On the Fučík Spectrum and Superlinear Elliptic Equations.

Tesi di Dottorato di Eugenio Massa[1]
Dottorato di Ricerca in Matematica, XV Ciclo[2]

Relatori:
Prof. Bernhard Ruf[3]
Prof. Djairo Guedes de Figueiredo[4]

Anno Accademico 2002-2003

18th May 2006

[1]Dipartimento di Matematica, Università degli studi di Milano, Via Saldini 50, 20133 Milano, Italy; eugenio@mat.unimi.it

[2]Consorzio: Università degli studi di Milano, Università di Milano Bicocca, Politecnico di Milano, Università degli studi di Trieste, Università Cattolica del Sacro Cuore di Brescia.

[3]Dipartimento di Matematica, Università degli studi di Milano, ruf@mat.unimi.it

[4]IMECC, UNICAMP, Campinas (SP), Brazil, djairo@ime.unicamp.br

Contents

1 Introduction

1.1 The problem

In this thesis we deal with nonlinear elliptic boundary value problems of the form

$$\begin{cases} -\Delta u = f(x,u) & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array}\right. & in\ \partial\Omega \end{cases}, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$.

It is well known that the solvability of (1.1) depends essentially on the interaction of the nonlinearity $f(x,s)$ with the spectrum of $(-\Delta)$.

In recent years it was discovered that for nonlinearities with different slopes at $+\infty$ and $-\infty$ also the so called Fučík spectrum plays an important role. This nonlinear spectrum is given by the set Σ of the couples $(\lambda^+, \lambda^-) \in \mathbb{R}^2$ such that the following equation (with Dirichlet or Neumann boundary conditions) has nontrivial solutions:

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array}\right. & in\ \partial\Omega \end{cases}, \tag{1.2}$$

where $u^+(x) = \max\{0, u(x)\}$ and $u^-(x) = \max\{0, -u(x)\}$.

The notion of Fučík spectrum was introduced in [Fuč76] and [Dan77]; for $\lambda^+ = \lambda^-$ the problem becomes linear and admits nontrivial solutions for $\lambda^+ = \lambda^- = \lambda_k$, being $\{\lambda_k\}_{k\in\mathbb{N}}$ the sequence of the usual eigenvalues of the operator; from these points arise curves belonging to the Fučík spectrum and in most cases it may be proven that the whole spectrum is composed by such curves.

To know the Fučík spectrum is important in many applications, for example in the study of problems with "jumping nonlinearities", that is nonlinearities which are asymptotically linear at both $+\infty$ and $-\infty$, but with different slopes.

Such problems were first considered in [AP72], where the nonlinearity is assumed to cross the first eigenvalue, that is the slope at $-\infty$ is below and the slope at $+\infty$ is above λ_1 (Ambrosetti-Prodi problem). The authors showed that such problems have, in dependence of a parameter, either no or two solutions.

Like for the usual spectrum it is important to have a variational characterization of the Fučík spectrum: this allows one to obtain interesting results for sublinear perturbations of the equation, since these characterizations are stable under such perturbations. Results of this kind may be found in [CG92, dFR93, dFG94, CdFG99], however these papers deal only with the first nontrivial curve of the Fučík spectrum or with the periodic case on an interval.

Here, in section 4, we give a variational characterization of parts of the Fučík spectrum for problem (1.2), in particular we prove:

Theorem 1.1. *Suppose that the point $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ with $\alpha^+ \geq \alpha^-$ is Σ-connected to the diagonal between λ_k and λ_{k+1} in the sense of definition 4.1 on page 33, then we can find and characterize one intersection of the Fučík spectrum with the halfline $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$, for each value of $r \in (0, 1]$.*

Moreover, some properties of this characterization will be proven, in particular we will show in section 4.3 that it characterizes a branch of the spectrum connected to the point $(\lambda_{k+1}, \lambda_{k+1})$ and that this branch is monotone decreasing.

In the one dimensional case we also prove that the intersection stated in theorem 1.1 is actually the first, that is the one with smallest t, on the halfline $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$ (see section 5.3).

To prove theorem 1.1 we will find a nontrivial solution of the Fučík problem as a critical point $\bar{u}$ of the following functional defined in the Hilbert space $H = H^1(\Omega)$ (resp. $H = H^1_0(\Omega)$ with Dirichlet boundary conditions):

$$J_\alpha(u) = \int_\Omega |\nabla u|^2 - \alpha^+ \int_\Omega (u^+)^2 - \alpha^- \int_\Omega (u^-)^2 \ , \tag{1.3}$$

constrained to the set

$$Q_r = \left\{ u \in H \quad s.t. \quad \int_\Omega (u^+)^2 + r(u^-)^2 = 1 \right\} \ ; \tag{1.4}$$

indeed by the Lagrange's multipliers rule this critical point $\bar{u}$ will satisfy the equation

$$-\Delta \bar{u} = \alpha^+ \bar{u}^+ - \alpha^- \bar{u}^- + t(\bar{u}^+ - r\bar{u}^-) \qquad in \ \Omega \tag{1.5}$$

with the considered boundary conditions.

A linking structure between a set homeomorphic to the boundary of a k-dimensional ball and another set homeomorphic to a subspace of H of codimension k, will prove (through a deformation lemma) the existence of such a critical point and that the Lagrange's multiplier t is positive. These sets will be obtained using a technique similar to the one described in [DR98].

The second main result of the thesis concerns the following superlinear equation with Neumann boundary conditions:

$$\begin{cases} -u'' = \lambda u + g(x, u) + h(x) \quad in \ (0, 1) \\ u'(0) = u'(1) = 0 \end{cases} \ , \tag{1.6}$$

where

$$\begin{aligned} & g \in \mathcal{C}^0([0,1] \times \mathbb{R}) \, , \\ & \lim_{s \to -\infty} \tfrac{g(x,s)}{s} = 0, \quad \lim_{s \to +\infty} \tfrac{g(x,s)}{s} = +\infty \end{aligned} \tag{H1}$$

uniformly with respect to $x \in [0, 1]$, and $h \in L^2(0, 1)$.

We will compare it to the Fučík problem

$$\begin{cases} -u'' = \lambda^+ u^+ - \lambda^- u^- \quad in \ (0, 1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{1.7}$$

and, taking advantage of the fact that in this case theorem 1.1 gives a characterization of the whole Fučík spectrum and that this may be explicitly calculated, we will prove, in section 5, existence results for problem (1.6). The proof uses the variational characterization above to make a comparison of these minimax levels with those of the functional associated to problem (1.6), in order to prove the existence of a linking structure for this last functional, too. A fundamental ingredient in the proof is the compact inclusion $H \subseteq \mathcal{C}^0([0,1])$.

Some additional hypotheses on the growth at infinity of the nonlinearity g will be needed to obtain the PS condition for the functional associated to problem (1.6): defining $G(x,s) = \int_0^s g(x,\xi)d\xi$, we ask

$$\exists \theta \in \left(0, \frac{1}{2}\right), \quad s_0 > 0 \quad s.t. \quad 0 < G(x,s) \leq \theta s g(x,s) \qquad \forall s > s_0\,; \tag{H2}$$

$$\exists s_1 > 0,\ C_0 > 0 \quad s.t. \quad G(x,s) \leq \frac{1}{2} s g(x,s) + C_0 \qquad \forall s < -s_1\,. \tag{H3}$$

For certain "resonant" values of λ also the following hypothesis will be needed:

$$\exists \rho_0 > 0,\ M_0 \in \mathbb{R} \quad s.t. \quad G(x,s) + h(x)s \leq M_0 \quad a.e.\ x \in [0,1],\ \forall s < -\rho_0\,. \tag{HR}$$

The exact statement of the results is this:

Theorem 1.2. *Under hypotheses (H1), (H2) and (H3), if $\lambda \in (\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4})$ for some $k \geq 1$, then there exists a solution of problem (1.6) for all $h \in L^2(0,1)$.*

Theorem 1.3. *Under hypotheses (H1), (H2), (H3) and (HR), with $h \in L^2(0,1)$, $\lambda = \frac{\lambda_{k+1}}{4}$ for some $k \geq 1$, then there exists a solution of problem (1.6).*

It is important to remark that the values $\frac{\lambda_k}{4}$ correspond to the asymptotes of the curves that compose the Fučík spectrum of problem (1.7).

In section 6 we also discuss how theorems 1.2 and 1.3 may be extended to the case of radial solutions on an annulus, with radial coefficients of course (theorems 6.2 and 6.3).

Then in section 7 and 8 we consider the same kind of problem for other operators, in particular in section 7 for the multi-Laplacian operator, that is the higher order operator $(-\Delta)^m$ with $m = 2, 3, ...$, while in section 8 for the p-Laplacian operator, that is the nonlinear operator $-\nabla \cdot [\psi(\nabla \cdot)]$ where $p > 1$ and $\psi(s) = \begin{cases} |s|^{p-2}s & s \neq 0 \\ 0 & s = 0 \end{cases}$.

For the multi-Laplacian operator (with suitable boundary conditions) we first adapt the variational characterization of the Fučík spectrum given in theorem 1.1 (theorem 7.7), then we obtain a result corresponding to theorem 1.2 with $k = 1$ (theorem 7.17), valid also for sets $\Omega \subseteq \mathbb{R}^N$ with $N > 1$ provided the relation between N and m is such that the space $H^m(\Omega)$ is included at least in $\mathcal{C}^0(\bar{\Omega})$); finally we consider the case $N = 1$ and $m = 2$ and we describe qualitatively the Fučík spectrum for this case (following the results in [CD01]) and with it we obtain again results corresponding to theorems 1.2 and 1.3 (theorems 7.34 and 7.35).

For the p-Laplacian operator we obtain results corresponding to theorems 1.1, 1.2 and 1.3 for the one dimensional Neumann problem, but only with $k = 2$ and $p \geq 2$ (theorems 8.23 and 8.24).

Finally in section 9 we give the complete proof of the PS condition for the functional associated to problem (1.6) and to its multi-Laplacian and p-Laplacian version for $p \geq 2$.

1.2 Bibliography

Theorem 1.2 extends the result obtained in [dFR91], where the existence is proved for $\lambda \in (0, \frac{\pi^2}{4})$, that is the case $k = 1$ of theorem 1.2.

Perera in [Per00] proved the existence of a solution for $\lambda \in (\frac{\pi^2}{4}, \lambda^*)$, where λ^* is some value in $(\frac{\pi^2}{4}, \frac{\pi^2}{2})$, and so theorem 1.2 extends this result, too.

We also mention that for periodic boundary conditions the equivalent of theorem 1.2 is proved in [dFR93].

Theorem 1.3 deals with some kind of resonance: the case $\lambda = \frac{\lambda_2}{4}$ was already discussed in [Per00], where the existence is proved under different hypotheses, while the case $\lambda = \frac{\lambda_1}{4}$ (that is $\lambda = 0$) is treated in [dFR91].

Concerning the variational characterizations of the Fučík spectrum we cite:

- [dFR93], where is characterized the whole spectrum for the one dimensional equation with periodic boundary conditions;
- [dFG94, CdFG99], where is characterized from below the part of the spectrum arising from the point (λ_2, λ_2), respectively for the Laplacian and the p-Laplacian operator;
- [MP01], where are characterized some pieces of the spectrum near to the diagonal, for the p-Laplacian.

Results similar to [dFR91] may be found in [Vil98], where the result is also extended to the p-Laplacian with $p > 1$ in any dimension $N < p$. Thus our result on the p-Laplacian is an extension of this work.

A far more extensive analysis of these works is given in section 3.

In section 2 and in the appendix we give a review of techniques, results and definitions used throughout the work.

1.3 Acknowledgments

The author wishes to thank his advisors Professor Bernhard Ruf and Professor Djairo Guedes de Figueiredo for the help and encouragement during the work, Professors D. G. Costa, J.-P. Gossez, M. Matzeu, K. Payne and P. Ubilla for the useful suggestions and comments, and his colleagues at Milan and Campinas for stimulating conversations.

The work was prepared part at Università degli studi di Milano and part at UNICAMP, Campinas (SP), Brazil.

The period at Campinas was partially supported by *Istituto Nazionale di Alta Matematica "F. Severi": Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* and *Progetto Giovani Ricercatori*.

2 Short introduction to the calculus of variations

The purpose of this chapter is to give a short review of the classical results that will be used in the following.

Moreover the definitions of some of the objects used here are given in the appendix.

Let us start by considering the model elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & in\ \ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \qquad or \\ u = 0 \end{array}\right. & in\ \ \partial\Omega \end{cases}, \tag{2.1}$$

where Ω is a bounded domain (that is, a non empty connected open set) in $\mathbb{R}^N$, $\partial\Omega$ denotes its boundary and n the unit outer normal; we just suppose for the moment that $\partial\Omega$ is Lipschitz continuous and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory, that is:

- the function $f(\cdot, s) : \Omega \to \mathbb{R} : x \mapsto f(x, s)$ is measurable for all $s \in \mathbb{R}$;
- the function $f(x, \cdot) : \mathbb{R} \to \mathbb{R} : s \mapsto f(x, s)$ is continuous for almost every $x \in \Omega$.

In the following we will call H the space $H^1(\Omega)$ when considering the Neumann problem (boundary condition $\frac{\partial u}{\partial n} = 0$) and $H^1_0(\Omega)$ when considering the Dirichlet problem (boundary condition $u = 0$).

We will define

Definition 2.1.

- **Classical solution**: *$u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ (or $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ for the Dirichlet case) satisfying pointwise the conditions in (2.1).*
- **Weak solution**: *$u \in H$ such that*

$$\int_\Omega \nabla u \nabla v = \int_\Omega f(x,u)v \quad for\ \ all \quad v \in H\,. \tag{2.2}$$

Actually (provided everything above is well defined) multiplying the equation in (2.1) by the function $v \in H$, integrating by parts and using the boundary condition (in the Neumann case) or the definition of the space H^1_0 (in the Dirichlet case) to get rid of the boundary term, it is clear that any classical solution is a weak solution too; we will see in section 2.2 that with some regularity conditions on Ω and f the converse is also true.

Note that the choice of the space H guarantees that $\int_\Omega \nabla u \nabla v$ exists and is finite, while in general some more hypotheses on f will be needed to give sense to the integral on the right hand side of the (2.2) for any $u, v \in H$: this is usually achieved by growth conditions at infinity like $|f(x,s)| \leq A + B|s|^\sigma$ where $\sigma = \frac{N+2}{N-2}$ (for $N \geq 3$), being N the dimension of the set Ω, by virtue of the Sobolev embedding theorems.

2.1 Variational approach

If the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u) \tag{2.3}$$

(where $F(x,s) = \int_0^s f(x,\xi)d\xi$) is a well defined $\mathcal{C}^1(H,\mathbb{R})$ functional, then we have a one to one relation between the weak solutions of (2.1) and the critical points of I: that is those $\bar{u} \in H$ such that the Fréchet derivative $I'(\bar{u}) = 0$.

Then the method of the calculus of variations consists in seeking solutions of (2.1), through the study of the geometry of the functional in (2.3).

2.1.1 Free critical points of minimum type

The first possibility is to find local minima of the functional: the existence of a global minimum may be guaranteed by the following theorem (see for example [Str96])

Theorem 2.2.

1. *E reflexive Banach space;*
2. *$I : E \to \mathbb{R} \cup \{+\infty\}$;*
3. *I coercive, that is:*
   $\lim_{\|u\|_E \to +\infty} I(u) = +\infty$;
4. *I sequentially weakly lower semicontinuous, that is:*
   if $\{u_n\} \subseteq E$ and $u_n \rightharpoonup u$, then $I(u) \le \liminf_{n\to+\infty} I(u_n)$;

Then there exists $\bar{u} \in E$ such that $I(\bar{u}) = \inf_{u\in E} I(u) > -\infty$.

Idea of the proof. One uses the coercivity to prove that a sequence which realizes the inf has to be bounded; then (by reflexivity) one extracts a subsequence weakly convergent to some $\bar{u} \in E$ and finally uses lower semicontinuity to assert that $I(\bar{u}) = \inf_{u\in E} I(u)$. □

2.1.2 Free critical points of minmax type

If the functional does not admit global minima, or if one is interested in finding other critical points, then different techniques should be used.

Let us define, for a functional $I \in \mathcal{C}^1(E,\mathbb{R})$, E being a Banach space,

Definition 2.3.

- $K_c = \{u \in E$ *such that* $I(u) = c$ *and* $I'(u) = 0\}$;
- $A_c = \{u \in E$ *such that* $I(u) \le c\}$.

Moreover define

Definition 2.4.

- **Palais-Smale (PS) sequence** *for I:*
  $\{u_n\} \subseteq E$ *such that* $|I(u_n)| \le C$ *and* $I'(u_n) \to 0$.

- *The functional I satisfies the* **PS condition***:*
  For each PS-sequence there exists a (strongly) convergent subsequence.

This last property (first introduced by Palais and Smale in [PS64, Pal63, Sma64]) is required to have some compactness in the problem: actually, if the functional is $\mathcal{C}^1$, it guarantees that if we are able to prove the existence of a sequence of "almost critical points at level c" (that is a PS-sequence), then there has to be a critical point at level c.

Remark 2.5. *In order to have a lighter notation, when passing to a subsequence, we will always continue to denote it with the same index as the previous sequence.*

The fundamental tool to prove the existence of a critical point will be the following deformation lemma (see for example in [Rab86]):

Lemma 2.6 (Deformation Lemma).

1. *E Banach space;*
2. *$I \in \mathcal{C}^1(E, \mathbb{R})$;*
3. *I satisfies the PS condition;*
4. *$c \in \mathbb{R}$; $\bar{\varepsilon} > 0$;*
5. *$K_c = \emptyset$.*

Then there exist $\varepsilon \in (0, \bar{\varepsilon})$ and $\eta \in \mathcal{C}([0,1] \times E, E)$ such that:

a. $\eta(0, u) = u \quad \forall u \in E$;

b. $\eta(t, u) = u \quad \forall (t, u)$ *such that* $I(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$;

c. $\eta(t, \cdot) : E \to E$ *is an homeomorphism* $\forall t \in [0, 1]$;

d. $\eta(1, A_{c+\varepsilon}) \subseteq A_{c-\varepsilon}$.

Using this Deformation Lemma one can prove the existence of critical points considering the geometry of the functional.

The general idea is:

- consider a class Γ of subsets of E,
- define $c = \inf_{A \in \Gamma} \sup_{u \in A} I(u)$,
- give conditions such that one can build the deformation η such that $\eta(1, A) \in \Gamma$ for all $A \in \Gamma$,
- finally obtain a contradiction between the infsup characterization of c and the fact that, if K_c were empty, one could find a $A \in \Gamma$ such that $\sup_{u \in \eta(1,A)} I(u) < c$.

The most classical example is the following (see [AR73]):

Theorem 2.7 (Mountain Pass Theorem).

1. *E Banach space;*

2. $I \in \mathcal{C}^1(E, \mathbb{R})$*;*

3. *I satisfies the PS condition;*

4. *I(0)=0;*

5. $\exists \rho, \alpha > 0$ *such that* $I(u) \geq \alpha$ *for all* u *such that* $\|u\|_E = \rho$*;*

6. $\exists e \in E$ *such that* $\|e\|_E > \rho$ *and* $I(e) < 0$*.*

Moreover, let

- $\Gamma = \{\gamma \in \mathcal{C}^0([0,1]; E)$ *such that* $\gamma(0) = 0$ *and* $\gamma(1) = e\}$*;*
- $c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t))$*.*

Then $c \geq \alpha$ *and* $K_c \neq \emptyset$*, that is there exists a critical point at level* c*.*

Idea of the proof. Since $\|0\|_E < \rho < \|e\|_E$, for all $\gamma \in \Gamma$ there exists $\bar{t}$ such that $\|\gamma(\bar{t})\|_E = \rho$ and then

$$\sup_{t \in [0,1]} I(\gamma(t)) \geq \alpha \qquad \forall \gamma \in \Gamma\,. \tag{2.4}$$

Now suppose $K_c = \emptyset$: choose $\bar{\varepsilon} < \frac{\alpha}{2}$ and apply the deformation lemma, obtaining a value $\varepsilon < \bar{\varepsilon}$ and the deformation η such that $\eta(1, A_{c+\varepsilon}) \subseteq A_{c-\varepsilon}$. Then select a $\gamma_\varepsilon \in \Gamma$ such that $\sup_{u \in \gamma_\varepsilon([0,1])} I(u) < c + \varepsilon$ and consider $\eta(1, \gamma_\varepsilon(\cdot))$:

- by property c. of the deformation lemma $\eta(1, \gamma_\varepsilon(\cdot)) \in \mathcal{C}^0([0,1]; E)$;
- by property b. of the deformation lemma (since $c - \bar{\varepsilon} > \frac{\alpha}{2} > 0$), we get $\eta(1, \gamma_\varepsilon(0)) = \gamma_\varepsilon(0) = 0$ and $\eta(1, \gamma_\varepsilon(1)) = \gamma_\varepsilon(1) = e$,
- by property d. of the deformation lemma $\sup_{u \in \eta(1, \gamma_\varepsilon([0,1]))} I(u) \leq c - \varepsilon$;

then $\eta(1, \gamma_\varepsilon(\cdot)) \in \Gamma$ and so the last inequality contradicts the definition of c. $\square$

A more general theorem is the following (see [Wil96])

Theorem 2.8.

1. *E Banach space;*

2. $I \in \mathcal{C}^1(E, \mathbb{R})$*;*

3. *I satisfies the PS condition;*

4. *K compact metric space,* K_0 *closed subset of* K*,* $f_0 \in \mathcal{C}(K_0, E)$ *such that*

$$\sup_{p \in K_0} I(f_0(p)) \leq 0\,; \tag{2.5}$$

5. $\Gamma = \{\gamma \in \mathcal{C}^0(K; E)$ *such that* $\gamma|_{K_0} = f_0\}$*;*

6. $c = \inf_{\gamma\in\Gamma} \sup_{p\in K} I(\gamma(p)) > 0$.

Then $K_c \neq \emptyset$, that is there exists a critical point at level c.

Proof. Consider the same idea as before, where the hypotheses $\sup_{p\in K_0} I(f_0(p)) \leq 0$ and $c > 0$ guarantee the possibility to make a deformation leaving unaffected $\gamma|_{K_0}$ so that $\eta(1, \gamma(\cdot)) \in \Gamma$ for all $\gamma \in \Gamma$. □

A classical sufficient condition to obtain the last hypothesis above is the following linking structure:

- $\exists W \subseteq E$ such that
  - * $I|_W \geq \alpha > 0$,
  - * $W \cap \gamma(K) \neq \emptyset$ for all $\gamma \in \Gamma$;

we say then that **the sets W and $f_0(K_0)$ link**.

2.1.3 Ekeland variational principle

Even more powerful tools to find PS sequences are the Ekeland variational principle and the minmax principle that follow (see [MW89]):

Theorem 2.9 (Ekeland variational principle).
M complete metric space;
$\phi : M \to \mathbb{R} \cup \{+\infty\}$ lower semicontinuous;
$c = \inf_{u\in M} \phi(u) \neq \pm\infty$.
Given any $\varepsilon > 0$ and $\bar{u} \in M$ such that

$$\phi(\bar{u}) \leq c + \varepsilon\,, \tag{2.6}$$

there exists $v \in M$ such that

$$\phi(v) \leq \phi(\bar{u})\,, \tag{2.7}$$

$$d(\bar{u}, v) \leq \sqrt{\varepsilon}\,, \tag{2.8}$$

$$\phi(w) > \phi(v) - \sqrt{\varepsilon} d(w, v) \quad for\ any \quad w \neq v,\ \ w \in M\,. \tag{2.9}$$

Moreover if M is a Banach space and $\phi \in \mathcal{C}^1(M, \mathbb{R})$ one gets from (2.9) that

$$\|\phi'(v)\|_{M'} \leq \sqrt{\varepsilon}\,. \tag{2.10}$$

Theorem 2.10.
E Banach space, $I \in \mathcal{C}^1(E, \mathbb{R})$;
K compact metric space, $K_0 \subseteq K$ and closed, $f_0 \in \mathcal{C}(K_0, E)$.
Assume that $\Gamma = \{\gamma \in \mathcal{C}(K, E)$ such that $\gamma|_{K_0} = f_0\}$ is a complete metric space.
Suppose

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma(K)} I(u) > c_1 = \max_{u\in f_0(K_0)} I(u)\,. \tag{2.11}$$

Given any $\varepsilon > 0$ and $\bar{\gamma} \in \Gamma$ such that

$$\max_{u \in \bar{\gamma}(K)} I(u) \leq c + \varepsilon \,, \tag{2.12}$$

there exists $v \in E$ such that

$$c - \varepsilon \leq I(v) \quad \leq \quad \max_{u \in \bar{\gamma}(K)} I(u) \,, \tag{2.13}$$

$$d(v, \bar{\gamma}(K)) \quad \leq \quad \sqrt{\varepsilon} \,, \tag{2.14}$$

$$\|I'(v)\|_{E'} \quad \leq \quad \sqrt{\varepsilon} \,. \tag{2.15}$$

2.1.4 Constrained critical points

The same constructions made in lemma 2.6 and in the following theorems, may be made if one restricts to a manifold in the space E defined by $M = \{J(u) = b\}$, J being a $\mathcal{C}^1(E, \mathbb{R})$ functional. For this topic see [CdFG99] and the references therein.

In this case the concept of critical (and "almost critical") point will be given by the Lagrange's multipliers rule:

Definition 2.11. *$\bar{u}$ is a* **critical point of I constrained to** *M:*
$\exists t, s \in \mathbb{R}$ not both zero such that $sI'(\bar{u}) = tJ'(\bar{u})$;

Remark 2.12. *If the constraint is such that $J'(u) \neq 0$ on M, then definition 2.11 may be replaced by the simpler one:*
$\exists t \in \mathbb{R}$ such that $I'(\bar{u}) = tJ'(\bar{u})$.
An equivalent formulation of this last condition is:
$inf_{t \in \mathbb{R}} \|I'(\bar{u}) - tJ'(\bar{u})\|_{E'} = 0$.

Definition 2.13. *$\{u_n\} \subseteq M$ is a* **PS-sequence for I constrained to** *M:*

- *$|I(u_n)| \leq C$,*
- *there exist two sequences $\{t_n\}, \{s_n\} \subseteq \mathbb{R}$, with the property that, for each n, $t_n = 1$ or $s_n = 1$, such that $s_n I'(u_n) - t_n J'(u_n) \to 0$.*

Remark 2.14. *Again for the case $J'(u) \neq 0$ on M the second condition reduces to:*
there exists a sequence $\{t_n\} \subseteq \mathbb{R}$ such that $I'(u_n) - t_n J'(u_n) \to 0$,
or equivalently $inf_{t \in \mathbb{R}} \|I'(u_n) - tJ'(u_n)\|_{E'} \to 0$.

With these definitions, a result analogous to lemma 2.6 guarantees that we may find a deformation $\eta \in \mathcal{C}([0,1] \times M, M)$ with the same properties given there, which then allows one to prove theorems analogous to 2.8 to find critical points of I constrained to M, while analogous to theorem 2.9 and 2.10 allow one to prove the existence of a point $v \in M$ with the property $\|I'(v)\|_{E'} \leq \sqrt{\varepsilon}$ replaced by $inf_{t \in \mathbb{R}} \|I'(v) - tJ'(v)\|_{E'} \leq \sqrt{\varepsilon}$.

2.2 Regularity of the weak solutions

Let us see now how one can obtain sufficient conditions in order to prove that a weak solution is a classical solution too.

We need the following result (see [Bre83]):

Definition 2.15. *Let:*
$Q = \{(x_1, x') : \quad x_1 \in \mathbb{R}, \quad x' \in \mathbb{R}^{N-1}, \quad |x_1| < 1, \quad |x'| < 1\}$,
$Q^+ = \{(x_1, x') \in Q : \quad x_1 > 0\}$,
$Q^0 = \{(x_1, x') \in Q : \quad x_1 = 0\}$.

We say that **the set** $\Omega \subseteq \mathbb{R}^N$ **is of class** $\mathcal{C}^m$*, where* $m \in \mathbb{N}$*, if for any* $x \in \partial\Omega$ *there exists a neighborhood* U *of* x *and a one to one map* $M : \bar{Q} \to \bar{U}$ *such that*

$$M \in \mathcal{C}^m(\bar{Q}), \quad M^{-1} \in \mathcal{C}^m(\bar{U}), \quad M(Q^+) = U \cap \Omega \quad \text{and} \quad M(Q^0) = U \cap \partial\Omega. \tag{2.16}$$

Lemma 2.16. *Let* $\bar{u}$ *be a weak solution of the problem*

$$\begin{cases} -\Delta u = h(x) & \text{in } \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \qquad or \\ u = 0 \end{array}\right. & \text{in } \partial\Omega \end{cases}, \tag{2.17}$$

where $h \in H^m(\Omega)$ *and* Ω *of class* $\mathcal{C}^{m+2}$ *for some* $m = 0, 1, 2, ..$

Then we have the estimate

$$\|\bar{u}\|_{H^{m+2}} \leq C(\|\bar{u}\|_{L^2} + \|h\|_{H^m}), \tag{2.18}$$

that is (for Ω *of class* $\mathcal{C}^{m+2}$*),* $h \in H^m \Rightarrow \bar{u} \in H^{m+2}$*.*

Remark 2.17. *If* Ω *does not have the required regularity then the estimate (2.18) has to be replaced by*

$$\|\bar{u}\|_{H^{m+2}(\omega)} \leq C(\omega) \left(\|\bar{u}\|_{L^2(\Omega)} + \|h\|_{H^m(\Omega)} \right), \tag{2.19}$$

where ω *is any open set such that* $\bar{\omega} \subseteq \Omega$*: the regularity is guaranteed only in the interior of* Ω*.*

In this case $\bar{u}$ *may not satisfy the differential equation in the classical sense, but the* H^2 *regularity in the interior of* Ω *allows one to integrate by parts in (2.2) when* $v \in \mathcal{C}_0^\infty(\Omega)$ *and so to obtain that the equation is satisfied almost everywhere in* Ω*.*

Now let us see how we may apply lemma 2.16 to problem (2.1): once we have proved the existence of a weak solution $\bar{u} \in H^1$ of (2.1), if the regularity of f is such that $f(x, u(x)) \in L^2$ for any $u \in H^1$ and Ω is regular enough, then the above lemma implies $\bar{u} \in H^2$.

This idea may be iterated until f and Ω are regular enough to guarantee that $f(x, u(x)) \in H^m$ for $u \in H^{m+1}$, obtaining $\bar{u} \in H^{m+2}$.

Finally if this "boot strap argument" may be iterated a sufficient number of times to conclude that $\bar{u} \in H^{\overline{m}}$ where $\overline{m}$ is such that $H^{\overline{m}}(\Omega) \subseteq \mathcal{C}^2(\bar{\Omega})$, then integration by parts and the use of appropriate test functions in (2.2) yields that $\bar{u}$ is a classical solution too.

2.3 Eigenvalues of the Laplacian

In the study of problems like (2.1) it is usually important to consider the spectrum of the operator, that is the set $\sigma \subseteq \mathbb{C}$ of those λ (**eigenvalues**) such that there exist non trivial solutions (**eigenfunctions**) of the problem

$$\begin{cases} -\Delta u = \lambda u & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array}\right. & in\ \partial\Omega \end{cases} . \tag{2.20}$$

We resume here some known results on the matter, considering only the case Ω bounded (see[Eva98]):

- The eigenvalues are all real and nonnegative and form a discrete set unbounded from above.
- For each eigenvalue λ, the set $N(\lambda)$ of the related eigenfunctions is a finite dimensional subspace of H. The dimension of this eigenspace is called the **multiplicity** of the eigenvalue. Distinct eigenvalues have orthogonal eigenspaces (in the H scalar product, and in the L^2 scalar product). Moreover (by a boot strap argument) the eigenfunctions are always $\mathcal{C}^\infty(\Omega)$ and, if Ω is regular, also $\mathcal{C}^\infty(\bar{\Omega})$.
- There exists a first eigenvalue, it is simple (that is its multiplicity is 1) and the related eigenspace is composed of multiples of a function strictly positive in Ω.
  We will use the convention to order the eigenvalues in a nondecreasing sequence $\{\lambda_k\}_{k=1,2,..}$, repeating each one of them according to its multiplicity, and to denote by ϕ_k one generator of the corresponding eigenspace, chosen such that $\phi_1 > 0$, $\|\phi_k\|_{L^2} = 1$ and $< \phi_k, \phi_h >_{L^2} = 0$ for $k \neq h$.

  Note that for the Neumann problem $\lambda_1 = 0$ and $\phi_1 = const$, while for the Dirichlet problem $\lambda_1 > 0$.

Another important property is that the sequence of the above chosen eigenfunctions is an orthogonal basis for the space H, that is, any $u \in H$ may be written in a unique way as $u = \sum_{i=1}^{+\infty} c_i\phi_i$ with $\{c_i\} \subseteq \mathbb{R}$. Moreover $\|u\|_{L^2}^2 = \sum_{i=1}^{+\infty} c_i^2$ and $\|\nabla u\|_{L^2}^2 = \sum_{i=1}^{+\infty} \lambda_i c_i^2$.

A classical result showing the importance of the interaction between the function $f(x, u)$ and the spectrum of the operator is the following linear result:

Theorem 2.18 (Frehdolm alternative).
Let $f(x, u) = \lambda u + h(x)$ with $h \in L^2(\Omega)$, then we have

- *if $\lambda \notin \sigma$, then there exists a unique solution of problem (2.1);*
- *if $\lambda \in \sigma$, then solutions exist if and only if $< f, \phi >_{L^2} = 0$ for all $\phi \in N(\lambda)$; moreover if $\bar{u}$ is a solution, then $\bar{u} + w$ is a solution too if and only if $w \in N(\lambda)$.*

We now give an example of what may happen in the nonlinear case:

Theorem 2.19.
Consider problem (2.1) with $f(x,u) = g(u) + h(x)$, $g \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$, $h \in L^2(\Omega)$, and such that

- $\lim_{s\to\pm\infty} \frac{g(s)}{s} = \lambda \in \mathbb{R}$,
- *there exists $M > 0$ such that $|g(s) - \lambda\, s| \leq M \quad \forall s \in \mathbb{R}$;*

then

- *if $\lambda \notin \sigma$ (nonresonant case) there exists a solution for any $h \in L^2$;*
  if moreover $g'(\mathbb{R}) \cap \sigma = \emptyset$ then the solution is unique (no interaction with the spectrum);
- *if $\lambda \in \sigma$ (resonant case) then, splitting $h = h^\perp + h_\lambda$ with $h_\lambda \in N(\lambda)$ and $h^\perp \in N(\lambda)^\perp$, one has that for any $h^\perp$, there exists a set $S(h^\perp) \subseteq N(\lambda)$ such that a solution exists if and only if $h_\lambda \in S(h^\perp)$.*
  *Moreover one can give sufficient conditions to have $h_\lambda \in S(h^\perp)$ (**nonresonance conditions**).*

The simplest case of resonance is $\lambda = \lambda_1$, where one has the result (obtained in [LL70]):

Theorem 2.20.
If:

- *there exist and are finite $M^\pm = \lim_{s\to\pm\infty} g(s) - \lambda\, s$,*
- *$M^+ < g(s) - \lambda\, s < M^- \quad \forall s \in \mathbb{R}$;*

then there exists a solution of problem (2.1) if and only if $\quad -M^- \int_\Omega \phi_1 < \int_\Omega h\phi_1 < -M^+ \int_\Omega \phi_1$, that is $S(h^\perp) = \{c\phi_1$ such that $c \in (-M^-, -M^+) \int_\Omega \phi_1\}$.

2.3.1 Variational characterization of the eigenvalues

The eigenvalues of the Laplacian may be characterized variationally: we will describe here one possible version of this characterization, because it is the one which will be adapted later to obtain new results.

The first eigenvalue may always be characterized as

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^2 : \qquad u \in H; \quad \|u\|_{L^2} = 1 \right\} . \tag{2.21}$$

For the other eigenvalues, if we consider a point $a \in (\lambda_k, \lambda_{k+1})$ and the functional $J_a : H \to \mathbb{R}$

$$J_a(u) = \int_\Omega |\nabla u|^2 - a \int_\Omega u^2 \ , \tag{2.22}$$

we have a natural splitting $H = V \oplus W$, where $V = span\{\phi_1, .., \phi_k\}$: taking ∂B_{L^2} to be the boundary of the unit ball in L^2 norm in H, one obtains that there exists $\mu > 0$ such that

$$J_a(u) \leq -\mu < 0 \qquad for\ all\ u \in \partial B_{L^2} \cap V \tag{2.23}$$
$$J_a(u) \geq \mu\, \|u\|_H^2 \geq 0 \qquad for\ all\ u \in W \ , \tag{2.24}$$

and that the two sets link.
The existence of this structure allows to characterize the eigenvalue λ_{k+1} as

$$\lambda_{k+1} = a + \inf_{\gamma\in\Gamma} \sup_{u\in\gamma(B^k)} J_a(u) \ , \tag{2.25}$$

where the family Γ is defined as

$$\Gamma = \{\gamma : B^k \to \partial B_{L^2} \ \ continuous \ \ s.t. \ \ \gamma|_{\partial B^k} \ \ is \ \ an \ \ homeomorphism \ \ onto \ \ \partial B_{L^2} \cap V\} \tag{2.26}$$

and $B^k = \{(x_1..., x_k) \in \mathbb{R}^k \ s.t. \ \sum_{i=1}^k x_i^2 \leq 1\}$.

2.4 The multi-Laplacian operator

Let us consider now a higher order problem: instead of the operator $-\Delta$ we consider an integer power of it, namely $(-\Delta)^m$; in dealing with such problems we will use the notation $\nabla^{2h}u = \Delta^h u$ and $\nabla^{2h+1}u = \nabla(\Delta^h u)$.

Here the natural definitions of solution will require more regularity than with the Laplacian, in particular:

- a **classical solution** must be at least $\mathcal{C}^{2m}(\Omega)$, in order to compute Δ^m pointwise,
- a **weak solution** will be in $H^m(\Omega)$ so that the integral $\int_\Omega |\nabla^m u|^2$ is well defined;

actually multiplying $(-\Delta)^m u$ by v, integrating by parts m times and supposing everything is regular enough to give sense to the passages, one gets

$$\int_\Omega (-1)^m \nabla^{2m} u \ v = (-1)^m \sum_{i=1}^m (-1)^{i-1} \int_{\partial\Omega} (\nabla^{2m-i} u \nabla^{i-1} v) \cdot n_{ext} + \int_\Omega \nabla^m u \nabla^m v \, . \tag{2.27}$$

For this problem one also has to give more boundary conditions: the natural sets of boundary conditions are those that make zero the boundary terms coming out from the integration by parts, then for each $i = 1, ..., m$ one may choose whether to impose

- when i is even $\nabla^{2m-i}u = 0$ or $\nabla^{i-1}u \cdot n_{ext} = 0$;
- when i is odd $\nabla^{2m-i}u \cdot n_{ext} = 0$ or $\nabla^{i-1}u = 0$.

The choice to impose a derivative of order higher than $m - 1$ automatically makes zero the corresponding term in equation (2.27) since these derivatives appear for the solution u, the other choice appears for the test function v in equation (2.27), but being on a derivative of order lower than m it may be imposed in the choice of the space which will then result to be

$$H^m_*(\Omega) = \{u \in H^m(\Omega) \quad such \quad that \quad Bu = 0\} \, , \tag{2.28}$$

where we call B the operator that maps u to the vector of the traces on $\partial\Omega$ of the derivatives of order strictly lower than m that we choose to impose.

2.5 The p-Laplacian operator

Here we want to recall another largely studied operator: the p-Laplacian; we will consider, for $p > 1$, the model problem

$$\begin{cases} -\Delta_p u := -\nabla \cdot [\psi(\nabla u)] = f(x,u) & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array}\right. & in\ \partial\Omega \end{cases} \tag{2.29}$$

where $\psi(s) = \begin{cases} |s|^{p-2}s & s \neq 0 \\ 0 & s = 0 \end{cases}$. Obviously for $p = 2$ we have again the usual linear operator Laplacian, while for $p \neq 2$ this operator is nonlinear (actually $-\Delta_p(au) \neq -a\Delta_p u$).

If we want to do the same kind of work as done in section 2.1 for the Laplacian we are led to define as weak solutions of problem (2.29) those functions $u \in W$ (where $W = W^{1,p}(\Omega)$ in the Neumann case and $W_0^{1,p}(\Omega)$ in the Dirichlet case) such that

$$\int_\Omega \psi(\nabla u)\nabla v = \int_\Omega f(x,u)v \quad for\ \ all \quad v \in W\,; \tag{2.30}$$

the space W is chosen in order to give sense to the integral in the left hand side, while as before some more hypotheses on the growth at infinity of f will be needed to guarantee the wellposedness of the right hand side.

The "natural" eigenvalue problem for this operator is

$$\begin{cases} -\nabla \cdot [\psi(\nabla u)] = \lambda\psi(u) & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array}\right. & in\ \partial\Omega \end{cases}, \tag{2.31}$$

actually the two sides of the equation have the same degree of homogeneity and so if $\bar{u}$ is a nontrivial solution then so is $t\bar{u}$ for each $t \in \mathbb{R}$. In this sense we will call "ψ-linear" the rate of growth of ψ and "ψ-superlinear" (resp. "ψ-sublinear") the higher (resp. lower) rates of growth.

Much less is known about this operator than in the case $p = 2$. Actually we lose many useful properties we had for $p = 2$: W is no longer a Hilbert space and so we have no notion of orthogonality, and while any multiple of an eigenfunction is still eigenfunction, this is no more true for the sum of two eigenfunctions related to the same eigenvalue.

For the Dirichlet problem it is known (see [Ana87] and [Lin90]) that there exists a first eigenvalue λ_1 for $(-\Delta_p, W)$, that it is simple and isolated and that the related eigenfunction ϕ_1 does not change sign.

This first eigenvalue may be characterized as

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^p : \qquad u \in W; \quad \|u\|_{L^p} = 1\right\}. \tag{2.32}$$

Then there exists a diverging sequence of eigenvalues which may be characterized variationally (see [MP01]), but it is not clear in general whether this sequence constitutes all of the eigenvalues or not.

The one dimensional case is studied in [Drá92], where is shown that both the usual and the Fučík spectrum has the same qualitative shape as in the linear case ($p = 2$); this is due to the possibility of using here too the uniqueness for the solution of the initial value problem.

3 Jumping nonlinearities and the Fučík spectrum

In this section we will give some results about the main theme of our work. In particular we will discuss the Fučík spectrum and problems involving the Laplacian operator and nonlinearities asymptotically linear at both $+\infty$ and $-\infty$ but with different slopes or asymptotically linear at $-\infty$ and superlinear at $+\infty$.

The notion of Fučík spectrum was introduced in [Fuč76] and [Dan77]; it is defined as the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a non trivial solution of the problem

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & in \;\; \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array} \right. & in \;\; \partial\Omega \end{cases} , \tag{3.1}$$

where $u^+(x) = \max\{0, u(x)\}$, $u^-(x) = \max\{0, -u(x)\}$ and Ω is a bounded domain with Lipschitz boundary.

For $\lambda^+ = \lambda^-$ the problem becomes linear and admits nontrivial solutions for $\lambda^+ = \lambda^- = \lambda_k$; from these points arise curves belonging to the spectrum and in most cases it may be proven that the whole spectrum is composed by such curves.

To know the Fučík spectrum is important in many applications, for example in the study of problems with "jumping nonlinearities", that is nonlinearities which are asymptotically linear at both $+\infty$ and $-\infty$, but with different slopes.

3.1 Computation of the Fučík spectrum in dimension one

In the one dimensional case the Fučík spectrum may be completely calculated.

