E_u \neq 0$.

The coefficients $\tilde{E}$, $\tilde{F}$, $\tilde{G}$ of the first fundamental form of $C_1(M)$ are equal to those of M on the LD. Let $\tilde{\delta} = \tilde{F}^2 - \tilde{E}\tilde{G}$. To prove that the LD on $C_1(M)$ splits this surface into a Riemannian and a Lorentzian part, we show that $\tilde{\delta}$ changes sign on the LD. For this, it is enough to show that $\tilde{\delta}_u \neq 0$. As $\tilde{\delta}_u = -\tilde{E}_u G$ on the LD, we show that $\tilde{E}_u \neq 0$.

The vector $\boldsymbol{x}_u$ is the unique lightlike direction in T_pM at $p \in LD$, therefore there exists a smooth function $\alpha(t)$ such that

$$\boldsymbol{x}_u \times \boldsymbol{x}_v = \alpha \boldsymbol{x}_u$$

on the LD. We have $\bar{l} = \langle \boldsymbol{x}_u \times \boldsymbol{x}_v, \boldsymbol{x}_{uu} \rangle$, so on the LD, $\bar{l} = \langle \alpha \boldsymbol{x}_u, \boldsymbol{x}_{uu} \rangle = \frac{1}{2}\alpha E_u$, and therefore

$$\alpha = \frac{2\bar{l}}{E_u}.$$

Differentiating equation (4) yields (on the LD)

$$\mu_u = -\frac{E_u}{\bar{l}}.$$

It follows from the identity $\langle \boldsymbol{x}_u, \boldsymbol{x}_u \times \boldsymbol{x}_v \rangle \equiv 0$ that $\langle \boldsymbol{x}_u, (\boldsymbol{x}_u \times \boldsymbol{x}_v)_u \rangle = -\langle \boldsymbol{x}_{uu}, \boldsymbol{x}_u \times \boldsymbol{x}_v \rangle$. In particular, $\langle \boldsymbol{x}_u, (\boldsymbol{x}_u \times \boldsymbol{x}_v)_u \rangle = -\alpha \langle \boldsymbol{x}_{uu}, \boldsymbol{x}_u \rangle$ on the LD.

Now, on the LD,

$$\boldsymbol{v}_u = (1 - \mu_u \alpha)\boldsymbol{x}_u = 3\boldsymbol{x}_u$$

and

$$\begin{array}{rcl} \boldsymbol{v}_{uu} & = & \boldsymbol{x}_{uu} - 2\mu_u(\boldsymbol{x}_u \times \boldsymbol{x}_v)_u - \mu_{uu}\boldsymbol{x}_u \times \boldsymbol{x}_v \\ & = & \boldsymbol{x}_{uu} - 2\mu_u(\boldsymbol{x}_u \times \boldsymbol{x}_v)_u - \mu_{uu}\alpha\boldsymbol{x}_u. \end{array}$$

Therefore, on the LD,

$$\begin{array}{rcl} \frac{1}{2}\tilde{E}_u & = & \langle \boldsymbol{v}_{uu}, \boldsymbol{v}_u \rangle \\ & = & \langle \boldsymbol{x}_{uu} - 2\mu_u(\boldsymbol{x}_u \times \boldsymbol{x}_v)_u - \mu_{uu}\alpha\boldsymbol{x}_u, 3\boldsymbol{x}_u \rangle \\ & = & 3\left(\langle \boldsymbol{x}_{uu}, \boldsymbol{x}_u \rangle - 2\mu_u \langle (\boldsymbol{x}_u \times \boldsymbol{x}_v)_u, \boldsymbol{x}_u \rangle\right) \\ & = & 3(1 + 2\mu_u\alpha)\langle \boldsymbol{x}_{uu}, \boldsymbol{x}_u \rangle \\ & = & -\frac{9}{2}E_u \neq 0. \end{array}$$

(2) The sheet $C_2(M)$ of the caustic is associated to the non-zero solution of equation (4). It is not difficult to show that the normal to $C_2(M)$ at its smooth points is parallel to the spacelike principal direction $-\bar{m}\boldsymbol{x}_u + \bar{l}\boldsymbol{x}_v$, so $C_2(M)$ is a Lorentzian surface at its smooth points. (The result on the smooth model of $C_2(M)$ follows from Theorem 4.1.)