Let us start by considering the Dirichlet case: solutions to the boundary value problem (BVP)

$$\begin{cases} -u'' = \lambda^+ u^+ - \lambda^- u^- & in \;\; (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{3.2}$$

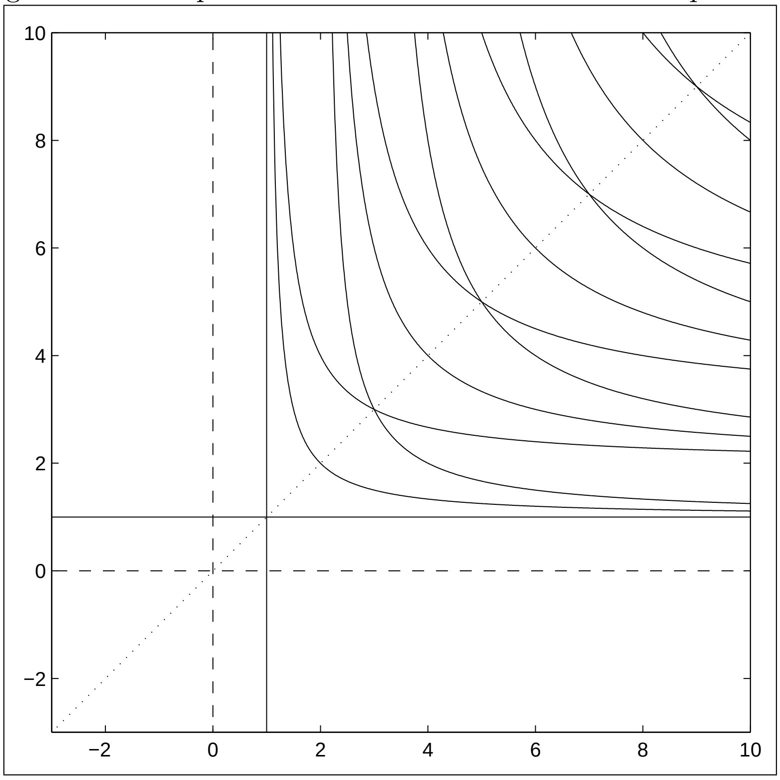
may be sought considering the initial value problem (IVP) $u(0) = 0$, $u'(0) = d_0$, for which we have existence and uniqueness of the solution.

Define $\psi_{\lambda,+}$ and $\psi_{\lambda,-}$ the solutions of the IVP with, respectively, $d_0 = 1$ and $d_0 = -1$, such that any other solution will be $d_0\psi_{\lambda,+}$ for $d_0 \geq 0$ and $d_0\psi_{\lambda,-}$ for $d_0 < 0$.

So $\psi_{\lambda,+}$ will be $\frac{1}{\sqrt{\lambda^+}} sin(\sqrt{\lambda^+}x)$ in $[0, \frac{\pi}{\sqrt{\lambda^+}}]$ and then $-\frac{1}{\sqrt{\lambda^-}} sin(\sqrt{\lambda^-}(x - \frac{\pi}{\sqrt{\lambda^+}}))$ in $[\frac{\pi}{\sqrt{\lambda^+}}, \frac{\pi}{\sqrt{\lambda^+}} + \frac{\pi}{\sqrt{\lambda^-}}]$ (such that it is differentiable in $\frac{\pi}{\sqrt{\lambda^+}}$) and then will continue with a sequence of analogous positive and negative bumps. $\psi_{\lambda,-}$ will be built in the same way, but starting with the negative bump $-\frac{1}{\sqrt{\lambda^-}} sin(\sqrt{\lambda^-}x)$.

So the Fučík spectrum will be composed by the points $\Sigma \subseteq \mathbb{R}^2$ such that $\psi_{\lambda,+}(1) = 0$ or $\psi_{\lambda,-}(1) = 0$, which gives (for $i = 1, 2, ..$) the following curves:

Figure 1: Fučík spectrum for one dimensional Dirichlet problem.



$$\Sigma_{2i} \quad : \quad \frac{i\pi}{\sqrt{\lambda^+}} + \frac{i\pi}{\sqrt{\lambda^-}} = 1\,, \tag{3.3}$$

$$\Sigma_{2i-1}^+ \quad : \quad \frac{i\pi}{\sqrt{\lambda^+}} + \frac{(i-1)\pi}{\sqrt{\lambda^-}} = 1\,, \tag{3.4}$$

$$\Sigma_{2i-1}^- \quad : \quad \frac{(i-1)\pi}{\sqrt{\lambda^+}} + \frac{i\pi}{\sqrt{\lambda^-}} = 1, \tag{3.5}$$

where Σ_{2i} corresponds to solutions with i positive and i negative bumps, starting either positive or negative, Σ_{2i-1}^+ corresponds to solutions with i positive and $i-1$ negative bumps, starting positive and Σ_{2i-1}^- to solutions with $i-1$ positive and i negative bumps, starting negative.

We plot in figure 1 this spectrum, where the axes have been moved to $\frac{\sqrt{\lambda^\pm}}{\pi}$.

The Neumann case can be built from the IVP $u(0) = c$, $u'(0) = 0$ and seeking solutions with $u'(1) = 0$, obtaining the curves in $\mathbb{R}^2$

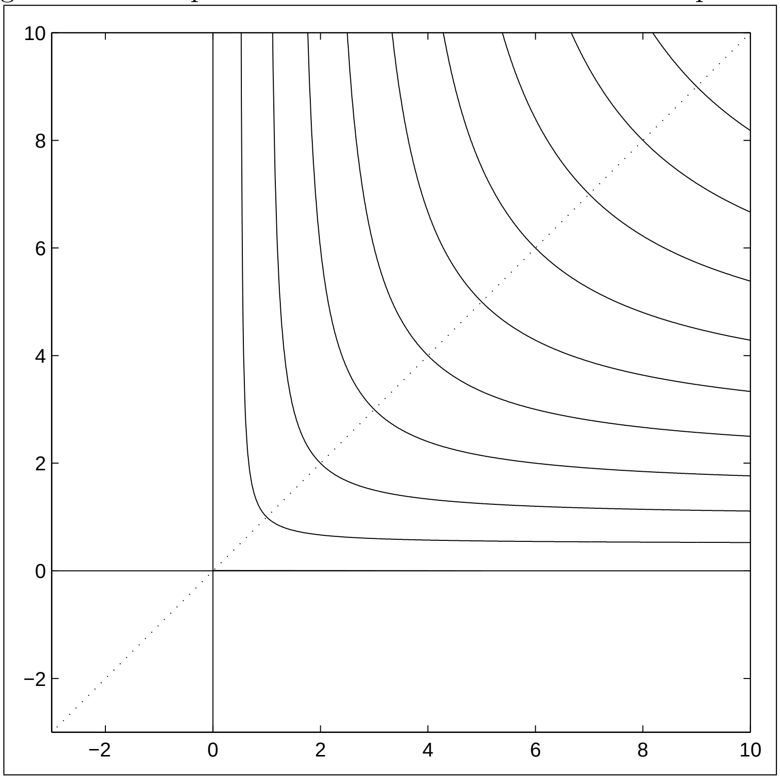
$$\Sigma_1^+ \quad : \quad \lambda^+ = 0\,, \tag{3.6}$$

$$\Sigma_1^- \quad : \quad \lambda^- = 0\,, \tag{3.7}$$

$$\Sigma_k \quad : \quad \frac{(k-1)\pi}{2\sqrt{\lambda^+}} + \frac{(k-1)\pi}{2\sqrt{\lambda^-}} = 1 \tag{3.8}$$

for $k = 2, 3, ..$, where $\Sigma_1^\pm$ correspond respectively to the positive and negative constant solutions,

Figure 2: Fučík spectrum for one dimensional Neumann problem.



while Σ_k for $k = 2, 3, ..$ to solutions with $(k-1)$ positive and $(k-1)$ negative half-bumps, starting either positive or negative.

This spectrum is plotted in figure 2, again with $\frac{\sqrt{\lambda^{\pm}}}{\pi}$ as axes.

Note that in both cases these spectra are composed by the two lines $\Sigma_1^+ : \{\lambda^+ = \lambda_1\}$ and $\Sigma_1^- : \{\lambda^- = \lambda_1\}$ (corresponding respectively to the nontrivial solutions ϕ_1 and $-\phi_1$), and then by other curves, all lying in the quadrant $\{\lambda^{\pm} > \lambda_1\}$, arising from each point (λ_j, λ_j), $j = 2, 3, ..$, which are continuous, symmetrical with respect to the line $\{\lambda^+ = \lambda^-\}$ and monotone decreasing.

The asymptotes of these curves are located, for the Dirichlet case, at the values

$$\lambda^- = \lambda_i \quad for \quad \Sigma_{2i-1}^-\,,\ \Sigma_{2i}\,,\ \Sigma_{2i+1}^+\,, \tag{3.9}$$

$$\lambda^+ = \lambda_i \quad for \quad \Sigma_{2i-1}^+\,,\ \Sigma_{2i}\,,\ \Sigma_{2i+1}^-\,, \tag{3.10}$$

and for the Neumann case at

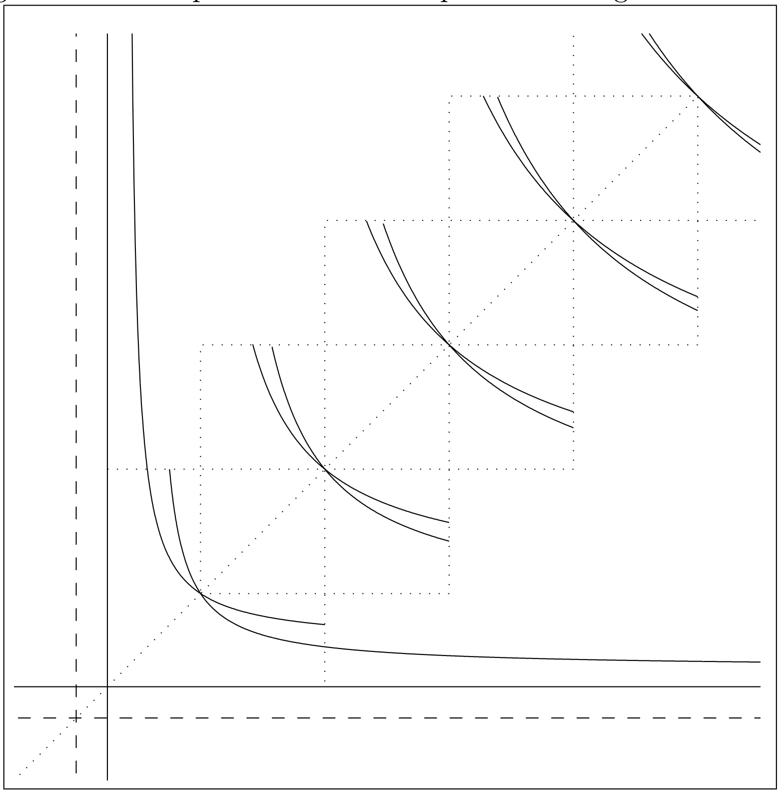
$$\lambda^- = \frac{\lambda_k}{4} \quad for \quad \Sigma_k\,, \tag{3.11}$$

$$\lambda^+ = \frac{\lambda_k}{4} \quad for \quad \Sigma_k\,. \tag{3.12}$$

3.2 Fučík spectrum in higher dimension

In the case of higher dimension less is known: Σ is always a closed set symmetrical with respect to the line $\{\lambda^+ = \lambda^-\}$; the lines $\{\lambda^+ = \lambda_1\}$ and $\{\lambda^- = \lambda_1\}$ are still in Σ while it cannot contain

Figure 3: Known parts of the Fučík spectrum in higher dimension.



any other point with $\lambda^+ < \lambda_1$ or $\lambda^- < \lambda_1$; moreover we still know (see for example [Dan77], [Ruf81], [GK81] and [Ma90]) that in each square $(\lambda_{k-1}, \lambda_{k+m+1})^2$, where $\lambda_{k-1} < \lambda_k = ... = \lambda_{k+m} < \lambda_{k+m+1}$, from the point (λ_k, λ_k) arises a continuum composed by a lower and a upper curve, both decreasing (may be coincident); other points in $\Sigma \cap (\lambda_{k-1}, \lambda_{k+m+1})^2$ can only lie between these two curves (and hence in the open squares $(\lambda_{k-1}, \lambda_k)^2$ and $(\lambda_{k+m}, \lambda_{k+m+1})^2$ there never are points of Σ). Something more can be said about the lower part of the continuum arising from (λ_2, λ_2): see [dFG94].

In [BNFS01] it is proved, under a non-degeneracy condition (which was first introduced in [Mic94] and [Pis97]) that the whole spectrum is composed by curves arising from a point (λ_k, λ_k), never intersecting and going to infinity; this non-degeneracy condition is discussed in [Pis97], where it is proved that it holds for 'almost all' (in a suitable sense) domains; however in general it seems not possible to arrive at the same conclusion.

For a larger bibliography about the Fučík spectrum see also [Sch00].

We sketch in figure 3 the known parts of the Fučík spectrum in the general multidimensional case.

3.3 Problems with jumping nonlinearities

Here we briefly discuss some results on the solvability of the nonlinear problem

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- + g(u) + h(x) & \textit{in } \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad \textit{or} \\ u = 0 \end{array} \right. & \textit{in } \partial\Omega \end{cases}, \tag{3.13}$$

where $|g(s)| \leq c_1 + c_2 |s|^\sigma$ with $\sigma \in [0, 1)$ and $h \in L^2(\Omega)$.

Remark 3.1. *If we call $f(u) = \lambda^+ u^+ - \lambda^- u^- + g(u)$ then the nonlinearity $f(u)$ satisfies*

$$\lim_{s \to -\infty} \frac{f(s)}{s} = \lambda^-, \qquad \lim_{s \to +\infty} \frac{f(s)}{s} = \lambda^+ \tag{3.14}$$

and is usually called a jumping nonlinearity.

Consider the Dirichlet case and define $T_\lambda : H_0^1 \to H_0^1$ such that $< T_\lambda u, v >_{H_0^1} = \lambda^+ \int_\Omega u^+ v - \lambda^- \int_\Omega u^- v$: note that for $(\lambda^+, \lambda^-) \notin \Sigma$ the equation $u - T_\lambda u = 0$ has only the trivial solution and this allows one to define the Leray-Schauder degree $d(u - T_\lambda u, B_R(0), 0)$ for any $R > 0$.

Dancer, in [Dan77], defines the following subsets of $\mathbb{R}^2 \backslash \Sigma$:

- $A_1 = \{(\lambda^+, \lambda^-) \in \mathbb{R}^2 \backslash \Sigma \text{ s.t. } d(u - T_\lambda u, B_R(0), 0) \neq 0\}$,
- $A_2 = \{(\lambda^+, \lambda^-) \in \mathbb{R}^2 \backslash \Sigma$ s.t. $\exists h \in L^2$ for which problem (3.13) with $g = 0$ has no solutions$\}$,

and proves that

- $A_1 \cap A_2 = \emptyset$, actually if the degree is not zero then one has a solution for any $h \in L^2$;
- A_1 and A_2 are open; A_1 is the union of components of $\mathbb{R}^2 \backslash \Sigma$, but is is not known whether $\mathbb{R}^2 \backslash \Sigma = A_1 \cup A_2$ or not;
- all the components of $\mathbb{R}^2 \backslash \Sigma$ which contain a segment of the diagonal $\{\lambda^+ = \lambda^-\}$ are in A_1;
- the two quarters of plane $\{\lambda^+ > \lambda_1,\ \lambda^- < \lambda_1, \}$ and $\{\lambda^+ < \lambda_1,\ \lambda^- > \lambda_1, \}$ are in A_2; actually the variational equation with test function ϕ_1 gives $\int \nabla u \nabla \phi_1 - \lambda^+ \int u^+ \phi_1 + \lambda^- \int u^- \phi_1 = (\lambda_1 - \lambda^+) \int u^+ \phi_1 + (\lambda^- - \lambda_1) \int u^- \phi_1 = \int h \phi_1$ and so (since $\phi_1 > 0$) the assumptions on (λ^+, λ^-) imply a necessary condition on the sign of $\int h \phi_1$.

  Remark 3.2. *Note that the same necessary condition on the sign of $\int h \phi_1$ arises if we consider $e^u + \lambda^- u$ with $\lambda^- < \lambda_1$ in place of $\lambda^+ u^+ - \lambda^- u^-$, that is a nonlinearity superlinear at $+\infty$ and asymptotically linear at $-\infty$ with slope smaller that λ_1.*

- if we add the sublinear perturbation g we have that:
  - for $(\lambda^+, \lambda^-) \in A_1$ there still exists a solution for any $h \in L^2$,
  - for $(\lambda^+, \lambda^-) \in A_2$ there still exists a $h \in L^2$ s.t. the problem has no solution.

The same results may be extended to the Neumann boundary conditions.

3.3.1 Dimension one

Going to the already seen one dimensional case one can deduce that

- in the Neumann case, only $\{\lambda^+ > \lambda_1, \lambda^- < \lambda_1,\}$ and $\{\lambda^+ < \lambda_1, \lambda^- > \lambda_1,\}$ are in A_2, since all the other components of $\mathbb{R}^2 \backslash \Sigma$ contain a segment of the diagonal,
- for the Dirichlet case
  - the regions between Σ_{2i} and the upper part of $\Sigma_{2i-1}^{\pm}$ or between Σ_{2i} and the lower part of $\Sigma_{2i+1}^{\pm}$ are in A_1 since they contain a segment of the diagonal,
  - the regions between Σ_{2i-1}^{+} and Σ_{2i-1}^{-} are in A_2.

Actually Dancer proves that:

Lemma 3.3. *Whenever (λ^+, λ^-) is such that $\psi_{\lambda,+}(1)\psi_{\lambda,-}(1) > 0$ (which by the way corresponds to the regions between Σ_{2i-1}^{+} and Σ_{2i-1}^{-}) there exists a $h \in L^2$ such that problem (3.13) with $g = 0$ has no solution.*

Idea of the proof.
First one observes that $\psi_{\lambda,+}(1)\psi_{\lambda,-}(1) > 0$ implies that there exists $x_0 \in (0, 1)$ such that

$$\psi_{\lambda,+}(x)\psi_{\lambda,-}(x) > 0 \quad \forall x \in [x_0, 1]\,; \tag{3.15}$$

in fact if we consider a point between Σ_{2i-1}^{+} and Σ_{2i-1}^{-} with $\lambda^+ > \lambda^-$ (the case $\lambda^+ < \lambda^-$ is analogous) we will have

$$\begin{cases} \frac{(i-1)\pi}{\sqrt{\lambda^+}} + \frac{i\pi}{\sqrt{\lambda^-}} > 1\,, \\ \frac{i\pi}{\sqrt{\lambda^+}} + \frac{(i-1)\pi}{\sqrt{\lambda^-}} < 1 \end{cases} , \tag{3.16}$$

that is $\psi_{\lambda,+}$ makes i positive bumps, $i-1$ negative ones and then a piece of a negative one, while $\psi_{\lambda,-}$ makes $i-1$ negative bumps, $i-1$ positive ones and does not complete the last negative bump; so we have $\psi_{\pm}(x) < 0$ in $(\frac{i\pi}{\sqrt{\lambda^+}} + \frac{(i-1)\pi}{\sqrt{\lambda^-}}, 1]$.

Now let $h = \chi([x_0, 1])$.

Since $h(x) = 0$ for $x \in [0, x_0)$, any solution of the IVP $u_d(0) = 0$, $u_d'(0) = d$ is exactly as in the homogeneous case in $[0, x_0]$ and so satisfies $u_d(x_0) \leq 0$.

After this one proves that there exists $\varepsilon_d > 0$ such that $u_d(x) < 0$ in $(x_0, x_0 + \varepsilon_d)$ and finally shows that in fact the equation implies $u_d(x) < 0$ in $(x_0, 1]$ $\forall d \in \mathbb{R}$. □

3.4 The PS condition and the Fučík spectrum

Another important property related to the Fučík spectrum is the following: if we want to solve variationally problem (3.13), we are led to consider the functional

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda^+}{2}\int_\Omega (u^+)^2 - \frac{\lambda^-}{2}\int_\Omega (u^-)^2 - \int_\Omega G(u) - \int_\Omega hu \ , \tag{3.17}$$

whre $G(s) = \int_0^s g(\xi)d\xi$.

We may prove that

Lemma 3.4. *The functional (3.17) satisfies the PS condition, provided* $(\lambda^+,\lambda^-) \notin \Sigma$

Proof. We take a sequence $\{u_n\} \subseteq H$, and $\varepsilon_n \to 0^+$ such that

$$\left|\frac{1}{2}\int_\Omega (\nabla u_n)^2 - \frac{\lambda^+}{2}\int_\Omega (u_n^+)^2 - \frac{\lambda^-}{2}\int_\Omega (u_n^-)^2 - \int_\Omega G(u_n) - \int_\Omega h u_n\right| \le C \tag{3.18}$$

$$\left|\int_\Omega \nabla u_n \nabla v - \lambda^+ \int_\Omega (u_n^+)v + \lambda^- \int_\Omega (u_n^-)v - \int_\Omega g(u_n)v - \int_\Omega hv\right| \quad \le \quad \varepsilon_n \|v\|_H\,, \quad \forall v \in H\ . \tag{3.19}$$

We will first prove that the sequence u_n is bounded in H: suppose the contrary, then we can assume $\|u_n\|_H \ge 1$, $\|u_n\|_H \to +\infty$ and define $z_n = \frac{u_n}{\|u_n\|_H}$, so that z_n is a bounded sequence in H and then we can select a subsequence such that $z_n \to z_0$ weakly in H and strongly in L^2.

Then if we consider $\frac{<F'(u_n),v>}{\|u_n\|_H}$ we get

$$\left|\int_\Omega \nabla z_n \nabla v - \lambda^+ \int_\Omega (z_n^+)v + \lambda^- \int_\Omega (z_n^-)v\right| \le \frac{|\int_\Omega g(u_n)v| + |\int_\Omega hv| + \varepsilon_n \|v\|_H}{\|u_n\|_H}, \tag{3.20}$$

where the whole right hand side goes to zero and then taking limit and using the weak convergence of z_n one obtains

$$\left|\int_\Omega \nabla z_0 \nabla v - \lambda^+ \int_\Omega (z_0^+)v + \lambda^- \int_\Omega (z_0^-)v\right| = 0 \quad \forall v \in H\,, \tag{3.21}$$

that is z_0 is a solution of the Fučík problem, which implies $z_0 = 0$ if $(\lambda^+,\lambda^-) \notin \Sigma$.

But this is not possible since considering $\frac{<F'(u_n),z_n>}{\|u_n\|_H}$ we get

$$\left|\int_\Omega |\nabla z_n|^2 - \lambda^+ \int_\Omega (z_n^+)^2 - \lambda^- \int_\Omega (z_0^-)^2\right| \to 0 \tag{3.22}$$

which now implies $\int_\Omega |\nabla z_n|^2 \to 0$, but this is a contradiction since it would give $1 = \|z_n\|_H^2 \to 0$.

Thus u_n is bounded and so there exists a subsequence such that $u_n \to u$ weakly in H and strongly in L^2.

Finally, with $v = u_n - u$ we get

$$\int_\Omega \nabla u_n \nabla(u_n - u) - \int_\Omega (\lambda^+ u_n^+ - \lambda^- u_n^-)(u_n - u) - \int_\Omega g(u_n)(u_n - u) - \int_\Omega h\,(u_n - u) \to 0, \tag{3.23}$$

where now all terms except the first go to zero.

We conclude that $\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$ and then $u_n \to u$ strongly in H. □

3.5 The Ambrosetti-Prodi problem

One interesting and largely studied problem with jumping nonlinearities is the so called Ambrosetti-Prodi problem, that is problem (3.13) with $\lambda^- < \lambda_1 < \lambda^+$: when the nonlinearity interacts with the first eigenvalue.

We have already seen that in this case there are functions for which there is no solution, but one can find more.

The first results about this problem were obtained in [AP72]; later Dancer, in [Dan78], extended such results proving by topological degree techniques that splitting $h = h^\perp + t\,\phi_1$ with $\langle h^\perp, \phi_1\rangle_{L^2} = 0$, one has that for any $h^\perp$, there exists a $\bar{t}(h^\perp)$ such that

- for $t > \bar{t}(h^\perp)$ there exists no solution,
- for $t = \bar{t}(h^\perp)$ there exists at least one solution,
- for $t < \bar{t}(h^\perp)$ there exist at least two solutions;

moreover if $f(u) = \lambda^+ u^+ + \lambda^- u^- + g(u)$ is strictly convex and $\lambda^+ \leq \lambda_2$ one obtains respectively exactly one and exactly two solutions in the last two cases.

In [dF84] and [dFS84] the same kind of results are proved by variational techniques:

Idea of the variational proof. Consider the functional associated to problem (3.13) (let us consider only the case $g = 0$):

$$J(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{\lambda^+}{2}\int (u^+)^2 - \frac{\lambda^-}{2}\int (u^-)^2 - \int hu\,. \tag{3.24}$$

Applying the derivative of the functional to ϕ_1 one gets

$$\begin{aligned} \langle J'(u), \phi_1\rangle &= \int \nabla u \nabla \phi_1 - \lambda^+ \int u^+\phi_1 + \lambda^- \int u^-\phi_1 - \int h\phi_1 \\ &= (\lambda_1 - \lambda^+)\int u^+\phi_1 + (\lambda^- - \lambda_1)\int u^-\phi_1 - \int h\phi_1 \end{aligned} \tag{3.25}$$

and so it is clear that for $\int h\phi_1 > 0$ this component of the derivative is never zero, and so no solution can exist.

For $\int h\phi_1 < 0$ a first solution is found using sub and super solutions techniques and is proved to be a local minimum of the functional, then a mountain pass solution is found since one shows that

$$\begin{aligned} \lim_{t\to+\infty} J(t\phi_1) &= \lim_{t\to+\infty}\left[t^2\left(\frac{1}{2}\int |\nabla\phi_1|^2 - \frac{\lambda^+}{2}\int \phi_1^2\right) - t\int h\phi_1\right] = \\ &= \lim_{t\to+\infty}\left[t^2\left(\frac{\lambda_1 - \lambda^+}{2}\int \phi_1^2\right) - t\int h\phi_1\right] = -\infty\,. \end{aligned} \tag{3.26}$$

□

3.6 Problems linear at $-\infty$ and superlinear at $+\infty$

Now we give a review of results about the case in which the nonlinearity is asymptotically linear at $-\infty$ and superlinear at $+\infty$; we will write the equation as

$$-\Delta u = \lambda u + g(u) + h(x) \tag{3.27}$$

where $\lim_{s\to-\infty} \frac{g(s)}{s} = 0$ and $\lim_{s\to+\infty} \frac{g(s)}{s} = +\infty$, so that the behavior at $-\infty$ is given by the parameter λ.

The results of the previous section (that is with $\lambda < \lambda_1$) may be extended to this superlinear case, however Dancer's approach needs strict hypotheses on the growth at $+\infty$, while the variational approach allows one to deal with superlinearities growing faster, but still requires some technical hypotheses to guarantee the PS condition.

Observe that this problem may be seen as the limiting case of the asymptotically linear problem with coefficients (λ^+, λ^-) in the quarter of plane $\{\lambda^+ > \lambda_1, \lambda^- < \lambda_1, \}$, when $\lambda^+ \to +\infty$, that is when the nonlinearity crosses all the eigenvalues. Remember that this quarter of plane is a component of the set A_2, and so the existence of solution only for suitable forcing term h is coherent with this interpretation.

A different problem is when the nonlinearity crosses all but a finite number of eigenvalues, that is the case $\lambda \in (\lambda_k, \lambda_{k+1})$.

There are several works in which particular cases are analyzed, so that one can see the kind of phenomena that may happen.

- In [RS86b] the authors analyze the equation (in dimension one) $-u'' = \lambda u + (u^+)^p + t\phi_1$ with Dirichlet boundary conditions, where $p > 1$; the result is

  For $\lambda < \lambda_1$ and $t \leq 0$ there exist at least 2 solutions.
  For $\lambda \in (\lambda_k, \lambda_{k+1})$ and $t \geq 0$ there exist at least $2k + 2$ solutions.

  In [dP89] a similar result is achieved for radial solutions in a ball.

- In [RS86a] and [CS85] the case $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ and with $h(x) = t$ is considered (still in dimension one with Dirichlet boundary conditions), that is with constant forcing term; the main result is

  For any $\lambda \in R$, chosen $n \in \mathbb{N}$ there exists t_n such that for $t < t_n$ one gets at least n solutions.

- Finally [dF88] considers the case $\lambda \in (\lambda_k, \lambda_{k+1})$ (here the nonlinearity g is a continuous function of both $x \in [0, 1]$ and $u \in \mathbb{R}$ and the limits for $u \to \pm\infty$ are supposed to hold uniformly in x).

  Splitting again $h = h^\perp + t\phi_1$, he first proves that

  If $\lambda > \lambda_1$, then there exists a $\bar{t}(h^\perp) > 0$ such that problem (3.27) has a (negative) solution for $t \geq \bar{t}(h^\perp)$.

  Then the problem in dimension $N \geq 2$ is treated variationally and so requires suitable hypotheses on the behavior of the nonlinearity at both $+\infty$ and $-\infty$ to guarantee the wellposedness of the functional and the PS condition; moreover the nonlinearity is required to be $\mathcal{C}^1$ and such that its derivative do no interacts with the eigenvalues $\lambda_1 ... \lambda_k$ (that is $\lambda + g'(s) > \lambda_k + \varepsilon$ for a suitable $\varepsilon > 0$).

  The result is

  Under the hypotheses stated above, with $\lambda \in (\lambda_k \lambda_{k+1})$, there exists a $\widehat{t}(h^\perp) \geq \bar{t}(h^\perp)$ such that for $t \geq \widehat{t}(h^\perp)$ there exist at least two solutions of problem (3.27).

Idea of the proof. The author analyzes a modification of the problem and proves the existence of a linking structure for the functional associated to this modification.

In particular consider the splitting $H_0^1 = V \oplus W$ with $V = span\{\phi_1 ... \phi_k\}$: the author finds suitable $\rho > 0$, $L > 0$, $e \in W$ such that

- $W \cap \partial B_\rho(0)$ and the relative boundary of the $k+1$ dimensional rectangle $\{v + te$ with $v \in V$, $\|v\| \leq L$, $t \in [0,1]\}$ link,

- J is positive and bounded away from zero on the first of the two sets and non positive on the second;

this allows one to get another solution by the linking theorem.

However the point $e \in W$ must be an unbounded function and so its existence relies on the fact that $H_0^1 \not\subseteq L^\infty$, that is $N \geq 2$.

□

3.6.1 Existence for arbitrary forcing term

Note that all the results for superlinear problems of the previous section are just existence results for suitable forcing terms.

Now we analyze other situations in which one may prove existence of solutions for arbitrary forcing terms.

In [dFR91] the Neumann problem in dimension one is considered (here too the nonlinearity g is a continuous function of both $x \in [0,1]$ and $u \in \mathbb{R}$ and the limits for $u \to \pm\infty$ are supposed to hold uniformly in x).

The result is the existence of a solution for any $h \in L^2$ when $\lambda \in (0, \frac{\pi^2}{4}) \equiv (\lambda_1, \frac{\lambda_2}{4})$.

It is obtained variationally and so needs an additional hypothesis on the behavior of the nonlinearity at $+\infty$ in order to obtain the PS condition.

The solution is found as a mountain pass critical point: the functional associated to the problem is such that:

- it is bounded from below in the set $N = \{u \in H^1(0,1)$ such that $sup_{x\in[0,1]} u(x) = 0\}$, for any $\lambda < \frac{\pi^2}{4}$,

- $\lim_{t\to\pm\infty} J(t\phi_1) = -\infty$, provided $\lambda > 0$;

finally, since $H^1(0,1) \subseteq \mathcal{C}([0,1])$, the set N splits $H^1(0,1)$ into two components and $\pm\phi_1$ lie on the opposite sides of it, giving the required mountain pass structure. We remark that here is where one uses the hypothesis of being in dimension one.

A very similar result, with slightly different hypotheses, is given in [Vil98], where it is also remarked that the same proof may work for the p-Laplacian analogs of the problem, giving a solution for any $h \in L^q$ (q being the dual exponent of p) provided $\lambda \in (0, \lambda^*)$ for a suitable $\lambda^* > 0$; moreover in the p-Laplacian case the result is still valid in dimension $N < p$, since this is the condition that implies $W^{1,p}(\Omega) \subseteq \mathcal{C}(\bar{\Omega})$.

The value $\frac{\pi^2}{4}$ that limits the validity of these results is characterized as

- $\inf \left\{ \int_0^1 |u'|^2 \text{ with } u \in H^1(0,1);\ \|u\|_{L^2}^2 + \|u\|_{L^\infty}^2 = 1;\ \int_0^1 u\phi_1 = 0 \right\}$ in [dFR91],

- $\inf \left\{ \frac{\int_0^1 |u'|^2}{\int_0^1 u^2} \text{ with } u \in H^1(0,1);\ u \not\equiv 0;\ sup_{x\in[0,1]} u(x) = 0 \right\}$ in [Vil98], but the most interesting property is that it is the asymptote of the curve Σ_2 of the Fučík spectrum, and so a natural interpretation of the result is to consider it as the limiting asymptotically linear problem with coefficients between Σ_1 and Σ_2, when $\lambda^+ \to +\infty$; actually we have already seen that this asymptotically linear problem is solvable for arbitrary forcing term too.

In [dFR93] the analogous problem with periodic boundary conditions is considered and is proved the existence of a solution for any $h \in L^2$ when $\lambda \in (\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4})$.

Since the Fučík spectrum of the periodic case is qualitatively the same of the Neumann one, this means that λ must lie between the asymptotes of two consecutive curves; then the result may again be interpreted as the limiting asymptotically linear problem with coefficients between Σ_k and Σ_{k+1}, when $\lambda^+ \to +\infty$.

Moreover the proof is indeed obtained through a variational characterization of the Fučík spectrum, that furnishes the estimates needed to apply the deformation lemma and so to find a critical point of the related functional.

Unfortunately this characterization makes use of the invariance of the eigenspaces of the operator with respect to translation, and so may not be adapted to other boundary conditions.

Going back to Neumann boundary conditions, a small step forward is given in [Per00], where it is proved that the same result of [dFR91] is still valid if $\lambda \in (\frac{\pi^2}{4}, \lambda^*)$ for a suitable $\lambda^* > \frac{\pi^2}{4}$; this λ^* is obtained by compactness argument and so there is no estimate about its value. To prove this result the author finds a subset of N with codimension 2 in H, where the functional is still bounded from below if $\lambda \in (\frac{\pi^2}{4}, \lambda^*)$ and then finds a second set linking with the first where it is lower.

To our knowledge no results of this kind for larger values of λ are available.

We just cite [AV95] [Per00], where similar problems are analyzed, but where the existence of a first trivial solution is guaranteed by more restrictive hypotheses and so the interest is in finding nontrivial ones.

3.7 Variational characterizations of the Fučík spectrum

As for the usual spectrum it is important to have a variational characterization of the Fučík spectrum: this allows one to obtain interesting results for sublinear perturbations of the considered problem, since these characterizations are stable under such perturbations.

We have already cited the variational characterization of the Fučík spectrum in dimension one with periodic boundary conditions given in [dFR93] and its application to the superlinear problem.

In [dFG94] and [CdFG99] the lower part of the first nontrivial curve of the Fučík spectrum is characterized for, respectively, the Laplacian and the p-Laplacian; the characterization is then used both to obtain a better description of the spectrum and to find existence results for a nonlinear problem where the nonlinearity lies asymptotically in a square between (λ_1, λ_1) and a point of the obtained curve.

In [Sch00] the lower and the upper curves coming out from an eigenvalue (λ_k, λ_k) are characterized for the Laplacian in any space dimension, but just in the square $(\lambda_{k-1}, \lambda_{k+m+1})^2$ being $m + 1$ the multiplicity of λ_k, so that it may not be applied to superlinear problems.

Finally in [MP01] some pieces of the Fučík spectrum of the p-Laplacian near to the diagonal are characterized.

4 A variational characterization of the Fučík spectrum

In this section we want to obtain a variational characterization of parts of the Fučík spectrum for the Laplacian with Neumann or Dirichlet boundary conditions, in any spatial dimension.

Recalling the variational characterization of the eigenvalues of the Laplacian described in section 2.3.1, what we intend to do now is to build suitable sets to play the same role played there by $\partial B_{L^2} \cap V$ and W, but now with the functional

$$J_\alpha(u) = \int_\Omega |\nabla u|^2 - \alpha^+ \int_\Omega (u^+)^2 - \alpha^- \int_\Omega (u^-)^2 \ , \tag{4.1}$$

with $(\alpha^+, \alpha^-) \in \mathbb{R}^2$, constrained to the set

$$Q_r = \left\{ u \in H \ \ s.t. \ \ \int_\Omega (u^+)^2 + r(u^-)^2 = 1 \right\} , \tag{4.2}$$

with $r \in (0, 1]$.

This procedure will indeed result in the following characterization of a point in the Fučík spectrum:

Theorem. 1.1. *Suppose that the point $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ with $\alpha^+ \geq \alpha^-$ is Σ-connected to the diagonal between λ_k and λ_{k+1} in the sense of definition 4.1, then we can find and characterize one intersection of the Fučík spectrum with the halfline $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$, for each value of $r \in (0, 1]$.*

The new sets mentioned above will be obtained in section 4.1 as a deformation of the previous ones, using a technique similar to the one described in [DR98].

Then the variational characterization will be done in section 4.2.

4.1 Construction of the linking structure

Let $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ be Σ-connected to the diagonal between λ_k and λ_{k+1}, that is:

Definition 4.1. *$(\alpha^+, \alpha^-) \notin \Sigma$ is Σ-connected to the diagonal between λ_k and λ_{k+1} if: $\exists a \in (\lambda_k, \lambda_{k+1})$ and a $\mathcal{C}^1$ function $\alpha : [0, 1] \to \mathbb{R}^2$ such that:*

a) $\alpha(0) = (a, a)$, $\alpha(1) = (\alpha^+, \alpha^-)$;

b) $\alpha([0, 1]) \cap \Sigma = \emptyset$.

Remark 4.2. *Since Σ is closed and $\alpha([0, 1])$ is compact, definition 4.1 implies the property*

b') $\exists d > 0$ such that $N_{\alpha,d} \cap \Sigma = \emptyset$, where $N_{\alpha,d} = \{p \in \mathbb{R}^2$ such that $d(p, \alpha([0, 1])) \leq d\}$.

This property will be used in the following proofs.

Now consider the Hilbert space H with the norm $\|u\|_H^2 = \int_\Omega |\nabla u|^2 + \int_\Omega |u|^2$, and the functional

$$\begin{aligned} J_{\alpha(t)}(u) &= \int_\Omega |\nabla u|^2 - \alpha^+(t) \int_\Omega (u^+)^2 - \alpha^-(t) \int_\Omega (u^-)^2 \\ &= \|u\|_H^2 - (\alpha^+(t) + 1) \int_\Omega (u^+)^2 - (\alpha^-(t) + 1) \int_\Omega (u^-)^2 \ , \end{aligned} \tag{4.3}$$

where $\alpha(t) = (\alpha^+(t), \alpha^-(t))$; then splitting as in section 2.3.1 $H = V \oplus W$ with $V = span\{\phi_1, .., \phi_k\}$, we have

$$J_{\alpha(0)}(u) \le -\mu \, \|u\|_H^2 \qquad \forall \, u \in V \; , \tag{4.4}$$

$$J_{\alpha(0)}(u) \ge \mu \, \|u\|_H^2 \qquad \forall \, u \in W \; , \tag{4.5}$$

for some $\mu > 0$.

Our aim is to obtain an analogous property for $J_{\alpha(1)}$.

We first need a technical lemma:

Lemma 4.3 (from lemma 2.3 of [DR98]).
If (α^+, α^-) is as in definition 4.1, we can find $\eta \in (0, \mu)$ and $\delta_\eta > 0$ such that:
$\forall t \in [0,1], \quad u \in H$ *with* $\|u\|_H = 1$*:*

$$\textit{if} \quad J_{\alpha(t)}(u) \in [-\eta, \eta] \quad \textit{then} \quad \left\|\nabla_u J_{\alpha(t)}(u)\right\|_H^2 - \left\langle \nabla_u J_{\alpha(t)}(u), \, u \right\rangle_H^2 \ge \delta_\eta \; .$$

Proof. Consider a fixed $\eta > 0$ and suppose by contradiction the existence of a sequence $\{t_n\} \subseteq [0,1]$ and $\{u_n\} \subseteq H$, with $\|u_n\|_H = 1$ such that

$$-\eta \le J_{\alpha(t_n)}(u_n) \le \eta \quad \textit{and} \quad \|\nabla_u J_{\alpha(t_n)}(u_n)\|_H^2 - \left\langle \nabla_u J_{\alpha(t_n)}(u_n), \, u_n \right\rangle_H^2 \to 0 \tag{4.6}$$

as $n \to +\infty$.

Define $j_n = \left\langle \nabla_u J_{\alpha(t_n)}(u_n), \, u_n \right\rangle_H = 2 J_{\alpha(t_n)}(u_n) \in [-2\eta, 2\eta]$; from Pythagoras' theorem deduce that

$$\|\nabla_u J_{\alpha(t_n)}(u_n)\|_H^2 - \left\langle \nabla_u J_{\alpha(t_n)}(u_n), \, u_n \right\rangle_H^2 = \|\nabla_u J_{\alpha(t_n)}(u_n) - j_n u_n\|_H^2 \, , \tag{4.7}$$

then evaluating the norm in the right hand side considering the points in H as operators on H one concludes that

$$(1 - j_n)\langle u_n, v_n \rangle_H - (\alpha^+(t_n) + 1) \int_\Omega u_n^+ v_n + (\alpha^-(t_n) + 1) \int_\Omega u_n^- v_n \to 0 \tag{4.8}$$

for any bounded sequence $v_n \subseteq H$.

Up to a subsequence we may say that $j_n \to j \in [-2\eta, 2\eta]$, $t_n \to t_0 \in [0,1]$ and $u_n \rightharpoonup u \in H$ (strongly in L^2); taking the limit of (4.8) with $v_n = u_n$ gives

$$1 - j = (\alpha^+(t_0) + 1) \int_\Omega (u^+)^2 + (\alpha^-(t_0) + 1) \int_\Omega (u^-)^2 \; , \tag{4.9}$$

where $j \le 2\eta < 1$ and then u is not trivial.

From equation (4.8) with arbitrary test function and using the weak convergence of u_n, we get

$$(1 - j)\langle u, v \rangle_H = (1 - j) \int_\Omega \nabla u \nabla v + uv = (\alpha^+(t_0) + 1) \int_\Omega u^+ v - (\alpha^-(t_0) + 1) \int_\Omega u^- v \, , \tag{4.10}$$

that is u is a solution of the Fučík problem with coefficient $\left(\frac{\alpha^+(t_0)+j}{1-j}, \frac{\alpha^-(t_0)+j}{1-j}\right)$, but this contradicts remark 4.2 for small enough choices of η, since $|j| \le 2\eta$, $lim_{j \to 0}\left(\frac{a+j}{1-j}\right) = a$ and t_0 takes values in a compact set.

□

Then as in [DR98] we consider the ordinary differential equation for the unknown function $\sigma : [0,1] \times H \to H : (t,u) \mapsto \sigma_t(u)$:

$$\begin{cases} \frac{d}{dt}\sigma_t(u) = M\, F_t(\sigma_t(u)) \\ \sigma_0(u) = u \end{cases} , \tag{4.11}$$

where

- M is a suitable positive constant, defined as $M = 2KS^2/\delta_\eta$, with
  - * $K = \sup_{t\in[0,1]}(|\alpha^+(t)'| + |\alpha^-(t)'|)$,
  - * $S = (\lambda_1 + 1)^{-\frac{1}{2}} = \sup_{u\in H\setminus\{0\}} \frac{\|u\|_{L^2}}{\|u\|_H}$;
- $F_t : H \to H$ is defined such that
  - * it is locally Lipschitz,
  - * there exists a constant $L > 0$ such that $\|F_t(u)\|_H \le L\,\|u\|_H$ for all $t \in [0,1]$, $u \in H$,
  - *

$$\begin{cases} F_t(u) = \nabla_u J_{\alpha(t)}(u) & \textit{where } \frac{J_{\alpha(t)}(u)}{\|u\|_H^2} \ge \eta/2 \\ F_t(u) = -\nabla_u J_{\alpha(t)}(u) & \textit{where } \frac{J_{\alpha(t)}(u)}{\|u\|_H^2} \le -\eta/2 \end{cases} . \tag{4.12}$$

4.1.1 Construction of $F_t(u)$

Be S the unit sphere in H and

$$A_1 = \{(t,u) \in [0,1] \times S : \quad J_{\alpha(t)}(u) \le -\eta/2\}, \tag{4.13}$$
$$A_2 = \{(t,u) \in [0,1] \times S : \quad J_{\alpha(t)}(u) \ge \eta/2\}; \tag{4.14}$$

then define $\chi : [0,1] \times S \to [-1,1]$ as

$$\chi(t,u) = \frac{d((t,u),A_1) - d((t,u),A_2)}{d((t,u),A_1) + d((t,u),A_2)} \tag{4.15}$$

so that

$$\chi(t,u) = \begin{cases} -1 & \textit{for} \quad (t,u) \in A_1 \\ 1 & \textit{for} \quad (t,u) \in A_2 \\ s \in (-1,1) & \textit{otherwise} \end{cases} . \tag{4.16}$$

Moreover

Lemma 4.4 (from lemma 2.4 of [DR98]).
χ is Lipschitz continuous.

Proof. All the distances are bounded and Lipschitz continuous, so we just need to prove that the denominator in (4.15) is bounded away from zero.