To simplify the calculations, we take a local parametrisation as in Theorem 2.1(2). We suppose that $\bar{l} \neq 0$ so that the solution

$$\mu_2 = \frac{\bar{l}G}{\bar{m}^2 - \bar{l}\bar{n}}$$

of equation (4) is not zero at p. Here too the caustic is parametrised by $\boldsymbol{v}(u,v) = \boldsymbol{x}(u,v) - \mu(u,v)\boldsymbol{x}_u \times \boldsymbol{x}_v(u,v)$ for some smooth function μ with $\mu = \mu_2$ on the LD. The image of the LD on $C_2(M)$ is then parametrised by

$$\gamma(t) = (\boldsymbol{x} - \mu\boldsymbol{x}_u \times \boldsymbol{x}_v)(u(t), v(t)),$$

where $(u(t), v(t))$ is a local parametrisation of the LD (in U). We simplify notation by dropping the coordinates $(u(t), v(t))$ of points on the LD. We can suppose $(u', v') = (-E_v, E_u)$ as this is a non-zero tangent vector to the LD. We seek to prove that $\langle \gamma', \gamma' \rangle = 0$. For this we need some preliminary results, where differentiation is carried out along the LD with respect to the parameter t.

We have $E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle = 0$ on the LD, so on this curve

$$\langle \boldsymbol{x}_u, -E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv} \rangle = 0. \tag{7}$$

From the proof of Theorem 4.2 (1), there exists a smooth function $\alpha(t)$ such that $\boldsymbol{x}_u \times \boldsymbol{x}_v = \alpha\boldsymbol{x}_u$ on the LD. Therefore, on the LD,

$$\bar{l} = \alpha\langle \boldsymbol{x}_u, \boldsymbol{x}_{uu} \rangle, \quad \bar{m} = \alpha\langle \boldsymbol{x}_u, \boldsymbol{x}_{uv} \rangle, \quad \bar{n} = \alpha\langle \boldsymbol{x}_u, \boldsymbol{x}_{vv} \rangle.$$

Differentiating both sides of the equality $\boldsymbol{x}_u \times \boldsymbol{x}_v = \alpha\boldsymbol{x}_u$, we get

$$(-E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv}) \times \boldsymbol{x}_v + \boldsymbol{x}_u \times (-E_v\boldsymbol{x}_{uv} + E_u\boldsymbol{x}_{vv}) = \alpha'\boldsymbol{x}_u + \alpha(-E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv}).$$

Taking the pseudo scalar product of both sides of the above equality with $\boldsymbol{x}_u$ and using (7) yield

$$\begin{array}{rcl} \langle (-E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv}) \times \boldsymbol{x}_v, \boldsymbol{x}_u \rangle & = & \alpha\langle -E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv}, \boldsymbol{x}_u \rangle \\ & = & 0. \end{array}$$

It follows that $-E_v\langle \boldsymbol{x}_{uu} \times \boldsymbol{x}_v, \boldsymbol{x}_u \rangle + E_u\langle \boldsymbol{x}_{uv} \times \boldsymbol{x}_v, \boldsymbol{x}_u \rangle = 0$ on the LD, that is,

$$-E_v\bar{l} + E_u\bar{m} = 0. \tag{8}$$

We have also on the LD, $F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0$. We get by differentiating along this curve

$$\langle -E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv}, \boldsymbol{x}_v \rangle = -\langle \boldsymbol{x}_u, -E_v\boldsymbol{x}_{uv} + E_u\boldsymbol{x}_{vv} \rangle = -\frac{1}{\alpha}(-E_v\bar{m} + E_u\bar{n}). \tag{9}$$

It follows from (7) that $-E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv} \in T_pM$ ($\boldsymbol{x}_u$ is a normal lightlike vector to T_pM). Hence, $-E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv} = a\boldsymbol{x}_u + b\boldsymbol{x}_v$, for some scalar a and b. Then, from (9), $\langle -E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv}, \boldsymbol{x}_v \rangle = -\frac{1}{\alpha}(-E_v\bar{m} + E_u\bar{n}) = bG$, and from this we get $b = -(-E_v\bar{m} + E_u\bar{n})/(\alpha G)$ and

$$\langle -E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv}, -E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv} \rangle = b^2G = \frac{(-E_v\bar{m} + E_u\bar{n})^2}{\alpha^2G}. \tag{10}$$