If it were not so, we could get sequences $\{(t_n^i, u_n^i)\} \subseteq A_i$ $(i = 1, 2)$, with $|t_n^1 - t_n^2| \to 0$ and $\left\|u_n^1 - u_n^2\right\|_H \to 0$; but this gives

$$\begin{aligned}\eta \quad \leq \quad & J_{\alpha(t_n^2)}(u_n^2) - J_{\alpha(t_n^1)}(u_n^1) \leq \left| \left\|u_n^2\right\|_H^2 - \left\|u_n^1\right\|_H^2 \right| + \max_{\tau\in[0,1]}\{|\alpha(\tau)|\} \left| \left\|u_n^2\right\|_{L^2}^2 - \left\|u_n^1\right\|_{L^2}^2 \right| \\ & \leq 2\left| \left\|u_n^2\right\|_H - \left\|u_n^1\right\|_H \right| + 2\max_{\tau\in[0,1]}\{|\alpha(\tau)|\} \left| \left\|u_n^2\right\|_{L^2} - \left\|u_n^1\right\|_{L^2} \right| \leq C\left\|u_n^2 - u_n^1\right\|_H \to 0 \;:\end{aligned}$$

contradiction.

□

Now let

$$F_t(u) = \begin{cases} \chi(t, \frac{u}{\|u\|_H})\nabla_u J_{\alpha(t)}(u) & for \quad u \neq 0 \\ 0 & for \quad u = 0 \end{cases} \quad : \tag{4.17}$$

Lemma 4.5 (from lemma 2.5 of [DR98]).
F is locally Lipschitz continuous in the two variables $(t, u) \in [0, 1]\times H$ and there exists a constant $L > 0$ such that $\|F_t(u)\|_H \leq L\|u\|_H$ for all $t \in [0, 1]$, $u \in H$.

Proof. First note that both statements are true for the function $\nabla_u J_{\alpha(t)}(u)$ since it is linear in u and $\alpha(t) \in \mathcal{C}^1([0, 1])$:

- $\left\|\nabla_u J_{\alpha(t)}(u)\right\|_H = \sup_{\|v\|_H=1}\langle\nabla_u J_{\alpha(t)}(u), v\rangle \leq (1 + \max_{\tau\in[0,1]}\{|\alpha(\tau)|\})\|u\|_H = L_\nabla\|u\|_H$,
- let $\|u\|_H, \|v\|_H \leq R$ and $t, s \in [0, 1]$, then

$$\left\|\nabla_u J_{\alpha(t)}(u) - \nabla_v J_{\alpha(s)}(v)\right\|_H \leq \left(1 + \max_{\tau\in[0,1]}\{|\alpha(\tau)|\}\right)\|u - v\|_H + \max_{\tau\in[0,1]}\{|\alpha'(\tau)|\}R^2|t - s|\,.$$

Now let $0 < \|u\|_H \leq \|v\|_H \leq R$:
then $\left\|\frac{u}{\|u\|_H} - \frac{v}{\|v\|_H}\right\|_H \|u\|_H \leq \|u - v\|_H$
and so by lemma 4.4

$$\left|\chi\left(t, \frac{u}{\|u\|_H}\right) - \chi\left(s, \frac{v}{\|v\|_H}\right)\right| \leq C\left(\frac{\|u - v\|_H}{\|u\|_H} + |t - s|\right).$$

Then we evaluate

$$\begin{aligned}\|F_t(u) - F_s(v)\|_H \quad = \quad & \left\|\chi\left(t, \frac{u}{\|u\|_H}\right)\nabla_u J_{\alpha(t)}(u) - \chi\left(s, \frac{v}{\|v\|_H}\right)\nabla_u J_{\alpha(t)}(u) + \right. \\ & \left. +\chi\left(s, \frac{v}{\|v\|_H}\right)\nabla_u J_{\alpha(t)}(u) - \chi\left(s, \frac{v}{\|v\|_H}\right)\nabla_v J_{\alpha(s)}(v)\right\|_H \\ \leq \quad & \left|\chi\left(t, \frac{u}{\|u\|_H}\right) - \chi\left(s, \frac{v}{\|v\|_H}\right)\right| \left\|\nabla_u J_{\alpha(t)}(u)\right\|_H \\ & + \left\|\nabla_u J_{\alpha(t)}(u) - \nabla_v J_{\alpha(s)}(v)\right\|_H \\ \leq \quad & CL_\nabla\left(\frac{\|u - v\|_H}{\|u\|_H} + |t - s|\right)\|u\|_H + D(R)(\|u - v\|_H + |t - s|)\,.\end{aligned}$$

which implies that $F_t(u)$ is Lipschitz in sets where $\|u\|_H$ is bounded.

The case $u = 0$ is equivalent to $\|F_t(v)\|_H \leq L\,\|v\|_H$.

□

Now by the given properties of F it follows that (4.11) generates a continuous flow $\sigma_t(u)$ with the properties:

- $\sigma_t(0) = 0$ and $\sigma_t(u) \neq 0\ \forall u \neq 0$,
- $\forall t, \quad \sigma_t : H \to H$ is an homeomorphism.

Moreover

Lemma 4.6 (from lemma 2.6 of [DR98]).
Defining $\Theta_t(u) = \frac{J_{\alpha(t)}(\sigma_t(u))}{\|\sigma_t(u)\|_H^2}$, *we have that, fixing* u,
$\Theta_t(u)$ *is increasing (resp. decreasing) in the variable* t *in any interval* $[t_1, t_2]$ *such that*

$$\eta/2 \leq \Theta_t(u) \leq \eta,\ \forall t \in [t_1, t_2] \qquad \textit{(resp. } -\eta \leq \Theta_t(u) \leq -\eta/2,\ \forall t \in [t_1, t_2]\textit{)}.$$

Proof. Consider the case $\eta/2 \leq \Theta_t(u) \leq \eta$: then the flow is defined by

$$\frac{d}{dt}\sigma_t(u) = M\nabla_u J_{\alpha(t)}(\sigma_t(u)) \tag{4.18}$$

for all $t \in [t_1, t_2]$.

Then we have (we will omit the dependence on u in the notation)

$$\begin{aligned}
\frac{d\Theta_t}{dt} &= \frac{1}{\|\sigma_t\|_H^2}\left[\frac{\partial J_{\alpha(t)}(\sigma_t)}{\partial t} + \langle \nabla_u J_{\alpha(t)}(\sigma_t), \frac{d}{dt}\sigma_t\rangle_H\right] + J_{\alpha(t)}(\sigma_t)\frac{d}{dt}\left(\frac{1}{\|\sigma_t\|_H^2}\right) \\
&= \frac{1}{\|\sigma_t\|_H^2}\left[-\alpha^+(t)'\int_\Omega (\sigma_t^+)^2 - \alpha^-(t)'\int_\Omega (\sigma_t^-)^2 + \langle \nabla_u J_{\alpha(t)}(\sigma_t), M\nabla_u J_{\alpha(t)}(\sigma_t)\rangle_H\right] + \\
&\quad + \frac{\langle \nabla_u J_{\alpha(t)}(\sigma_t), \sigma_t\rangle_H}{2}\left(-\frac{2}{\|\sigma_t\|_H^4}\langle \sigma_t, \frac{d}{dt}\sigma_t\rangle_H\right) \\
&\geq -KS^2 + M\left(\frac{\|\nabla_u J_{\alpha(t)}(\sigma_t)\|_H^2}{\|\sigma_t\|_H^2} - \frac{\langle \nabla_u J_{\alpha(t)}(\sigma_t), \sigma_t\rangle_H^2}{\|\sigma_t\|_H^4}\right) \geq -KS^2 + M\delta_\eta\,.
\end{aligned}$$

By the choice made above $M > KS^2/\delta_\eta$, the proof of the first part is done.

In the case $-\eta \leq \Theta_t(u) \leq -\eta/2$ the proof follows the same ideas. □

Finally denote $\sigma_1(u)$ with $\tau_\alpha(u)$ (to remember its dependence on α), to obtain

Lemma 4.7 (from equation (2.9) and lemma (2.7) of [DR98]).

$$J_{\alpha(1)}(\tau_\alpha(u)) \leq -\eta\,\|\tau_\alpha(u)\|_H^2 \quad for\ all\ u \in V\ ; \tag{4.19}$$

$$J_{\alpha(1)}(\tau_\alpha(u)) \geq \eta\,\|\tau_\alpha(u)\|_H^2 \quad for\ all\ u \in W\ ; \tag{4.20}$$

$\forall R > 0$, $\tau_\alpha(W)$ links with $R\,\tau_\alpha(\partial B_V^k)$ where B_V^k is the unit ball, in the H-norm, of V.

Proof. Equations (4.19) and (4.20) follow easily from lemma 4.6.

For the linking property we need to prove that:
$\forall \gamma \in \Gamma = \{\gamma : R\tau_\alpha(B_V^k) \to H \text{ continuous and s.t. } \gamma(u) = u \text{ for } u \in R\tau_\alpha(\partial B_V^k)\}$, there exists a point $\bar{u} \in \gamma(R\tau_\alpha(B_V^k)) \cap \tau_\alpha(W)$.

We start by proving that

$$\xi\tau_\alpha(u) \neq \tau_\alpha(v) \tag{4.21}$$

for any $u \in \partial B_V^k$, $v \in W$ and $\xi > 0$: actually if it were not so, from equations (4.19) and (4.20) we would get $\eta \|\tau_\alpha(v)\|_H^2 \le J_{\alpha(1)}(\tau_\alpha(v)) = J_{\alpha(1)}(\xi\tau_\alpha(u)) = \xi^2 J_{\alpha(1)}(\tau_\alpha(u)) \le -\eta\xi^2 \|\tau_\alpha(u)\|_H^2$ which implies $u = v = 0$: contradiction since $u \in \partial B_V^k$.

Now define P to be the orthogonal projection of H onto V and consider the map $H_t = P \circ \tau_\alpha^{-1} \circ (1 + (R-1)t)\tau_\alpha$: property (4.21) implies that $H_t \neq 0$ on ∂B_V^k for any $t \in [0,1]$ and then $deg(H_1, B_V^k, 0) = deg(H_0, B_V^k, 0) = deg(Id, B_V^k, 0) = 1$.

Now for any $\gamma \in \Gamma$, $deg(P \circ \tau_\alpha^{-1} \circ \gamma \circ R\tau_\alpha, B_V^k, 0) = 1$ since on ∂B_V^k the function is exactly H_1, and then there is a point $p \in B_V^k$ such that $\gamma(R\tau_\alpha(p)) \in \tau_\alpha(W)$.

□

Finally we prove one more property that we will need later:

Lemma 4.8. *If $u \in V$ or $u \in W$, and $\xi > 0$ then $\tau_\alpha(\xi u) = \xi\tau_\alpha(u)$.*

Proof. From lemma 4.6 and 4.7 and equations (4.11) and (4.12) we have that in these two cases the equation just contains the gradient of $J_{\alpha(t)}$.

If we take $u \in V$, then the flow is defined by

$$\begin{cases} \frac{d}{dt}\sigma_t(u) = -M\nabla_u J_{\alpha(t)}(\sigma_t(u)) \\ \sigma_0(u) = u \in V \end{cases} . \tag{4.22}$$

Consider then the change of variable $\sigma = k\pi$ with $k > 0$: equation (4.22) becomes

$$\begin{cases} k\frac{d}{dt}\pi_t(u) = -M\nabla_u J_{\alpha(t)}(k\pi_t(u)) \\ k\pi_0(u) = u \in V \end{cases} \tag{4.23}$$

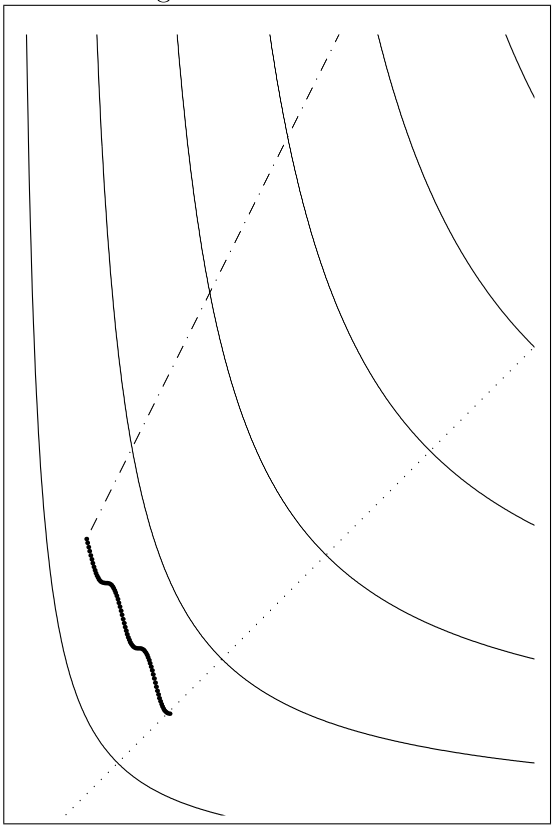
and considering the linear positive homogeneity of $\nabla_u J_{\alpha(t)}$ it can be simplified to obtain

$$\begin{cases} \frac{d}{dt}\pi_t(u) = -M\nabla_u J_{\alpha(t)}(\pi_t(u)) \\ \pi_0(u) = u/k \in V \end{cases} , \tag{4.24}$$

which is the same equation as (4.22) with a different initial condition: then $\sigma_t(u) = k\pi_t(u) = k\sigma_t(u/k)$.

The case $u \in W$ is treated in the same way. □

Figure 4: Theorem 1.1.



4.2 Construction of the variational characterization

Here we use the results of section 4.1 to obtain a variational characterization of some parts of the Fučík spectrum.

The result is the one stated in theorem 1.1, clearly the cases $\alpha^+ \leq \alpha^-$ and $r \in [1, +\infty)$ can be done in an equivalent way.

Note that in the one dimensional case, since the spectrum is known, (α^+, α^-) may be taken anywhere between the continuous curves arising from a point (λ_k, λ_k) and the ones arising from $(\lambda_{k+1}, \lambda_{k+1})$ (see figures 1 and 2 on pages 22 and 23).

In the multi dimensional case one has to be more careful, but Σ-connection may be assured at least for (α^+, α^-) in the square $(\lambda_{k-1}, \lambda_{k+m+1})^2$ (being $\lambda_{k-1} < \lambda_k = ... = \lambda_{k+m} < \lambda_{k+m+1}$) when it is not between (or on) the lower and the upper curve arising from (λ_k, λ_k) (see figure 3 on page 24).

In figure 4 we sketch graphically, in the one dimensional Neumann case, the meaning of theorem 1.1: the bold curve is $\alpha([0, 1])$ and the dashdotted half line is $\{(\alpha^+ + t, \alpha^- + rt),\, t > 0\}$ (we are considering λ^+ on the vertical axes and λ^- on the horizontal one).

We will obtain the characterization imitating that of λ_{k+1} described in section 2.3.1.

We fix a point (α^+, α^-) Σ-connected to the diagonal between λ_k and λ_{k+1} and with $\alpha^+ \geq \alpha^-$, then we apply the results of section 4.1 obtaining the deformation τ_α in H, we choose $r \in (0, 1]$, we split again $H = V \oplus W$ with $V = span\{\phi_1, .., \phi_k\}$ and we consider:

- The set
$$Q_r = \left\{ u \in H \quad s.t. \quad \int_\Omega (u^+)^2 + r(u^-)^2 = 1 \right\} . \tag{4.25}$$

- The radial projection on Q_r of the set obtained in section 4.1 by the deformation of ∂B_V^k, that is
$$L_{\alpha,r} = P^r(\tau_\alpha(\partial B_V^k)) , \tag{4.26}$$
where $P^r : u \mapsto \frac{u}{\sqrt{\int_\Omega (u^+)^2 + r\int_\Omega (u^-)^2}}$.

- The class of maps
$$\Gamma_{\alpha,r} = \{\gamma : B^k \to Q_r \;\; continuous \;\; s.t. \;\; \gamma|_{\partial B^k} \;\; is \;\; an \;\; homeomorphism \;\; onto \;\; L_{\alpha,r}\} , \tag{4.27}$$
where $B^k = \{(x_1..., x_k) \in \mathbb{R}^k \; s.t. \; \sum_{i=1}^k x_i^2 \leq 1\}$.

- The functional
$$J_\alpha(u) = \int_\Omega (\nabla u)^2 - \alpha^+ \int_\Omega (u^+)^2 - \alpha^- \int_\Omega (u^-)^2 \quad . \tag{4.28}$$

The idea now is to consider
$$d_{\alpha,r} = \inf_{\gamma \in \Gamma_{\alpha,r}} \sup_{u \in \gamma(B^k)} J_\alpha(u) \tag{4.29}$$

and to prove that it is assumed by a nontrivial solution of the Fučík problem (1.2), which then corresponds to a point in Σ.

We first have to prove that the above definitions are well posed and derive some properties of the defined sets:

Lemma 4.9. *For $u \in Q_r$ we have that $1 \leq \int_\Omega u^2 \leq 1/r$.*

Proof. $1 = \int_\Omega (u^+)^2 + r(u^-)^2 \leq \int_\Omega (u^+)^2 + (u^-)^2 = \int_\Omega u^2 \leq (\int_\Omega (u^+)^2 + r(u^-)^2)/r = 1/r$. □

Lemma 4.10.

(i) The set $L_{\alpha,r}$ is homeomorphic to ∂B^k.

(ii) $L_{\alpha,r} \subseteq \tau_\alpha(V)$.

Proof. (i) Since ∂B_V^k is homeomorphic to ∂B^k and τ_α is an homeomorphism, we just need to prove that P^r is an homeomorphism when restricted to $\tau_\alpha(\partial B_V^k)$.

τ_α on ∂B_V^k has the property (see lemma 4.8) that $\forall \xi > 0$, $\tau_\alpha(\xi u) = \xi \tau_\alpha(u)$, then P^r is one to one on $\tau_\alpha(\partial B_V^k)$ and so can be inverted.

Finally P^r is continuous together with its inverse because, since ∂B_V^k is a compact set which does not contain the origin, $\int_\Omega (u^+)^2 + r\int_\Omega (u^-)^2$ is continuous, bounded and bounded away from zero on it.

(ii) The second point is a trivial consequence of lemma 4.8.

□

Lemma 4.11. *$\tau_\alpha(W)$ links with $L_{\alpha,r}$.*

Proof. From lemma 4.7 $\tau_\alpha(W)$ links with $\tau_\alpha(\partial B^k_V)$.

Then the claim could be false only if for some $u \in L_{\alpha,r}$, $\xi > 0$, and $v \in \tau_\alpha(W)$ we had $\xi u = v$. But by the homogeneity property of τ_α in V and W (lemma 4.8) this would imply $\xi(\tau_\alpha)^{-1}(u) = (\tau_\alpha)^{-1}(v)$ and then $u = v = 0$, which is impossible since $u \in P^r(\tau_\alpha(B^k_V))$. □

In the next three lemmas are verified the conditions for the "Linking Theorem" which will imply the criticality of $d_{\alpha,r}$.

Lemma 4.12. *The functional $J_\alpha(u)$ constrained to Q_r satisfies the PS condition.*

Proof. Consider the sequences $\{u_n\} \subseteq Q_r$, $\{\beta_n\} \subseteq \mathbb{R}$ (Lagrange's multipliers) and $\varepsilon_n \to 0^+$ such that

$$\left|\int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2\right| \le C \tag{4.30}$$

$$\begin{aligned}\Bigg|\int_\Omega \nabla u_n \nabla v - \alpha^+ \int_\Omega (u_n^+) v + \alpha^- \int_\Omega (u_n^-) v + \\ + \beta_n \left(\int_\Omega u_n^+ v - r u_n^- v\right)\Bigg| \quad \le \quad \varepsilon_n \|v\|_H\,, \quad \forall v \in H\ .\end{aligned} \tag{4.31}$$

Since $\{u_n\} \subseteq Q_r$, it is a bounded sequence in L^2, and then equation (4.30) implies that it is also a bounded sequence in H. Then there exists a subsequence converging weakly in H and strongly in L^2 to some u.
The L^2 convergence implies that $u \in Q_r$.
Taking $v = u_n$ we get that

$$\beta_n + \left(\int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2\right) \to 0\,. \tag{4.32}$$

Then, with $v = u_n - u$ we have

$$\begin{aligned}&\int_\Omega \nabla u_n \nabla (u_n - u) - \alpha^+ \int_\Omega (u_n^+)(u_n - u) + \alpha^- \int_\Omega (u_n^-)(u_n - u) + \\ &- \left(\int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2\right)\left(\int_\Omega (u_n^+ - r u_n^-)(u_n - u)\right) \to 0,\end{aligned}$$

where all terms except the first go to zero. Then we conclude that $\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$ and then $u_n \to u$ strongly in H. □

Lemma 4.13. $\sup_{u \in \gamma(\partial B^k)} J_\alpha(u) \le 0 \quad \forall \gamma \in \Gamma_{\alpha,r}$.

Proof. By lemma 4.7, since $\gamma(\partial B^k) = L_{\alpha,r} \subseteq \tau_\alpha(V)$ and then $J_\alpha(u) \le -\eta \|u\|_H^2 < 0$. □

Lemma 4.14. $+\infty > \sup_{u \in \gamma(B^k)} J_\alpha(u) \ge \eta > 0$ *for each* $\gamma \in \Gamma_{\alpha,r}$.

Proof. By lemma 4.11 there is always a point $u \in \gamma(B^k) \cap \tau_\alpha(W)$, and by lemma 4.7 we have in that point $J_\alpha(u) \ge \eta \|u\|_H^2$; considering lemma 4.9 and that $u \in Q_r$, this becomes $\ge \eta$.

Finally it is less than $+\infty$ since each $\gamma(B^k)$ is a compact set.

□

At this point we can state the following standard "Linking Theorem"

Proposition 4.15. *The level $d_{\alpha,r} \geq \eta > 0$ is a critical value for $J_\alpha(u)$ constrained to Q_r.*

The importance of the criticality of the level $d_{\alpha,r}$ is clarified in the following proposition:

Proposition 4.16. *The critical points associated to the critical value $d_{\alpha,r}$ are non trivial solutions of the Fučík problem (1.2) with coefficients (λ^+, λ^-), where $\lambda^+ - \alpha^+ = d_{\alpha,r}$ and $\lambda^- - \alpha^- = rd_{\alpha,r}$.*

Proof. Criticality of u implies that there exists a Lagrange's multiplier $\beta \in \mathbb{R}$ such that

$$\int_\Omega \nabla u \nabla v - \alpha^+ \int_\Omega (u^+)v + \alpha^- \int_\Omega (u^-)v + \beta \left(\int_\Omega u^+ v - r u^- v \right) = 0 \quad \forall v \in H \ , \tag{4.33}$$

but testing against u we get $\beta = -d_{\alpha,r}$ and so u solves

$$-\Delta u = \alpha^+ u^+ - \alpha^- u^- + d_{\alpha,r} u^+ - d_{\alpha,r} r u^- = (\alpha^+ + d_{\alpha,r})u^+ - (\alpha^- + rd_{\alpha,r})u^- \tag{4.34}$$

in Ω, with the considered boundary conditions.
Finally u is not trivial since it is in Q_r. □

Proposition 4.15 and 4.16 imply that the point $(\alpha^+ + d_{\alpha,r},\ \alpha^- + rd_{\alpha,r})$ belongs to the halfline $\{(\alpha^+ + t, \alpha^- + rt),\, t > 0\}$ (since $d_{\alpha,r} > 0$) and also to the Fučík spectrum; thus theorem 1.1 is proved.

Up to this point it is not clear whether this point corresponds to the first intersection (that is the one with smallest t) of the halfline with Σ.
However this is the case when the problem is linear, that is for $\alpha^+ = \alpha^-$ and $r = 1$:

Lemma 4.17. *If $\alpha = (a, a)$ with $a \in (\lambda_k, \lambda_{k+1})$ and $r = 1$, then $d_{\alpha,r} = \lambda_{k+1} - a$, that is the characterized point $(\alpha^+ + d_{\alpha,r},\ \alpha^- + rd_{\alpha,r})$ is indeed $(\lambda_{k+1}, \lambda_{k+1})$.*

Proof. We just have to exhibit a map $\widehat{\gamma} \in \Gamma_{(a,a),1}$ such that $\sup_{u \in \widehat{\gamma}(B^k)} J_{(a,a)}(u) = \lambda_{k+1} - a$.
Note that in this case Q_1 is the boundary of the unit ball in norm L^2, $L_{(a,a),1}$ is simply $V \cap Q_1$ and $J_{(a,a)}(u) = \int_\Omega (\nabla u)^2 - a \int_\Omega u^2$.
Then if we consider the map

$$\widehat{\gamma} : B^k \to Q_1 : (x_1, ..., x_k) \mapsto \sum_1^k x_i \phi_i + \left(1 - \sum_1^k x_i^2 . \right)^{\frac{1}{2}} \phi_{k+1} \tag{4.35}$$

we have:

- $\widehat{\gamma} \in \Gamma_{(a,a),1}$ since for $\sum_1^k x_i^2 = 1$ one has $\widehat{\gamma}(x_1, ..., x_k) \in V \cap Q_1$,
- $J_{(a,a)}(u)|_{\widehat{\gamma}(B^k)} \leq \lambda_{k+1} - a$, since $\widehat{\gamma}(B^k) \subseteq span\{\phi_1, .., \phi_{k+1}\} \cap Q_1$.

□

4.3 Properties of the variational characterization.

Here we want to prove some properties of the variational characterization obtained in the previous section.

We will make use of the continuity of the deformation τ_α with respect to the variable α and of the projection P_r with respect to the variable r to prove the continuity of $d_{\alpha,r}$; then since when $\alpha^+ = \alpha^-$ and $r = 1$ the characterization was proven to give the eigenvalue λ_{k+1} we will obtain:

Proposition 4.18. *Having fixed $r \in (0,1]$ and α as in definition 4.1, the point in the Fučík spectrum determined by the variational characterization in theorem 1.1, that is $\alpha(1)+(d_{\alpha(1),r}, rd_{\alpha(1),r})$, lies in a continuum of Σ which contains the point $\alpha(0) + (d_{\alpha(0),1}, d_{\alpha(0),1})$, that is $(\lambda_{k+1}, \lambda_{k+1})$.*

Moreover, through the monotonicity of the projection P_r with respect to the variable r, we will prove:

Proposition 4.19. *Having fixed α as in definition 4.1, the curves in $\mathbb{R}^2$: $(\alpha^+ + d_{\alpha,r}, \alpha^- + rd_{\alpha,r})$ with $r \in (0,1]$, are non increasing.*

Actually $s > r$ implies $d_{\alpha,r} \geq d_{\alpha,s}$ and $rd_{\alpha,r} \leq sd_{\alpha,s}$.

4.3.1 Continuity

First note that looking at the definitions in equations (4.26) and (4.27), it is clear that the projection map: $P_r^s : Q_r \to Q_s : u \mapsto \frac{u}{\sqrt{\int_\Omega (u^+)^2 + s\int_\Omega (u^-)^2}}$ gives a one to one relation between the elements of the two families $\Gamma_{\alpha,r}$ and $\Gamma_{\alpha,s}$:

$$\tilde{P}_r^s : \Gamma_{\alpha,r} \to \Gamma_{\alpha,s} : \gamma \mapsto P_r^s \circ \gamma. \tag{4.36}$$

Now we assert:

Lemma 4.20. *Having fixed α as in definition 4.1, the function of r: $d_{\alpha,r} : (0,1] \to \mathbb{R}$ is continuous.*

Proof. If we consider a sequence $r_n \to r$, with $r_n, r \in (0,1]$, then we want to prove that for any subsequence there exists a further subsequence such that $d_{\alpha,r_n} \to d_{\alpha,r}$.

Having fixed the subsequence, up to a further subsequence, $\exists c \in [0,+\infty]$ such that $d_{\alpha,r_n} \to c$: we will prove that $c = d_{\alpha,r}$.

- Claim: $d_{\alpha,r} = \inf_{\gamma\in\Gamma_{\alpha,r}} \sup_{u\in\gamma(B^k)} J_\alpha(u) \geq c$.

  Let us suppose that, contrary to the claim, there exists $\gamma \in \Gamma_{\alpha,r}$ such that

$$d = \sup_{u\in\gamma(B^k)} J_\alpha(u) < c\,. \tag{4.37}$$

  Then consider

$$\sup_{u\in P_r^{r_n}\circ\gamma(B^k)} J_\alpha(u) = \sup_{u\in\gamma(B^k)} \frac{J_\alpha(u)}{\int_\Omega (u^+)^2 + r_n \int_\Omega (u^-)^2} \;: \tag{4.38}$$

since $u \in Q_r$, the denominator in (4.38) is $1+(r_n-r)\int(u^-)^2$ and then is bounded between 1 and $1+(r_n-r)/r$, which tends to 1 for $n \to +\infty$; then, since $P_r^{r_n} \circ \gamma \in \Gamma_{\alpha,r_n}$, for any $\varepsilon > 0$ we could find $\bar{n}$ such that for $n > \bar{n}$:

$$d_{\alpha,r_n} \le \sup_{u \in P_r^{r_n} \circ \gamma(B^k)} J_\alpha(u) < d + \varepsilon \tag{4.39}$$

which, if we choose ε such that $d+\varepsilon < c$, contradicts that $d_{\alpha,r_n} \to c$.

- Claim: $d_{\alpha,r} \le c$.

  Let us suppose that, contrary to the claim,

$$d_{\alpha,r} = \inf_{\gamma \in \Gamma_{\alpha,r}} \sup_{u \in \gamma(B^k)} J_\alpha(u) > c\,. \tag{4.40}$$

  Then for any $\varepsilon > 0$ we could find $\bar{n}$ such that for all $n > \bar{n}$:

  - $d_{\alpha,r_n} < c + \varepsilon$ (since $d_{\alpha,r_n} \to c$),
  - there exists a $\gamma_{\varepsilon,n} \in \Gamma_{\alpha,r_n}$ such that

$$\sup_{u \in \gamma_{\varepsilon,n}(B^k)} J_\alpha(u) < c + 2\varepsilon\,. \tag{4.41}$$

  Then consider (for each one of these n)

$$\sup_{u \in P_{r_n}^r \circ \gamma_{\varepsilon,n}(B^k)} J_\alpha(u) = \sup_{u \in \gamma_{\varepsilon,n}(B^k)} \frac{J_\alpha(u)}{\int_\Omega (u^+)^2 + r\int_\Omega (u^-)^2} \ : \tag{4.42}$$

  as before (since $r_n \to r$) we can find $\bar{n}_2$ such that for $n > \bar{n}_2$:

$$\sup_{u \in P_{r_n}^r \circ \gamma_{\varepsilon,n}(B^k)} J_\alpha(u) < c + 3\varepsilon\,, \tag{4.43}$$

  but this, since $P_{r_n}^r \circ \gamma_{\varepsilon,n} \in \Gamma_{\alpha,r}$, is in contradiction with the definition of $d_{\alpha,r}$ if we choose ε such that $c + 3\varepsilon < d_{\alpha,r}$.

$\square$

Now note that the properties of the homeomorphisms $\tau_\alpha = \sigma_1 : H \to H$ obtained in section 4.1, hold also for σ_t at any $t \in [0,1]$, that is lemma 4.7 and 4.8 are still valid using σ_t and $J_{\alpha(t)}$ in place of τ_α and $J_{\alpha(1)}$.

Then we can think to make the variational characterization in each point along the curve $\alpha(t)$ obtaining the corresponding critical values $d_{\alpha(t),r}$.

Now we want to prove:

Lemma 4.21. *Having fixed $r = 1$ and the path $\alpha(t)$ as in definition 4.1, the function of t, $d_{\alpha(t),1} : [0,1] \to \mathbb{R}$ is continuous.*

Before giving the proof we need some preliminary lemmas; hereafter we will make estimates using some constants which will all be denoted by C.

First we need the following estimate for the solution of problem (4.11):

Lemma 4.22. *There exists a constant C such that*

$$\|\sigma_t(u) - \sigma_s(u)\|_H \leq C \|u\|_H \ |t-s| \tag{4.44}$$

and

$$\left\|u - \sigma_s \circ \sigma_t^{-1}(u)\right\|_H \leq C \|u\|_H \ |t-s| \tag{4.45}$$

for any $u \in H$ and $t, s \in [0,1]$.

Proof. Taking the norm of the differential equation in (4.11) gives

$$\left\|\frac{d}{dt}\sigma_t(u)\right\|_H = M \|F_t(\sigma_t(u))\|_H \tag{4.46}$$

and so

$$\|\sigma_t(u) - \sigma_s(u)\|_H \leq M \sup_{\tau\in[0,1]} \| F_\tau(\sigma_\tau(u))\|_H \ |t-s| \,. \tag{4.47}$$

Taking the scalar product with $\sigma_t(u)$ we get

$$\left|\frac{d}{dt} \|\sigma_t(u)\|_H^2\right| = |2M \langle F_t(\sigma_t(u)), \sigma_t(u)\rangle| \leq 2M \|F_t(\sigma_t(u))\|_H \|\sigma_t(u)\|_H \tag{4.48}$$

so that using the estimate $\|F_t(u)\|_H \leq L \|u\|_H$ we arrive at

$$-2ML \|\sigma_t(u)\|_H^2 \leq \frac{d}{dt} \|\sigma_t(u)\|_H^2 \leq 2ML \|\sigma_t(u)\|_H^2 \ , \tag{4.49}$$

which implies by Gronwall's lemma

$$e^{-MLt} \|\sigma_0(u)\|_H \leq \|\sigma_t(u)\|_H \leq e^{MLt} \|\sigma_0(u)\|_H \ . \tag{4.50}$$

Finally

$$\sup_{\tau\in[0,1]} \| F_\tau(\sigma_\tau(u))\|_H \leq L \sup_{\tau\in[0,1]} \|\sigma_\tau(u)\|_H \leq Le^{ML} \|u\|_H \tag{4.51}$$

and so from (4.47)

$$\|\sigma_t(u) - \sigma_s(u)\|_H \leq MLe^{ML} \|u\|_H \ |t-s| \,. \tag{4.52}$$

For the second inequality one simply uses the first one with $\sigma_t^{-1}(u)$ in place of u obtaining

$$\left\|u - \sigma_s \circ \sigma_t^{-1}(u)\right\|_H \leq C \left\|\sigma_t^{-1}(u)\right\|_H \ |t-s| \tag{4.53}$$

and then use the left part of equation (4.50) with $\sigma_t^{-1}(u)$ in place of u (remember that $\sigma_0(u) = u$) to obtain

$$\left\|\sigma_t^{-1}(u)\right\|_H \leq e^{ML} \|u\|_H \ . \tag{4.54}$$

□

Note that we also have the estimates

$$\begin{aligned} \left| \|\sigma_t(u)\|_{L^2} - \|\sigma_s(u)\|_{L^2} \right| &\leq \|\sigma_t(u) - \sigma_s(u)\|_{L^2} \leq \|\sigma_t(u) - \sigma_s(u)\|_H \\ &\leq C\,\|u\|_H\,|t-s| \end{aligned} \tag{4.55}$$

and

$$\left| \|u\|_{L^2} - \left\|\sigma_s \circ \sigma_t^{-1}(u)\right\|_{L^2} \right| \leq C\,\|u\|_H\,|t-s|\,. \tag{4.56}$$

Now note, considering again the definitions in equations (4.26) and (4.27), that given $t, s \in [0,1]$ we may define a one to one correspondence between $\Gamma_{\alpha(t),1}$ and $\Gamma_{\alpha(s),1}$, considering the map

$$T_t^s : \Gamma_{\alpha(t),1} \to \Gamma_{\alpha(s),1} : \gamma_t \mapsto P^1 \circ \sigma_s \circ \sigma_t^{-1} \circ \gamma_t\,, \tag{4.57}$$

where $P^1 : H \to Q_1 : u \mapsto \frac{u}{\|u\|_{L^2}}$.

Moreover we define the map

$$S_t^s : Q_1 \to \mathbb{R} : u \mapsto J_{\alpha(s)}\left(P^1 \circ \sigma_s \circ \sigma_t^{-1}(u)\right) - J_{\alpha(t)}(u)\,. \tag{4.58}$$

Lemma 4.23. *Having fixed a constant $D > 1$, let $A_D = \{u \in Q_1$ such that $\|u\|_H \leq D\}$, then for any $t \in [0,1]$ fixed, we have $S_t^t(u) = 0$ and $\lim_{s\to t} S_t^s(u) = 0$ uniformly in A_D.*

Proof. $S_t^t(u) = 0$ is trivial by the definition.

Now let $u \in A_D$ and write

$$\left\|u - P^1 \circ \sigma_s \circ \sigma_t^{-1}(u)\right\|_H \leq \left\|u - \sigma_s \circ \sigma_t^{-1}(u)\right\|_H + \left\|\sigma_s \circ \sigma_t^{-1}(u) - P^1 \circ \sigma_s \circ \sigma_t^{-1}(u)\right\|_H\;: \tag{4.59}$$

- the first part is estimated in (4.45), from which

$$\left\|u - \sigma_s \circ \sigma_t^{-1}(u)\right\|_H \leq CD|t-s| \quad for \quad u \in A_D\,; \tag{4.60}$$

- the second part is

$$\left\|\sigma_s \circ \sigma_t^{-1}(u)\right\|_H \left|1 - \frac{1}{\left\|\sigma_s \circ \sigma_t^{-1}(u)\right\|_{L^2}}\right|\,, \tag{4.61}$$

  where

  - using (4.50) and arguing as for equation (4.54),

$$\left\|\sigma_s \circ \sigma_t^{-1}(u)\right\|_H \leq e^{LM}\left\|\sigma_t^{-1}(u)\right\|_H \leq e^{2LM}\,\|u\|_H \leq CD \quad for \quad u \in A_D\,; \tag{4.62}$$

  - using that $|1 - \frac{1}{x}| \leq 2|x-1|$ for $|x-1| < \frac{1}{2}$, since $1 = \|u\|_{L^2}$ and using equation (4.56) we get

$$\left|1 - \frac{1}{\left\|\sigma_s \circ \sigma_t^{-1}(u)\right\|_{L^2}}\right| \leq 2\left|\,\left\|\sigma_s \circ \sigma_t^{-1}(u)\right\|_{L^2} - \|u\|_{L^2}\,\right| \leq 2C\,\|u\|_H\,|t-s| \leq 2CD|t-s| \tag{4.63}$$

    for $|t-s| < \frac{1}{2CD}$ and $u \in A_D$.

So joining (4.60), (4.62) and (4.63) we get that

$$\left\|u - P^1 \circ \sigma_s \circ \sigma_t^{-1}(u)\right\|_H \le C_1 D^2 |t-s| \quad for \quad u \in A_D \quad and \quad |t-s| < \frac{1}{2CD}\,. \tag{4.64}$$

Moreover we deduce from (4.64) that

$$P^1 \circ \sigma_s \circ \sigma_t^{-1}(u) \in A_{D+1} \quad if \quad u \in A_D \quad and \quad |t-s| < \frac{1}{C_1 D^2}\,. \tag{4.65}$$

Finally $J_{\alpha(t)}(u)$ is

- continuous in the variable u for any fixed t, uniformly if we consider it in A_{D+1} (actually $|J_{\alpha(t)}(u) - J_{\alpha(t)}(v)| \le |\int \nabla(u+v)\nabla(u-v)| + \max\{\alpha^+(t), \alpha^-(t)\}|\int (u+v)(u-v)| \le (1+\max\{\alpha^+(t),\alpha^-(t)\})2(D+1)\,\|u-v\|_H$);
- uniformly continuous in the variable t for any $u \in Q_1$ (actually $|J_{\alpha(t)}(u) - J_{\alpha(s)}(u)| \le \max_{t\in[0,1]}\{|\alpha^+(t)'| + |\alpha^-(t)'|\}\,\|u\|_{L^2}\,|t-s|$ where $\|u\|_{L^2} = 1$);

then $J_{\alpha(t)}(u)$ is uniformly continuous in $[0,1] \times A_{D+1}$ and then

$$S_t^s(u) \le C_2(|t-s| + \left\|u - P^1 \circ \sigma_s \circ \sigma_t^{-1}(u)\right\|_H) \le C_2(1 + C_1 D^2)|t-s| \tag{4.66}$$

provided $|t-s| < \min\{\frac{1}{C_1 D^2}, \frac{1}{2CD}\}$ and $u \in A_D$. □

Now we can give

Proof of lemma 4.21.
If we consider a sequence $t_n \to t$, with $t_n, t \in [0,1]$, then we want to prove that for any subsequence there exists a further subsequence such that $d_{\alpha(t_n),1} \to d_{\alpha(t),1}$.

Having fixed the subsequence, up to a further subsequence, $\exists c \in [0, +\infty]$ such that $d_{\alpha(t_n),1} \to c$: we will prove that $c = d_{\alpha(t),1}$.

- Claim: $d_{\alpha(t),1} = \inf_{\gamma\in\Gamma_{\alpha(t),1}} \sup_{u\in\gamma(B^k)} J_{\alpha(t)}(u) \ge c$.

  Let us suppose that, contrary to the claim, there exists $\gamma \in \Gamma_{\alpha(t),1}$ such that

$$d = \sup_{u\in\gamma(B^k)} J_{\alpha(t)}(u) < c\,. \tag{4.67}$$

  This implies that $\|u\|_H$ is bounded in $\gamma(B^k)$, that is $\gamma(B^k) \subseteq A_D$ for a suitable D.

  Then consider

$$\sup_{u\in T_t^{t_n}\circ\gamma(B^k)} J_{\alpha(t_n)}(u) = \sup_{u\in\gamma(B^k)} J_{\alpha(t_n)}(P^1 \circ \sigma_{t_n} \circ \sigma_t^{-1}(u)) = \sup_{u\in\gamma(B^k)} (J_{\alpha(t)}(u) + S_t^{t_n}(u)) \ : \tag{4.68}$$

  using lemma 4.23 we would get, since $T_t^{t_n} \circ \gamma(B^k) \in \Gamma_{\alpha(t_n),1}$, that for any $\varepsilon > 0$ we could find $\bar{n}$ such that for $n > \bar{n}$:

$$d_{\alpha(t_n),1} \le \sup_{u\in T_t^{t_n}\circ\gamma(B^k)} J_{\alpha(t_n)}(u) < d + \varepsilon \tag{4.69}$$

  which, if we choose ε such that $d + \varepsilon < c$, contradicts that $d_{\alpha(t_n),1} \to c$.

- Claim: $d_{\alpha(t),1} \leq c$.

  Let us suppose that, contrary to the claim, $d_{\alpha(t),1} = \inf_{\gamma\in\Gamma_{\alpha(t),1}} \sup_{u\in\gamma(B^k)} J_{\alpha(t)}(u) > c$. Then for any $\varepsilon > 0$ we could find $\bar{n}$ such that for all $n > \bar{n}$:

  - $d_{\alpha(t_n),1} < c + \varepsilon$ (since $d_{\alpha(t_n),1} \to c$),
  - there exists a $\gamma_{\varepsilon,n} \in \Gamma_{\alpha(t_n),1}$ such that

  $$\sup_{u\in\gamma_{\varepsilon,n}(B^k)} J_{\alpha(t_n)}(u) < c + 2\varepsilon\,. \tag{4.70}$$

  Again this implies that for a suitable D, $\gamma_{\varepsilon,n}(B^k) \subseteq A_D$ for all $n > \bar{n}$.

  Then consider (for each one of these n)

  $$\sup_{u\in T^t_{t_n}\circ\gamma_{\varepsilon,n}(B^k)} J_{\alpha(t)}(u) = \sup_{u\in\gamma_{\varepsilon,n}(B^k)} J_{\alpha(t)}(P^1\circ\sigma_t\circ\sigma_{t_n}^{-1}(u)) = \sup_{u\in\gamma_{\varepsilon,n}(B^k)} (J_{\alpha(t_n)}(u) + S^t_{t_n}(u))\ : \tag{4.71}$$

  as before (since $t_n \to t$) by lemma 4.23 we could find $\bar{n}_2$ such that for $n > \bar{n}_2$:

  $$\sup_{u\in T^t_{t_n}\circ\gamma_{\varepsilon,n}(B^k)} J_{\alpha(t)}(u) < c + 3\varepsilon\,; \tag{4.72}$$

  but this, since $T^t_{t_n}\circ\gamma_{\varepsilon,n} \in \Gamma_{\alpha(t),1}$, is in contradiction with the definition of $d_{\alpha(t),1}$ if we choose ε such that $c + 3\varepsilon < d_{\alpha(t),1}$.

□

Joining the previous lemmas we may conclude:

Proof of proposition 4.18.
One just uses lemma 4.17, 4.20 and 4.21, considering first to move along $\alpha([0,1])$ with $r = 1$ fixed and then to change r with $\alpha = \alpha(1)$ fixed.

□

4.3.2 Monotonicity

Proof of proposition 4.19.
By the infsup characterization we have that, for any $\varepsilon > 0$, there exists $\gamma_\varepsilon \in \Gamma_{\alpha,r}$ such that

$$d_{\alpha,r} \leq z_r = \sup_{u\in\gamma_\varepsilon(B^k)} J_\alpha(u) \leq d_{\alpha,r} + \varepsilon\,; \tag{4.73}$$

now for $s > r$ we get (since $P^s_r\circ\gamma_\varepsilon \in \Gamma_{\alpha,s}$)

$$d_{\alpha,s} \leq z_s = \sup_{u\in P^s_r\circ\gamma_\varepsilon(B^k)} J_\alpha(u) = \sup_{u\in\gamma_\varepsilon(B^k)} \frac{J_\alpha(u)}{1+(s-r)\int(u^-)^2}\,; \tag{4.74}$$

since for $u \in Q_r$, $\frac{r}{s} = \frac{1}{1+\frac{s-r}{r}} \leq \frac{1}{1+(s-r)\int(u^-)^2} \leq 1$, we get $z_r\frac{r}{s} \leq z_s \leq z_r$.

Then we conclude $d_{\alpha,s} \leq z_s \leq z_r \leq d_{\alpha,r} + \varepsilon$ for any $\varepsilon > 0$, that is $d_{\alpha,s} \leq d_{\alpha,r}$.

Now do the converse: by the inf sup characterization we have that, for any $\varepsilon > 0$, there exists $\gamma_\varepsilon \in \Gamma_{\alpha,s}$ such that

$$d_{\alpha,s} \leq w_s = \sup_{u \in \gamma_\varepsilon(B^k)} J_\alpha(u) \leq d_{\alpha,s} + \varepsilon\,; \tag{4.75}$$

now for $s > r$ we get (since $P_s^r \circ \gamma_\varepsilon \in \Gamma_{\alpha,r}$)

$$d_{\alpha,r} \leq w_r = \sup_{u \in P_s^r \circ \gamma_\varepsilon(B^k)} J_\alpha(u) = \sup_{u \in \gamma_\varepsilon(B^k)} \frac{J_\alpha(u)}{1 + (r-s)\int (u^-)^2}\,; \tag{4.76}$$

since for $u \in Q_s$, $\quad 1 \leq \frac{1}{1+(r-s)\int (u^-)^2} \leq \frac{1}{1+\frac{r-s}{s}} = \frac{s}{r}$, we get now $w_s \leq w_r \leq w_s \frac{s}{r}$.

Then we conclude $r\, d_{\alpha,r} \leq r\, w_r \leq s\, w_s \leq s(d_{\alpha,s} + \varepsilon)$ for any $\varepsilon > 0$, that is $r\, d_{\alpha,r} \leq s\, d_{\alpha,s}$. $\square$

5 The superlinear problem

Now, as announced in the introduction, we will consider the Sturm-Liouville equation in dimension one with Neumann boundary conditions:

$$\begin{cases} -u'' = \lambda u + g(x,u) + h(x) \quad in \ (0,1) \\ u'(0) = u'(1) = 0 \end{cases}, \tag{1.6}$$

where

$$\begin{aligned} & g \in \mathcal{C}^0([0,1]\times\mathbb{R})\,, \\ & \lim_{s\to-\infty} \tfrac{g(x,s)}{s} = 0, \quad \lim_{s\to+\infty} \tfrac{g(x,s)}{s} = +\infty \end{aligned} \tag{H1}$$

uniformly with respect to $x \in [0,1]$, and $h \in L^2(0,1)$.

We will compare it to the Fučík problem

$$\begin{cases} -u'' = \lambda^+ u^+ - \lambda^- u^- \quad in \ (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{1.7}$$

and, taking advantage of the fact that in the one dimensional case the Fučík spectrum may be exactly calculated, we will prove existence results for problem (1.6). The proof uses the variational characterization of the previous section to make a comparison of these minimax levels with those of the functional associated to problem (1.6), in order to prove the existence of a linking structure for this last functional.

Some hypotheses on the growth at infinity of the nonlinearity g will be needed to obtain the PS condition for the functional associated to problem (1.6): defining $G(x,s) = \int_0^s g(x,\xi)d\xi$, we ask

$$\exists\theta \in \left(0, \frac{1}{2}\right), \quad s_0 > 0 \quad s.t. \quad 0 < G(x,s) \leq \theta s g(x,s) \qquad \forall s > s_0\,; \tag{H2}$$

$$\exists s_1 > 0,\ C_0 > 0 \quad s.t. \quad G(x,s) \leq \frac{1}{2} s g(x,s) + C_0 \qquad \forall s < -s_1\,. \tag{H3}$$

For certain "resonant" values of λ the following hypothesis will be needed as well:

$$\exists\rho_0 > 0,\ M_0 \in \mathbb{R} \quad s.t. \quad G(x,s) + h(x)s \leq M_0 \quad a.e.\ x \in [0,1],\ \forall s < -\rho_0\,. \tag{HR}$$

The exact statement of the results is this:

Theorem. 1.2. *Under hypotheses (H1), (H2) and (H3), if $\lambda \in (\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4})$ for some $k \geq 1$, then there exists a solution of problem (1.6) for all $h \in L^2(0,1)$.*

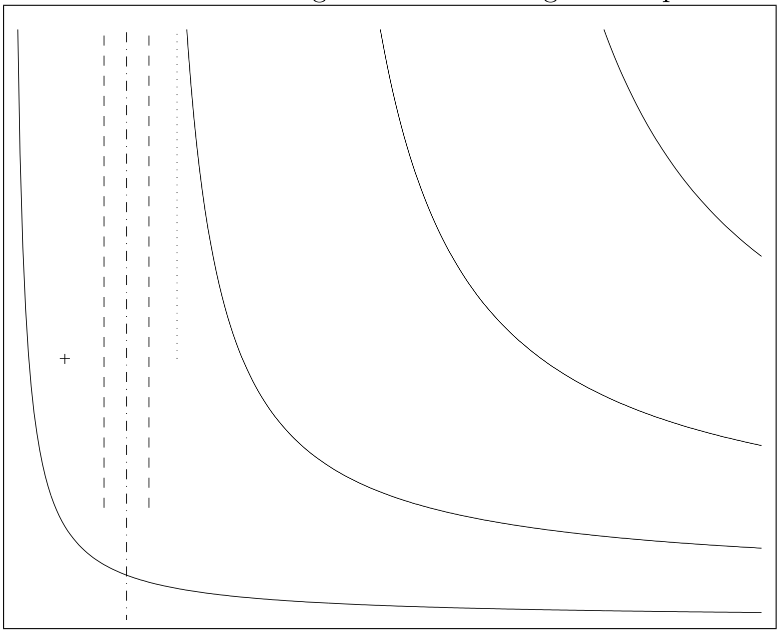
Theorem. 1.3. *Under hypotheses (H1), (H2), (H3) and (HR), with $h \in L^2(0,1)$, $\lambda = \frac{\lambda_{k+1}}{4}$ for some $k \geq 1$, then there exists a solution of problem (1.6).*

Remark 5.1. *The hypotheses (H1) to (H3) are satisfied for example by the function $g(x,s) = e^s$; in this case, in order to satisfy (HR) we will also need $h(x) \geq 0$ a.e.*

Another example of a nonlinearity satisfying also (HR) and where there is some more freedom on h, is when g behaves at $-\infty$ as $|s|^\delta$ with $\delta \in (0,1)$, so that h may be chosen arbitrarily in $L^\infty(0,1)$.

Remark 5.2. *Observe that here again the result in theorem 1.2 may be interpreted as the limiting asymptotically linear problem with coefficients between Σ_k and Σ_{k+1} when $\lambda^+ \to +\infty$; actually these problems too have a solution for arbitrary $h \in L^2$.*

Figure 5: The setting for the proof of theorem 1.2.



$$\begin{cases} + & : \ (\alpha^+, \alpha^-) \\ -\cdot- & : \ \{\lambda^- = \lambda\} \\ --- & : \ \{\lambda^- = \lambda \pm \delta\} \\ \cdots\cdots & : \ \{\lambda^- = \frac{\lambda_{k+1}}{4}\} \end{cases}$$

5.1 Proof of theorem 1.2

Consider the superlinear problem (1.6): the idea here is to prove the existence of a non constrained critical point of the functional

$$F(u) = \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda}{2}\int_0^1 u^2 - \int_0^1 G(x,u) - \int_0^1 hu \ , \tag{5.1}$$

which corresponds to a solution of the problem.

We will follow a strategy inspired by [dFR93].

A key role in the proof will be played by the fact that $H^1(0,1) \subseteq \mathcal{C}^0([0,1])$ with compact inclusion (see, later, the estimate in equation (5.10)); moreover recall that in this case the asymptotes of each Σ_k with $k \geq 2$ are at $\lambda^- = \frac{\lambda_k}{4}$ and that Σ_k lies entirely in $\lambda^- > \frac{\lambda_k}{4}$ (see figure 2 on page 23).

This structure of Σ implies that, having fixed $\lambda \in (\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4})$, $k \geq 1$, it is always possible to find:

- a point (α^+, α^-) Σ-connected to the diagonal between λ_k and λ_{k+1} and such that $\alpha^- < \lambda$,
- a $\delta > 0$ such that $\alpha^- < \lambda - \delta$ and $\lambda + \delta < \frac{\lambda_{k+1}}{4}$.

This construction is sketched in figure 5.

Now, using the notation of section 4.2, we define, for $R > 0$, the family of maps

$$\Gamma^R_{\alpha,\bar{r}} = \{\gamma^* : B^k \to H \ \ continuous\ \ s.t.\ \ \gamma^*|_{\partial B^k}\ \ is\ \ an\ \ homeomorphism\ \ onto\ \ RL_{\alpha,\bar{r}}\} \ . \tag{5.2}$$

We want to prove that, for a suitable $R > 0$, the level

$$f = \inf_{\gamma^* \in \Gamma^R_{\alpha,\bar{r}}} \sup_{u \in \gamma^*(B^k)} F(u) \tag{5.3}$$

is a critical value for the functional F.

Remark 5.3. *In the definition of $\Gamma^R_{\alpha,\bar{r}}$ the choice of $\bar{r} \in (0,1]$ has no importance: it can be chosen arbitrarily.*

Since $h \in L^2$ and using hypothesis (H1), we can find constants C_1, C_2 and C_3 as follows:

- $C_1(\delta, h)$ such that
$$\left| \int_0^1 hu \right| \leq \frac{\delta}{4} \|u\|^2_{L^2} + C_1(\delta, h)\,; \tag{5.4}$$
- $C_2(\delta, g)$ such that
$$\left| \int_0^1 G(x, -u^-) \right| \leq \frac{\delta}{4} \|u\|^2_{L^2} + C_2(\delta, g)\,; \tag{5.5}$$
- for any M, $C_3(M, g)$ such that
$$\int_0^1 G(x, u^+) \geq \frac{M}{2} \left\|u^+\right\|^2_{L^2} - C_3(M, g)\,. \tag{5.6}$$

To find a Generalized Mountain Pass structure we first need

Lemma 5.4. *$\forall C \in \mathbb{R}$ we can find $R > 0$ such that*

$$\sup_{u \in \gamma^*(\partial B^k)} F(u) < C \quad \forall \gamma^* \in \Gamma^R_{\alpha,\bar{r}}\,. \tag{5.7}$$

Proof. We evaluate, for $u \in L_{\alpha,\bar{r}}$ and $\rho > 0$,

$$\begin{aligned}
\frac{F(\rho u)}{\rho^2} &= \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda}{2}\int_0^1 u^2 - \frac{\int_0^1 G(x,\rho u)}{\rho^2} - \frac{\int_0^1 h\rho u}{\rho^2} \\
&\leq \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda}{2}\int_0^1 u^2 + \frac{|\int_0^1 G(x,-\rho u^-)|}{\rho^2} - \frac{\int_0^1 G(x,\rho u^+)}{\rho^2} + \frac{|\int_0^1 h\rho u|}{\rho^2} \\
&\leq \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda}{2}\int_0^1 u^2 + \left(\frac{\delta}{4}\int_0^1 u^2 + \frac{C_2(\delta,g)}{\rho^2}\right) \\
&\quad - \left(\frac{M}{2}\int_0^1 (u^+)^2 - \frac{C_3(M,g)}{\rho^2}\right) + \left(\frac{\delta}{4}\int_0^1 u^2 + \frac{C_1(\delta,h)}{\rho^2}\right) \\
&\leq \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda-\delta}{2}\int_0^1 u^2 - \frac{M}{2}\int_0^1 (u^+)^2 + \frac{C_1(\delta,h)+C_2(\delta,g)+C_3(M,g)}{\rho^2} \\
&= \frac{1}{2}J_\alpha(u) - \frac{\lambda-\delta+M-\alpha^+}{2}\int_0^1 (u^+)^2 - \frac{\lambda-\delta-\alpha^-}{2}\int_0^1 (u^-)^2 \\
&\quad + \frac{C_1+C_2+C_3(M,g)}{\rho^2}\,.
\end{aligned}$$

Now if we fix $M = \alpha^+ - \alpha^-$ and consider that $J_\alpha(u) \le 0$ and $\int_0^1 u^2 \ge 1$ on $L_{\alpha,\bar{r}}$, we get

$$\frac{F(\rho u)}{\rho^2} \le -\frac{\lambda - \delta - \alpha^-}{2} + \frac{\tilde{C}(\delta, \alpha, g, h)}{\rho^2} \ , \tag{5.8}$$

where the first part is negative by the choice made for δ and then we can find the required R, namely $R > \sqrt{\frac{2(\tilde{C}(\delta,\alpha,g,h)-C)}{\lambda-\delta-\alpha^-}}$. □

Next we need

Lemma 5.5.
$$\sup_{u\in\gamma^*(B^k)} F(u) \ge -C_1(\delta, h) - C_2(\delta, g) - 1 \quad \forall \gamma^* \in \Gamma^R_{\alpha,\bar{r}} \, . \tag{5.9}$$

Proof. Fix a $\gamma^* \in \Gamma^R_{\alpha,\bar{r}}$.

Since $\gamma^*(B^k)$ is a compact set in a space of continuous functions, we can find

$$b(\gamma^*) = \max\{|u(x)| : x \in [0, 1], u \in \gamma^*(B^k)\} \tag{5.10}$$

and then there exists $\mu_{\gamma^*} > 0$ such that

$$G(x, s) \le 1 + \frac{\mu_{\gamma^*}}{2} s^2 \quad for \ \ all \quad s \in [0, b(\gamma^*)] \ . \tag{5.11}$$

Then

$$\begin{aligned}\int_0^1 G(x, u) + \int_0^1 hu \quad \le \quad & \frac{\delta}{4}\int_0^1 u^2 + C_1(\delta, h) + \\ & +\frac{\delta}{4}\int_0^1 u^2 + C_2(\delta, g) + \frac{\mu_{\gamma^*}}{2}\int_0^1 (u^+)^2 + \int_0^1 1 \ ,\end{aligned} \tag{5.12}$$

and so

$$\begin{aligned}\sup_{u\in\gamma^*(B^k)} F(u) \quad \ge \quad & \frac{1}{2} \sup_{u\in\gamma^*(B^k)} \left(\int_0^1 (u')^2 - (\lambda + \delta)\int_0^1 u^2 - \mu_{\gamma^*}\int_0^1 (u^+)^2\right) + \\ & -C_1(\delta, h) - C_2(\delta, g) - 1 \ .\end{aligned} \tag{5.13}$$

Now if $0 \in \gamma^*(B^k)$ the sup in the right hand side is clearly nonnegative.

Otherwise we can rearrange the terms in the sup on the right, adding and subtracting $\alpha^+ \int_0^1 (u^+)^2 + \alpha^- \int_0^1 (u^-)^2$, defining $r_{\gamma^*} = \frac{\lambda+\delta-\alpha^-}{\lambda+\delta+\mu_{\gamma^*}-\alpha^+}$ and collecting $\int_0^1 (u^+)^2 + r_{\gamma^*} \int_0^1 (u^-)^2 > 0$, obtaining

$$\sup_{u\in\gamma^*(B^k)} \left[\left(\frac{J_\alpha(u)}{\int_0^1 (u^+)^2 + r_{\gamma^*}\int_0^1 (u^-)^2} - (\lambda + \delta + \mu_{\gamma^*} - \alpha^+)\right)\left(\int_0^1 (u^+)^2 + r_{\gamma^*}\int_0^1 (u^-)^2\right)\right] \ . \tag{5.14}$$

Now if the sup of the first part is nonnegative, then is so all of the sup.

But $\sup_{u\in\gamma^*(B^k)} \frac{J_\alpha(u)}{\int_0^1 (u^+)^2 + r_{\gamma^*}\int_0^1 (u^-)^2}$ is equivalent to $\sup_{u\in\gamma(B^k)} J_\alpha(u)$ for some $\gamma \in \Gamma_{\alpha,r_{\gamma^*}}$ (compare equation (4.27) and (5.2), considering the definition (4.26)); then it is not lower than the value of $d_{\alpha,r_{\gamma^*}}$ obtained in proposition 4.15.

However by theorem 1.1, $d_{\alpha,r_{\gamma^*}} = \lambda^+_{\gamma^*} - \alpha^+ = \frac{\lambda^-_{\gamma^*}-\alpha^-}{r_{\gamma^*}}$ where $(\lambda^+_{\gamma^*}, \lambda^-_{\gamma^*})$ is a point in Σ_h for some $h \geq k+1$ and $\frac{\lambda^-_{\gamma^*}-\alpha^-}{\lambda^+_{\gamma^*}-\alpha^+} = r_{\gamma^*}$, so we obtain

$$\sup_{u\in\gamma(B^k)} J_\alpha(u) \geq \lambda^+_{\gamma^*} - \alpha^+ = \frac{\lambda^-_{\gamma^*} - \alpha^-}{r_{\gamma^*}}\,. \tag{5.15}$$

Then remains the calculation

$$(\lambda^+_{\gamma^*} - \alpha^+) - (\lambda+\delta+\mu_{\gamma^*} - \alpha^+) = ((\lambda^-_{\gamma^*} - \alpha^-) - (\lambda+\delta-\alpha^-))/r_{\gamma^*} = (\lambda^-_{\gamma^*} - (\lambda+\delta))/r_{\gamma^*}\ , \tag{5.16}$$

which is positive for the choice made for δ, since the curves Σ_h with $h \geq k+1$ have all points with $\lambda^- > \frac{\lambda_{k+1}}{4}$.

To conclude note that in this way we lost the dependence on γ^* (and on the values which depended upon it: r_{γ^*}, $\lambda^+_{\gamma^*}$ and $\lambda^-_{\gamma^*}$) in the estimates, hence the lemma is proved. □

Remark 5.6. *In the above proof we did not make use of the results of proposition 4.18; actually the unique information that we really need about the characterized intersection between Σ and $\{(\alpha^+ + t, \alpha^- + r_{\gamma^*}t),\ t > 0\}$ is that it belongs to a branch Σ_h with $h \geq k+1$.*

The PS condition for F was proved (using hypothesis (H2)) in [dFR91] for $\lambda \in (0, \frac{\pi^2}{4})$, and in [dFR93] (using also (H3)) for any $\lambda > 0$ in the case of periodic boundary conditions, however it can be extended to the Neumann case. The complete proof is given, in a more general setting, in section 9.

Using lemma 5.4 with $C < -C_1(\delta,h) - C_2(\delta,g) - 1$, lemma 5.5 and the PS condition, we are in the conditions to apply a linking theorem that proves the criticality of the level f defined in equation (5.3), and then theorem 1.2 is proved.

5.2 Proof of theorem 1.3

For the values $\lambda = \frac{\lambda_{k+1}}{4}$ one has a kind of resonance which creates difficulties for some of the estimates; actually the proof of lemma 5.4 can be done in the same way, choosing $\delta > 0$ such that $\alpha^- < \lambda - \delta$, but for lemma 5.5 it would not help to make the same estimates since no choice of $\delta > 0$ would allow to conclude that the expression in (5.16) is not negative.

Thus in this case we need to impose also the hypothesis (HR) and we proceed using the following estimates:

$$\begin{aligned}
\int_{u<-\rho_0} G(x,u) + hu &\leq M_0 \int_0^1 1\,,\\
\int_{u\in[-\rho_0,0]} G(x,u) + hu &\leq \sup_{s\in[-\rho_0,0],x\in[0,1]} G(x,s) \int_0^1 1 + \rho_0 \int_0^1 |h| = C_4(h,g)\,,\\
\int_{u>0} G(x,u) + hu &\leq \frac{\mu_{\gamma^*}}{2} \int_0^1 (u^+)^2 + \int_0^1 1 + \frac{1}{2}\int_0^1 (u^+)^2 + \frac{1}{2}\int_0^1 |h|^2\,, \qquad \forall\, u \in \gamma^*(B^k)\,;
\end{aligned}$$

then we get, in place of (5.12), that

$$\int_0^1 G(x,u) + \int_0^1 hu \;\leq\; \frac{\mu_{\gamma^*}+1}{2} \int_0^1 (u^+)^2 + M_0 + C_4(h,g) + 1 + \frac{1}{2}\int_0^1 |h|^2$$

and then we can estimate the sup as done in (5.13) by

$$\begin{aligned}\sup_{u\in\gamma^*(B^k)} F(u) \quad \geq \quad & \frac{1}{2}\sup_{u\in\gamma^*(B^k)}\left(\int_0^1 (u')^2 - \lambda\int_0^1 u^2 - (\mu_{\gamma^*}+1)\int_0^1 (u^+)^2\right) + \\ & -M_0 - 1 - \frac{1}{2}\int_0^1 |h|^2 - C_4(h,g)\ .\end{aligned} \tag{5.17}$$

After this, we make the same calculations we did before, now with $r_{\gamma^*} = \frac{\lambda-\alpha^-}{\lambda+\mu_{\gamma^*}+1-\alpha^+}$, to conclude that there is a point $(\lambda^+_{\gamma^*}, \lambda^-_{\gamma^*}) \in \Sigma_h$ with $h \geq k+1$ and $\frac{\lambda^-_{\gamma^*}-\alpha^-}{\lambda^+_{\gamma^*}-\alpha^+} = r_{\gamma^*}$ such that the sup is not negative if the following expression is not negative too:

$$(\lambda^+_{\gamma^*} - \alpha^+) - (\lambda + \mu_{\gamma^*} + 1 - \alpha^+) = ((\lambda^-_{\gamma^*} - \alpha^-) - (\lambda - \alpha^-))/r_{\gamma^*} = (\lambda^-_{\gamma^*} - \lambda)/r_{\gamma^*}\ ; \tag{5.18}$$

but this is actually positive since all points in Σ_h with $h \geq k+1$ have $\lambda^- > \lambda$.

5.3 One more property of the variational characterization in dimension one

Here we will use the same ideas used in the previous proofs to obtain one more property of the variational characterization made in section 4:

Proposition 5.7. *In the one dimensional case, with both Neumann and Dirichlet boundary conditions, fix $r \in (0,1]$ and α as in definition 4.1, then the point in the Fučík specrtum determined by the variational characterization in theorem 1.1, that is $\alpha(1) + (d_{\alpha(1),r}, rd_{\alpha(1),r})$, is the first intersection of the halfline $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$ with the Fučík spectrum, that is the one with smallest t.*

When there is only one curve coming out from the point $(\lambda_{k+1}, \lambda_{k+1})$, that is in the Neumann case and for k odd in the Dirichlet one, proposition 5.7 is a trivial consequence of proposition 4.18; in the Dirichlet case with k even, that is when two curves (Σ^+_{k+1} and Σ^-_{k+1}) come out from $(\lambda_{k+1}, \lambda_{k+1})$, we will show that if it were not the first intersection, then one could prove the existence of a solution of problem (3.13) with (λ^+, λ^-) in the region between Σ^+_{k+1} and Σ^-_{k+1}, $g = 0$ and any $h \in L^2$, contradicting the result of [Dan77] given in lemma 3.3.

Let us work, without loss of generality, with $\lambda^+ \geq \lambda^-$, so that the lower curve is Σ^+_{k+1} and the upper one is Σ^-_{k+1}, and let us take (λ^+, λ^-) in the region between them.

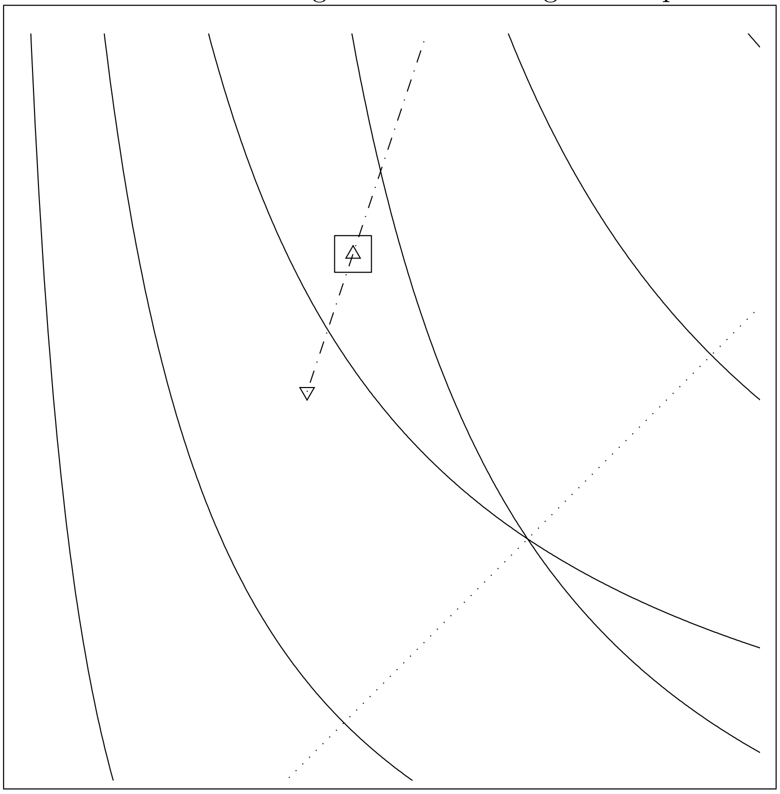
Take any point (α^+, α^-) with $\alpha^+ \geq \alpha^-$ and:

- (α^+, α^-) Σ-connected to the diagonal between λ_k and λ_{k+1},
- $\alpha^\pm < \lambda^\pm$,
- $\frac{\lambda^- - \alpha^-}{\lambda^+ - \alpha^+} \in (0,1]$.

Then choose a $\delta > 0$ such that:

- $(\lambda^+ + \delta, \lambda^- + \delta)$ is still below the higher curve (Σ^-_{k+1}),
- $\lambda^\pm - \delta > \alpha^\pm$.

Figure 6: The setting for the proof of theorem 5.7.



$$\begin{cases} \nabla & : \ (\alpha^+, \alpha^-) \\ \Delta & : \ (\lambda^+, \lambda^-) \\ \square & : \ (\lambda^+ \pm \delta, \lambda^- \pm \delta) \end{cases}$$

This construction is sketched in figure 6.

Now define the functional associated to the problem:

$$F(u) = \frac{1}{2}\left(\int (u')^2 - \lambda^+ \int (u^+)^2 - \lambda^- \int (u^-)^2\right) - \int hu \ , \tag{5.19}$$

find $C_1(\delta, h)$, as in section 5.1, such that $|\int hu| \leq \frac{\delta}{2}\|u\|_{L^2}^2 + C_1(\delta, h)$ and use it as in lemma 5.4; that is, one proves (for fixed $\bar{r} \in (0, 1]$) that

Lemma 5.8. *$\forall C \in \mathbb{R}$ we can find $R > 0$ such that*

$$\sup_{u \in \gamma^*(\partial B^k)} F(u) < C \quad \forall \gamma^* \in \Gamma^R_{\alpha, \bar{r}} \, . \tag{5.20}$$

Proof. As in lemma 5.4, for $u \in L_{\alpha,\bar{r}}$ and $\rho > 0$:

$$\begin{aligned}
\frac{F(\rho u)}{\rho^2} &= \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda^+}{2}\int_0^1 (u^+)^2 - \frac{\lambda^-}{2}\int_0^1 (u^-)^2 - \frac{\int_0^1 h\rho u}{\rho^2} \\
&\leq \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda^+}{2}\int_0^1 (u^+)^2 - \frac{\lambda^-}{2}\int_0^1 (u^-)^2 + \left(\frac{\delta}{2}\int_0^1 u^2 + \frac{C_1(\delta,h)}{\rho^2}\right) \\
&= \frac{1}{2}\int_0^1 (u')^2 - \frac{\lambda^+ - \delta}{2}\int_0^1 (u^+)^2 - \frac{\lambda^- - \delta}{2}\int_0^1 (u^-)^2 + \frac{C_1(\delta,h)}{\rho^2} \\
&\leq \frac{1}{2}J_\alpha(u) - \frac{min\{\lambda^+ - \delta - \alpha^+, \lambda^- - \delta - \alpha^-\}}{2}\int_0^1 u^2 + \frac{C_1(\delta,h)}{\rho^2} \\
&\leq -\frac{min\{\lambda^+ - \delta - \alpha^+, \lambda^- - \delta - \alpha^-\}}{2} + \frac{C_1(\delta,h)}{\rho^2}
\end{aligned}$$

and so again by the choice made for δ the first part is negative and so we can find the required R. □

Then notice that

Lemma 5.9. *If the intersection of the halfline were a point $(\beta^+, \beta^-) \in \Sigma^-_{k+1}$, then we could prove that*

$$\sup_{u \in \gamma^*(B^k)} F(u) \geq -C_1(\delta,h) \quad \forall \gamma^* \in \Gamma^R_{\alpha,\bar{r}} . \tag{5.21}$$

Proof. With the same argument of lemma 5.5:
first estimate the sup as in equation (5.13):

$$\sup_{u\in\gamma^*(B^k)} F(u) \geq \frac{1}{2}\sup_{u\in\gamma^*(B^k)}\left(\int (u')^2 - (\lambda^+ + \delta)\int (u^+)^2 - (\lambda^- + \delta)\int (u^-)^2\right) - C_1(\delta,h) \ , \tag{5.22}$$

then define $r = \frac{\lambda^- + \delta - \alpha^-}{\lambda^+ + \delta - \alpha^+}$ and consider as before (equation (5.14))

$$\sup_{u\in\gamma^*(B^k)}\left[\left(\frac{J_\alpha(u)}{\int_0^1 (u^+)^2 + r\int_0^1 (u^-)^2} - (\lambda^+ + \delta - \alpha^+)\right)\left(\int_0^1 (u^+)^2 + r\int_0^1 (u^-)^2\right)\right] . \tag{5.23}$$

But now we would have in place of (5.15)

$$\sup_{u\in\gamma(B^k)} J_\alpha(u) \geq \beta^+ - \alpha^+ \tag{5.24}$$

where (β^+, β^-) is such that $\frac{\beta^- - \alpha^-}{\beta^+ - \alpha^+} = r$, and we are supposing it to be in Σ^-_{k+1}, so that $\beta^\pm > \lambda^\pm + \delta$.

So finally (5.16) becomes

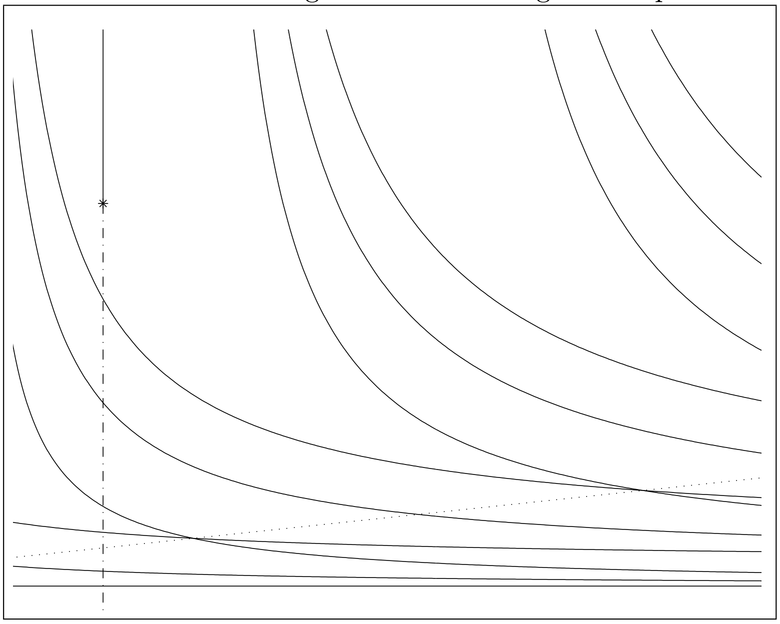
$$(\beta^+ - \alpha^+) - (\lambda^+ + \delta - \alpha^+) = ((\beta^- - \alpha^-) - (\lambda^- + \delta - \alpha^-))/r = (\beta^- - (\lambda^- + \delta))/r \ , \tag{5.25}$$

which would be positive for the choice made for δ, proving the lemma.

□

To conclude, since (λ^+, λ^-) is not in the Fučík spectrum, F satisfies the PS-condition (see lemma 3.4) and then one could conclude by a linking theorem the existence of a solution for any $h \in L^2$, giving the contradiction that concludes the proof of proposition 5.7.

Figure 7: The setting for the proof of theorem 5.10.



$$\begin{cases} * & : \ (\overline{\lambda^+}, \lambda) \\ - \cdot - & : \ \{\lambda^- = \lambda\} \\ \text{——} & : \ H_\lambda \\ \cdots\cdots & : \ \{\lambda^+ = \lambda^-\} \end{cases}$$

5.4 Non existence of solutions for superlinear problem in the Dirichlet case

In theorem 1.2, we proved that under hypotheses (H1) to (H3) the problem (1.6) had a solution for any $h \in L^2(0,1)$, provided $\lambda > \lambda_1$ and not resonant.

Now we want to show, with a counterexample, that a similar result cannot be achieved for the Dirichlet problem.

We first observe that (consider $\lambda^+ > \lambda^-$) for any $\lambda \neq \lambda_1$ there always exists $\overline{\lambda^+}$ such that the halfline $H_\lambda = \{(\lambda^*, \lambda)$ with $\lambda^* \geq \overline{\lambda^+}\}$ is contained in one of the zones between Σ^+_{2i-1} and Σ^-_{2i-1}.

This situation is sketched in figure 7.

Moreover if we fix, for the point $(\overline{\lambda^+}, \lambda)$, the point x_0 defined in the proof of lemma 3.3, then this x_0 satisfies (3.15) for all the points $(\lambda^*, \lambda) \in H_\lambda$, since $\frac{i\pi}{\sqrt{\lambda^*}} + \frac{(i-1)\pi}{\sqrt{\lambda}}$ is a decreasing function of λ^*; this implies that $h = \chi([x_0, 1])$ gives nonexistence of solutions for every pair of coefficients $(\lambda^*, \lambda) \in H_\lambda$.

Now we can prove:

Theorem 5.10. *If $\lambda \neq \lambda_1$, $\overline{\lambda^+}$ such that $\{(\lambda^*, \lambda)$ with $\lambda^* \geq \overline{\lambda^+}\}$ is contained in the zone between Σ^+_{2i-1} and Σ^-_{2i-1}, then the problem*

$$\begin{cases} -w'' = \overline{\lambda^+} w^+ - \lambda w^- + (e^{w^+} - 1) + h \quad in \ (0,1) \\ w(0) = 0 \\ w(1) = 0 \end{cases} \tag{5.26}$$

has no solution for $h = \chi([x_0, 1])$, with $x_0 \in (\frac{i\pi}{\sqrt{\overline{\lambda^+}}} + \frac{(i-1)\pi}{\sqrt{\lambda}}, 1)$.

Remark 5.11. *If we set $g(s) = (\overline{\lambda^+} - \lambda)s^+ + (e^{s^+} - 1)$ then it satisfies the hypotheses from (H1) to (H3) and the equation in (5.26) reads as in (1.6).*

Then this is indeed a counterexample that shows that theorem 1.2 cannot have an analogue for the Dirichlet problem; once again this result is coherent with the interpretation of the superlinear problem as the limiting asymptotically linear problem when $\lambda^+ \to +\infty$; actually for the Dirichlet problem if we fix $\lambda \neq \lambda_1$, then the point (λ^+, λ) lies, for λ^+ big enough, in the set A_2, where solutions exist only for suitable $h \in L^2$.

Proof. Consider the initial value problem $w_d(0) = 0$, $w_d'(0) = d$: we have that in $[0, x_0]$ w_d has negative bumps of length $\frac{\pi}{\sqrt{\lambda}}$ and positive bumps of a length between 0 and $\frac{\pi}{\sqrt{\overline{\lambda^+}}}$.

Actually consider a positive bump, that is compare

$$\begin{cases} -w'' = \overline{\lambda^+} w^+ - \lambda w^- + (e^{w^+} - 1) \\ w(0) = 0 \\ w'(0) = d > 0 \end{cases} \tag{5.27}$$

with

$$\begin{cases} -u'' = \overline{\lambda^+} u^+ - \lambda u^- \\ u(0) = 0 \\ u'(0) = d > 0 \end{cases} ; \tag{5.28}$$

multiply the first by u, the second by w, integrate by parts in $[0, \frac{\pi}{\sqrt{\overline{\lambda^+}}}]$ and subtract: since $u \geq 0$ in this interval, if we suppose that $w \geq 0$ too we get

$$(u'w - w'u)|_{\frac{\pi}{\sqrt{\overline{\lambda^+}}}} = \int_0^{\frac{\pi}{\sqrt{\overline{\lambda^+}}}} (e^{w^+} - 1)u\,; \tag{5.29}$$

since $u\left(\frac{\pi}{\sqrt{\overline{\lambda^+}}}\right) = 0$ and $u'\left(\frac{\pi}{\sqrt{\overline{\lambda^+}}}\right) = -d$, we obtain (observe that $w \not\equiv 0$ since $w'(0) > 0$)

$$-d\, w\left(\frac{\pi}{\sqrt{\overline{\lambda^+}}}\right) > 0\,, \tag{5.30}$$

which contradicts the assumption $w \geq 0$ and so implies that the bump of w is shorter.

Now, whatever are the lengths of the positive bumps, they correspond to $\frac{\pi}{\sqrt{\lambda^*}}$ for some $\lambda^* > \overline{\lambda^+}$ and so by the choice made of $\overline{\lambda^+}$ we still have $w_d(x_0) \leq 0$ and then (since where $w \leq 0$, the equation in (5.26) is the same as the one in (3.13) with $g = 0$), one again concludes that $w_d(1) < 0 \quad \forall d \in \mathbb{R}$ as in lemma 3.3. $\square$

6 Radial problem in higher dimension

Here we consider the same problem of section 5, in dimension greater than one:

$$\begin{cases} -\Delta u = \lambda u + g(x,u) + h(x) & in\ \Omega \\ \frac{\partial u}{\partial n} = 0 & in\ \partial\Omega \end{cases} \tag{6.1}$$

where Ω is the set $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ with $R_2 > R_1 \geq 0$, and h and g depend only on $|x|$ and u.

If we seek radial solutions, this means that we are looking at the equivalent problem

$$\begin{cases} -(r^{N-1}u'(r))' = r^{N-1}(\lambda u(r) + \tilde{g}(r,u) + \tilde{h}(r)) & in\ (R_1, R_2) \\ u'(R_1) = u'(R_2) = 0 \end{cases} \tag{6.2}$$

where $\tilde{g}(|x|, s) = g(x,s)$ and $\tilde{h}(|x|) = h(x)$.

In order to apply the same argument used in section 5 we need to find some space between the asymptotes of the Fučík spectrum originating from two consecutive eigenvalues.

6.1 The radial Fučík spectrum

The Fučík problem for this case is

$$\begin{cases} -(r^{N-1}u'(r))' = r^{N-1}(\lambda^+ u^+(r) - \lambda^- u^-(r)) & in\ (R_1, R_2) \\ u'(R_1) = u'(R_2) = 0 \end{cases} ; \tag{6.3}$$

the spectrum has been calculated in [AC95] and [RW99]: we report here the results we are interested in, from the second reference.

First consider the equation

$$-(r^{N-1}u'(r))' = r^{N-1}\lambda u(r) \qquad in\ (R_1, R_2)\,, \tag{6.4}$$

and call

- λ_i^{NN} the i^{th} eigenvalue of equation (6.4) with boundary conditions $u'(R_1) = 0$, $u'(R_2) = 0$ (this is also the i^{th} eigenvalue of the problem we are considering),
- λ_i^{DD} the i^{th} eigenvalue of equation (6.4) with boundary conditions $u(R_1) = 0$, $u(R_2) = 0$,
- λ_i^{DN} the i^{th} eigenvalue of equation (6.4) with boundary conditions $u(R_1) = 0$, $u'(R_2) = 0$,
- λ_i^{ND} the i^{th} eigenvalue of equation (6.4) with boundary conditions $u'(R_1) = 0$, $u(R_2) = 0$.

It is known that these eigenvalues are all simple and that each one is related to an eigenfunction with $i-1$ simple zeros in the interior of the interval.

From the point $(\lambda_1^{NN}, \lambda_1^{NN})$ arise as usual the two lines $\{\lambda^+ = \lambda_1^{NN}\}$ and $\{\lambda^- = \lambda_1^{NN}\}$ belonging to the Fučík spectrum of problem (6.3); then from each point $(\lambda_k^{NN}, \lambda_k^{NN})$ with $k \geq 2$ originate two monotone curves whose asymptotes are:

- in the case $R_1 > 0$, let $i = 1, 2, ...$:
  - if k=2i: λ_i^{DN} and λ_i^{ND},
  - if k=2i+1: λ_i^{DD} and λ_{i+1}^{NN};
- in the case $R_1 = 0$, let $i = 1, 2, ...$:
  - if k=2i: λ_i^{NN} and λ_i^{ND},
  - if k=2i+1: λ_i^{ND} and λ_{i+1}^{NN}.

In the case $R_1 = 0$, then the higher asymptote of the curves originating from (λ_k, λ_k) always coincides with the lower one of those from $(\lambda_{k+1}, \lambda_{k+1})$, hence we do not have the needed space between them.

In the case $R_1 > 0$, on the other hand, this space always exists:

Lemma 6.1. *If $R_1 > 0$ we have:*

- $\lambda_1^{NN} < (\lambda_1^{DN}$ *and* $\lambda_1^{ND})$
- $(\lambda_i^{DN}$ *and* $\lambda_i^{ND}) < (\lambda_i^{DD}$ *and* $\lambda_{i+1}^{NN}) < (\lambda_{i+1}^{DN}$ *and* $\lambda_{i+1}^{ND})$ *for* $i = 1, 2, ...$

Proof. As we noted before each eigenvalue λ_i is simple and related to an eigenfunction with $i-1$ simple zeros in the interior of the interval.

Now consider the inequality $\lambda_i^{DN} < \lambda_i^{DD} < \lambda_{i+1}^{DN}$:

- λ_i^{DN} corresponds to an eigenfunction ϕ_i^{DN} that we may choose to have $\phi_i^{DN}(R_1) = 0$ and $\phi_i^{DN}(R_1)' = 1$, to have $i - 1$ zeros in (R_1, R_2) and $\phi_i^{DN}(R_2) \neq 0$ (if it were zero, since $(\phi_i^{DN}(R_2))' = 0$ then ϕ_i^{DN} would be identically zero).
- λ_{i+1}^{DN} corresponds to an eigenfunction ϕ_{i+1}^{DN} that we may choose to satisfy the same conditions in R_1, have i zeros in (R_1, R_2) and then $\phi_{i+1}^{DN}(R_2)\phi_i^{DN}(R_2) < 0$ (since ϕ_{i+1}^{DN} has one zero more than ϕ_i^{DN}).