Now, differentiating $\gamma = \boldsymbol{x} - (\mu\alpha)\boldsymbol{x}_u$, we get

$$\gamma' = -(E_v + (\mu\alpha)')\boldsymbol{x}_u + E_u\boldsymbol{x}_v - (\mu\alpha)(-E_v\boldsymbol{x}_{uu} + E_u\boldsymbol{x}_{uv}).$$

Therefore,

$$\begin{array}{rcl}\langle\gamma',\gamma'\rangle & = & E_u^2G + 2\mu E_u(-E_v\bar{m} + E_u\bar{n}) + \frac{\mu^2}{G}(-E_v\bar{m} + E_u\bar{n})^2 \text{ (using (7), (9), (10))}\\ & = & E_u^2G - 2\mu E_u(E_v\bar{m} - E_u\bar{n}) + \frac{\mu^2}{G}(E_v\bar{m} - E_u\bar{n})^2\\ & = & \frac{1}{G}(E_uG - \mu(E_v\bar{m} - E_u\bar{n}))^2\\ & = & \frac{E_u^2}{G}(G - \mu(\frac{E_v}{E_u}\bar{m} - \bar{n}))^2\\ & = & \frac{E_u^2}{G}(G - \mu(\frac{\bar{m}^2}{\bar{l}} - \bar{n}))^2 \text{(using (8))}\\ & = & \frac{(\bar{m}^2-\bar{l}\bar{n})^2E_u^2}{\bar{l}G}(\frac{\bar{l}G}{\bar{m}^2-\bar{l}\bar{n}} - \mu)^2\\ & = & 0.\end{array}$$

□

Remark 4.3 *The principal curvature κ_i $(i = 1, 2)$ tends to infinity and $||\boldsymbol{x}_u \times \boldsymbol{x}_v||$ tends to zero along a sequence of points on $M \setminus LD$ converging to a point on the LD. Theorem 4.2 states that $1/(\kappa_i||\boldsymbol{x}_u \times \boldsymbol{x}_v||)$ has a finite limit μ_i and the focal set of $M \setminus LD$ can be extended to M (giving the caustic of M).*

We turn now to the case when $p \in LD$ is a folded singular point of the lightlike curves. Recall from [6] that the LPL meets the LD tangentially at such points.

Theorem 4.4 *Suppose that $p \in M$ is folded singular point of the lightlike curves. Then p is an A_3-singularity of the distance squared function. The LPL, LD and the ridge curve are tangential at p and the ridge curve can be either in the Riemannian or in the Lorentzian part of M. There are four possible configurations for the caustic as shown in Figure 6.*

Proof We follow the same steps of the proof of Theorem 3.3. We denote by $\bar{\phi}$ the left hand side of equation (4). The surface $\bar{S} = \bar{\phi}^{-1}(0)$ is generically smooth and we can assume, without loss of generality, that $\bar{\phi}_v \neq 0$ and parametrise S by $(u, v(u,\mu), \mu)$. Then the caustic is parametrised by $\psi(u,\mu) = \boldsymbol{x}(u, v(u,\mu)) - \mu\boldsymbol{x}_u \times \boldsymbol{x}_v(u, v(u,\mu))$.

On the LD near p, equation (4) has two roots $\mu_1 = 0$ and $\mu_2 = (\bar{n}E - 2\bar{m}F + \bar{l}G)/(\bar{m}^2 - \bar{l}\bar{n})$, which coincide at p. Thus the LD lifts to two curves on $\bar{S}$, one is the LD itself and corresponds to the root $\mu = \mu_1$ and the other corresponds to the root $\mu = \mu_2$. The latter maps to a lightlike curve on the caustic (Theorem 4.2(2); dotted curve in Figure 6). We denote this curve by $(\mathcal{L})$. It is generically smooth and transverse to the LD on $\bar{S}$.