Then if we consider the initial value problem

$$\begin{cases} -(r^{N-1}u_\lambda'(r))' = \lambda r^{N-1}u_\lambda(r) \quad in \quad (R_1, R_2) \\ u_\lambda(R_1) = 0, \;\; u_\lambda'(R_1) = 1 \end{cases} \tag{6.5}$$

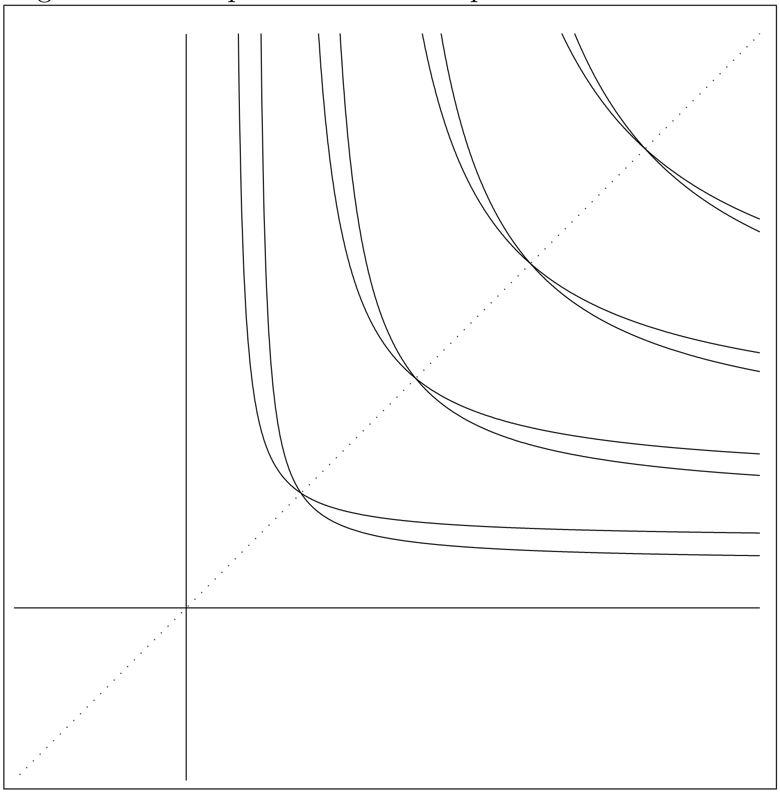
with $\lambda \in [\lambda_i^{DN}, \lambda_{i+1}^{DN}]$ we have that $u_\lambda(R_2)$ must be a continuous function of the variable λ that changes sign; then there exists a first zero in $(\lambda_i^{DN}, \lambda_{i+1}^{DN})$, which corresponds to a non trivial solution of the equation in (6.5) with Dirichlet conditions at both ends; moreover this solution must still have $i - 1$ zeros in $(0, 1)$ since for the continuity of the dependence on λ and the uniqueness of initial value problem, a zero may not appear or disappear from the interior of the interval (remember that no nontrivial solution may be null with zero derivative in any point); then this zero of $u_\lambda(R_2)$ in $(\lambda_i^{DN}, \lambda_{i+1}^{DN})$ is λ_i^{DD}.

The same kind of considerations give the remaining inequalities.

□

We sketch in figure 8 and 9 the qualitative behavior of the spectrum in the cases $R_1 > 0$ and $R_1 = 0$ respectively.

Figure 8: Fučík spectrum for radial problem on an annulus.



6.2 The superlinear problem

Solutions of equation (6.2) can be associated to critical points of the functional

$$F(u) = \frac{1}{2}\int_{R_1}^{R_2} r^{N-1}(u')^2 - \frac{\lambda}{2}\int_{R_1}^{R_2} r^{N-1}u^2 - \int_{R_1}^{R_2} r^{N-1}\tilde{G}(r,u) - \int_{R_1}^{R_2} r^{N-1}\tilde{h}u\ . \qquad (6.6)$$

Since r takes values in $[R_1, R_2]$ and so is bounded and bounded away from zero, the functional is well defined in the space $H^1([R_1, R_2])$, where one can use the equivalent scalar product $< u, v >^N_{H^1} = \int r^{N-1}(u'v' + uv)$; then all the work done in section 4 and 5 can be applied here (use in L^2 the scalar product $< u, v >^N_{L^2} = \int r^{N-1}uv$).

The above observations imply that we may obtain the same kind of result:

Theorem 6.2. *Under hypothesis (H1-R), (H2) and (H3), with g and h depending radially on $x \in \Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ and $R_1 > 0$, if λ is such that*

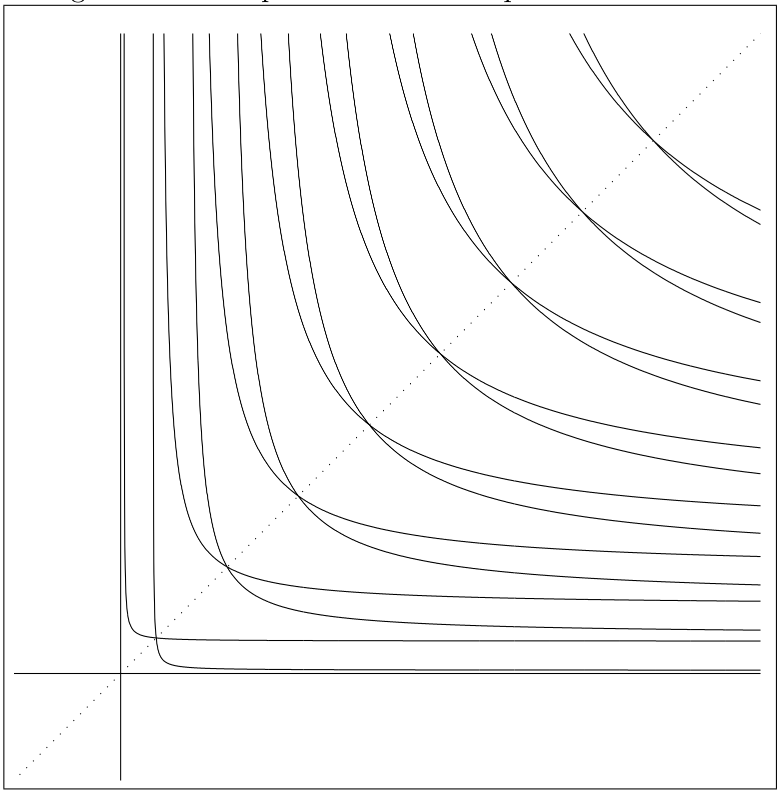
- $\lambda_1^{NN} < \lambda < (\lambda_1^{DN}$ *and* $\lambda_1^{ND})$

or

- $(\lambda_i^{DN}$ *and* $\lambda_i^{ND}) < \lambda < (\lambda_i^{DD}$ *and* $\lambda_{i+1}^{NN})$ *or* $(\lambda_i^{DD}$ *and* $\lambda_{i+1}^{NN}) < \lambda < (\lambda_{i+1}^{DN}$ *and* $\lambda_{i+1}^{ND})$ *for some* $i = 1, 2, \dots$,

then there exists a radial solution of problem (6.1) for any $h \in L^2_{rad}(\Omega)$.

Figure 9: Fučík spectrum for radial problem on a ball.



Theorem 6.3. *Under hypothesis (H1-R), (H2) (H3) and (HR-R) with $h \in L^2(\Omega)$, g and h depending radially on $x \in \Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$ and $R_1 > 0$, if λ is such that*

- $\lambda = \min\{\lambda_i^{DN}; \lambda_i^{ND}\}$

or

- $\lambda = \min\{\lambda_i^{DD}; \lambda_{i+1}^{NN}\}$,

for some $i = 1, 2, ...$, then there exists a radial solution of problem (6.1).

The new hypotheses introduced above reads:

$$\begin{aligned} &g \in \mathcal{C}^0(\bar{\Omega} \times \mathbb{R}), \\ &\lim_{s\to-\infty} \tfrac{g(x,s)}{s} = 0, \quad \lim_{s\to+\infty} \tfrac{g(x,s)}{s} = +\infty \end{aligned} \tag{H1-R}$$

uniformly with respect to $x \in \bar{\Omega}$;

$$\exists \rho_0 > 0,\ M_0 \in \mathbb{R} \quad s.t. \quad G(x,s) + h(x)s \le M_0 \quad a.e.\ x \in \Omega,\ \forall s < -\rho_0 . \tag{HR-R}$$

Remark 6.4. *In [RW99] it is shown that if one substitutes the equation in (6.2) with*

$$-(r^\alpha u'(r))' = r^\alpha(\lambda u(r) + \tilde{g}(r,u) + \tilde{h}(r)) \tag{6.7}$$

for $\alpha \ge 0$, then one has the same qualitative behavior of the Fučík spectrum; thus it is clear from the above proof that the result still holds in this case, provided $R_1 > 0$.

7 Problems of higher order

In this section we will consider the problems with the multi-Laplacian operator

$$\begin{cases} (-\Delta)^m u = \lambda u + g(x,u) + h(x) & in\ \ \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = \dots = \frac{\partial \Delta^{m-1} u}{\partial n} = 0 & on\ \ \partial\Omega \end{cases} \tag{7.1}$$

and

$$\begin{cases} (-\Delta)^m u = \lambda u + g(x,u) + h(x) & in\ \ \Omega \\ u = \Delta u \dots = \Delta^{m-1} u = 0 & on\ \ \partial\Omega \end{cases} \tag{7.2}$$

with $\Omega \subseteq \mathbb{R}^N$ a bounded domain of class $\mathcal{C}^m$, $g \in \mathcal{C}(\bar\Omega \times \mathbb{R}\ \mathbb{R})$ and $h \in L^2(\Omega)$.

We will assume for certain results the hypotheses:

$$N < 2m \quad so \quad that \quad H^m(\Omega) \subseteq \mathcal{C}^0(\bar\Omega) \quad with \quad compact \quad inclusion\,; \tag{HN}$$

$$N < 2(m-1) \quad so \quad that \quad H^m(\Omega) \subseteq \mathcal{C}^1(\bar\Omega) \quad with \quad compact \quad inclusion\,; \tag{HD}$$

in particular (HN) will be assumed for problem (7.1) and (HD) for problem (7.2).

We will still assume the hypotheses

$$\begin{array}{l} g \in \mathcal{C}^0(\bar\Omega \times \mathbb{R})\,, \\ \lim_{s\to-\infty} \frac{g(x,s)}{s} = 0, \quad \lim_{s\to+\infty} \frac{g(x,s)}{s} = +\infty \end{array} \tag{H1-m}$$

uniformly with respect to $x \in \bar\Omega$;

$$\exists \theta \in \left(0, \frac{1}{2}\right), \quad s_0 > 0 \quad s.t. \quad 0 < G(x,s) \le \theta s g(x,s) \quad \forall s > s_0 \tag{H2-m}$$

where $G(x,s) = \int_0^s g(x,\xi)d\xi$, and for some of the results also

$$\exists s_1 > 0,\ C_0 > 0 \quad s.t. \quad G(x,s) \le \frac{1}{2} s g(x,s) + C_0 \quad \forall s < -s_1\,; \tag{H3-m}$$

$$\exists \rho_0 > 0,\ M_0 \in \mathbb{R} \quad s.t. \quad G(x,s) + h(x)s \le M_0 \quad a.e.\ x \in \Omega,\ \forall s < -\rho_0\,. \tag{HR-m}$$

Moreover for λ equal to the first eigenvalue of the problem we will assume

$$g(x,s) > 0, \qquad \lim_{s\to-\infty} g(x,s) = 0 \tag{HR0-m}$$

uniformly with respect to $x \in \bar\Omega$.

We choose the above sets of boundary conditions since, as will be clear in the following, they allow us to use our approach; we will refer to the first type of boundary conditions as the case (N) and to the second as the case (D), moreover we will usually write the results for the case (N) and when needed remark in parentheses what is different for the case (D).

Let B_N (resp. B_D) be the operator that maps u to the vector of the traces on $\partial\Omega$ of the derivatives of order strictly less than m (as done in section 2.4) which are imposed in problem

(7.1) (resp. (7.2)): then the problem in variational form will be

$$u \in H^m_*(\Omega) \quad such \quad that$$
$$\int_\Omega \nabla^m u \nabla^m v - \lambda \int_\Omega uv - \int_\Omega g(x,u)v - \int_\Omega hv = 0 \qquad for \quad all \quad v \in H^m_*(\Omega)\,, \tag{7.3}$$

where with H^m_* we have denoted H^m_N or H^m_D when considering respectively (7.1) or (7.2), namely

$$H^m_N(\Omega) = \{u \in H^m(\Omega) \quad such \quad that \quad B_N u = 0\}\,, \tag{7.4}$$

$$H^m_D(\Omega) = \{u \in H^m(\Omega) \quad such \quad that \quad B_D u = 0\}\,; \tag{7.5}$$

observe that for $m = 1$ these are $H^1_N(\Omega) = H^1(\Omega)$ and $H^1_D(\Omega) = H^1_0(\Omega)$.

In order to find a solution of problem (7.1) (resp. (7.2)) we will consider the functional

$$F(u) = \frac{1}{2}\int_\Omega |\nabla^m u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega G(x,u) - \int_\Omega hu\ , \tag{7.6}$$

defined on the space $H^m_N(\Omega)$ (resp. $H^m_D(\Omega)$); actually $F \in \mathcal{C}^1(H^m_*)$ and so if $F'(u)[v] = 0$ $\forall v \in H^m_*(\Omega)$ then u is a weak solution of problem (7.1) (resp. (7.2)).

7.1 Some useful lemmas about the spaces $H^m_*(\Omega)$

In this section we will obtain some results about the properties of the spaces we will work with, in particular we will show that if the set Ω is regular enough, then the space $H^1_*(\Omega)$ may be normed with a norm which has the structure of the first part of the functional (7.6); this will help in making estimates on this functional.

We remark that this result is a consequence of the particular sets of boundary conditions chosen.

Lemma 7.1. *For $m \geq 1$, if $u \in H^m_*(\Omega)$ and Ω is of class $\mathcal{C}^1$ then*

$\int_\Omega |\nabla^m u|^2 = 0$ implies $u = const$ a.e;

in particular in the case $u \in H^m_D(\Omega)$ this constant is zero.

Proof. For $m = 1$ the lemma reads:

if $u \in H^1(\Omega)$ (resp. $u \in H^1_0(\Omega)$) and $\int_\Omega |\nabla u|^2 = 0$ then $u = const$ a.e,

which is true for $\mathcal{C}^1$ domains.

Now suppose $m \geq 2$ and the lemma to hold for $m-1$, then compute

$$\int_\Omega |\nabla^{m-1} u|^2 = -\int_\Omega \nabla^m u \nabla^{m-2} u + \int_{\partial\Omega} (\nabla^{m-1} u \nabla^{m-2} u) \cdot n\,, \tag{7.7}$$

where $u \in H^m_N$ (resp. H^m_D) implies that the boundary term is zero and then, if $\nabla^m u = 0$ a.e, also $\nabla^{m-1} u = 0$ a.e, which by induction hypothesis (since $H^m_N \subseteq H^{m-1}_N$ and $H^m_D \subseteq H^{m-1}_D$) implies $u = const$ a.e.

□

Proposition 7.2. *If Ω is of class $\mathcal{C}^m$, then $\left(\|\nabla^m u\|^2_{L^2} + \|u\|^2_{L^2}\right)^{\frac{1}{2}}$ is an equivalent norm for $H^m_*(\Omega)$.*

Proof. Since $|\nabla^h u|^2$ is the product of two finite sums of derivatives of u of order h, then $|\nabla^h u|^2 \leq C(h)\sum_{|\alpha|=h}(D^\alpha u)^2$ and so the claimed norm can be controlled by the usual one.

Let us show the converse: for $m = 0, 1$ the lemma is trivially true since the claimed norm is indeed the usual one.

Let us suppose $m \geq 2$ and the proposition to hold for $m-2$ and $m-1$ and take $u \in H^m_*(\Omega)$, this implies that $\Delta u \in H^{m-2}(\Omega)$: that is u satisfies

$$\begin{cases} -\Delta u = h & in\ \ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array}\right. & on\ \ \partial\Omega \end{cases} \tag{7.8}$$

where $h \in H^{m-2}(\Omega)$.

Then (by the regularity of Ω) one may apply lemma 2.16 to obtain

$$\|u\|^2_{H^m} \leq C(\|h\|^2_{H^{m-2}} + \|u\|^2_{L^2})\,. \tag{7.9}$$

But using $h = -\Delta u$ and induction hypothesis this reads

$$\|u\|^2_{H^m} \leq C(\|\nabla^m u\|^2_{L^2} + \left\|\nabla^2 u\right\|^2_{L^2} + \|u\|^2_{L^2}) \tag{7.10}$$

where (for $m \geq 3$):

$$\left\|\nabla^2 u\right\|^2_{L^2} \leq C\left(\|u\|^2_{L^2} + \left\|\nabla^{m-1} u\right\|^2_{L^2}\right) \tag{7.11}$$

by the assumption that the right hand side forms a norm for H^{m-1}_*, then as in equation (7.7) we may estimate

$$\begin{aligned} \left\|\nabla^{m-1} u\right\|^2_{L^2} &\leq \left\|\nabla^{m-2} u\right\|_{L^2} \|\nabla^m u\|_{L^2} &(7.12)\\ &\leq \varepsilon^2 \left\|\nabla^{m-2} u\right\|^2_{L^2} + \frac{1}{\varepsilon^2}\|\nabla^m u\|^2_{L^2} &(7.13)\end{aligned}$$

and using again the induction hypothesis

$$\left\|\nabla^{m-1} u\right\|^2_{L^2} \leq C\varepsilon^2(\|u\|^2_{L^2} + \left\|\nabla^{m-1} u\right\|^2_{L^2}) + \frac{1}{\varepsilon^2}\|\nabla^m u\|^2_{L^2}\ , \tag{7.14}$$

from which, choosing $0 < \varepsilon < C^{1/2}$ and collecting the terms $\left\|\nabla^{m-1} u\right\|^2_{L^2}$ in the left hand side, one gets

$$\left\|\nabla^{m-1} u\right\|^2_{L^2} \leq D(\|u\|^2_{L^2} + \|\nabla^m u\|^2_{L^2})\,. \tag{7.15}$$

Joining the estimates (or directly from (7.10) in the case $m = 2$) we obtain that $\|u\|^2_{H^m}$ can be controlled by $\|\nabla^m u\|^2_{L^2} + \|u\|^2_{L^2}$.

□

7.2 The spectrum of the multi-Laplacian

In this section we will obtain a description and some properties of the spectrum of the operator $(-\Delta)^m$ with the boundary conditions (N) and (D).

Namely consider the problem

$$\begin{cases} (-\Delta)^m u = \lambda u & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = ... = \frac{\partial \Delta^{m-1} u}{\partial n} = 0 \\ \quad or \\ u = \Delta u ... = \Delta^{m-1} u = 0 \end{array}\right. & on\ \partial\Omega \end{cases} : \tag{7.16}$$

we will prove in the following that

Proposition 7.3. *Provided Ω is of class $\mathcal{C}^m$, the eigenvalues of (7.16) are the m-th power of those of the Laplacian with Neumann (resp. Dirichlet) boundary conditions, while the eigenfunctions are the same of those cases.*

Remark 7.4. *In view of proposition 7.3, we will maintain the notation λ_k for the eigenvalues of the Laplacian so that those of $(-\Delta)^m$ will be λ_k^m.*

First observe that:

- If λ is not real or negative then (7.16) cannot have nontrivial solutions: actually multiplying by u and integrating by parts m times we get $\int_\Omega |\nabla^m u|^2 = \lambda \int_\Omega u^2$ which would imply $u = 0$ a.e.
- For $\lambda = 0$ the same equation implies $\int_\Omega |\nabla^m u|^2 = 0$ and then (by lemma 7.1) u is a constant; in particular:
  - in the case (N) 0 is an eigenvalue and its eigenspace has dimension 1,
  - in the case (D) the constant must be zero and so 0 is not an eigenvalue.

To study the case $\lambda > 0$ note that problem (7.16) may be written as

$$\begin{cases} -\Delta u_1 = \mu u_2 & in\ \Omega \\ -\Delta u_2 = \mu u_3 & in\ \Omega \\ .... & \\ -\Delta u_{m-1} = \mu u_m & in\ \Omega \\ -\Delta u_m = \mu u_1 & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u_1}{\partial n} = ... = \frac{\partial u_m}{\partial n} = 0 \\ \quad or \\ u_1 = ... = u_m = 0 \end{array}\right. & on\ \partial\Omega \end{cases} \tag{7.17}$$

with $\mu^m = \lambda$ and $\mu > 0$.

Then we will compare problem (7.16) with the known problem

$$\begin{cases} -\Delta u = \mu u \qquad in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \qquad or \qquad on\ \partial\Omega \\ u = 0 \end{array} \right. \end{cases} . \tag{7.18}$$

Lemma 7.5. *If μ, u are eigenvalue and eigenfunction of (7.18) then μ^m, u are eigenvalue and eigenfunction of (7.16).*

Proof. Since μ, u are eigenvalue and eigenfunction of (7.18), u is $\mathcal{C}^\infty$ at least in the interior of Ω, and then we have

$$(-\Delta)^m u = (-\Delta)^{m-1}(-\Delta u) = \mu(-\Delta)^{m-1}u = ... = \mu^m u \quad \forall x \in \Omega\,. \tag{7.19}$$

Now if Ω has enough regularity we also have

$$\frac{\partial(-\Delta)^h u}{\partial n} = \mu^h \frac{\partial u}{\partial n} = 0 \quad \forall x \in \partial\Omega \quad for \quad h = 0, ..., m-1 \tag{7.20}$$

$$(resp. \quad (-\Delta)^h u = \mu^h u = 0 \quad \forall x \in \partial\Omega \quad for \quad h = 0, ..., m-1)\,. \tag{7.21}$$

If this is not the case, we need to prove that the boundary conditions in (7.16) are satisfied in the weak sense, actually consider $v \in H^m_*(\Omega)$: we have by equation (7.19) that $\int_\Omega (-\Delta)^m u\ v = \mu^m \int_\Omega uv$, but integrating by parts m times we get the terms

$$\int_\Omega \nabla \cdot (\nabla^{2m-i} u \nabla^{(i-1)} v) = \int_{\partial\Omega} (\nabla^{2m-i} u \nabla^{(i-1)} v) \cdot n_{ext} \qquad for \quad i = 1, .., m \tag{7.22}$$

that, using (7.19), are of the form

$$\int_\Omega \nabla \cdot (\mu^{m-j} u \nabla^{(i-1)} v) \qquad for \quad i = 2j \tag{7.23}$$

$$\int_\Omega \nabla \cdot (\mu^{m-j} \nabla u \nabla^{(i-1)} v) \qquad for \quad i = 2j-1 \tag{7.24}$$

and then give rise to the boundary terms

$$\mu^{m-j} \int_\Omega (u \nabla^{(i-1)} v) \cdot n_{ext} \qquad for \quad i = 2j \tag{7.25}$$

$$\mu^{m-j} \int_\Omega (\nabla u \nabla^{(i-1)} v) \cdot n_{ext} \qquad for \quad i = 2j-1 \tag{7.26}$$

which are zero by the choice of v or by the boundary conditions in (7.18) which are satisfied at least in the weak sense.

So what remains is

$$\int_\Omega \nabla^m u \nabla^m v = \mu^m \int_\Omega uv \quad for \quad all \quad v \in H^m_*(\Omega)\,, \tag{7.27}$$

which is indeed the variational formulation of (7.16). □

Lemma 7.6. *If μ^m, u are eigenvalue and eigenfunction of (7.16) then μ, u are eigenvalue and eigenfunction of (7.18).*

Proof. Since μ^m, u are eigenvalue and eigenfunction of (7.16), u is $\mathcal{C}^\infty$ at least in the interior of Ω, and then we may define

$$\begin{aligned} u_1 &= u\,, \\ u_{h+1} &= -\frac{\Delta u_h}{\mu} = \left(\frac{-\Delta}{\mu}\right)^h u_1 \quad for \quad h = 1, .., m-1\,, \end{aligned} \tag{7.28}$$

and so obtain from $(-\Delta)^m u = \mu^m u$ that

$$-\Delta u_m = \mu u_1 \quad \forall x \in \Omega\,. \tag{7.29}$$

Then again if Ω has enough regularity we also have

$$\frac{\partial u_h}{\partial n} = \frac{\partial \left(\frac{-\Delta}{\mu}\right)^{h-1} u}{\partial n} = 0 \quad \forall x \in \partial\Omega \quad for \quad h = 0, ..., m-1 \tag{7.30}$$

$$(resp. \qquad u_h = \left(\frac{-\Delta}{\mu}\right)^{h-1} u = 0 \quad \forall x \in \partial\Omega \quad for \quad h = 0, ..., m-1)\,. \tag{7.31}$$

If this is not the case, we need to prove that the boundary conditions in (7.17) are satisfied in the weak sense, actually consider $\psi \in H^1_*(\Omega)$: we have by equation (7.28) and (7.29)

$$\int_\Omega (-\Delta u_h)\psi = \mu \int_\Omega u_{[h+1]}\psi \tag{7.32}$$

(where we denoted by $[h]$ the remainder class modulus m of the integer h).

But integrating by parts we get

$$\int_\Omega \nabla u_h \nabla \psi - \int_\Omega \nabla \cdot (\nabla u_h \psi) = \mu \int_\Omega u_{[h+1]}\psi \tag{7.33}$$

where the divergence term may be written by equation (7.28) as

$$\int_\Omega \nabla \cdot \left(\nabla \left[\left(\frac{-\Delta}{\mu}\right)^{h-1} u\right] \psi\right) = \frac{1}{\mu^{h-1}} \int_{\partial\Omega} \left(\nabla \left[\left(-\Delta\right)^{h-1} u\right] \psi\right) \cdot n \tag{7.34}$$

which then is zero by the choice of ψ in the case (D) or by the boundary conditions in (7.16) which are satisfied at least in the weak sense.

Then the vector $(u_1, .., u_m) \in [H^1_*(\Omega)]^m$ satisfies problem (7.17) in the weak sense, that is

$$\int_\Omega \sum_{h=1}^m \nabla u_h \nabla \psi_h - \mu \int_\Omega \sum_{h=1}^m u_{[h+1]}\psi_h = 0 \qquad \forall (\psi_1, .., \psi_m) \in [H^1_*(\Omega)]^m\,. \tag{7.35}$$

Now let $\theta = e^{\frac{2\pi i}{m}}$ and consider equation (7.35) with test functions $\psi_h = \theta^h \sum_{l=1}^m \theta^{-l} u_l$:

$$\begin{aligned} \int_\Omega \sum_{h=1}^m \theta^h \nabla u_h \left(\sum_{l=1}^m \theta^{-l} \nabla u_l\right) &= \mu \int_\Omega \sum_{h=1}^m \theta^h u_{[h+1]} \left(\sum_{l=1}^m \theta^{-l} u_l\right) = & (7.36) \\ = \int_\Omega \left(\sum_{h=1}^m \theta^h \nabla u_h\right)\left(\sum_{l=1}^m \theta^{-l} \nabla u_l\right) &= \mu\theta^{-1} \int_\Omega \left(\sum_{h=1}^m \theta^{[h+1]} u_{[h+1]}\right)\left(\sum_{l=1}^m \theta^{-l} u_l\right)\,. & (7.37) \end{aligned}$$

Since the sums in l are the conjugates of those in h, then the last equation reads

$$\int_\Omega \left|\sum_{h=1}^m \theta^h \nabla u_h\right|^2 = \mu\, \theta^{-1} \int_\Omega \left|\sum_{h=1}^m \theta^h u_h\right|^2 \tag{7.38}$$

which implies

$$\left|\sum_{h=1}^m \theta^h \nabla u_h\right| = \left|\sum_{h=1}^m \theta^h u_h\right| = 0 \quad a.e. \tag{7.39}$$

since otherwise the left hand side of (7.38) would be real and positive while the right hand side would not be.

The above procedure may be repeated with θ^j in place of θ for any $j = 1,..,m-1$, obtaining the $m-1$ equations $\sum_{h=1}^m \theta^{jh} u_h = 0$ a.e.

This may be written in linear system form $Ax = b$ with

$$\begin{aligned} A &= [\theta^{j*h}]_{j,h=1,..,m-1}\,, & (7.40)\\ x &= \{u_h\}_{h=1,...,m-1}\,, & (7.41)\\ b &= -\{u_m\}_{h=1,...,m-1}. & (7.42)\end{aligned}$$

Since $\sum_{h=1}^{m-1} \theta^{jh} = -1$ for any $j = 1, ..., m-1$, the vector $u_1 = u_2 = ... = u_m$ solves the system; in fact this is the unique solution since the matrix A is nonsingular: actually dividing the j^{th} row by θ^j one obtains the Vandermonde matrix $[\theta^{j(h-1)}]_{j,h=1,..,m-1}$, whose determinant is $\prod_{h<j}(\theta^j - \theta^h) \neq 0$.

Inserting $u_1 = u_2$ in (7.35) with $\psi_1 \in H^m_*(\Omega)$ and $\psi_h = 0$ for $h = 2,..,m$ gives

$$\int_\Omega \nabla u_1 \nabla \psi_1 - \mu \int_\Omega u_1 \psi_1 = 0 \qquad \forall \psi_1 \in H^1_*(\Omega)\,. \tag{7.43}$$

which is indeed the variational formulation of problem (7.18).

□

Now we may assert that all the properties claimed for the spectrum of the Laplacian are still valid in this case, actually since we proved that the eigenfunctions are the same we still have:

- The first eigenvalue is simple and related to a positive eigenfunction,
- The eigenvalues are all real and nonnegative and form a discrete set unbounded from above.
- To each eigenvalue corresponds a finite dimensional eigenspace,
- Eigenfunctions related to different eigenvalues are orthogonal in the L^2 scalar product.

Moreover it is easy to see that the eigenfunctions are orthogonal also in the H^m scalar product and that they form a basis for it.

Proof. By the variational equation for eigenfunction ϕ_i tested against ϕ_k we get

$$\int_\Omega \nabla^m \phi_i \nabla^m \phi_k = \lambda_i \int_\Omega \phi_i \phi_k \,, \tag{7.44}$$

so that the orthogonality in L^2 implies that in H^m.

Since $H^m_* \subseteq L^2$ we have for any $u \in H^m_*(\Omega)$ that $u = \sum_{i=1}^{+\infty} c_i \phi_i$ in L^2: we need to prove that this is true in H^m too, namely that $\left\| u - \sum_{i=1}^{N} c_i \phi_i \right\|_{H^m} \to 0$ for $N \to +\infty$.

Since $\|u\|^2_{H^m} = \|\nabla^m u\|^2_{L^2} + \|u\|^2_{L^2}$, the condition $u \in H^m$ implies $\left\|\nabla^m \sum_{i=1}^{+\infty} c_i \phi_i\right\|^2_{L^2} < +\infty$, that is
$\left\langle \nabla^m \sum_{i=1}^{+\infty} c_i \phi_i, \quad \nabla^m \sum_{i=1}^{+\infty} c_i \phi_i \right\rangle_{L^2} = \lim_{M,N \to +\infty} \left\langle \nabla^m \sum_{i=1}^{M} c_i \phi_i, \quad \nabla^m \sum_{i=1}^{N} c_i \phi_i \right\rangle_{L^2};$
now since the series are finite one may pass ∇^m inside and compute the product using $\langle \nabla^m \phi_i, \nabla^m \phi_j \rangle_{L^2} = \lambda_i^m \delta_{i,j}$, obtaining $\lim_{M,N\to+\infty} \left(\sum_{i=1}^{min\{M,N\}} c_i^2 \lambda_i^m \right) = \sum_{i=1}^{+\infty} c_i^2 \lambda_i^m < +\infty$.

On the other hand, reasoning in the same way,
$\left\| u - \sum_{i=1}^{N} c_i \phi_i \right\|^2_{H^m} = \left\| \sum_{i=N+1}^{+\infty} c_i \phi_i \right\|^2_{H^m} = \sum_{i=N+1}^{+\infty} c_i^2 \lambda_i^m + \sum_{i=N+1}^{+\infty} c_i^2$:
since it is the tail of a converging series with nonnegative terms it has to tend to zero for $N \to +\infty$.

□

For what concerns the variational characterization of the eigenvalues we still have

$$\lambda_1^m = \inf \left\{ \int_\Omega |\nabla^m u|^2 : \qquad u \in H^m_*(\Omega); \quad \|u\|_{L^2} = 1 \right\} , \tag{7.45}$$

actually we have $\lambda_1^m \geq \inf \left\{ \int_\Omega |\nabla^m u|^2 \right\}$ since $\int_\Omega |\nabla^m \phi_1|^2 = \lambda_1^m$; but any minimizing sequence converges weakly to a minimizer satisfying, by the Lagrange's multipliers rule, $\int_\Omega \nabla^m u \nabla^m v - \alpha \int_\Omega uv = 0$ for all $v \in H^m_*$, and as usual testing with $v = u$ one gets $\alpha = \inf \left\{ \int_\Omega |\nabla^m u|^2 \right\}$ and so u is an eigenfunction, implying that $\inf \left\{ \int_\Omega |\nabla^m u|^2 \right\} \geq \lambda_1^m$.

For the characterization of the following eigenvalues, since we saw that the structure of the space is the same as in the case of the Laplacian, one may proceed as in section 2.3.1 with

$$J_a(u) = \int_\Omega |\nabla^m u|^2 - a \int_\Omega u^2 \tag{7.46}$$

(see also the proof in lemma 4.17).

7.3 Variational characterization of the Fučík spectrum for the multi-Laplacian

Consider now the Fučík problem

$$\begin{cases} (-\Delta)^m u = \lambda^+ u^+ - \lambda^- u^- & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = ... = \frac{\partial \Delta^{m-1} u}{\partial n} = 0 \\ \qquad or \\ u = \Delta u ... = \Delta^{m-1} u = 0 \end{array} \right. & on\ \partial\Omega \end{cases} , \tag{7.47}$$

where $u^+(x) = \max\{0, u(x)\}$ and $u^-(x) = \max\{0, -u(x)\}$.

In analogy with the Laplacian case we define the Fučík spectrum as the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a non trivial solution of the above problem.

In section 7.1 and 7.2 we showed that, provided Ω is of class $\mathcal{C}^m$, the space $H_*^m(\Omega)$ equipped with the norm $\left(\|\nabla^m u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{\frac{1}{2}}$ has the same properties of $H_*^1(\Omega)$ equipped with the norm $\left(\|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{\frac{1}{2}}$: this implies that the same argument used in section 4 may be repeated substituting the terms of the kind $\int_\Omega \nabla u \nabla v$ with the corresponding term $\int_\Omega \nabla^m u \nabla^m v$.

Then we may assert that

Theorem 7.7. *Let Σ be the Fučík spectrum corresponding to problem (7.47), suppose that Ω is of class $\mathcal{C}^m$ and the point $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ with $\alpha^+ \geq \alpha^-$ is Σ-connected to the diagonal between λ_k^m and λ_{k+1}^m in the sense of definition 4.1, then we can find and characterize one intersection of the Fučík spectrum with the halfline $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$, for each value of $r \in (0, 1]$.*

Moreover also the properties of the variational characterization proved in section 4.3 may be extended to this case.

7.4 The superlinear problem under hypothesis (HN) (resp. (HD))

In this section we will show the existence of a linking structure for functional (7.6) in order to prove the existence of a solution for problems (7.1) and (7.2), for suitable values of the parameter λ.

The approach is inspired by the one used in [dFR91] and [Vil98] for the Laplacian in dimension one.

Given $u \in H_*^m(\Omega)$ with m satisfying hypothesis (HN) (resp. (HD)), we define:

$$c(u) = \sup_{x \in \Omega} \frac{u(x)}{\phi_1(x)}\,. \tag{7.48}$$

Remark 7.8. *In the case (N), ϕ_1 is the constant function and so (suppose without loss of generality $|\Omega| = 1$) $c(u) = \sup_{x\in\Omega}[u(x)]$, which is finite by the inclusion $H^m(\Omega) \subseteq \mathcal{C}^0(\bar{\Omega})$.*

In the case (D), ϕ_1 is the first eigenfunction of the Laplacian, which is known to have the property that $\inf_{x\in\partial\Omega} \frac{\partial \phi_1}{\partial n_{int}}(x) > 0$; this property and the inclusion $H^m(\Omega) \subseteq \mathcal{C}^1(\bar{\Omega})$ implies that $c(u)$ is finite also in this case.

Then we define

$$E = \left\{ u \in H_*^m(\Omega) : \int_\Omega u\phi_1 = 0 \right\}, \tag{7.49}$$

$$S_0 = \{ u \in H_*^m(\Omega) : \quad c(u) = 0 \}, \tag{7.50}$$

$$\gamma = \inf \left\{ \frac{\int_\Omega |\nabla^m u|^2}{\int_\Omega u^2} \quad \textit{with} \quad u \in S_0 \backslash \{0\} \right\}. \tag{7.51}$$

First we will prove some properties of the objects defined above:

Lemma 7.9. *$c : H_*^m(\Omega) \to \mathbb{R} : u \mapsto c(u)$ is a continuous function.*

Proof. In the case (N) we have

$$|c(u) - c(v)| \leq \|u - v\|_{L^\infty(\Omega)} \leq C \|u - v\|_{H^m_N(\Omega)} \tag{7.52}$$

by hypothesis (HN).

In the case (D) we have

$$|c(u) - c(v)| \leq \left\| \frac{u - v}{\phi_1} \right\|_{L^\infty(\Omega)} . \tag{7.53}$$

To estimate the last norm, note that since ϕ_1 is $\mathcal{C}^1(\bar{\Omega})$ and vanishes on the $\mathcal{C}^1$ boundary $\partial\Omega$ and since $\eta = \inf_{\xi \in \partial\Omega} \frac{\partial \phi_1}{\partial n_{int}}(\xi) > 0$, we may estimate

$$\phi_1(x) \geq \frac{\eta}{2} d(x, x_0) \quad for \quad d(x, x_0) < \delta(x_0) \tag{7.54}$$

where $x_0 \in \partial\Omega$ is such that $d(x, \partial\Omega) = d(x, x_0)$.

Then let

$$\delta = \min_{\xi \in \partial\Omega} (\delta(\xi)) \tag{7.55}$$

and define

$$\omega_\delta = \{x \in \Omega : d(x, \partial\Omega) > \delta\}, \qquad \partial\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\} \tag{7.56}$$

and

$$\mu = \inf_{\xi \in \omega_\delta} \phi_1(\xi) . \tag{7.57}$$

Now we may estimate for $w \in H^m_D(\Omega)$

$$\left\| \frac{w}{\phi_1} \right\|_{L^\infty(\Omega)} \leq \left\| \frac{w}{\phi_1} \right\|_{L^\infty(\partial\Omega_\delta)} + \left\| \frac{w}{\phi_1} \right\|_{L^\infty(\omega_\delta)} , \tag{7.58}$$

where, using $w(x) \leq \|w\|_{\mathcal{C}^1(\bar{\Omega})} d(x, \partial\Omega)$,

$$\left\| \frac{w}{\phi_1} \right\|_{L^\infty(\partial\Omega_\delta)} \leq \frac{\|w\|_{\mathcal{C}^1(\bar{\Omega})}}{(\eta/2)} , \tag{7.59}$$

$$\left\| \frac{w}{\phi_1} \right\|_{L^\infty(\omega_\delta)} \leq \frac{\|w\|_{\mathcal{C}^0(\Omega)}}{\mu} ; \tag{7.60}$$

then, since the constants δ, η and μ depend only on ϕ_1 and hence only on Ω, we conclude

$$|c(u) - c(v)| \leq \left\| \frac{u - v}{\phi_1} \right\|_{L^\infty(\Omega)} \leq C_1 \|u - v\|_{\mathcal{C}^1(\bar{\Omega})} \leq C_2 \|u - v\|_{H^m_D(\Omega)} \tag{7.61}$$

by the hypothesis (HD).

□

Lemma 7.10. *The set S_0 is homeomorphic to E, moreover S_0 divides $H^m_*(\Omega)$ into two components containing respectively $\{t\phi_1 : \quad t > 0\}$ and $\{t\phi_1 : \quad t < 0\}$.*

Proof. The map $M : E \to S_0 : u \mapsto u - c(u)\phi_1$ is continuous by the previous lemma and has the orthogonal projection on E as its inverse, so it is homeomorphism.

Moreover, it is clear by the definitions that $H^m_*(\Omega)$ is divided in the two components $\{u \in H^m_*(\Omega) : \quad c(u) > 0\}$ and $\{u \in H^m_*(\Omega) : \quad c(u) < 0\}$. $\square$

Lemma 7.11. *$\gamma > \lambda^m_1$ and there exists $u \in S_0 \backslash \{0\}$ such that $\gamma = \frac{\int_\Omega |\nabla^m u|^2}{\int_\Omega u^2}$.*

Proof. Let us take a minimizing sequence $\{u_n\} \subseteq S_0 \backslash \{0\}$. By the homogeneity of the definition of γ and S_0 we may assume $\|u_n\|_{L^2} = 1$; since $\int_\Omega |\nabla^m u_n|^2 \to \gamma$, u_n is bounded in H^m_* and we can extract a subsequence such that $u_n \to u$ weakly in H^m_* and strongly in L^2 and in $\mathcal{C}^0(\bar{\Omega})$ (resp. in $\mathcal{C}^1(\bar{\Omega})$) by hypothesis (HN) (resp. (HD)).

The strong convergences implies that $c(u) = 0$ and $\|u\|_{L^2} = 1$ and so $u \in S_0 \backslash \{0\}$.

Then $\int_\Omega |\nabla^m u|^2 \geq \gamma$ by the definition of γ, but by the weak convergence this implies $\int_\Omega |\nabla^m u|^2 = \gamma$ and so u realizes the value γ.

Finally $\gamma \geq \lambda^m_1$ by the variational characterization of λ^m_1 and if, by contradiction, $\gamma = \lambda^m_1$, then the minimizer would be a multiple of ϕ_1, which is a contradiction since $span\{\phi_1\} \cap S_0 = \{0\}$. $\square$

Now we proceed to prove the existence of the linking structure for the functional.

We will use the same estimates used in section 5.1 on page 53.

Lemma 7.12. $\lim_{\rho \to +\infty} F(\rho\phi_1) = -\infty$.

Proof. Remembering that $\phi_1 > 0$ in Ω we estimate

$$\begin{aligned} \frac{F(\rho\phi_1)}{\rho^2} &= \frac{1}{2}\int_\Omega |\nabla^m \phi_1|^2 - \frac{\lambda}{2}\int_\Omega \phi_1^2 - \int_\Omega \frac{G(x,\rho\phi_1)}{\rho^2} - \int_\Omega \frac{h\rho\phi_1}{\rho^2} \\ &\leq \frac{\lambda^m_1 - \lambda}{2}\int_\Omega \phi_1^2 - \left(\frac{M}{2}\int_\Omega \phi_1^2 - \frac{C_3(M,g)}{\rho^2}\right) + \left(\frac{\delta}{2}\int_\Omega \phi_1^2 + \frac{C_1(\delta,h)}{\rho^2}\right) \\ &\leq \frac{\lambda^m_1 - \lambda - M + \delta}{2} + \frac{C_1(\delta,h) + C_3(M,g)}{\rho^2}; \end{aligned}$$

then choosing $M > \lambda^m_1 - \lambda + \delta$ the lemma is proved. $\square$

Lemma 7.13. *If*

- $\lambda > \lambda^m_1$

or

- $\lambda = \lambda^m_1$, $\int_\Omega h\phi_1 < 0$ *and hypothesis (HR0-m) holds,*

then $\lim_{\rho \to +\infty} F(-\rho\phi_1) = -\infty$.

Proof. Estimating as before we now get for $\lambda > \lambda^m_1$

$$\begin{aligned} \frac{F(-\rho\phi_1)}{\rho^2} &= \frac{1}{2}\int_\Omega |\nabla^m \phi_1|^2 - \frac{\lambda}{2}\int_\Omega \phi_1^2 - \int_\Omega \frac{G(x,-\rho\phi_1)}{\rho^2} - \int_\Omega \frac{-h\rho\phi_1}{\rho^2} \\ &\leq \frac{\lambda^m_1 - \lambda}{2}\int_\Omega \phi_1^2 + \left(\frac{\delta}{4}\int_\Omega \phi_1^2 + \frac{C_2(\delta,g)}{\rho^2}\right) + \left(\frac{\delta}{4}\int_\Omega \phi_1^2 + \frac{C_1(\delta,h)}{\rho^2}\right) \\ &\leq \frac{\lambda^m_1 - \lambda + \delta}{2} + \frac{C_1(\delta,h) + C_2(\delta,g)}{\rho^2}; \end{aligned}$$

then choosing $\delta < \lambda - \lambda_1^m$ the first part of the lemma is proved.

For $\lambda = \lambda_1^m$ we need a finer estimate.

Since $\lim_{s\to-\infty} g(x,s) = 0$ we may estimate:

for any $\varepsilon > 0$ there exists C_ε such that
$|g(x,s)| \le \varepsilon + \frac{C_\varepsilon}{|s-1|^2}$, $\qquad \forall s \le 0,$

and then also

$$|G(x,s)| \le \varepsilon|s| + \frac{C_\varepsilon}{|s-1|}, \qquad \forall s \le 0.$$

Then

$$\left|\int_\Omega \frac{G(x,-\rho\phi_1)}{\rho}\right| \le \int_\Omega \varepsilon\phi_1 + \frac{C_\varepsilon}{\rho(1+\rho\phi_1)} \le \left(\varepsilon + \frac{C_\varepsilon}{\rho}\right)|\Omega| \tag{7.62}$$

and so

$$\limsup_{\rho\to+\infty}\left|\int_\Omega \frac{G(x,-\rho\phi_1)}{\rho}\right| \le \varepsilon|\Omega| \tag{7.63}$$

for any choice of ε, that is it is zero.

Then we conclude

$$\lim_{\rho\to+\infty}\frac{F(-\rho\phi_1)}{\rho} = \rho\frac{\lambda_1^m - \lambda}{2} + \int_\Omega h\phi_1 \tag{7.64}$$

which for $\lambda = \lambda_1^m$ and $\int_\Omega h\phi_1 < 0$ implies that this last limit is negative and so the second part of the lemma is proved too.

$\square$

Lemma 7.14. *For $\lambda < \gamma$, $F|_{S_0}$ is bounded from below.*

Proof. For $u \in S_0$ we have $u(x) \le 0$ and $\int_\Omega |\nabla^m u|^2 \ge \gamma \|u\|_{L^2}^2$, then we may estimate:

$$\begin{aligned}
F(u) &= \frac{1}{2}\int_\Omega |\nabla^m u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega G(x,u) - \int_\Omega hu && (7.65)\\
&\ge \frac{\gamma-\lambda}{2}\|u\|_{L^2}^2 - \left(\frac{\delta}{4}\int_\Omega u^2 + C_2(\delta,g)\right) - \left(\frac{\delta}{4}\int_\Omega u^2 + C_1(\delta,h)\right) && (7.66)\\
&\ge \frac{\gamma-\lambda-\delta}{2}\int_\Omega u^2 - C_2(\delta,g) - C_1(\delta,h) && (7.67)
\end{aligned}$$

and so it is enough to choose $\delta < \gamma - \lambda$ to obtain $F(u) \ge -C_2(\delta,g) - C_1(\delta,h)$.

$\square$

Finally in section 9.2 we will prove the following

Lemma 7.15. *For Ω of class $\mathcal{C}^m$, under hypotheses (HN) (resp. (HD)), (H1-m) and (H2-m), with $h \in L^2(\Omega)$, the functional (7.6) defined in $H_N^m(\Omega)$ (resp. in $H_D^m(\Omega)$) satisfies the PS condition for $\lambda \in (\lambda_1^m, \gamma)$.*

Moreover under hypothesis (HR0-m) and $\int_\Omega h\phi_1 < 0$ it satisfies the PS condition also for $\lambda = \lambda_1^m$.

The previous lemmas allow one to apply the generalized mountain pass theorem to get a solution of problem (7.1) and (7.2).

In fact, define

$$f = \inf_{\gamma \in \Gamma_R} \sup_{u \in \gamma([0,1])} F(u) \tag{7.68}$$

where

$$\Gamma_R = \{\gamma : [0,1] \to H^m_*(\Omega) \quad \textit{continuous} \quad \textit{s.t.} \quad \gamma(0) = -R\phi_1 \quad \textit{and} \quad \gamma(1) = R\phi_1\} : \tag{7.69}$$

provided R is large enough to have $F(\pm R\phi_1) < -C_2(\delta, g) - C_1(\delta, h)$ where δ is the value fixed in the proof of lemma 7.14, one may apply the deformation lemma and then prove that f is a free critical value for F.

In particular, the condition $\int_\Omega h\phi_1 < 0$ for $\lambda = \lambda_1^m$ is necessary: considering the variational equation with test function ϕ_1 one gets

$$\int_\Omega \nabla^m u \nabla^m \phi_1 - \frac{\lambda_1^m}{2} \int_\Omega u\phi_1 - \int_\Omega g(x,u)\phi_1 - \int_\Omega h\phi_1 = 0, \tag{7.70}$$

that is $-\int_\Omega g(x,u)\phi_1 - \int_\Omega h\phi_1 = 0$ which by hypothesis (HR0-m) implies $\int_\Omega h\phi_1 < 0$.

Then the results achieved are:

Theorem 7.16. *For Ω of class $\mathcal{C}^m$, under hypotheses (HN) (resp. (HD)), (H1-m), (H2-m) and (HR0-m), if $h \in L^2(\Omega)$ and $\lambda = \lambda_1^m$, then there exists a solution of problem (7.1) (resp. (7.2)) if and only if $\int_\Omega h\phi_1 < 0$.*

Theorem 7.17. *For Ω of class $\mathcal{C}^m$, under hypotheses (HN) (resp. (HD)), (H1-m) and (H2-m), if $\lambda \in (\lambda_1^m, \gamma)$, then there exists a solution of problem (7.1) (resp. (7.2)) for all $h \in L^2(\Omega)$; where γ is defined in equation (7.51).*

Remark 7.18. *In the case (N) hypothesis (HN) allows $m = 1$ provided $N = 1$, actually in this case theorem 7.16 and 7.17 correspond to the result in [dFR91].*

In the case (D) hypothesis (HD) implies $m \geq 2$, even in dimension one.

7.4.1 The fourth order one dimensional case

In dimension one and with $m = 2$ we can find the minimizing functions of (7.51), and then the value of γ; we will proceed in a way similar to [Vil98].

Let $\Omega = (0,1)$: we will start considering the case (N):

- Claim: the minimizer of (7.51) satisfies $u(x) < 0 \quad \forall x \in (0,1)$.

  Proof of the claim. In dimension 1 we have that $H^2_N(0,1) \subseteq \mathcal{C}^1([0,1])$, so if $u(x_0) = 0$ with $x_0 \in (0,1)$, since $u \in S_0$ then x_0 is a maximum and so $u'(x_0) = 0$; this implies that $u_l(x) = u(x_0 x)$ and $u_r(x) = u(1-(1-x_0)(1-x))$ with $x \in (0,1)$ are both in S_0, and it can be seen that one of them realizes a lower value of $\frac{\int_0^1 |u''|^2}{\int_0^1 u^2}$.

Actually observe that

$$\int_0^1 u_l^2 = \frac{1}{x_0}\int_0^{x_0} u^2\,, \tag{7.71}$$

$$\int_0^1 |u_l''|^2 = x_0^3\int_0^{x_0} |u''|^2\,, \tag{7.72}$$

and analogous equations hold for u_r with coefficients $\frac{1}{1-x_0}$ and $(1-x_0)^3$.

So if u_l or u_r is identically zero then the ratio realized by the other is smaller by a factor x_0^4 or $(1-x_0)^4$ than that realized by u.

If both u_l and u_r are different from zero, remark that given the reals $a, b, c, d > 0$ one has

$$\frac{a+b}{c+d} \geq \min\left\{\frac{a}{c}, \frac{b}{d}\right\} : \tag{7.73}$$

indeed suppose $\frac{a+b}{c+d} \leq \min\left\{\frac{a}{c}, \frac{b}{d}\right\}$, then one obtain $\frac{a+b}{a} \leq \frac{c+d}{c}$ and $\frac{a+b}{b} \leq \frac{c+d}{d}$ and so $\frac{b}{a} \leq \frac{d}{c}$ and $\frac{a}{b} \leq \frac{c}{d}$ which implies $\frac{a}{c} = \frac{b}{d} = \frac{a+b}{c+d}$.

So we have

$$\frac{\int_0^1 |u''|^2}{\int_0^1 u^2} = \frac{\int_0^{x_0} |u''|^2 + \int_{x_0}^1 |u''|^2}{\int_0^{x_0} u^2 + \int_{x_0}^1 u^2} \tag{7.74}$$

$$= \frac{x_0^{-3}\int_0^1 |u_l''|^2 + (1-x_0)^{-3}\int_0^1 |u_r''|^2}{x_0\int_0^1 u_l^2 + (1-x_0)\int_0^1 u_r^2} \tag{7.75}$$

$$\geq \min\left\{x_0^{-4}\frac{\int_0^1 |u_l''|^2}{\int_0^1 u_l^2}, (1-x_0)^{-4}\frac{\int_0^1 |u_r''|^2}{\int_0^1 u_r^2}\right\} \tag{7.76}$$

and so, since $x_0 \in (0,1)$, one of the two ratios in the minimum is strictly less than the left hand side.

□

- The previous claim implies that the minimizer needs to satisfy $u(0) = 0$ or $u(1) = 0$ and so, by symmetry, we may look for a minimizer with $u(1) = 0$.

  In particular we consider the problem

$$\delta = \inf\left\{\frac{\int_0^1 |u''|^2}{\int_0^1 u^2} \quad \textit{with} \quad u \in H^2_{N0}(0,1)\backslash\{0\}\right\}, \tag{7.77}$$

  where

$$H^2_{N0}(0,1) = \{u \in H^2_N(0,1) : \quad u(1) = 0\} : \tag{7.78}$$

  if we show that the minimizer of (7.77) is in $S_0\backslash\{0\}$ then it is also the minimizer we are looking for and so $\delta = \gamma$.

The minimizer of (7.77) must satisfy the eigenvalue problem

$$\begin{cases} u'''' = \delta u & in\ (0,1) \\ u'(0) = u'''(0) = 0 & \\ u(1) = u'(1) = 0 & \end{cases}, \tag{7.79}$$

in fact by reformulating problem (7.77) as $\delta = \inf \left\{ \int_0^1 |u''|^2 : \quad u \in H^2_{N0}(0,1), \quad \int_0^1 u^2 = 1 \right\}$, by Lagrange's multipliers rule we get

$$\int_0^1 u''v'' = \delta \int_0^1 uv \quad \forall v \in H^2_{N0}(0,1)\,, \tag{7.80}$$

where by a boot strap argument one gets that u is smooth and so integrating by parts we get $\int_0^1 u''''v = \delta \int_0^1 uv + [u'''v]_0^1 - [u''v']_0^1$ where the conditions on v kills all the boundary terms except $u'''(0)v(0)$, and so equation (7.80) implies $u'''(0) = 0$.

- Setting $q^4 = \delta$ with $q > 0$, the solutions of (7.79) are of the form

$$A\cos(qx) + B\sin(qx) + C\sinh(qx) + D\cosh(qx)\,; \tag{7.81}$$

from $u'(0) = u'''(0) = 0$ we get $B = C = 0$ and forcing the remaining conditions we get

$$\frac{A}{D} = -\frac{\cosh(q)}{\cos(q)} = \frac{\sinh(q)}{\sin(q)}\,. \tag{7.82}$$

To have the minimal value of δ we get the first positive solution of $\tanh(q) = -\tan(q)$, which is in $(\frac{\pi}{2}, \pi)$, so $\sin(q) > 0$ and the resulting minimizer is

$$\tilde{u} = A\left(\cos(qx) + \cosh(qx)\frac{\sin(q)}{\sinh(q)}\right) : \qquad A < 0\,. \tag{7.83}$$

Observe that the zeros of $\tilde{u}$ are solutions of $\frac{\cos(qx)}{\sin(q)} = -\frac{\cosh(qx)}{\sinh(q)}$ and so since we chose q to be the first positive solution of $\tanh(q) = -\tan(q)$ we have no zeros in $(0,1)$ and so $\tilde{u} \in S_0 \backslash \{0\}$.

We conclude that

Proposition 7.19. *In the case (N), with $m = 2$ and $\Omega = (0,1)$, we have $\gamma = q^4$ where q is the first positive solution of $\tanh(q) = -\tan(q)$; moreover $\tilde{u}$ in (7.83) is a minimizer for (7.51).*

An approximate value for γ is $0.32\pi^4$ ($q = 0.753\pi$).

Now consider the case (D):

- As before we have that the minimizer satisfies $u(x) < 0 \quad \forall x \in (0,1)$; in fact, as before, if it were not so we still would be able to find another function in S_0 realizing a lower value of $\frac{\int_0^1 |u''|^2}{\int_0^1 u^2}$.

- Since $u<0$ in $(0,1)$, but $u\in S_0$ implies that $\sup_{x\in(0,1)}(u(x)+\varepsilon\phi_1(x))>0$ for any $\varepsilon>0$, we deduce that $u'=0$ in 0 or in 1; then, by symmetry, we may look for a minimizer with $u'(1)=0$.

  In particular we consider the problem

$$\delta = \inf\left\{\frac{\int_0^1 |u''|^2}{\int_0^1 u^2} \quad with \quad u\in H^2_{D0}(0,1)\backslash\{0\}\right\}, \tag{7.84}$$

  where

$$H^2_{D0}(0,1)=\{u\in H^2_D(0,1):\quad u'(1)=0\}\,. \tag{7.85}$$

  Again if we show that the minimizer of (7.84) is in $S_0\backslash\{0\}$ then it is also the minimizer we are looking for and so $\delta=\gamma$.

  Proceeding as in the case (N) one sees that the minimizer of (7.84) needs to satisfy the eigenvalue problem

$$\begin{cases} u''''=\delta u & in\ (0,1) \\ u(0)=u''(0)=0 \\ u(1)=u'(1)=0 \end{cases}. \tag{7.86}$$

- Finally imposing the boundary conditions to (7.81) we obtain $A=D=0$ by the conditions $u(0)=u''(0)=0$ and forcing the remaining ones

$$\frac{B}{C}=-\frac{\sinh(q)}{\sin(q)}=-\frac{\cosh(q)}{\cos(q)}\,, \tag{7.87}$$

  so we have to look for the first positive solution of $\tanh(q)=\tan(q)$, which will be in $(\pi,\frac{3\pi}{2})$, so $\cos(q)<0$ and the resulting minimizer is

$$\tilde u = B\left(\sin(qx)-\sinh(qx)\frac{\cos(q)}{\cosh(q)}\right):\qquad B<0\,. \tag{7.88}$$

  Again the zeros of $\tilde u$ are solutions of $\frac{\sin(qx)}{\cos(q)}=\frac{\sinh(qx)}{\cosh(q)}$ and so there are no zeros in $(0,1)$ implying that $\tilde u\in S_0\backslash\{0\}$.

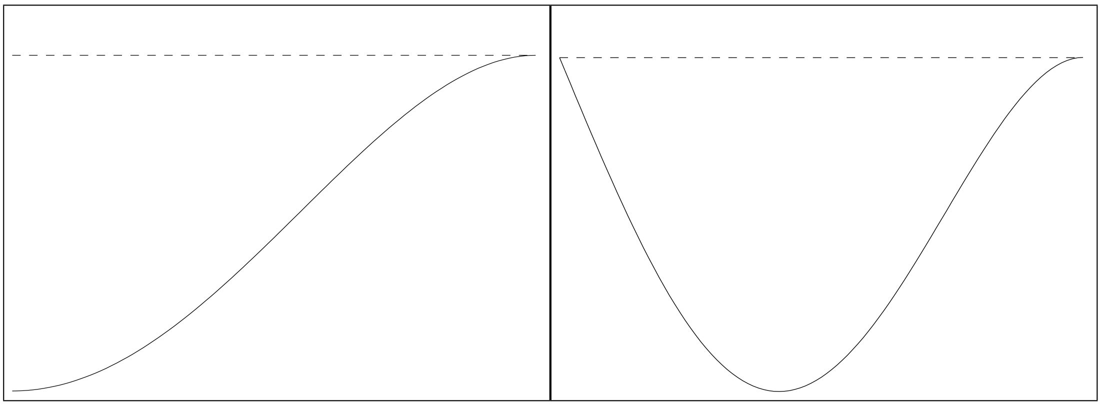
Then we conclude:

Proposition 7.20. *In the case (D), with $m=2$ and $\Omega=(0,1)$, we have $\gamma=q^4$ where q is the first positive solution of* $\tanh(q)=\tan(q)$*; moreover $\tilde u$ in (7.88) is a minimizer for (7.51).*
An approximate value for γ is $2.44\pi^4$ ($q=1.2499\pi$).

In figure 10, we plot the shape of the minimizers $\tilde u$ for the case (N) (on the left) and the case (D) (on the right).

We remark that in both cases $\gamma\in(\lambda_1^2,\lambda_2^2)$, which is $(0,\pi^4)$ in the case (N) and $(\pi^4,16\pi^4)$ in the case (D).

Figure 10: Minimizers of (7.51) in the fourth order one dimensional case (case (N) and case (D)).



7.5 Fučík spectrum for the fourth order operator on an interval

In this section we will discuss the Fučík spectrum on an interval for the operator $(-\Delta)^2$, that is the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a non trivial solution of the problem

$$\begin{cases} u'''' = \lambda^+ u^+ - \lambda^- u^- \quad in \;\; (0,1) \\ \left[\begin{array}{l} u'(0) = u'''(0) = 0 \\ u'(1) = u'''(1) = 0 \\ \quad or \\ u(0) = u''(0) = 0 \\ u(1) = u''(1) = 0 \end{array} \right. \end{cases} , \tag{7.89}$$

where $u^+(x) = \max\{0, u(x)\}$ and $u^-(x) = \max\{0, -u(x)\}$; as before we will refer to the first type of boundary conditions as (N) and to the second as (D).

The results of this section will allow us to extend the results in theorem 7.17 and 7.16 to larger values of the parameter λ, using the techniques of section 5 (theorem 7.34 and 7.35).

The case (D) has been considered in [CD01]; in the following we will follow that work and show the corresponding results for the case (N).

First observe that:

Lemma 7.21. *Any weak solution $\tilde{u}$ of problem (7.89) is a classical solution too.*

Proof. Let $h = \lambda^+ \tilde{u}^+ - \lambda^- \tilde{u}^-$: since $\tilde{u} \in H^2$ we have $h \in H^1$; since it is a weak solution, $\tilde{u} \in H^2_*$ satisfies

$$\int_0^1 \tilde{u}'' \phi'' = \int_0^1 h\phi \quad for \quad all \quad \phi \in H^2_* . \tag{7.90}$$

Define $\quad v = -\tilde{u}'' \quad$ in the weak sense, that is

$$\int_0^1 \tilde{u}' \psi' = \int_0^1 v\psi \quad for \quad all \quad \psi \in H^1_* ; \tag{7.91}$$

since $\tilde{u} \in H^2_*$, $v \in L^2$ and then (7.90) may be written as

$$\int_0^1 -v\phi'' = \int_0^1 h\phi \quad for \quad all \quad \phi \in H^2_* \,. \tag{7.92}$$

Let now $w \in H^1_*$ be a weak solution of $\; -w'' = h \;$ with the given boundary conditions, that is

$$\int_0^1 w'\phi' = \int_0^1 h\phi \quad for \quad all \quad \phi \in H^1_* \,; \tag{7.93}$$

then for $\phi \in H^2_*$ we may integrate by parts in (7.93) and subtract from (7.92) to obtain

$$\int_0^1 (w-v)\phi'' = 0 \quad for \quad all \quad \phi \in H^2_* \,; \tag{7.94}$$

this means that $(v-w)'' = 0$ in $(H^2_*)'$, but since in the chain of embeddings $H^2_* \hookrightarrow H^1_* \hookrightarrow L^2 \hookrightarrow (H^1_*)' \hookrightarrow (H^2_*)'$ the function zero belongs to all of the spaces, this implies $\|(v-w)''\|_{H^2_*} = 0$, and so $v - w = ax + b$ in H^2_*, which is then an arbitrary constant in the case (N) and 0 in the case (D).

Since $w \in H^1_*$ this gives $v \in H^1_*$ too and then integrating by parts (7.92), we may apply lemma 2.16 to get $v \in H^3$ and then by (7.91) and using again lemma 2.16 one has $\tilde{u} \in H^5 \subseteq \mathcal{C}^4([0,1])$. □

To build the Fučík spectrum we will first build a related set in $\mathbb{R}^3$: define, in the case (N),

$$\widetilde{\Sigma}^{\pm} = \left\{ \begin{array}{c} (\lambda^+, \lambda^-, s) \in \mathbb{R}^3 \quad such \quad that \quad the \quad solution \quad of \quad the \quad IVP \\ u'''' = \lambda^+ u^+ - \lambda^- u^- \,, \qquad (u, u', u'', u''')(0) = (\pm 1, 0, s, 0) \\ satisfies \quad u'(1) = u'''(1) = 0 \end{array} \right\} ; \tag{7.95}$$

in the case (D) this will be

$$\widetilde{\Sigma}^{\pm} = \left\{ \begin{array}{c} (\lambda^+, \lambda^-, s) \in \mathbb{R}^3 \quad such \quad that \quad the \quad solution \quad of \quad the \quad IVP \\ u'''' = \lambda^+ u^+ - \lambda^- u^- \,, \qquad (u, u', u'', u''')(0) = (0, \pm 1, 0, s) \\ satisfies \quad u(1) = u''(1) = 0 \end{array} \right\} . \tag{7.96}$$

Then we will denote by

$$\Sigma^{\pm} = \{(\lambda^+, \lambda^-) \in \mathbb{R}^2 \quad such \quad that \quad \exists s \in \mathbb{R} : \quad (\lambda^+, \lambda^-, s) \in \widetilde{\Sigma}^{\pm}\} \,, \tag{7.97}$$

and so $\Sigma = \Sigma^+ \cup \Sigma^-$.

In particular we only need to study one of the two components (say $\widetilde{\Sigma}^+$), since the other is analogous if we exchange λ^+ and λ^-.

As usual it is simple to see that the lines $\{\lambda^+ = \lambda_1^2\}$ and $\{\lambda^- = \lambda_1^2\}$ are in Σ (since the eigenfunction ϕ_1 does not change sign), while the rest of Σ lies in the quadrant $\{\lambda^{\pm} > \lambda_1^2\}$ and corresponds to nontrivial solutions which change sign.

7.5.1 Qualitative description

In the following we will try to describe $\widetilde{\Sigma}$ and Σ. In particular we will use the implicit function theorem to describe them locally (lemma 7.23) and then we will give a qualitative but global description making use of the local information obtained and of some topological properties of these sets (proposition 7.27).

First observe that any solution of (7.89) which vanishes on a set of measure zero, may be seen as satisfying the linear equation $u'''' = c(x)u$ where $c(x) = \lambda^+\chi_{\{u>0\}}(x) + \lambda^-\chi_{\{u<0\}}(x)$ is a $L^\infty(0,1)$, a.e. positive function (we denoted by $\chi_A(x)$ the characteristic function of the set A).

Because of the above observations the following lemma will be useful:

Lemma 7.22. *Let $c \in L^\infty(0,1)$, $c(x) > 0$ a.e.*

Let u be a nontrivial solution of the boundary value problem (BVP)

$$\begin{cases} u'''' = c(x)u & in \quad (0,1) \\ u'(0) = u'''(0) = 0 \\ u'(1) = u'''(1) = 0 \end{cases}, \tag{7.98}$$

then

- $u(x)\ u''(x) < 0$ *for* $x \in \{0;1\}$;
- $u(x) = 0 \Rightarrow u'(x) \neq 0$ *for* $x \in (0,1)$.

Similarly let u be a nontrivial solution of the BVP

$$\begin{cases} u'''' = c(x)u & in \quad (0,1) \\ u(0) = u''(0) = 0 \\ u(1) = u''(1) = 0 \end{cases}, \tag{7.99}$$

then

- $u'(x)\ u'''(x) < 0$ *for* $x \in \{0;1\}$;
- $u(x) = 0 \Rightarrow u'(x) \neq 0$ *for* $x \in (0,1)$.

Proof. We will consider the first case, the second is similar and the proof is given in [CD01].

Consider $x = 0$: if $u(0) = 0$ then $u''(0) \neq 0$ or u would be identically zero by uniqueness of the IVP.

By linearity we may suppose $u''(0) > 0$, and then for some $\varepsilon > 0$

$u(x) > 0$ and $u'(x) > 0$ for all $x \in (0,\varepsilon)$.

Let t_0 be the first zero of u' in $(0,1]$, then $u > 0$ in $(0,t_0)$ and integrating the differential equation we get

$$u'(t_0) = u''(0)t_0 + \int_0^{t_0} d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} c(\xi_3)u(\xi_3)d\xi_3 > 0 \tag{7.100}$$

and so u may not satisfy $u'(1) = 0$.

Then $u(0) \neq 0$, by linearity we may suppose $u(0) > 0$; using again (7.100) we get that if u remains positive then $u''(0) < 0$ in order to satisfy $u'(1) = 0$, otherwise let now t_0 be the first zero of u, then

$$0 = u(t_0) = u(0) + u''(0)\frac{t_0^2}{2} + \int_0^{t_0} d\xi_1 \int_0^{\xi_1} d\xi_2 \int_0^{\xi_2} d\xi_3 \int_0^{\xi_3} c(\xi_4)u(\xi_4)d\xi_4 \qquad (7.101)$$

and since the other terms are positive we get again $u''(0) < 0$ as claimed.

The case $x = 1$ is analogous.

Now let $x \in (0,1)$, $u(x) = 0$ and, by contradiction, $u'(x) = 0$: if $u''(x) \geq 0$ and $u'''(x) \geq 0$ then $u, u' > 0$ until the first zero of u' larger than x, but then for $t > x$,

$$u'(t) = u''(x)(t-x) + u'''(x)\frac{(t-x)^2}{2} + \int_x^t d\xi_1 \int_x^{\xi_1} d\xi_2 \int_x^{\xi_2} c(\xi_3)u(\xi_3)d\xi_3 \ > 0 \qquad (7.102)$$

and so u may not satisfy $u'(1) = 0$; in the case $u''(x) \geq 0$ and $u'''(x) \leq 0$ we obtain the same kind of contradiction, actually $u, u' > 0$ between x and the first zero of u' smaller than x, but then for $t < x$,

$$u'(t) = u''(x)(t-x) + u'''(x)\frac{(t-x)^2}{2} + \int_x^t d\xi_1 \int_x^{\xi_1} d\xi_2 \int_x^{\xi_2} c(\xi_3)u(\xi_3)d\xi_3 \ < 0 \qquad (7.103)$$

and so u may not satisfy $u'(0) = 0$; the remaining combinations are analogous by linearity. □

Now we may prove:

Lemma 7.23. *Given $(\overline{\lambda^+}, \overline{\lambda^-}, \overline{s}) \in \widetilde{\Sigma}^+$ with $\overline{\lambda^+}, \overline{\lambda^-} > \lambda_1^2$, then $\widetilde{\Sigma}^+$ is locally of the form $(\lambda^+(\lambda^-), \lambda^-, s(\lambda^-))$, where (for a suitable $\varepsilon > 0$) $\lambda^+, s : (\overline{\lambda^-} - \varepsilon, \overline{\lambda^-} + \varepsilon) \to \mathbb{R}$ are analytic functions of λ^-.*

Moreover the related nontrivial solutions have all the same number of (simple) zeros.

Proof. Again we will give the proof for the case (N), which is similar to that for the case (D) which was done in [CD01].

We will denote by $u\,[\lambda^+, \lambda^-, s]\,(x)$ the solution of the IVP

$$\begin{cases} u'''' = \lambda^+ u^+ - \lambda^- u^- \qquad in \quad (0,1) \\ (u, u', u'', u''')(0) = (1, 0, s, 0) \end{cases} \qquad (7.104)$$

and we will apply the implicit function theorem to the system

$$\begin{cases} u'\,[\lambda^+, \lambda^-, s]\,(1) = 0 \\ u'''\,[\lambda^+, \lambda^-, s]\,(1) = 0 \end{cases} , \qquad (7.105)$$

that is we want to solve locally the set of its zeros with respect to the variable λ^-.

We remark that $u'\,[\lambda^+, \lambda^-, s]\,(x)$ and $u'''\,[\lambda^+, \lambda^-, s]\,(x)$ are $\mathcal{C}^1$ functions of the four variables $(\lambda^+, \lambda^-, s) \in \bar{N}$ and $x \in [0,1]$, where $\bar{N}$ is a suitable neighborhood of the point $(\overline{\lambda^+}, \overline{\lambda^-}, \overline{s})$; actually the derivatives may be calculated through the differential equation, where the nonlinearity $\lambda^+ u^+ + \lambda^- u^-$ is a $\mathcal{C}^1$ function of the variables $\lambda^{\pm}$.

Denote by $\overline{u} = u\left[\overline{\lambda^+}, \overline{\lambda^-}, \overline{s}\right]$; since its zeros are simple by lemma 7.22, we may restrict the neighborhood $\bar{N}$ such that this property is maintained for all the $u\,[\lambda^+, \lambda^-, s]$ with $(\lambda^+, \lambda^-, s) \in \bar{N}$. We remark also that $\overline{u}$ changes sign since $\overline{\lambda^+}, \overline{\lambda^-} > \lambda_1^2$.

Now let $c(x) = \overline{\lambda^+}\chi_{\{\overline{u}>0\}} + \overline{\lambda^-}\chi_{\{\overline{u}<0\}}$: $\overline{u}$ satisfies

$$\begin{cases} \overline{u}'''' = c(x)\overline{u} \qquad in \quad (0,1) \\ (\overline{u}, \overline{u}', \overline{u}'', \overline{u}''')(0) = (1, 0, \overline{s}, 0) \end{cases} . \tag{7.106}$$

Then let $v(x) = \frac{\partial}{\partial s} u\left[\overline{\lambda^+}, \overline{\lambda^-}, \overline{s}\right](x)$: differentiating (7.106) with respect to s we get (the dependence on s is just in the boundary condition):

$$\begin{cases} v'''' = c(x)v \qquad in \quad (0,1) \\ (v, v', v'', v''')(0) = (0, 0, 1, 0) \end{cases} ; \tag{7.107}$$

note here that $v(x) > 0$ and $v'(x) > 0$ in $(0,1]$, by the same argument used in lemma 7.22.

Now multiply the equation by $\overline{u}$ and integrate by parts four times obtaining from $\int v''''\overline{u} = \int c(x)v\overline{u}$:

$$\int_0^1 v\overline{u}'''' + [v'''\overline{u} - v''\overline{u}' + v'\overline{u}'' - v\overline{u}''']_0^1 = \int_0^1 c(x)v\overline{u} \; : \tag{7.108}$$

since $\overline{u}$ is solution of the BVP too, it satisfies

$$\begin{cases} \overline{u}'(0) = \overline{u}'''(0) = 0 \\ \overline{u}'(1) = \overline{u}'''(1) = 0 \end{cases} \tag{7.109}$$

and so in equation (7.108) only the following term survives

$$(v'''\overline{u} + v'\overline{u}'')(1) = 0\,. \tag{7.110}$$

In the same way let $w(x) = \frac{\partial}{\partial \lambda^+} u\left[\overline{\lambda^+}, \overline{\lambda^-}, \overline{s}\right](x)$: differentiating (7.106) with respect to λ^+ we get (the dependence on λ^+ is in the coefficient $c(x)$):

$$\begin{cases} w'''' = c(x)w + \overline{u}^+ \qquad in \quad (0,1) \\ (w, w', w'', w''')(0) = (0, 0, 0, 0) \end{cases} ; \tag{7.111}$$

again multiplying the equation by $\overline{u}$ and integrating by parts four times we obtain from $\int w''''\overline{u} = \int c(x)w\overline{u} + (\overline{u}^+)^2$:

$$\int_0^1 w\overline{u}'''' + [w'''\overline{u} - w''\overline{u}' + w'\overline{u}'' - w\overline{u}''']_0^1 = \int_0^1 c(x)w\overline{u} + (\overline{u}^+)^2\,, \tag{7.112}$$

where using the boundary conditions the only term which survives is

$$(w'''\overline{u} + w'\overline{u}'')(1) = \int_0^1 (\overline{u}^+)^2 > 0 \,. \tag{7.113}$$

We deduce by the above computations that the vector $(v'(1), v'''(1))$ is not null and is orthogonal to $(\overline{u}''(1), \overline{u}(1))$, while $(w'(1), w'''(1))$ is not orthogonal to it; then

$$det \begin{bmatrix} v'(1) & v'''(1) \\ w'(1) & w'''(1) \end{bmatrix} \neq 0 \,, \tag{7.114}$$

which is indeed the condition we need to apply the implicit function theorem.

Now we need to prove that the obtained functions are also analytic.

Having fixed $\varepsilon > 0$ small and such that $\overline{u}$ has no other zero in $(1, 1+\varepsilon]$, by redefining $\bar{N}$ we may guarantee that for all those $u\,[\lambda^+, \lambda^-, s]$ with $(\lambda^+, \lambda^-, s) \in \bar{N}$ that have a zero in $[1, 1+\varepsilon]$, these are simple too.

Since the zeros of $u\,[\lambda^+, \lambda^-, s]$ are simple and between two zeros the equation in (7.104) is analytic, then the coordinate of each zero is an analytic function of $(\lambda^+, \lambda^+, s)$; we also deduce from this that the number of zeros in $[0, 1+\varepsilon]$ does not change in $\bar{N}$ and so it is constant along the piece of $\widetilde{\Sigma}^+$ we are solving.

This also implies that $u\,[\lambda^+, \lambda^-, s]\,(x)$ is analytic in the four variables in a neighborhood of the points where it is not zero.

For the boundary conditions (N) this is sufficient to conclude that the system (7.105) is analytic since by lemma 7.22 $\overline{u}(1) \neq 0$.

For the boundary conditions (D) one needs to proceed as in [CD01], modifying the definition of $u\,[\lambda^+, \lambda^-, s]$ such that in a neighborhood of 1 it satisfies $u'''' = au$ instead of the nonlinear equation in (7.104), where a is chosen to be λ^+ if $\overline{u}(1^-) > 0$ or λ^- in the opposite case.

In this way the system

$$\begin{cases} u\,[\lambda^+, \lambda^-, s]\,(1) = 0 \\ u''\,[\lambda^+, \lambda^-, s]\,(1) = 0 \end{cases} , \tag{7.115}$$

becomes analytic but the set of its zeros does not change. □

Lemma 7.24. *Let $\lambda^+(\lambda^-)$ be the function found in lemma 7.23; then $\frac{d\lambda^+}{d\lambda^-}(\overline{\lambda^-}) < 0$.*

Proof. Let $y(x) = \frac{\partial}{\partial\lambda^-} u\left[\overline{\lambda^+}, \overline{\lambda^-}, \overline{s}\right](x)$: differentiating (7.106) with respect to λ^- we get

$$\begin{cases} y'''' = c(x)y - u^- \qquad in \quad (0,1) \\ (y, y', y'', y''')(0) = (0,0,0,0) \end{cases} , \tag{7.116}$$

again multiplying the equation by $\overline{u}$ and integrating by parts four times we obtain from $\int y''''\overline{u} = \int c(x)y\overline{u} - (\overline{u}^+)^2$

$$\int_0^1 y\overline{u}'''' + [y'''\overline{u} - y''\overline{u}' + y'\overline{u}'' - y\overline{u}''']_0^1 = \int_0^1 c(x)y\overline{u} + (\overline{u}^-)^2 \,, \tag{7.117}$$

where using the boundary conditions the only term which survives is

$$(y'''\overline{u} + y'\overline{u}'')(1) = \int_0^1 (\overline{u}^-)^2 > 0\,. \tag{7.118}$$

Now differentiate the system (7.105) with respect to λ^-, remembering that we defined $v(x) = \frac{\partial}{\partial s} u\left[\overline{\lambda^+}, \overline{\lambda^-}, \overline{s}\right](x)$, $w(x) = \frac{\partial}{\partial \lambda^+} u\left[\overline{\lambda^+}, \overline{\lambda^-}, \overline{s}\right](x)$ and $y(x) = \frac{\partial}{\partial \lambda^-} u\left[\overline{\lambda^+}, \overline{\lambda^-}, \overline{s}\right](x)$:

$$\begin{cases} v'(1)\frac{ds}{d\lambda^-} + w'(1)\frac{d\lambda^+}{d\lambda^-} + y'(1) = 0 \\ v'''(1)\frac{ds}{d\lambda^-} + w'''(1)\frac{d\lambda^+}{d\lambda^-} + y'''(1) = 0 \end{cases}, \tag{7.119}$$

where the derivatives $\frac{ds}{d\lambda^-}$ and $\frac{d\lambda^+}{d\lambda^-}$ are calculated in the point $\overline{\lambda^-}$.

Multiplying the first line by $\overline{u}''(1)$ and the second by $\overline{u}(1)$ and then summing we get

$$\frac{ds}{d\lambda^-}\left(v'\overline{u}'' + v'''\overline{u}\right)(1) + \frac{d\lambda^+}{d\lambda^-}\left(w'\overline{u}'' + w'''\overline{u}\right)(1) + \left(y'\overline{u}'' + y'''\overline{u}\right)(1) \;=\; 0\,, \tag{7.120}$$

which, using equations (7.110), (7.113) and (7.117), becomes

$$\frac{d\lambda^+}{d\lambda^-}\int_0^1 (\overline{u}^+)^2 + \int_0^1 (\overline{u}^-)^2 = 0\,; \tag{7.121}$$

this implies $\frac{d\lambda^+}{d\lambda^-}(\overline{\lambda^-}) < 0$ since $\overline{u}$ changes sign.

In the case (D) the proof is analogous. $\square$

Lemma 7.25. *Given* $\{(\lambda_n^+, \lambda_n^-, s_n)\} \subseteq \widetilde{\Sigma}^+$ *with* $\lambda_n^\pm \to \lambda_0^\pm \in \mathbb{R}$*, there exists a subsequence* $s_n \to s_0$ *such that* $(\lambda_0^+, \lambda_0^-, s_0) \in \widetilde{\Sigma}^+$*.*

Moreover if the sequence u_n *of the corresponding nontrivial solutions is composed of functions all with the same number of (simple) zeros, then* z_0 *too has this number of (simple) zeros.*

Proof. As seen before $u_n \in H^4(0,1) \subseteq \mathcal{C}^3([0,1])$.

Let $z_n = \frac{u_n}{\|u_n\|_{H^4}}$: up to a subsequence $z_n \to z_0$ weakly in $H^4(0,1)$ and strongly in $\mathcal{C}^3([0,1])$.

The variational equation for z_n is $\int_0^1 z_n''v'' = \int_0^1 (\lambda_n^+ z_n^+ - \lambda_n^- z_n^-)v$ for all $v \in H^2_*$; taking the limit, since $\lambda_n^\pm$ are bounded, gives

$$\int_0^1 z_0''v'' = \int_0^1 (\lambda_0^+ z_0^+ - \lambda_0^- z_0^-)v \quad for \quad all \quad v \in H^2_*\,, \tag{7.122}$$

that is z_0 is a solution of (7.89) with coefficients $(\lambda_0^+, \lambda_0^-)$.

Since the solution is strong too, we have

$$\int_0^1 (z_n'''' - z_0'''')^2 = \int_0^1 (\left[(\lambda_n^+ z_n^+ - \lambda_n^- z_n^-) - (\lambda_0^+ z_0^+ - \lambda_0^- z_0^-)\right]^2\,, \tag{7.123}$$

where the right hand side goes to zero and so $z_n \to z_0$ strongly in H^4; this implies that $\|z_0\|_{H^4} = 1$ and so it is a non trivial solution, that is $(\lambda_0^+, \lambda_0^-, \frac{z_0''(0)}{z_0(0)}) \in \widetilde{\Sigma}^+$.

We have $z_0(0) \geq 0$ since $u_n(0) = 1$, in fact $z_0(0) > 0$ by lemma 7.22 since $z_0'(0) = 0$; moreover by the convergence in $\mathcal{C}^3$, $z_n''(0) \to z_0''(0)$ and $z_n(0) \to z_0(0)$; since $s_n = u_n''(0) = \frac{z_n''(0)}{z_n(0)}$ its limit exists and it is indeed $\frac{z_0''(0)}{z_0(0)}$, so we proved the first part of the lemma.

To conclude, observe that $z_n(x) = u_n(x)\, z_n(0)$ and we saw that $z_n(0)$ is bounded away from zero; this implies that if the sequence u_n is composed of functions all with the same number of simple zeros then the same is true for z_n and, by the $\mathcal{C}^1$ convergence, also for z_0.

The case (D) is analogous.

□

Corollary 7.26. *The set Σ^+ is closed.*

Proof. In fact the previous lemma implies that if a sequence $\{(\lambda_n^+, \lambda_n^-)\} \subseteq \Sigma^+$ is such that $\lambda_n^\pm \to \lambda_0^\pm \in \mathbb{R}$, then $(\lambda_0^+, \lambda_0^-) \in \Sigma^+$. □

Now we may obtain a first qualitative description of the set $\widetilde{\Sigma}^+$:

Proposition 7.27. *Let C be a connected component of $\widetilde{\Sigma}^+$ containing some point with $\lambda^\pm > \lambda_1^2$; then there exist $\lambda_\infty^+, \lambda_\infty^- \in [\lambda_1^2, +\infty)$ such that C is of the form $C = \{(\lambda^+(\lambda^-), \lambda^-, s(\lambda^-))\}$ where:*

1. *$\lambda^+, s : (\lambda_\infty^-, +\infty) \to \mathbb{R}$ are analytic functions of λ^-,*
2. *$\lim_{\lambda^- \to \lambda_\infty^-} \lambda^+(\lambda^-) = +\infty$,*
3. *$\lim_{\lambda^- \to +\infty} \lambda^+(\lambda^-) = \lambda_\infty^+$,*
4. *the related nontrivial solutions have all the same number of (simple) zeros,*
5. *$\frac{d\lambda^+}{d\lambda^-}(\lambda^-) < 0$,*
6. *$\exists!\ \lambda_*^- \in (\lambda_\infty^-, +\infty)$ such that $\lambda^+(\lambda_*^-) = \lambda_*^-$, that is $\lambda_*^- = \lambda_k^2$ for some $k \geq 2$.*

Proof. The fact that C may be solved as a function of λ^-, that λ^- and $\lambda^+(\lambda^-)$ take values in open intervals and the points 4 and 5 follow from lemma 7.23 and 7.24.

Moreover point 4 implies that $C \subseteq \left\{\lambda^\pm > \lambda_1^2\right\}$ and then $\lambda_\infty^+, \lambda_\infty^- \in [\lambda_1^2, +\infty)$.

Finally the other limits in points 2 and 3 need be $+\infty$ since the projection of C in the plane $\left\{(\lambda^+, \lambda^-) \in \mathbb{R}^2\right\}$ is a closed set too, by corollary 7.26.

The last property in the proposition is obvious since $\lambda^+(\lambda^-)$ is continuous, monotone decreasing and has a vertical and a horizontal asymptote.

□

Remark 7.28. *Since for a usual eigenvalue λ_k^2 there exists a unique value s such that the point $(\lambda_k^2, \lambda_k^2, s) \in \widetilde{\Sigma}^+$ and by lemma 7.23 through a point $(\lambda^+, \lambda^-, s)$ may pass only one connected component of $\widetilde{\Sigma}^+$, we have that any connected component in $\widetilde{\Sigma}^+$ may be identified by the eigenvalue it passes through. Then we will use the notation $\widetilde{\Sigma}_k^+$ for the component corresponding to λ_k^2 and Σ_k^+ for its projection in the plane $\left\{(\lambda^+, \lambda^-) \in \mathbb{R}^2\right\}$.*

Obviously the same holds for $\widetilde{\Sigma}^-$ and so through a point $(\lambda_k^2, \lambda_k^2) \in \mathbb{R}^2$ may pass at most two curves in Σ: Σ_k^+ and Σ_k^-, which may or may not coincide.

7.5.2 Asymptotic behavior

Now we intend to find the asymptotic behavior of the curves in Σ. In order to do this we will analyze the behavior of the nontrivial solutions corresponding to a point $(\lambda^+, \lambda^-) \in \Sigma_k^\pm$ when this point moves toward the asymptote of the curve. In particular we will prove that one may define a "limiting function" u_∞ whose properties will give us the value of the limits $\lambda_\infty^\pm$ in proposition 7.27.

In the following we will call a "positive bump" (resp. "negative bump") of the function u, an interval (a, b) such that $u > 0$ (resp. $u < 0$) in (a, b) and $u(a) = u(b) = 0$.

Lemma 7.29. *Let $\{(\lambda_n^+, \lambda_n^-)\} \subseteq \Sigma_k^+$ or Σ_k^- for some $k \geq 2$, let $\{\lambda^- = \lambda_\infty^-\}$ be its asymptote and $\lambda_n^- \to \lambda_\infty^-$ (and so $\lambda_n^+ \to +\infty$) and finally let u_n be the related nontrivial solutions chosen with $\|u_n\|_{L^\infty} = 1$.*

Then there exists a subsequence $u_n \to u_\infty$ in $\mathcal{C}^2([0,1])$ in the case (N), while in the case (D) the convergence is in $\mathcal{C}^2([\varepsilon, 1-\varepsilon]) \cap \mathcal{C}^1([0,1])$ for any $\varepsilon > 0$.

Moreover u_∞ is such that:

(i) $u_\infty \leq 0$ but $u_\infty \not\equiv 0$;

(ii) u_∞ satisfies $u_\infty'''' = \lambda_\infty^- u_\infty$ where $u_\infty < 0$;
moreover in the case (N) also satisfies the boundary condition $u_\infty'''(p) = 0$ at the boundary points p where $u_\infty(p) < 0$;

(iii) $\{x \in [0,1] : \quad u_\infty(x) = 0\}$ does not contain intervals, and hence the positive bumps of the u_n collapse to points;

(iv) u_∞ has the same number of negative bumps as the u_n in the sequence;

(v) in the case (D), if the u_n in the sequence start with a negative bump then the claimed convergence is in fact in $\mathcal{C}^2([0, 1-\varepsilon])$ and if the u_n end with a negative bump then the claimed convergence is in fact in $\mathcal{C}^2([\varepsilon, 1])$.