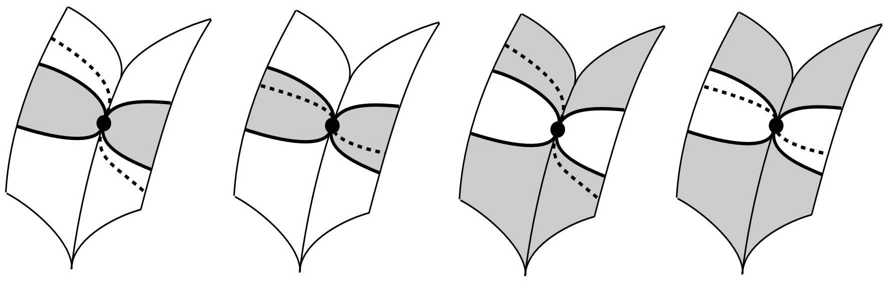

Figure 6: The metric structure on the caustic at a folded singularity of the lightlike curves. The thick curve is the LDC and the dotted curve is the lightlike curve in Theorem 4.2(2).

Calculation (similar to those in the proofs of Proposition 3.2 and Theorems 3.3) show that the ridge is a solution of the system of equations

$$\left\{\begin{array}{l}(F^2-EG)-\mu(\bar{n}E-2\bar{m}F+\bar{l}G)+\mu^2(\bar{m}^2-\bar{l}\bar{n})=0,\\ F^2-EG+\mu(F\bar{m}-G\bar{l})+\mu v_u(F\bar{n}-G\bar{m})=0.\end{array}\right. \tag{11}$$

The first equation of (11) is $\bar{\phi}=0$ and we denote the left hand side of the second equation by $\bar{H}$. The solutions of (11) are the projections to the (u,v)-space of the intersection set of $\bar{S}=\bar{\phi}^{-1}(0)$ with $\bar{H}^{-1}(0)$. We consider the map germ $J(u,\mu)=\bar{H}(u,v(u,\mu),\mu))$, where $(u,v(u,\mu),\mu))$ is the local parametrisation of $\bar{S}$. A calculation shows that J has an A_1^--singularity at the origin, so $J^{-1}(0)$ consists of a pair of transverse curve. One of them maps to the LD on $\bar{S}$ and the other to the lift of the ridge curve on $\bar{S}$.

We can use J to calculate the tangent directions to the lifts of the relevant curves on $\bar{S}$. To simplify the notation, we take a parametrisation of M at p as in Theorem 2.1(2) so that the tangent direction to the LD at p is along the lightlike direction $\boldsymbol{x}_u$. Then at p, $E=F=\bar{l}=E_u=0$. We found that the tangent directions to the lift of the relevant curves on $\bar{S}$ are as follows

the lift of the LD	$(1,0,0)$,
the lift of the ridge	$(1,0,\frac{2(-F_uE_v+E_{uu}G-2F_u^2)}{\bar{m}E_v})$,
the lift of the LPL	$(1,0,-\frac{2\bar{m}F_u-\bar{l}_uG}{2\bar{m}^2})$,
the lift of the curve $\boldsymbol{\mathcal{L}}$	$(1,0,-\frac{2\bar{m}F_u-\bar{l}_uG}{\bar{m}^2})$.

These curves are generically transverse and all possible configurations can occur. The projections of the above curves to the (u,v)-plane are tangential curves and have generically pairwise ordinary tangency at the origin. A maple calculation shows that the ridge curve is between the LD and the LPL if and only if the singularity of $d^2_{\boldsymbol{v}}$ is of type A_3^-. Then the ridge lies in the Lorentzian part of the surface. The ridge lies in the Riemannian part of the surface (the LD is between the ridge and the LPL) if and only if the singularity of $d^2_{\boldsymbol{v}}$ is of type A_3^+. □

Remark 4.5 *Define the colour of a ridge as white if its associated principal direction is timelike and grey if it is spacelike. The ridge does not change colour at an* A_3*-singularity of* $d^2_{\boldsymbol{v}}$ *on the LD (Figure 6). However, it changes colour at an* A_3*-singularity on the* LPL *and away from the LD (Figure 3, right). The ridges also change colour at a* $D_4^{\pm}$*-singularity (Figures 4, 5).*

Remark 4.6 *The distance squared function* $d^2_{\boldsymbol{v}}$ *can have a swallowtail singularity* (A_4) *at isolated points on a generic surface* M*. These points are generically neither on the* LPL *nor on the* LD *of* M*. They occur either on the Riemannian part of* M*, in which case the caustic is Lorentzian at its regular points, or in the Lorentzian part of* M *away from the* LPL*. In the latter case the caustic is either Riemannian or Lorentzian at its regular points (this depends on whether its normal direction is parallel to the timelike or spacelike principal direction).*

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