Proof. In this proof we will denote by E a positive constant whose value may be increased during the course of the proof.

Testing the differential equation with ϕ_1 we get
$\int_0^1 u_n''\phi_1'' = \lambda_1^2 \int_0^1 u_n\phi_1 = \lambda_n^+ \int_0^1 u_n^+\phi_1 - \lambda_n^- \int_0^1 u_n^-\phi_1$, that is

$$(\lambda_n^+ - \lambda_1^2) \int_0^1 u_n^+\phi_1 - (\lambda_n^- - \lambda_1^2) \int_0^1 u_n^-\phi_1 = 0\,; \tag{7.124}$$

here $\left\{\int_0^1 u_n^-\phi_1\right\}$ is bounded by $\int_0^1 \phi_1$ and so (since $\{\lambda_n^-\}$ is bounded) we conclude that $\left\{(\lambda_n^+ - \lambda_1^2) \int_0^1 u_n^+\phi_1\right\}$ is bounded too, in particular

$$\int_0^1 |u_n''''|\phi_1 = \int_0^1 (\lambda_n^+ u_n^+ + \lambda_n^- u_n^-)\phi_1 < E\ : \tag{7.125}$$

- in the case (N), since ϕ_1 is a constant function, $\{u_n''''\}$ is bounded in $L^1(0,1)$:

$$\int_0^1 |u_n''''| \leq E\,, \tag{7.126}$$

- in the case (D) we have the same rsult but only with bounds in $L^1(\varepsilon, 1-\varepsilon)$ for any $\varepsilon > 0$ since ϕ_1 goes to zero in 0 and 1:

$$\int_\varepsilon^{1-\varepsilon} \left|u_n''''\right| \leq E\,. \tag{7.127}$$

Now we want to estimate u_n'' through Green's formulas (see section A.4). Observe that $-(-u_n'')'' = \lambda_n^+ u_n^+ - \lambda_n^- u_n^-$ and $(u_n'')'(0) = (u_n'')'(1) = 0$ (resp. $u_n''(0) = u_n''(1) = 0$), that is $(-u_n'')$ satisfies the second order problem above with Neumann (resp. Dirichlet) boundary conditions.

Observe that equations (A.11) and (A.8) imply that

$$|G(x,y)| \leq K\phi_1(y) \tag{7.128}$$

for a suitable K, both in the case (N) and (D).

Then we have from equations (A.10) and (A.7): in the case (N):

$$-u_n''(x) = -u_n''(1) + \int_0^1 (\lambda_n^+ u_n^+(y) - \lambda_n^- u_n^-(y))G(x,y)dy\,; \tag{7.129}$$

in the case (D):

$$-u_n''(x) = \int_0^1 (\lambda_n^+ u_n^+(y) - \lambda_n^- u_n^-(y))G(x,y)dy\,. \tag{7.130}$$

The integrals may be estimated using (7.125) and (7.128), to estimate $u_n''(1)$ observe that since in the case (N) $u_n'(1) = u_n'''(1) = 0$ we have

$$u_n(x) = u_n(1) + u_n''(1)\frac{(x-1)^2}{2} + \int_1^x d\xi_1 \int_1^{\xi_1} d\xi_2 \int_1^{\xi_2} d\xi_3 \int_1^{\xi_3} u_n''''(\xi_4)d\xi_4\,, \tag{7.131}$$

where $u_n(x)$ and the integral are uniformly bounded and hence so is $u_n''(1)$ is.

Then in both cases we conclude

$$\left\|u_n''\right\|_{L^\infty} \leq E\,. \tag{7.132}$$

Now we estimate u_n''':

- for the case (N) $|u_n'''(x)| = \left|u_n'''(0) + \int_0^x u_n''''(\xi)d\xi\right| \leq 0 + E$,
- in the case (D) we have $|u_n'''(x)| = \left|u_n'''(p) + \int_p^x u_n''''(\xi)d\xi\right| \leq |u_n'''(p)| + E$, provided $p, x \in [\varepsilon, 1-\varepsilon]$: to estimate $u_n'''(p)$ consider $|u_n''(x)| = \left|u_n''(p) + u_n'''(p)(p-x) + \int_p^x d\xi_1 \int_p^{\xi_1} u_n''''(\xi_2)d\xi_2\right|$: this implies that $u_n'''(p)$ is bounded, since all of thr other terms are;

then we have, for any $\varepsilon > 0$,

$$\sup_{x\in[0,1]} |u_n'''(x)| \leq E \qquad in \quad case \quad (N)\,, \tag{7.133}$$

$$\sup_{x\in[\varepsilon,1-\varepsilon]} |u_n'''(x)| \leq E \qquad in \quad case \quad (D)\,. \tag{7.134}$$

To conclude, since $|u_n'(x)| = \left|u_n'(0) + \int_0^x u_n''(\xi)d\xi\right|$ and $|u_n(x)| = \left|u_n(0) + u_n'(0)x + \int_0^x d\xi_1 \int_0^{\xi_1} u_n''(\xi_2)d\xi_2\right|$, we have by the uniform boundedness of both u_n and u_n'' that of u_n' too.

So we have now, for any $\varepsilon > 0$,

$$\|u_n\|_{\mathcal{C}^3([0,1])} \leq E \qquad in \quad case \quad (N)\,, \tag{7.135}$$

$$\|u_n\|_{\mathcal{C}^3([\varepsilon,1-\varepsilon])} + \|u_n\|_{\mathcal{C}^2([0,1])} \leq E \qquad in \quad case \quad (D) \tag{7.136}$$

and then, up to a subsequence, strong convergence in $\mathcal{C}^2[0,1]$ in the case (N) and in $\mathcal{C}^2[\varepsilon, 1-\varepsilon] \cap \mathcal{C}^1[0,1]$ in the case (D): we call u_∞ the limit.

Now let us prove the claimed properties of u_∞.

(i) From equation (7.124) we have

$$\int_0^1 u_n^+\phi_1 = \frac{\lambda_n^- - \lambda_1^2}{\lambda_n^+ - \lambda_1^2}\int_0^1 u_n^-\phi_1 < \frac{E}{\lambda_n^+ - \lambda_1^2} \to 0 \tag{7.137}$$

and then $\int_0^1 u_\infty^+\phi_1 = 0$, that is $u_\infty \leq 0$; however by the $\mathcal{C}^0$ convergence $\|u_\infty\|_{L^\infty} = 1$ and so it is not identically zero, in particular $\inf_{x\in[0,1]} u_\infty(x) = -1$.

(ii) Let $v \in H^2_*(0,1)$ and $supp(v) \subseteq \{x \in [0,1] : \ u_\infty(x) < 0\}$. By the $\mathcal{C}^0$ convergence we have $u_n(x) < 0$ in $supp(v)$ for $n > \bar{n}$ and so $\int_0^1 u_n''v'' = \int_0^1 \lambda_n^- u_n^- v$ and taking the limit yields

$$\int_0^1 u_\infty''v'' = \int_0^1 \lambda_\infty^- u_\infty^- v\,. \tag{7.138}$$

The same calculation may be done allowing $v(0) \neq 0$, in the case (N) if $u_\infty(0) < 0$ (or $v(1) \neq 0$ if $u_\infty(1) < 0$).

Then equation (7.138) with the chosen test functions implies the claim.

(iii) Suppose

* $u_\infty \equiv 0$ in the non trivial interval $[a,b] \subseteq [0,1)$,
* $u_\infty < 0$ in $(b, b+\varepsilon)$ for some $\varepsilon > 0$;

(the symmetric case goes in the same way), then (since u_∞ is $\mathcal{C}^2$ in a neighborhood of b)

$$u_\infty(b) = u_\infty'(b) = u_\infty''(b) = 0\,. \tag{7.139}$$

But we have seen that if $u_\infty < 0$ in (b,c) then it satisfies $u_\infty'''' = \lambda_\infty^- u_\infty$ in (b,c) and so it is also $\mathcal{C}^\infty(b,c)$; moreover since $\lim_{t\to 0^+} u_\infty(b+t) = 0$ we have $\lim_{t\to 0^+} u_\infty''''(b+t) = 0$ and this implies that there exists and is finite $\lim_{t\to 0^+} u_\infty'''(b+t) = -\eta$; in fact $\eta \geq 0$ since $u_\infty < 0$ in $(b, b+\varepsilon)$.

Now if $\eta = 0$ then u_∞ is in fact $\mathcal{C}^4$ in a neighborhood of b and

$$u_\infty'(b+t) = \int_b^{b+t} d\xi_1 \int_b^{\xi_1} d\xi_2 \int_b^{\xi_2} \lambda_\infty^- u_\infty(\xi_3)d\xi_3 < 0\,. \tag{7.140}$$

Hence u_∞ cannot satisfy $u_\infty(1) = 0$ nor $u'_\infty(1) = 0$, which is a contradiction since it is the $\mathcal{C}^1$ limit of functions satisfying one of the two.

If instead $\eta > 0$ then we have, for $\delta > 0$ small enough,

$$\begin{aligned} u_\infty(b+\delta) &\leq -\frac{\eta}{2}\frac{\delta^3}{6} < 0\,, && (7.141)\\ u'_\infty(b+\delta) &\leq -\frac{\eta}{2}\frac{\delta^2}{2} < 0\,, && (7.142)\\ u''_\infty(b+\delta) &\leq -\frac{\eta}{2}\delta < 0\,, && (7.143)\\ u'''_\infty(b+\delta) &\leq -\frac{\eta}{2} < 0 && (7.144) \end{aligned}$$

and then

$$u'_\infty(b+\delta+t) < \int_{b+\delta}^{b+\delta+t} d\xi_1 \int_{b+\delta}^{\xi_1} d\xi_2 \int_{b+\delta}^{\xi_2} \lambda_\infty^- u_\infty(\xi_3) d\xi_3 < 0\,, \tag{7.145}$$

which again gives a contradiction.

To conclude, since we have $\mathcal{C}^0$ convergence to a function that is non negative only in a set which does not contain intervals, we have that the positive bumps need to collapse to points.

(iv) Let us start by supposing that a negative bump collapses to an interior point (this implies that there exists a positive bump both before and after this negative bump).

In particular let

$s_n, t_n \to t_0 \in (0,1)$ such that $u_n(s_n) = u_n(t_n) = 0$ and $u_n < 0$ in (s_n, t_n).

Since $u'_n(s_n) < 0$ and $u'_n(t_n) > 0$ there exists

$m_n \in (s_n, t_n)$ such that $u'_n(m_n) = 0$ and $u''_n(m_n) \geq 0$;

moreover consider the following positive bump: there exists a maximum point $M_n > t_n$ where $u'_n(M_n) = 0$ and $u''_n(M_n) \leq 0$ and hence there exists a point

$p_n \in [m_n, M_n]$ such that $u''_n(p_n) = 0$.

However, since the positive bump collapses, all these points converge to t_0, and then by the $\mathcal{C}^2$ convergence $u_\infty(t_0) = u'_\infty(t_0) = u''_\infty(t_0) = 0$ which is a contradiction as seen before.

The same proof may be adapted to a negative bump near to the boundary in the case (N), since in this case $u'_n(0) = 0$ (the case in $t = 1$ is analogous) and by the same argument if $t_n \to 0$ is such that $u_n(t_n) = 0$ and $u_n < 0$ in $[0, t_n)$ then $u_\infty(0) = u'_\infty(0) = u''_\infty(0) = 0$.

In the case (D) we do not have $\mathcal{C}^2$ convergence near the boundary and so this argument does not work: let us suppose

$t_n \to 0$ such that $u_n(t_n) = 0$ and $u_n < 0$ in $(0, t_n)$;

as before there exists

$m_n \in (0, t_n)$ such that $u'_n(m_n) = 0$ and $u''_n(m_n) \geq 0$:

let us estimate $u_n''(m_n)$ by Green's functions:

$$0 \geq -u_n''(m_n) = \int_0^1 (\lambda_n^+ u_n^+(y) - \lambda_n^- u_n^-(y))G(m_n, y)dy \qquad (7.146)$$
$$\geq -\int_0^1 \lambda_n^- u_n^-(y)G(m_n, y)dy \geq -E\int_0^1 G(m_n, y)dy$$

where the last inequality comes from the boundedness of $\{\lambda_n^- u_n^-\}$; now consider the sequence of functions $G_n(y) = G(m_n, y)$: by equation (A.9), since $m_n \to 0$, we get $G_n(y) \to 0$ in $\mathcal{C}^0([0,1])$ and so we conclude $u_n''(m_n) \to 0$.

Now multiplying the differential equation for u_n by u_n' we get $u_n''''u_n' - \lambda_n^+ u_n^+ u_n' + \lambda_n^- u_n^- u_n' = 0$, where $u_n''''u_n' = (u_n'''u_n')' - u_n'''u_n'' = (u_n'''u_n')' - \left(\frac{(u_n'')^2}{2}\right)'$ and $(u_n^\pm u_n') = \pm\left(\frac{(u_n^\pm)^2}{2}\right)'$; so integrating we get

$$2u_n'''u_n' - (u_n'')^2 - \lambda_n^+(u_n^+)^2 - \lambda_n^-(u_n^-)^2 = C_n\,; \qquad (7.147)$$

if we compute C_n in an absolute minimum we have $u_n = -1$ and $u_n' = 0$, and so $C_n \leq -\lambda_n^- \leq -\lambda_1^2 < 0$, but if we compute it in m_n we get $C_n \to 0$ since $u_n''(m_n) \to 0$ and $u_n(m_n) \to 0$ by the $\mathcal{C}^0$ convergence, giving a contradiction.

Finally we need prove that a negative bump may not split in two distinct bumps, actually even if $u_\infty(x_0) = 0$, if $u_n < 0$ in a neighborhood of x_0, then the argument used to prove that u_∞ satisfies $u_\infty'''' = \lambda_\infty^- u_\infty$ still applies and so $u_\infty(x_0) = u_\infty'(x_0) = 0$ gives a contradiction as in lemma 7.22.

(v) We have just seen that in this case the bump does not collapse, then $u_n < 0$ in $(0, \varepsilon)$ (or in $(1-\varepsilon, 1)$) and so here too $\{u_n''''\}$ is bounded since $\{\lambda_n^-\}$ is: then we get in place of (7.127) $\int_0^{1-\varepsilon} |u_n''''| \leq E$ or $\int_\varepsilon^1 |u_n''''| \leq E$, and then we may proceed as before to obtain the claim.

□

Now consider the problems:

$$\begin{cases} u'''' = u & in\ (0, A) \\ u(0) = u'(0) = 0 \\ u(A) = u'(A) = 0 \end{cases}, \qquad (7.148)$$

$$\begin{cases} u'''' = u & in\ (0, A) \\ u(0) = u''(0) = 0 \\ u(A) = u'(A) = 0 \end{cases}, \qquad (7.149)$$

$$\begin{cases} u'''' = u & in\ (0, A) \\ u'(0) = u'''(0) = 0 \\ u(A) = u'(A) = 0 \end{cases}: \qquad (7.150)$$

each one of them admits a positive (or negative) solution for a unique value of A; let us call these values respectively A_1, A_2 and A_3.

The last two problems have already been analyzed (in a slightly different formulation) in section 7.4.1: actually if we set $v(x) = u(qx)$ we obtain for v the equation $v'''' = q^4 v$ with the boundary conditions imposed in 0 and in $\frac{A}{q}$; then with $q = A$ and $\delta = q^4$ we obtain equation (7.79) from (7.150) and (7.86) from (7.149).

Then we may conclude that A_2 is the first positive solution of $\tanh(A_2) = \tan(A_2)$ and A_3 is the first positive solution of $\tanh(A_3) = -\tan(A_3)$.

To compute A_1 we may observe that, by the symmetry of the boundary conditions in (7.148), the solution is symmetric and then satisfies $u'(A/2) = u'''(A/2) = 0$; this implies that it is nothing but the solution of (7.150) joined to its symmetrical, giving $A_1 = 2A_3$.

Remember also that we already saw that $A_2 \in (\pi, \frac{3\pi}{2})$ and $A_3 \in (\frac{\pi}{2}, \pi)$; moreover a better estimate gives that since $\tanh(A_3) < 1$, then $\tan(A_3) > -1$ and so $A_3 \in (\frac{3\pi}{4}, \pi)$ and $A_1 \in (\frac{3\pi}{2}, 2\pi)$.

We may conclude then that

$$\pi < A_2, \quad A_3 < A_2 \quad \textit{and} \quad A_2 < A_1 < 2A_2\,. \tag{7.151}$$

Now we may state:

Theorem 7.30. *Let* $\left\{\lambda^- = \lambda^-_{\infty,k,\pm}\right\}$ *be the asymptote of the curve* $\Sigma^\pm_k$*: then we have, for* $k \geq 2$*:*

- *in the case (N)* $\lambda^-_{\infty,k,\pm} = \left(\frac{k-1}{2} A_1\right)^4 = ((k-1)A_3)^4$*;*
- *in the case (D)*
  - *if k is even* $\lambda^-_{\infty,k,\pm} = \left(A_2 + \left(\frac{k}{2} - 1\right) A_1\right)^4$,
  - *if k is odd*
    - $\lambda^-_{\infty,k,+} = \left(\frac{k-1}{2} A_1\right)^4$,
    - $\lambda^-_{\infty,k,-} = \left(2A_2 + \left(\frac{k+1}{2} - 2\right) A_1\right)^4$.

In particular we have

- *in the case (N)*
  $0 = \lambda_1^2 < \lambda^-_{\infty,2,+} = \lambda^-_{\infty,2,-}$,
  $\lambda^-_{\infty,k,+} = \lambda^-_{\infty,k,-} < \lambda^-_{\infty,k+1,+} = \lambda^-_{\infty,k+1,-}$ *for* $k \geq 2$*;*
- *in the case (D)*
  $\lambda_1^2 < \lambda^-_{\infty,2,+} = \lambda^-_{\infty,2,-} < \lambda^-_{\infty,3,+} < \lambda^-_{\infty,3,-}$,
  $\lambda^-_{\infty,k-1,+} < \lambda^-_{\infty,k-1,-} < \lambda^-_{\infty,k,+} = \lambda^-_{\infty,k,-} < \lambda^-_{\infty,k+1,+} < \lambda^-_{\infty,k+1,-}$ *for* $k \geq 4$ *even.*

Proof. Consider the sequences $\{(\lambda^+_n, \lambda^-_n)\}$ and $u_n \to u_\infty$ of lemma 7.29 and denote as there $\lambda^-_\infty = \lim_{n\to\infty} \lambda^-_n$.

By lemma 7.29 the positive bumps and halfbumps in the sequence $\{u_n\}$ collapse to a point for u_∞, the interior negative bumps (p_1, p_2) of u_∞ satisfy

$$\begin{cases} u_\infty'''' = \lambda_\infty^- u_\infty & in\ (p_1, p_2) \\ u(p_1) = u'(p_1) = 0 & \\ u(p_2) = u'(p_2) = 0 & \end{cases}, \tag{7.152}$$

and so $p_2 - p_1 = A_1(\lambda_\infty^-)^{-1/4}$.

In the case (N) the boundary negative halfbumps of u_∞ satisfy (we treat the case of the left end point, the other case is analogous)

$$\begin{cases} u_\infty'''' = \lambda_\infty^- u_\infty & in\ (0, p_2) \\ u'(0) = u'''(0) = 0 & \\ u(p_2) = u'(p_2) = 0 & \end{cases}, \tag{7.153}$$

and so $p_2 = A_3(\lambda_\infty^-)^{-1/4}$.

In the case (D) if the u_n start (or end) with negative bumps we have seen that the $\mathcal{C}^2$ convergence is achieved up to the boundary and so since all u_n satisfy $u_n''(0) = 0$ and $u_n''(1) = 0$ we get that the boundary negative bumps of u_∞ satisfy (again we treat the case on the left hand end point)

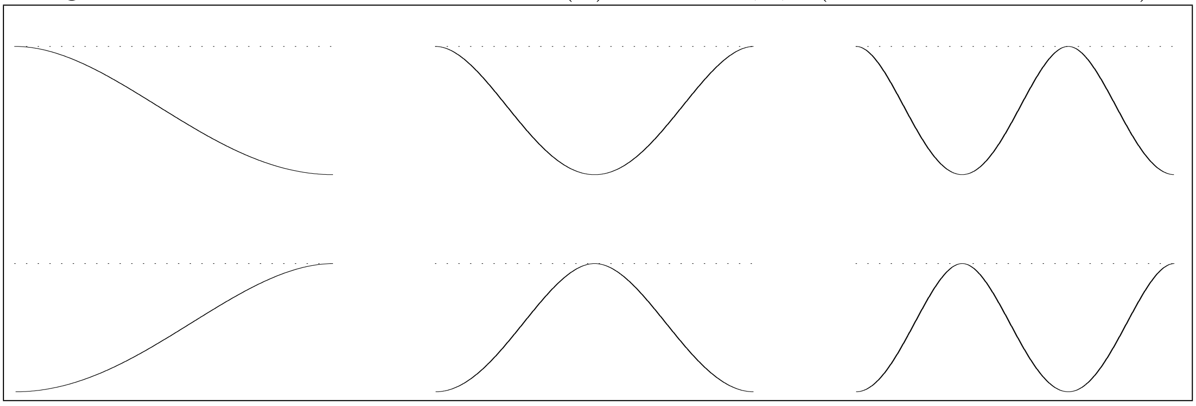
$$\begin{cases} u_\infty'''' = \lambda_\infty^- u_\infty & in\ (0, p_2) \\ u(0) = u''(0) = 0 & \\ u(p_2) = u'(p_2) = 0 & \end{cases}, \tag{7.154}$$

and so $p_2 = A_2(\lambda_\infty^-)^{-1/4}$.

Then we have:

- in the case (N) each u_n in the sequence is composed of $2(k-1)$ halfbumps of which $k-1$ are negative; since the negative halfbumps at the boundary tend to halfbumps of length $\frac{A_3}{\sqrt[4]{\lambda_\infty^-}}$ and the bumps in the interior tend to bumps of length $\frac{A_1}{\sqrt[4]{\lambda_\infty^-}} = 2\frac{A_3}{\sqrt[4]{\lambda_\infty^-}}$ we conclude that each negative halfbump tends to a halfbump of length $\frac{A_3}{\sqrt[4]{\lambda_\infty^-}}$, giving the condition for the existence of a non trivial solution $(k-1)\frac{A_3}{\sqrt[4]{\lambda_\infty^-}} = 1$;
- in the case (D)
  - if k is even each u_n in the sequence is composed of $\frac{k}{2}$ positive bumps and $\frac{k}{2}$ negative ones, of which one is at the boundary, then the condition for the existence of a non trivial solution is $\left(\frac{k}{2} - 1\right)\frac{A_1}{\sqrt[4]{\lambda_\infty^-}} + \frac{A_2}{\sqrt[4]{\lambda_\infty^-}} = 1$;
  - if k is odd then
    * if the u_n in the sequence start positive then we have $\frac{k+1}{2}$ positive bumps and $\frac{k-1}{2}$ negative ones, all interior, giving the condition $\left(\frac{k-1}{2}\right)\frac{A_1}{\sqrt[4]{\lambda_\infty^-}} = 1$,

Figure 11: Sketch of the u_∞ for the case (N) with $k = 2, 3, 4$ (Σ^+ above and Σ^- below).



- $*$ if finally the u_n in the sequence start negative, then we have $\frac{k-1}{2}$ positive bumps and $\frac{k+1}{2}$ negative ones, of which two are at the boundary, giving the condition $\left(\frac{k+1}{2} - 2\right) \frac{A_1}{\sqrt[4]{\lambda_\infty^-}} + 2\frac{A_2}{\sqrt[4]{\lambda_\infty^-}} = 1$.

The inequalities for the case (N) are trivial; let us see those for the case (D):

- $\lambda_1^2 < \lambda^-_{\infty,2,\pm}$ gives $\pi < A_2$;
- $\lambda^-_{\infty,k-1,+} < \lambda^-_{\infty,k-1,-}$ gives $\frac{(k-1)-1}{2} A_1 < 2A_2 + \left(\frac{(k-1)+1}{2} - 2\right) A_1$, which simplifying gives $A_1 < 2A_2$;
- $\lambda^-_{\infty,k-1,-} < \lambda^-_{\infty,k,+}$ gives $2A_2 + \left(\frac{(k-1)+1}{2} - 2\right) A_1 < A_2 + \left(\frac{k}{2} - 1\right) A_1$, which simplifying gives $A_2 < A_1$;
- $\lambda^-_{\infty,k,-} < \lambda^-_{\infty,k+1,+}$ gives $A_2 + \left(\frac{k}{2} - 1\right) A_1 < \frac{(k+1)-1}{2} A_1$, which simplifying gives again $A_2 < A_1$;
- last inequality is analogous to the first.

Then all inequalities are verified by equation (7.151).

$\square$

In figure 11 and 12 we sketch the limiting functions u_∞ for the first curves of the Fučík spectra of the case (N) and (D) respectively.

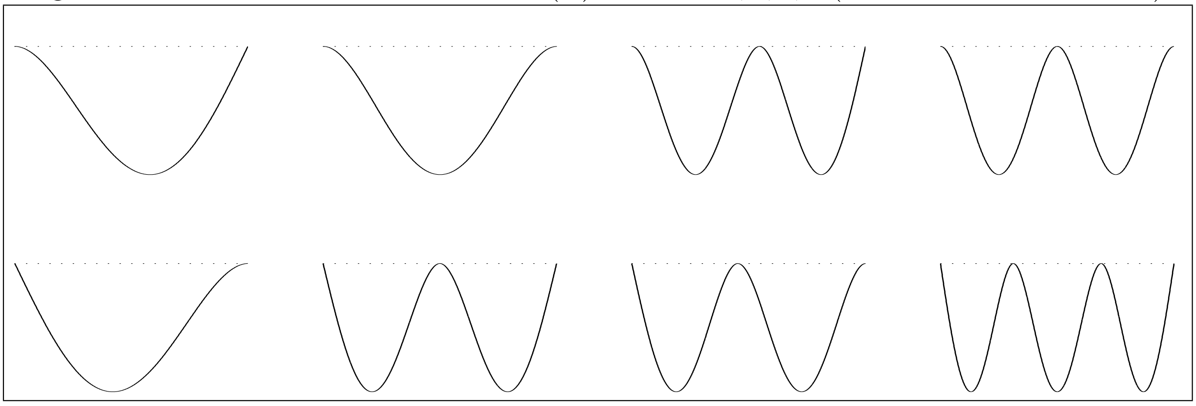
7.5.3 Relationship between the curves $\Sigma_k^\pm$

To conclude the qualitative description of these Fučík spectra we prove two lemmas dealing with the possible intersections between the curves $\Sigma_k^\pm$.

Lemma 7.31. *If $k \neq h$ then $\Sigma_k^\pm \cap \Sigma_h^\pm = \emptyset$.*

Proof. Let $\lambda^+(\lambda^-)$ describe Σ_k^+ or Σ_k^-: then $\lambda^+(\lambda_k^2) = \lambda_k^2$ and since it is decreasing $\lambda^+(\lambda_{k+1}^2) < \lambda_k^2 < \lambda_{k+1}^2$, then for $\lambda^- = \lambda_{k+1}^2$ the curves $\Sigma_k^\pm$ are lower than the $\Sigma_{k+1}^\pm$.

Figure 12: Sketch of the u_∞ for the case (D) with $k = 2, 3, 4, 5$ (Σ^+ above and Σ^- below).



Then it is enough to prove that $\Sigma_k^{\pm} \cap \Sigma_{k+1}^{\pm} = \emptyset$ to imply the claim.

By contradiction, suppose $(\lambda^+, \lambda^-) \in \Sigma_k^{\pm} \cap \Sigma_{k+1}^{\pm}$, then we have the corresponding nontrivial solutions u_k and u_{k+1}, where the second changes sign once more than the first one and so in one of the two endpoints (suppose in 0) the sign is the same and we may choose them such that $u_k(0) = u_{k+1}(0)$ and $u_k'(0) = u_{k+1}'(0)$.

Then let $\delta = u_k - u_{k+1}$: we have $\delta(0) = \delta'(0) = 0$ and

$$\begin{aligned} \delta'''' = (u_k - u_{k+1})'''' &= \lambda^+(u_k^+ - u_{k+1}^+) - \lambda^-(u_k^- - u_{k+1}^-) \\ &= \left(\lambda^+\chi_{++}(x) + \lambda^-\chi_{--}(x) + c_1(x)\chi_{+-}(x) + c_2(x)\chi_{-+}(x)\right)\delta\,, \end{aligned} \tag{7.155}$$

where

$$\chi_{\pm_1,\pm_2}(x) = \chi_{\{\pm_1 u_k>0,\ \pm_2 u_{k+1}>0\}}(x)\,, \tag{7.156}$$

$$c_1(x) = \frac{\lambda^+ u_k^+ + \lambda^- u_{k+1}^-}{u_k^+ + u_{k+1}^-}\chi_{+-}(x)\,, \tag{7.157}$$

$$c_2(x) = \frac{-\lambda^+ u_{k+1}^+ - \lambda^- u_k^-}{-u_{k+1}^+ - u_k^-}\chi_{-+}(x)\,; \tag{7.158}$$

$$\tag{7.159}$$

since the function in brackets is $L^\infty(0,1)$ and positive a.e, by lemma 7.22 we get $\delta \equiv 0$, a contradiction since $\delta(1) \neq 0$.

□

Lemma 7.32. *In the case (N) $\Sigma_k^+ \equiv \Sigma_k^-$ for all $k \geq 2$.*

In the case (D)

- $\Sigma_k^+ \equiv \Sigma_k^-$ *for all even* $k \geq 2$,
- $\Sigma_k^+ \not\equiv \Sigma_k^-$ *for all odd* $k \geq 3$.

Proof. If k is even we have nontrivial solutions which start positive and end negative or viceversa.

Then let $(\lambda^+, \lambda^-) \in \Sigma_k^+$ and u_* be the corresponding nontrivial solution: we already know that $(\lambda^-, \lambda^+) \in \Sigma_k^-$ since $-u_*(x)$ satisfies the equation with coefficients (λ^-, λ^+) and starts negative; but in this case $-u_*(1-x)$ starts positive and satisfies the equation with coefficients (λ^-, λ^+), that is $(\lambda^-, \lambda^+) \in \Sigma_k^+$ too.

This gives $\Sigma_k^- \subseteq \Sigma_k^+$, the other inclusion follows in the same way.

If k is odd we have that the nontrivial solutions which start positive end positive and those which start negative end negative; moreover by equation (7.121) $\frac{d\lambda^+}{d\lambda^-}(\lambda_k^2) = \frac{\int_0^1 (\phi_k^-)^2}{\int_0^1 (\phi_k^+)^2}$ which in the case (D) changes if we choose ϕ_k starting positive or negative since it has a different number of positive and negative congruent bumps, and so implies that $\Sigma_k^+ \not\equiv \Sigma_k^-$.

In the case (N) let $(\lambda^+, \lambda^-) \in \Sigma_k^+$ and u_* be the corresponding nontrivial solution with $u_*(0) = 1$, then $v(x) = u_*(1-x)$ starts positive and is a nontrivial solution corresponding to (λ^+, λ^-) too, however if we rescale it (if necessary) in order to have $v(0) = 1$ we obtain $v''(0) = u_*''(0)$ since otherwise we would have another branch of $\widetilde{\Sigma}_k^+$ which is excluded by remark 7.28.

By uniqueness for the initial value problem this implies that $v \equiv u_*$, that is u_* is in fact symmetric and so $u_*'(1/2) = u_*'''(1/2) = 0$; moreover $u_*(1) = u_*(0) = 1$ and $u_*''(1) = u_*''(0)$, and so we may consider the function $\tilde{u}$ defined in $[0,2]$ gluing $u_*(x) : x \in [0,1]$ to $u_*(x-1) : x \in [1,2]$.

By the above considerations $w(x) = \tilde{u}(x+1/2) : x \in [0,1]$ is another nontrivial solution of the problem; now we have:

- if $\tilde{u}(1/2) < 0$ then $w(x)$ is a non trivial solution starting negative and so $-w(x)$ is a non trivial solution starting positive corresponding to the problem with coefficients (λ^-, λ^+), that is $(\lambda^-, \lambda^+) \in \Sigma_k^+$ and so $\Sigma_k^- \subseteq \Sigma_k^+$.
- if $\tilde{u}(1/2) > 0$ then we may conclude as before (since w is another nontrivial solution starting positive corresponding to the problem with coefficients (λ^+, λ^-)) that in fact $\tilde{u}(1/2) = 1$ and $\tilde{u}''(1/2) = \tilde{u}''(0)$ and so that $\tilde{u}(1/2 - x) = \tilde{u}(x)$ for $x \in [0, 1/2]$, then $u_*'(1/4) = u_*'''(1/4) = 0$ and we may repeat the argument.

The procedure ends by finding a negative point of the form $1/2^i$, and then proving that $\Sigma_k^- \subseteq \Sigma_k^+$, since otherwise we would prove the existence of infinitely many symmetries for u_*, which is a contradiction for a function which has a finite (and non zero) number of zeros.

□

7.5.4 Conclusion

By the information obtained in the lemmas of this section we get a good idea of the shape of the Fučík spectrum: in particular in the case (N) the shape is very similar to that of the Neumann problem with the second order operator on an interval (see figure 2 on page 23); in the case (D) the shape may be similar to that of the Dirichlet problem with the second order operator on an interval (see figure 1 on page 22), but with the important difference that there is always some space between the asymptotes of two consecutive curves in the spectrum; moreover we did not prove whether the two distinct curves Σ_k^+ and Σ_k^- with k odd have common points other than $(\lambda_k^2, \lambda_k^2)$.

7.6 The superlinear fourth order problem in one dimension

Now that we know the qualitative shape of the Fučík spectrum for the one dimensional case with the fourth order operator and we have also its variational characterization (see theorem 7.7), we may apply the same ideas of section 5.

Actually by the above lemmas we have that Σ_k and Σ_{k+1} never intersect and have different asymptotes at which they arrive in a monotone way, then any point between them is Σ-connected to the diagonal between λ_k^2 and λ_{k+1}^2 and so we may obtain existence results for problem (7.1) and (7.2) provided λ is between the larger asymptote of Σ_k and the smaller one of Σ_{k+1} or corresponds to this last one with a suitable nonresonance condition.

In section 9.1 we will prove

Lemma 7.33. *For Ω of class $\mathcal{C}^m$, under hypotheses (HN), (H1-m), (H2-m) and (H3-m) with $h \in L^2(\Omega)$, the functional (7.6) satisfies the PS condition in $H_N^m(\Omega)$ (resp. in $H_D^m(\Omega)$) for any $\lambda > \lambda_1^m$.*

With this lemma and the above observations we may state (the values $\lambda_{\infty,k,\pm}^-$ are those obtained in theorem 7.30):

Theorem 7.34. *Under hypotheses (H1-m), (H2-m) and (H3-m), if $\lambda \in (\lambda_{\infty,k,-}^-, \lambda_{\infty,k+1,+}^-)$ for some $k \geq 2$, then there exists a solution of problem (7.1) (resp. (7.2)) with $\Omega = (0,1)$ and $m = 2$, for all $h \in L^2(0,1)$.*

Theorem 7.35. *Under hypotheses (H1-m), (H2-m), (H3-m) and (HR-m), with $h \in L^2(0,1)$, $\lambda = \lambda_{\infty,k+1,+}^-$ for some $k \geq 1$, then there exists a solution of problem (7.1) (resp. (7.2)) with $\Omega = (0,1)$ and $m = 2$.*

Remark 7.36. *Observe that the asymptote $\lambda_{\infty,2,\pm}^-$ is in both cases the value we got for γ in section 7.4.1, then the case $\lambda \in (\lambda_1^2, \lambda_{\infty,2,\pm}^-)$ corresponds to theorem 7.17, where hypothesis (H3-m) was not needed.*

8 Problem with the p-Laplacian operator

Here we intend to consider the problem

$$\begin{cases} -[\psi(u')]' = \lambda\psi(u) + g(x,u) + h(x) & in\ \ (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{8.1}$$

where $\psi(s) = \begin{cases} |s|^{p-2}s & s \neq 0 \\ 0 & s = 0 \end{cases}$,

$$\begin{aligned} & g \in \mathcal{C}^0([0,1]\times\mathbb{R})\,, \\ & \lim_{s\to-\infty} \tfrac{g(x,s)}{\psi(s)} = 0, \quad \lim_{s\to+\infty} \tfrac{g(x,s)}{\psi(s)} = +\infty \end{aligned} \tag{H1-p}$$

uniformly in $[0,1]$ and $h \in L^q([0,1])$ where $1/p + 1/q = 1$.

As for the Laplacian we will need suitable hypotheses on the growth at infinity of g in order to obtain the PS condition: let $G(x,s) = \int_0^s g(x,\xi)d\xi$, we ask:

$$\exists\theta \in (0,\frac{1}{p}), s_0 > 0\ \ s.t.\ \ 0 < G(x,s) \leq \theta s g(x,s) \quad \forall s > s_0\,; \tag{H2-p}$$

$$\exists s_1 > 0,\ \ C_0 > 0\ \ s.t.\ \ G(x,s) \leq \frac{1}{p} s g(x,s) + C_0 \quad \forall s < -s_1\,. \tag{H3-p}$$

Moreover for some values of the parameter λ we will need the nonresonance condition

$$\exists\rho_0 > 0,\ \ M_0 \in \mathbb{R} \quad s.t. \quad G(x,s) + h(x)s \leq M_0 \quad a.e.\ x \in [0,1],\ \forall s < -\rho_0\,. \tag{HR-p}$$

In order to study problem (8.1) we will consider also the following Fučík problem with Neumann boundary conditions in dimension 1:

$$\begin{cases} -[\psi(u')]' = \lambda^+\psi(u^+) - \lambda^-\psi(u^-) & in\ \ (0,1) \\ u'(0) = u'(1) = 0 \end{cases}\,, \tag{8.2}$$

where $u^+(x) = \max\{0, u(x)\}$ and $u^-(x) = \max\{0, -u(x)\}$.

8.1 Some useful lemmas

As noted in section 2.5, in dealing with this kind of operator we are led to work in the spaces $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, respectively in the Neumann and Dirichlet case; we will denote the space considered by W.

Let us prove here some useful properties; from now on we will denote by $q = \frac{p}{p-1}$ the dual exponent of p, that is the one such that $1/p + 1/q = 1$.

Lemma 8.1. *$u \in L^p(\Omega)$ implies $\psi(u) \in L^q(\Omega)$.*
Moreover $\|\psi(u)\|_{L^q} = \|u\|_{L^p}^{p-1}$.

Proof. $\|\psi(u)\|_{L^q} = \left(\int_\Omega |\psi(u)|^q\right)^{1/q} = \left(\int_\Omega |u|^{p-1\frac{p}{p-1}}\right)^{\frac{p-1}{p}} = \left(\int_\Omega |u|^p\right)^{\frac{p-1}{p}} = \|u\|_{L^p}^{p-1}$. □

Corollary 8.2. *For $u, v \in L^p(\Omega)$, we have $\psi(u)v \in L^1$ and we may estimate (using Hölder's inequality)*

$$\left|\int_\Omega \psi(u)v\right| \leq \|u\|_{L^p}^{p-1} \|v\|_{L^p} . \tag{8.3}$$

Moreover

Lemma 8.3. *$u_n \to u$ in $L^p(\Omega)$ implies $\int_\Omega \psi(u_n)v \to \int_\Omega \psi(u)v$ for all $v \in L^p$.*

Proof. Since $u_n \to u$ in L^p, up to a subsequence we have convergence almost everywhere and we may find a function $k \in L^p$ such that $|u_n| \leq k$ a.e, so that $|\psi(u_n)v| \leq |k|^{p-1}|v|$ which is a L^1 function by the previous lemma, and so the dominated convergence theorem gives $\int_\Omega \psi(u_n)v \to \int_\Omega \psi(u)v$. This procedure may be applied to any subsequence and then the result is true also without passing to a subsequence. □

In the course of the following proofs we will use the known fact that the operator $T : W \to W^*$ defined by $\langle Tu, v\rangle = \int_\Omega \psi(\nabla u)\nabla v$ satisfies the following property S^+ (see [Neč83]):

Definition 8.4. *The operator $T : E \to E^*$ has the property S^+ if*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty}\langle Tu_n - Tu, u_n - u\rangle \leq 0 \text{ implies } u_n \to u.$$

We remark that condition $\limsup_{n\to+\infty}\langle Tu_n - Tu, u_n - u\rangle \leq 0$ may be replaced by $\limsup_{n\to+\infty}\langle Tu_n, u_n - u\rangle \leq 0$ since by weak convergence $\lim_{n\to+\infty}\langle Tu, u_n - u\rangle = 0$.

We give here the proof for sake of completeness, following [Neč83].

Proof of the property S^+ for the p-Laplacian.
The inequality above reads
$\limsup_{n\to+\infty} \int (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot (\nabla u_n - \nabla u) \leq 0$.
First we claim that

for $a, b \in \mathbb{R}^N$, $(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) \geq 0$
and one has equality if and only if $a = b$.

In fact:
$(|a|^{p-2}a - |b|^{p-2}b) \cdot (a - b) = |a|^p + |b|^p - (|a|^{p-2} + |b|^{p-2})(a \cdot b) \geq |a|^p + |b|^p - (|a|^{p-2} + |b|^{p-2})|a||b|$
and one has equality if and only if a and b are collinear.
Now choose, without loss of generality, $0 \leq |a| = c \leq c + \delta = |b|$:

$$\begin{aligned}
|a|^p + |b|^p - (|a|^{p-2} + |b|^{p-2})|a||b| &= c^p + (c+\delta)^p - (c^{p-2} + (c+\delta)^{p-2})c(c+\delta) &&= \\
&= (c+\delta)^p - c\delta\, c^{p-2} - c(c+\delta)(c+\delta)^{p-2} &&= \\
&= (c+\delta)^{p-2}((c+\delta)^2 - c^2 - c\delta) - c\delta c^{p-2} &&= \\
&= (c+\delta)^{p-2}(\delta^2 + c\delta) - c\delta(c^{p-2}) &&= \\
&= (c+\delta)^{p-2}\delta^2 + c\delta((c+\delta)^{p-2} - c^{p-2}) &&\geq 0
\end{aligned}$$

and one has equality if and only if $\delta = 0$.

So we get $(|a|^{p-2}a - |b|^{p-2}b) \cdot (a-b) \geq 0$ where one has equality if and only if a and b are collinear and $|a| = |b|$.

Note also that the expression is continuous in a and b and so if we have $b_n \to b$ then $(|a|^{p-2}a - |b_n|^{p-2}b_n) \cdot (a-b_n) \to (|a|^{p-2}a - |b|^{p-2}b) \cdot (a-b)$; this implies that:

$$if \quad b_n \to b \quad and \quad (|a|^{p-2}a - |b_n|^{p-2}b_n) \cdot (a-b_n) \to 0 \quad then \quad a = b\,. \tag{8.4}$$

Now suppose

$$u_n \rightharpoonup u \quad in \quad W^{1,p}(\Omega)\,, \tag{8.5}$$
$$f_n = (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot (\nabla u_n - \nabla u)\,, \tag{8.6}$$
$$\limsup_{n\to+\infty} \int_\Omega f_n \leq 0\,. \tag{8.7}$$

Since $f_n \geq 0$ we have $\int_\Omega f_n \to 0$ and so there exists a subsequence, a set E with $|\Omega \backslash E| = 0$ and a function $C(x) \in L^1(\Omega)$ such that for any $x \in E$:

$$f_n(x) \to 0 \quad and \quad |f_n| \leq C(x)\,, \tag{8.8}$$
$$u_n(x) \to u(x)\,. \tag{8.9}$$

From (8.6) and (8.8) we get

$$|\nabla u_n(x)|^p \leq C(x) - |\nabla u(x)|^p + |\nabla u(x)|^{p-1}|\nabla u_n(x)| + |\nabla u_n(x)|^{p-1}|\nabla u(x)| \tag{8.10}$$

which implies $|\nabla u_n(x)| \leq D(x)$; so for any fixed $x \in E$, given a subsequence there exists a further subsequence which is convergent, and the limit has to be $\nabla u(x)$ by (8.4), thus we have that $\nabla u_n(x) \to \nabla u(x)$ pointwise in E.

Using Young's inequality $ab \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}$ we get, for arbitrary $\varepsilon > 0$ and nonnegative a, b,

$$ab^{p-1} \leq \varepsilon a^p + C(\varepsilon) b^p, \qquad a^{p-1}b \leq \varepsilon a^p + C(\varepsilon) b^p, \tag{8.11}$$

and so from (8.10)

$$|\nabla u_n|^p \leq f_n + |\nabla u|^p + |\nabla u_n|^{p-1}|\nabla u| + |\nabla u|^{p-1}|\nabla u_n| \leq f_n + \frac{1}{2}|\nabla u_n|^p + C|\nabla u|^p\,, \tag{8.12}$$

that is $|\nabla u_n|^p \leq F(x) \in L^1$ a.e.

Then by the absolute continuity of the integral we have that

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ such that } \int_{N_\delta} |\nabla u_n(x)|^p < \int_{N_\delta} F(x) \leq \varepsilon \quad \forall N_\delta \text{ with } |N_\delta| < \delta:$$

note that here δ may be chosen independently of n, and such that the same property holds for $\int_{N_\delta} |\nabla u(x)|^p$ too.

Finally by Egorov's theorem there exists a set N_δ with $|N_\delta| < \delta$ such that $\nabla u_n(x) \to \nabla u(x)$ uniformly in $\Omega \backslash N_\delta$; then

$$\begin{aligned}\int_\Omega |\nabla u_n - \nabla u|^p &= \int_{\Omega\backslash N_\delta} |\nabla u_n - \nabla u|^p + \int_{N_\delta} |\nabla u_n - \nabla u|^p \\ &\leq \int_{\Omega\backslash N_\delta} |\nabla u_n - \nabla u|^p + 2\varepsilon \to 2\varepsilon\end{aligned} \tag{8.13}$$

for any choice of ε, and so (since we already had $u_n \to u$ in $L^p(\Omega)$) this implies $u_n \to u$ in $W^{1,p}(\Omega)$.

As before, this procedure may be applied to any subsequence and then the result is true also without passing to a subsequence.

□

8.2 The usual spectrum for the p-Laplacian

Here we deal with the problem

$$\begin{cases} -\nabla \cdot [\psi(\nabla u)] = \lambda \psi(u) & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array}\right. & in\ \partial\Omega \end{cases} . \tag{8.14}$$

As noted in section 2.5 for the Dirichlet problem it is known (see [Ana87] and [Lin90]) that there exists a first eigenvalue λ_1, that it is simple and isolated and that the corresponding eigenfunction ϕ_1 does not change sign, so that we may take it to be positive and with $\|\phi_1\|_{L^p} = 1$.

Moreover this first eigenvalue may be characterized as

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^p : \qquad u \in W; \quad \|u\|_{L^p} = 1 \right\} . \tag{8.15}$$

The same proofs may be adapted to work in the Neumann case.

Neumann case. Since any constant satisfies the equation $\nabla \cdot [\psi(\nabla u)] = 0$ we have that 0 is an eigenvalue; actually it is the first since $\int_\Omega |\nabla u|^p = \lambda \int_\Omega |u|^p$ with $\lambda < 0$ implies $u = 0$ a.e; then equation (8.15) is trivially satisfied by $\lambda_1 = 0$.

Moreover this eigenvalue is simple since $\int_\Omega |\nabla u|^p = 0$ implies, for a function $u \in W^{1,p}$, that $u = const$ a.e.

Finally the proof of the isolatedness of $\lambda_1 = 0$ may be adapted from [Ana87]: suppose it were not isolated, that is, that there exist sequences $\lambda_n \to 0^+$ and $\{u_n\} \subseteq W^{1,p}$ with $\lambda_n \neq 0$ and $\|u_n\|_{L^p} = 1$, such that

$$\int_\Omega \psi(\nabla u_n)\nabla v = \lambda_n \int_\Omega \psi(u_n) v \quad \forall v \in W^{1,p} . \tag{8.16}$$

Taking $v = u_n$ we get that $\|\nabla u_n\|_{L^p} \to 0$, then $\|u_n\|_{W^{1,p}}$ is bounded and so up to a subsequence we have $u_n \to u$ weakly in $W^{1,p}$ and strongly in L^p.

Taking $v = u_n - u$ we get

$$\int_\Omega \psi(\nabla u_n)\nabla(u_n - u) = \lambda_n \int_\Omega \psi(u_n)(u_n - u) \to 0\,, \tag{8.17}$$

and so $u_n \to u$ strongly in W by property S^+.

This allows to take the limit in equation (8.16) using lemma 8.3 and so gives

$$\int_\Omega \psi(\nabla u)\nabla v = 0 \quad \forall v \in W^{1,p}\,, \tag{8.18}$$

which again implies $u = \mathit{const}$ a.e. and this constant is not zero since $\|u\|_{L^p} = 1$.

Taking $v = 1$ in (8.16) we get

$$\int_\Omega \psi(u_n) = 0 \tag{8.19}$$

which implies that all u_n must change sign.

Supposing now $u > 0$ and taking $v = u_n^-$ in (8.16) we get

$$\int_\Omega |\nabla u_n^-|^p = \lambda_n \int_\Omega |u_n^-|^p\,. \tag{8.20}$$

Now choose any $\delta > 0$ such that $W^{1,p} \subseteq L^{p+\delta}$ with continuous inclusion, then by Hölder's inequality with the dual exponents $\frac{p+\delta}{p}$ and $\frac{p+\delta}{\delta}$ we may estimate

$$\|u_n^-\|_{L^p}^p = \int_{\Omega_n^-} |u_n^-|^p \le \left(\int_{\Omega_n^-} 1^{\frac{p+\delta}{\delta}}\right)^{\frac{\delta}{p+\delta}} \left(\int_\Omega |u_n^-|^{p\frac{p+\delta}{p}}\right)^{\frac{p}{p+\delta}} \tag{8.21}$$

where Ω_n^- is the set where $u_n \le 0$, and so

$$\|u_n^-\|_{L^{p+\delta}}^p \le C\,\|u_n^-\|_{W^{1,p}}^p = C(1+\lambda_n)\,\|u_n^-\|_{L^p}^p \le C(1+\lambda_n)\,|\Omega_n^-|^{\frac{\delta}{p+\delta}}\,\|u_n^-\|_{L^{p+\delta}}^p\,, \tag{8.22}$$

from which (since we saw that $\|u_n^-\| \ne 0$)

$$|\Omega_n^-| \ge (C(1+\lambda_n))^{-\frac{p+\delta}{\delta}} \ge (2C)^{-\frac{p+\delta}{\delta}}\,. \tag{8.23}$$

But the L^p convergence implies quasi uniform convergence and since the limit is a positive constant this implies that the u_n are positive outside of an arbitrary small set for n large enough, giving a contradiction. □

8.3 Variational characterization of parts of the Fučík spectrum of the p-Laplacian

The variational characterization of the Fučík spectrum made for the case $p = 2$ fails for $p \ne 2$ since the deformation obtained in section 4.1 relied on the structure of Hilbert space of H.

However we may recover a part of the result using different techniques.

We first consider the Fučík problem in any spatial dimension with both Neumann or Dirichlet boundary conditions, namely

$$\begin{cases} -\nabla\cdot[\psi(\nabla u)] = \lambda^+\psi(u^+) - \lambda^-\psi(u^-) & in\ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array}\right. & in\ \partial\Omega \end{cases} . \tag{8.24}$$

We will need some preliminary lemmas.
Consider, for a given point $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ and $r \in (0, 1]$, the functional

$$J_\alpha(u) = \int_\Omega |\nabla u|^p - \alpha^+ \int_\Omega (u^+)^p - \alpha^- \int_\Omega (u^-)^p \tag{8.25}$$

and the manifold

$$Q_r = \{u \in W \ \ s.t.\ \ V(u) = \int_\Omega (u^+)^p + r(u^-)^p = 1\} . \tag{8.26}$$

Remark 8.5. *Note that the functional (resp. the manifold) are of class $\mathcal{C}^2$ for $p > 2$, $\mathcal{C}^1$ but not $\mathcal{C}^{1,1}$ for $p \in (1, 2)$, while for $p = 2$ they are $\mathcal{C}^{1,1}$, but not $\mathcal{C}^2$ unless $\alpha^+ = \alpha^-$ (resp. $r = 1$).*

Definition 8.6. *For the derivative of the functional J_α restricted to Q_r we will consider the norm $\|J'_\alpha(u)\|_* = inf_{t\in\mathbb{R}} \|J'_\alpha(u) - tV'(u)\|_{W^*}$.*

Lemma 8.7. *When $u \in Q_r$ we have that $1 \le \int |u|^p \le 1/r$.*

Proof. $1 = \int(u^+)^p + r(u^-)^p \le \int(u^+)^p + (u^-)^p = \int |u|^p \le (\int(u^+)^p + r(u^-)^p)/r = 1/r$. □

We will also need some sort of PS condition: for $p < 2$ we need a stronger property (see [Bon93]), actually if Q_r is just of class $\mathcal{C}^1$ we need to prove the existence of a converging subsequence for any PS-sequence $\{u_n\}$ where $u_n \in Q_r^{\delta_n}$, δ_n being any sequence such that $\delta_n \to 0$ and
$Q_r^{\delta_n} = \{u \in W \ \ s.t.\ \ \int_\Omega (u^+)^p + r(u^-)^p = 1 + \delta_n\}$.

Lemma 8.8. *The functional J_α constrained to Q_r satisfies the PS condition.*

Proof. We take two sequences $\delta_n \to 0$ and $\varepsilon_n \to 0^+$, a sequence $\{u_n\} \subseteq Q_r^{\delta_n}$ and a sequence $\{\beta_n\} \subseteq \mathbb{R}$, such that

$$\left| \int |\nabla u_n|^p - \alpha^+ \int |u_n^+|^p - \alpha^- \int |u_n^-|^p \right| \le C \tag{8.27}$$

$$\left| \int \psi(\nabla u_n)\nabla v - \alpha^+ \int \psi(u_n^+)v + \alpha^+ \int \psi(u_n^-)v + \beta_n \left(\int \psi(u_n^+)v - r\psi(u_n^-)v \right) \right| \le \\ \le \varepsilon_n \|v\|_W , \quad \forall v \in W . \tag{8.28}$$

Since $\{u_n\} \subseteq Q_r^{\delta_n}$, it is bounded in L^p, and then by equation (8.27) it is also bounded in W. Then there exists a subsequence converging weakly in W and strongly in L^p to some u.
The L^p convergence implies that $u \in Q_r$.
Taking $v = u_n$ we get that

$$\left(\int |\nabla u_n|^p - \alpha^+ \int |u_n^+|^p - \alpha^- \int |u_n^-|^p\right) + (1+\delta_n)\beta_n \to 0\,. \tag{8.29}$$

Finally with $v = u_n - u$ we have

$$\begin{aligned} &\int \psi(\nabla u_n)\nabla(u_n - u) - \alpha^+ \int \psi(u_n^+)(u_n - u) + \alpha^- \int \psi(u_n^-)(u_n - u) \quad + \\ -&\left(\int |\nabla u_n|^p - \alpha^+ \int |u_n^+|^p - \alpha^- \int |u_n^-|^p\right)\left(\int (\psi(u_n^+) - r\psi(u_n^-))(u_n - u)\right) \qquad \to 0 \end{aligned} \tag{8.30}$$

where (estimating with inequality (8.3)) all terms except the first go to zero and then we conclude that $u_n \to u$ strongly in W by the property S^+ for the p-Laplacian. □

Finally it will be crucial in the following that:

Proposition 8.9. *The critical points at some level c of J_α constrained to Q_r are non trivial solutions of the Fučík problem with coefficients $(\alpha^+ + c, \alpha^- + rc)$, that is the criticality of c implies that $(\alpha^+ + c, \alpha^- + rc) \in \Sigma$.*

Proof. Criticality of u implies that there exists the Lagrange's multiplier $\beta \in \mathbb{R}$ such that

$$\int_\Omega \psi(\nabla u)\nabla v - \alpha^+ \int_\Omega \psi(u^+)v + \alpha^- \int_\Omega \psi(u^-)v + \beta\left(\int_\Omega \psi(u^+)v - r\psi(u^-)v\right) = 0 \quad \forall v \in W\ , \tag{8.31}$$

but testing against u we get $\beta = -c$ and so u solves

$$-\Delta_p u = \alpha^+\psi(u^+) - \alpha^-\psi(u^-) + c\psi(u^+) - cr\psi(u^-) = (\alpha^+ + c)\psi(u^+) - (\alpha^- + rc)\psi(u^-) \tag{8.32}$$

in Ω, with the given boundary conditions.
Finally u is not trivial since it is in Q_r. □

8.3.1 First nontrivial curve

First we will reformulate in a slightly different way the variational characterization of the second curve of the Fučík spectrum of the p-Laplacian, made in [CdFG99].

In this part, we can still work in any spatial dimension with both Neumann or Dirichlet boundary conditions.

Consider

$$d_{\lambda_1, r} = \inf_{\delta \in \Gamma_{\lambda_1, r}} \sup_{u \in \delta([0,1])} J_{\lambda_1}(u)\ , \tag{8.33}$$

where

$$J_{\lambda_1}(u) = \int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p\ , \tag{8.34}$$

$$\Gamma_{\lambda_1,r} = \{\delta : [0,1] \to Q_r \textit{ continuous s.t. } \delta(0) = \phi_1\,,\ \delta(1) = -\frac{\phi_1}{\sqrt[p]{r}}\}\,. \tag{8.35}$$

We first have:

Lemma 8.10. $\sup_{u\in\delta(\{0;1\})} J_{\lambda_1}(u) = 0$, $\forall \delta \in \Gamma_{\lambda_1,r}$.

Proof. One needs only to note that $J_{\lambda_1}(\phi_1) = 0$. □

Lemma 8.11. $+\infty > d_{\lambda_1,r} = \inf_{\delta\in\Gamma_{\lambda_1},r} \sup_{u\in\delta([0,1])} J_{\lambda_1}(u) > 0$.

Proof. It is less than $+\infty$ since each $\delta([0,1])$ is a compact set.

Proposition 8.9 implies that the only critical points at level 0 on Q_r are $z_1 = \phi_1$ and $z_2 = -\frac{\phi_1}{\sqrt[p]{r}}$: call d the distance between them.

Since $J_{\lambda_1}(u) \geq 0$ in Q_r by the variational characterization of λ_1, we have $d_{\lambda_1,r} \geq 0$.

Now suppose by contradiction that $d_{\lambda_1,r} = 0$: then for any sequence of positive reals $\varepsilon_n \to 0$ there would exist a sequence $\{\delta_n\} \subseteq \Gamma_{\lambda_1,r}$ such that

$$\sup_{u\in\delta_n([0,1])} J_{\lambda_1}(u) < \varepsilon_n\,, \tag{8.36}$$

and then also a sequence $\{u_n\} \subseteq Q_r$ such that

(1a) $u_n \in \delta_n([0,1])$, and then $J_{\lambda_1}(u_n) < \varepsilon_n$

(2a) $\|u_n - z_i\|_W > d/4$ for $i = 1, 2$.

Since $\inf_{u\in Q_r} J_{\lambda_1}(u) = 0$ we are in the conditions to apply the Ekeland variational principle (see theorem 2.9) to each u_n, obtaining a sequence $\{w_n\} \subseteq Q_r$ such that

(1b) $0 \leq J_{\lambda_1}(w_n) \leq J_{\lambda_1}(u_n) < \varepsilon_n$,

(2b) $\|u_n - w_n\|_W \leq \sqrt{\varepsilon_n}$,

(3b) $\|J'_{\lambda_1}(w_n)\|_* \leq \sqrt{\varepsilon_n}$.

But then w_n would be a PS sequence for J_α on Q_r and so would have a subsequence converging to one of the critical points at level 0 (z_1 or z_2), which is impossible considering properties (2a) and (2b).

□

Combining the previous two lemmas, the PS condition in lemma 8.8 and proposition 8.9, we can assert, by a classical linking theorem, that

Theorem 8.12. *The level $d_{\lambda_1,r}$ is critical for $J_{\lambda_1}(u)$ constrained to Q_r. That is the point $(\lambda_1 + d_{\lambda_1,r}, \lambda_1 + rd_{\lambda_1,r}) \in \Sigma$.*

As mentioned before, this is nothing other than a different formulation of the variational characterization in [CdFG99], however it is in a useful form to be used in the following.

8.3.2 Third (or higher) curve for the Neumann problem in one dimension

Now consider the one dimensional Neumann case: we want to make one more step in the characterization of the Fučík spectrum.

For the properties of the Fučík spectrum in this case we can refer to [RW99] and [Drá92], where we see that it has the same qualitative shape of the case p=2 (see figure 2 on page 23), with single curves (we will call them Σ_k) coming out from each (simple) eigenvalue (λ_k, λ_k), and having distinct asymptotes. Moreover to each point in the spectrum correspond only two families of nontrivial solutions: the positive multiples of the initially positive and of the initially negative ones. In particular the nontrivial solutions corresponding to a point in the curve Σ_2 are composed by a positive half-bump followed by a negative one and viceversa.

The idea we are going to apply is to "build" a suitable set homeomorphic to ∂B^2 to be used as $L_{\alpha,r}$ in equation (4.27) and so to repeat the characterization made in section 4.2.

8.3.2.1 Construction of the set L_{α,r_1}

We fix a point $\alpha = (\alpha^+, \alpha^-)$ on the curve Σ_2 with $\alpha^+ \geq \alpha^-$.

We define $r_1 = \frac{\alpha^- - \lambda_1}{\alpha^+ - \lambda_1} = \frac{\alpha^-}{\alpha^+}$, we call u_α one of the two solutions in Q_{r_1} of the Fučík problem (8.2) with coefficients (α^+, α^-), and $\overline{u}_\alpha$ the other one.

Then we consider the functional

$$J_\alpha(u) = \int |u'|^p - \alpha^+ \int (u^+)^p - \alpha^- \int (u^-)^p \ . \tag{8.37}$$

Remark 8.13. *Observe that for $u \in Q_{r_1}$ we have:*

$$J_\alpha(u) = J_{\lambda_1}(u) - (\alpha^+ - \lambda_1) \tag{8.38}$$

and so

$$\inf_{\delta \in \Gamma_{\lambda_1, r_1}} \sup_{u \in \delta([0,1])} J_\alpha(u) = d_{\lambda_1, r_1} - (\alpha^+ - \lambda_1) \geq 0 \,, \tag{8.39}$$

indeed, it is not less than zero since we chose $\alpha \in \Sigma_2$ and so by theorem 8.12 $d_{\lambda_1,r_1} \geq \alpha^+ - \lambda_1$; moreover we have

$$\sup_{u \in \delta(\{0;1\})} J_\alpha(u) = -(\alpha^+ - \lambda_1) < 0 \,. \tag{8.40}$$

Now we look for a particular $\delta \in \Gamma_{\lambda_1, r_1}$ such that $J_\alpha(u)|_{\delta([0,1])} \leq 0$: we will build the image of this δ as follows: take the path l on Q_{r_1}: $\overbrace{\phi_1 \tilde{u}^+}\, \overbrace{\tilde{u}^+ u}\, \overbrace{u(-\tilde{u}^-)}\, \overbrace{(-\tilde{u}^-)\frac{-\phi_1}{\sqrt[p]{r_1}}}$ where $u = u_\alpha$, $\tilde{u}^+ = \frac{u_\alpha^+}{\|u_\alpha^+\|_{L^p}}$, $\tilde{u}^- = \frac{u_\alpha^-}{\sqrt[p]{r_1}\|u_\alpha^-\|_{L^p}}$ and the arcs are taken projecting on Q_{r_1} the segment that joins the two vertices (note that these segments never pass through zero).

Lemma 8.14. $\sup_{u \in l}(J_\alpha(u)) = 0$.

Proof. Let us start by observing that the Fučík equation in variational form $\int \psi(u_\alpha')v' = \alpha^+ \int \psi(u_\alpha^+)v - \alpha^- \int \psi(u_\alpha^-)v$, with test functions u_α^+ and u_α^- gives

$$\int |(u_\alpha^\pm)'|^p = \alpha^\pm \int (u_\alpha^\pm)^p \,, \tag{8.41}$$

that is $J_\alpha(u_\alpha^\pm) = 0$; moreover the homogeneity of J_α allows us to ignore the projection on Q_{r_1} in the proof.

Then we look at the four arcs:

- $\overbrace{\phi_1\tilde{u}^+}$: call $v = t\phi_1 + (1-t)u^+$ so that $v' = (1-t)(u^+)'$:
  v is everywhere non negative and then (since $[t\phi_1 + (1-t)u^+] \geq (1-t)u^+$ everywhere):

$$\begin{aligned} J_\alpha(v) &= (1-t)^p \int |(u^+)'|^p - \alpha^+ \int [t\phi_1 + (1-t)u^+]^p \\ &\leq (1-t)^p\alpha^+ \int (u^+)^p - (1-t)^p\alpha^+ \int (u^+)^p = 0\,. \end{aligned}$$

- $\overbrace{(-\tilde{u}^-)(-\phi_1/\sqrt[p]{r_1})}$: in the same way: call $v = t(-\phi_1) + (1-t)(-u^-)$ so that $v' = (1-t)(-u^-)'$:
  v is everywhere non positive and then (since $[t\phi_1 + (1-t)u^-] \geq (1-t)u^-$ everywhere):

$$\begin{aligned} J_\alpha(v) &= (1-t)^p \int |(u^-)'|^p - \alpha^- \int [t\phi_1 + (1-t)u^-]^p \\ &\leq (1-t)^p\alpha^- \int (u^-)^p - (1-t)^p\alpha^- \int (u^-)^p = 0\,. \end{aligned}$$

- $\overbrace{\tilde{u}^+u}$: here $v = tu^+ + (1-t)u = u^+ + (1-t)(-u^-)$: obviously u^+ and u^- are non zero on disjoint sets, then

$$J_\alpha(v) = J_\alpha(u^+) + (1-t)^p J_\alpha(u^-) = 0\,.$$

- $\overbrace{-\tilde{u}^-u}$: here $v = t(-u^-) + (1-t)u = (-u^-) + (1-t)(u^+)$: as before

$$J_\alpha(v) = J_\alpha(u^-) + (1-t)^p J_\alpha(u^+) = 0\,.$$

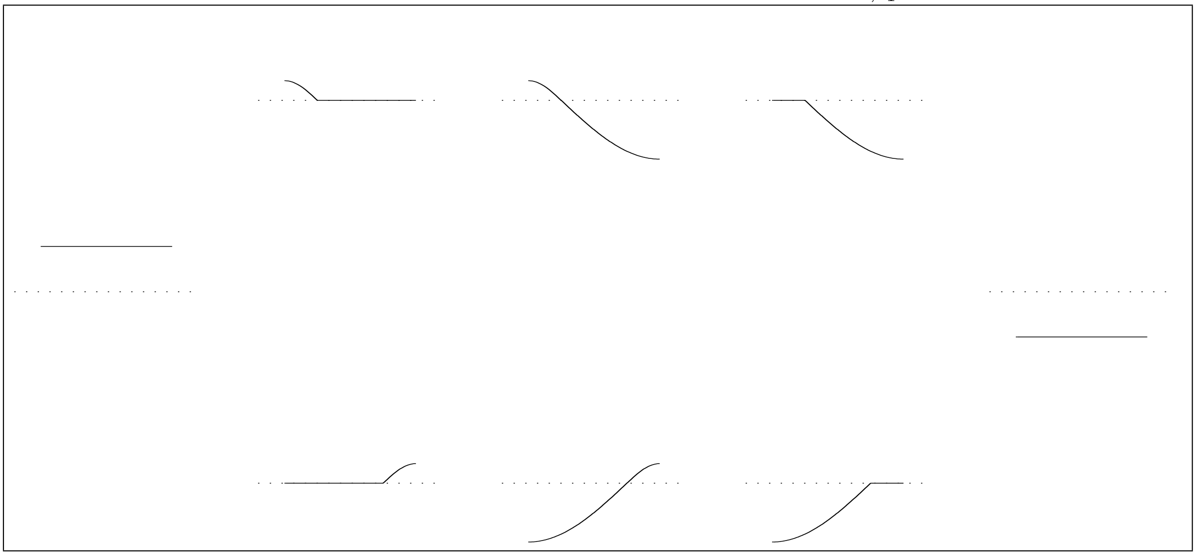
□

Now note that the functional $J_\alpha(u)$ is invariant under the transformation $x \mapsto 1-x$, and that the path defined in the proof is composed by non symmetrical functions with respect to this transformation, except for the two points in *span*$\{\phi_1\}$. Then we can consider the loop $L_{\alpha,r_1} \subseteq Q_{r_1}$ obtained joining l with its symmetrical path.

In figure 13 are sketched (in a qualitative way) the eight functions used to build the set L_{α,r_1}.

Remark 8.15. *At this point it is clear that the level $d_{\lambda_1,r}$ defined in (8.33) corresponds to the first intersection with the Fučík spectrum of the halfline $\{(\lambda_1+t, \lambda_1+r_1t),\, t>0\}$: it cannot be lower (if it were it would give a new solution of Fučík problem that we know does not exist) and we were able to give an example of a $\delta \in \Gamma_{\lambda_1,r_1}$ where* $\sup(J_\alpha(u)) = 0$*, that is* $\sup(J_{\lambda_1}(u)) = \alpha^+ - \lambda_1$*, and then $d_{\lambda_1,r} = \alpha^+ - \lambda_1$, where (α^+,α^-) was taken on the second curve.*

Figure 13: The functions used to compose L_{α,r_1}.



8.3.2.2 Linking structure

Now define the class

$$\Gamma_{\alpha,r_1} = \{\gamma : B^2 \to Q_{r_1} \ \ \textit{continuous s.t.}\ \ \gamma|_{\partial B^2} \ \textit{is an homeomorphism onto}\ L_{\alpha,r_1}\}\ . \tag{8.42}$$

Then by construction we have that

Lemma 8.16. $\sup_{u\in\gamma(\partial B^2)} J_\alpha(u) = 0 \quad \forall\gamma\in\Gamma_{\alpha,r_1}$.

Moreover

Lemma 8.17. $+\infty > d_{\alpha,r_1} = \inf_{\gamma\in\Gamma_{\alpha,r_1}} \sup_{u\in\gamma(B^2)} J_\alpha(u) > 0$.

Proof. It is less than $+\infty$ since each $\gamma(B^2)$ is a compact set.

Proposition 8.9 implies that the only critical points at level 0 on Q_{r_1} are $z_1 = u_\alpha$ and $z_2 = \overline{u}_\alpha$: call d the distance between them, and take $\hat{d} < d$ such that $B_{\hat{d}}(u_\alpha)$ and $B_{\hat{d}}(\overline{u}_\alpha)$ are disjoint and do not contain ϕ_1 nor $-\frac{\phi_1}{\sqrt[p]{r_1}}$.

Lemma 8.16 implies that $d_{\alpha,r_1} \geq 0$, so suppose by contradiction that $d_{\alpha,r_1} = 0$: then for any sequence of positive reals $\varepsilon_n \to 0$ there would exist a sequence $\{\gamma_n\} \subseteq \Gamma_{\alpha,r_1}$ such that

$$\sup_{u\in\gamma_n(B^2)} J_\alpha(u) < \varepsilon_n\,, \tag{8.43}$$

and then also a sequence of paths $\{\delta_n\} \subseteq \Gamma_{\lambda_1,r_1}$ such that

(1a) $\delta_n([0,1]) \subseteq \gamma_n(B^2)$, and then $0 \leq \sup_{u\in\delta_n([0,1])} J_\alpha(u) < \varepsilon_n$ (see equation (8.39)),

(2a) $d(\delta_n([0,1]), z_i) > \hat{d}$ for $i = 1, 2$.

Now we may apply to each δ_n the minimax principle derived from Ekeland's variational principle (see theorem 2.10).

In fact (see remarks 8.13 and 8.15),

$$\inf_{\delta\in\Gamma_{\lambda_1,r_1}} \sup_{u\in\delta([0,1])} J_\alpha(u) = d_{\lambda_1,r_1} - (\alpha^+ - \lambda_1) = 0\,, \tag{8.44}$$

$$\sup_{u\in\delta(\{0;1\})} J_\alpha(u) = -(\alpha^+ - \lambda_1) < 0 \tag{8.45}$$

and the sequence δ_n above is minimizing for the value $\sup_{u\in\delta([0,1])} J_\alpha(u)$ with $\delta \in \Gamma_{\lambda_1,r_1}$.

So we obtain a sequence $\{w_n\} \subseteq Q_{r_1}$ such that

(1b) $-\varepsilon_n \le J_\alpha(w_n) \le \sup_{u\in\delta_n([0,1])} J_\alpha(u) < \varepsilon_n$,

(2b) $d(\delta_n([0,1]), w_n) \le \sqrt{\varepsilon_n}$,

(3b) $\|J'_\alpha(w_n)\|_* \le \sqrt{\varepsilon_n}$.

But then w_n would be a PS sequence for J_α on Q_{r_1} and so would have a subsequence converging to one of the critical points at level 0 (z_1 or z_2), which is impossible considering properties (2a) and (2b).

□

8.3.2.3 Characterization of a point above Σ_2

Now, given a $r_2 \in (0,1]$ and considering $P_{r_1}^{r_2}$ the radial projection from Q_{r_1} to Q_{r_2}, we define

$$\Gamma_{\alpha,r_2} = \{\gamma = P_{r_1}^{r_2} \circ \widetilde{\gamma} \ \ s.t. \ \ \widetilde{\gamma} \in \Gamma_{\alpha,r_1}\} \tag{8.46}$$

and we get from the previous two lemmas, these corollaries:

Corollary 8.18. $\sup_{u\in\gamma(\partial B^2)} J_\alpha(u) \le 0 \quad \forall\gamma \in \Gamma_{\alpha,r_2}$.

Proof. The result of the projection is just multiplying by a positive scalar the point u and then the effect on $J_\alpha(u)$ is multiplying by the p_{th} power of this scalar, which does not change the sign. □

Corollary 8.19. $+\infty >= \inf_{\gamma\in\Gamma_{\alpha,r_2}} \sup_{u\in\gamma(B^2)} J_\alpha(u) > 0$.

Proof. As before: the effect of the projection is just multiplying by a number that (on Q_{r_1}) is positive, bounded and bounded away from zero, and then the result follows. □

From now on we can proceed as in the case of $p = 2$, that is we define

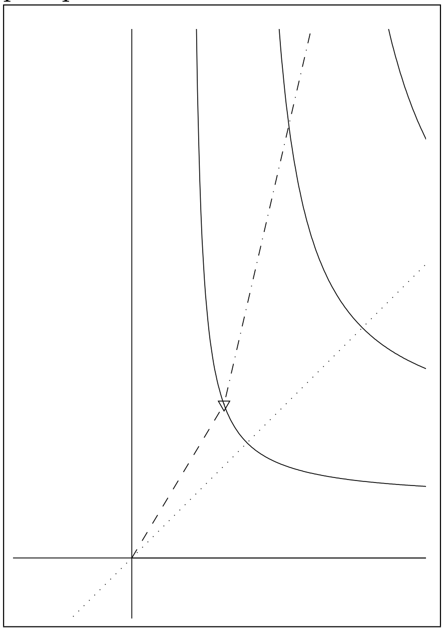
$$d_{\alpha,r_2} = \inf_{\gamma\in\Gamma_{\alpha,r_2}} \sup_{u\in\gamma(B^2)} J_\alpha(u) > 0 \ \ , \tag{8.47}$$

we deduce that it is a critical level for J_α constrained to Q_{r_2} and then that $(\alpha^+ + d_{\alpha,r_2}, \alpha^- + r_2 d_{\alpha,r_2}) \in \Sigma$: in particular we can assert:

Proposition 8.20. *For any point (α^+, α^-) on Σ_2 with $\alpha^+ \ge \alpha^-$ we can find and characterize one intersection with the Fučík spectrum of the halfline $\{(\alpha^+ + t, \alpha^- + r_2 t),\ t > 0\}$, for each value of $r_2 \in (0,1]$.*

The above construction is sketched in figure 14.

Figure 14: The construction for the variational characterization of a point above Σ_2 for the p-Laplacian.



$$\begin{cases} \nabla & : \ (\alpha^+, \alpha^-) \\ - \cdot - & : \ \{(\alpha^+ + t, \alpha^- + r_2 t), \quad t > 0\} \\ - - - & : \ \{(\lambda_1 + t, \lambda_1 + r_1 t), \quad t > 0\} \end{cases}$$

8.4 The “ψ-superlinear” problem

Since we reproduced the variational characterization as in section 4.2, we may apply it to the “ψ-superlinear” problem (8.1) when λ is between the asymptotes of Σ_2 and Σ_3 or (resonant case) coincides with one of them.

Actually using now the following estimates in place of those in section 5.1:

- $\exists\ C_1(\delta, h)$ such that $|\int hu| \le \frac{\delta}{2p}\|u\|^p_{L^p} + C_1(\delta, h)$;
- $\exists\ C_2(\delta, g)$ such that $|\int G(x, -u^-)| \le \frac{\delta}{2p}\|u\|^p_{L^p} + C_2(\delta, g)$;
- for any M, $\exists\ C_3(M, g)$ such that $\int G(x, u^+) \ge \frac{M}{p}\|u^+\|^p_{L^p} - C_3(M, g)$;
- $G(x, s) \le 1 + \frac{\mu_{\gamma^*}}{p} s^p \quad for\ \ all \quad s \in [0, b(\gamma^*)]$;

and in place of that in section 5.2

- $\int_0^1 G(x, u) + \int_0^1 hu \le \frac{\mu_{\gamma^*}+1}{p}\int_0^1 (u^+)^p + M_0 + C_4(h, g) + 1 + \frac{1}{q}\int_0^1 |h|^q$,

we may prove the equivalents of lemmas 5.4 and 5.5, for the functional associated to problem (8.1), namely:

$$F(u) = \frac{1}{p}\int_0^1 |u'|^p - \frac{\lambda}{p}\int_0^1 |u|^p - \int_0^1 G(x, u) - \int_0^1 hu\ . \tag{8.48}$$

Remark 8.21. *In proposition 8.20 it is not specified whether the characterized intersection is the first (that is the one with smallest t) of the halfline with Σ. However this information was not needed in the proof of lemma 5.5, as observed in remark 5.6.*

Then we assert:

Lemma 8.22. *For $p \geq 2$, under hypotheses (H1-p), (H2-p) and (H3-p) with $h \in L^q(0,1)$ where $\frac{1}{p} + \frac{1}{q} = 1$, the functional (8.48) satisfies the PS condition for any $\lambda > 0$.*

Proof. See section 9. □

Finally if we call λ_i^* the value of the asymptote of the curve Σ_i of the Fučík spectrum, we conclude that

Theorem 8.23. *Under hypotheses (H1-p), (H2-p) and (H3-p), if $p \geq 2$ and $\lambda \in (\lambda_2^*, \lambda_3^*)$, then there exists a solution of problem (8.1) for all $h \in L^q(0,1)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

and

Theorem 8.24. *Under hypotheses (H1-p), (H2-p), (H3-p) and (HR-p), with $p \geq 2$, $h \in L^q(0,1)$ where $\frac{1}{p} + \frac{1}{q} = 1$, if $\lambda = \lambda_i^*$ for $i = 2$ or $i = 3$, then there exists a solution of problem (8.1).*

9 Proof of the PS condition

In this section we will prove the PS condition for the functional (8.48) with $p \geq 2$ (and then also for the functional (5.1)).

This proof is adapted from that made in [dFR93] for the periodic problem on an interval, with the Laplacian operator.

The exact statement of the result is

Lemma. 8.22. *For $p \geq 2$, under hypotheses (H1-p), (H2-p) and (H3-p) with $h \in L^q(0,1)$ where $\frac{1}{p} + \frac{1}{q} = 1$, the functional (8.48) satisfies the PS condition for any $\lambda > 0$.*

First note that from hypothesis (H1-p) one can always make the estimates:
for any $\varepsilon > 0$, $\bar{s} \in \mathbb{R}$ and $M \in R$, there exist C_M, $C_\varepsilon \in \mathbb{R}$ (of course depending also on $\bar{s}$) such that

$$g(x,s) \geq M\psi(s) - C_M \quad for \quad s > \bar{s} \ , \tag{9.1}$$

$$|g(x,s)| \leq \varepsilon\psi(-s) + C_\varepsilon \quad for \quad s \leq \bar{s} \ . \tag{9.2}$$

Let now $\{u_n\} \subseteq W^{1,p}(0,1)$ be a PS sequence, i.e. there exist $T > 0$ and $\varepsilon_n \to 0^+$ such that

$$|F(u_n)| = \left| \frac{1}{p} \int_0^1 |u_n'|^p - \frac{\lambda}{p} \int_0^1 |u_n|^p - \int_0^1 G(x,u_n) - \int_0^1 hu_n \right| \leq T \ , \tag{9.3}$$

$$\begin{aligned} \left|\langle F'(u_n), v\rangle\right| = \left| \int_0^1 \psi(u_n')v' - \lambda \int_0^1 \psi(u_n)v - \int_0^1 g(x,u_n)v - \int_0^1 hv \right| \leq \\ \leq \varepsilon_n \|v\|_{W^{1,p}} \ , \quad \forall v \in W^{1,p} \ . \end{aligned} \tag{9.4}$$

1. Suppose u_n is not bounded, then we can assume $\|u_n\|_{W^{1,p}} \geq 1$, $\|u_n\|_{W^{1,p}} \to +\infty$ and define $z_n = \frac{u_n}{\|u_n\|_{W^{1,p}}}$, so that z_n is a bounded sequence in $W^{1,p}$ and we can select a subsequence such that $z_n \to z_0$ weakly in $W^{1,p}$ and strongly in $L^p(0,1)$ and $\mathcal{C}^0[0,1]$.

2. Claim: $z_0 \leq 0$.

   Proof of the claim. Consider $\left| \frac{\langle F'(u_n), z_0^+\rangle}{\|u_n\|^{p-1}_{W^{1,p}}} \right|$:

$$\left| \int_0^1 \psi(z_n')(z_0^+)' - \lambda \int_0^1 \psi(z_n)z_0^+ - \int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|^{p-1}_{W^{1,p}}} - \int_0^1 \frac{hz_0^+}{\|u_n\|^{p-1}_{W^{1,p}}} \right| \leq \frac{\varepsilon_n \left\|z_0^+\right\|_{W^{1,p}}}{\|u_n\|^{p-1}_{W^{1,p}}} \ , \tag{9.5}$$

   from which

$$\int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|^{p-1}_{W^{1,p}}} \leq \left| \int_0^1 \psi(z_n')(z_0^+)' \right| + \lambda \left| \int_0^1 \psi(z_n)z_0^+ \right| + \left| \int_0^1 \frac{hz_0^+}{\|u_n\|^{p-1}_{W^{1,p}}} \right| + \frac{\varepsilon_n \left\|z_0^+\right\|_{W^{1,p}}}{\|u_n\|^{p-1}_{W^{1,p}}} \ . \tag{9.6}$$

   Now for any $\bar{x}$ such that $z_0^+(\bar{x}) > 0$, we have that $u_n(\bar{x}) > 0$ for n large enough and then we can use the estimate (9.1) to obtain

$$\frac{g(\bar{x},u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} \geq M\psi(z_n(\bar{x})) - \frac{C_M}{\|u_n\|^{p-1}_{W^{1,p}}} \ ; \tag{9.7}$$

taking lim inf we get

$$\liminf_{n\to+\infty} \frac{g(\bar{x},u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} \geq M\psi(z_0(\bar{x})) \tag{9.8}$$

for any choice of M and then

$$\lim_{n\to+\infty} \frac{g(\bar{x},u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} = +\infty \ . \tag{9.9}$$

Joining equations (9.1) and (9.2) with $\bar{s}=0$ and divided by $\|u_n\|^{p-1}_{W^{1,p}}$ we get

$$\begin{aligned} \frac{g(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} &\geq M\psi(z_n) - \tfrac{C_M}{\|u_n\|^{p-1}_{W^{1,p}}} \quad &\textit{where } z_n > 0 \\ &\geq -\varepsilon\psi(-z_n) - \tfrac{C_\varepsilon}{\|u_n\|^{p-1}_{W^{1,p}}} \quad &\textit{where } z_n \leq 0 \end{aligned}$$

and so

$$\frac{g(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} \geq -\varepsilon\psi(|z_n|) - \frac{C_{M,\varepsilon}}{\|u_n\|^{p-1}_{W^{1,p}}} \ ; \tag{9.10}$$

since z_n is uniformly bounded by its $\mathcal{C}^0$ convergence and $\|u_n\|_{W^{1,p}} \geq 1$, this implies that the functions $\frac{g(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}}}$ are bounded below uniformly so that we can use Fatou's Lemma and get from (9.6) and supposing $z_0^+ \not\equiv 0$

$$\begin{aligned} +\infty &= \int_0^1 \lim_{n\to+\infty} \frac{g(x,u_n)z_0^+}{\|u_n\|^{p-1}_{W^{1,p}}} \leq \liminf_{n\to+\infty} \int_0^1 \frac{g(x,u_n)z_0^+}{\left\|u_n^{p-1}\right\|_{W^{1,p}}} \\ &\leq \liminf_{n\to+\infty} \left(\left|\int_0^1 \psi(z_n')(z_0^+)'\right| + \lambda \left|\int_0^1 \psi(z_n)z_0^+\right| + \left|\int_0^1 \frac{hz_0^+}{\|u_n\|^{p-1}_{W^{1,p}}}\right| + \frac{\varepsilon_n \left\|z_0^+\right\|_{W^{1,p}}}{\|u_n\|^{p-1}_{W^{1,p}}} \right) \ , \end{aligned} \tag{9.11}$$

but the right hand side can be estimated since the first two terms are bounded by $(1+\lambda)\|z_n\|^{p-1}_{W^{1,p}} \left\|z_0^+\right\|_{W^{1,p}} \leq 1+\lambda$ and the last two clearly go to zero; then equation (9.11) gives rise to a contradiction unless $z_0 \leq 0$.

□

3. Claim: Using hypotheses (H2-p) and (H3-p) we obtain a constant A such that

$$\int_{u_n > s_0} u_n g(x,u_n) \leq A\|u_n\|_{W^{1,p}} \ , \tag{9.12}$$

at least for n large enough.

For $p \geq 2$ this implies

$$\int_{u_n > s_0} u_n g(x,u_n) \leq A\|u_n\|^{p-1}_{W^{1,p}} \ . \tag{9.13}$$

Proof of the claim. Consider first $|pF(u_n) - \langle F'(u_n), u_n\rangle|$:

$$\left| \int_0^1 -pG(x,u_n) + g(x,u_n)u_n + (1-p)\int_0^1 hu_n \right| \leq pT + \varepsilon_n \|u_n\|_{W^{1,p}} \ , \tag{9.14}$$

from which

$$\begin{aligned} \int_{u_n > s_0} g(x,u_n)u_n - pG(x,u_n) \quad &\leq \quad \int_{u_n \leq s_0} pG(x,u_n) - g(x,u_n)u_n + \\ &+(p-1)\left|\int_0^1 hu_n\right| + pT + \varepsilon_n \|u_n\|_{W^{1,p}} \ . \end{aligned} \tag{9.15}$$

The right hand side may be estimated as follows:

- $$\int_{-s_1 \leq u_n \leq s_0} pG(x,u_n) - g(x,u_n)u_n \leq sup_{\left\{\begin{array}{l} x \in [0,1], \\ s \in [-s_1, s_0] \end{array}\right\}} \Big(pG(x,s) - g(x,s)s \Big) \ , \tag{9.16}$$

- using hypothesis (H3-p)

$$\int_{u_n \leq -s_1} pG(x,u_n) - g(x,u_n)u_n \leq pC_0 \ , \tag{9.17}$$

- $\left|\int_0^1 hu_n\right| \leq \|h\|_{L^q} \|u_n\|_{L^p} \leq \|h\|_{L^q} \|u_n\|_{W^{1,p}}$.

For the left hand side we use hypothesis (H2-p) to obtain

$$(1-p\theta)\int_{u_n > s_0} g(x,u_n)u_n \leq \int_{u_n > s_0} g(x,u_n)u_n - pG(x,u_n) \tag{9.18}$$

and then, since $(1-p\theta) > 0$, joining all estimates from (9.15) to (9.18), we get

$$\int_{u_n > s_0} g(x,u_n)u_n \leq \frac{A}{2}\|u_n\|_{W^{1,p}} + \frac{A}{2} \leq A\|u_n\|_{W^{1,p}} \tag{9.19}$$

for some constant A.

Since we are supposing $\|u_n\|_{W^{1,p}} \geq 1$, this implies (9.13) for $p \geq 2$.

$\square$

4. Claim: under hypothesis (H3-p),

$$\lim_{n \to +\infty} \int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{W^{1,p}}^{p-1}} = 0 \ . \tag{9.20}$$

Proof of the claim. Fix $\varepsilon > 0$ and k such that $\frac{A}{k} \le \varepsilon$ and $k > s_0$.
Estimate (9.2) shows that

$$\int_{u_n \le k} \frac{|g(x,u_n)|}{\|u_n\|_{W^{1,p}}} \le \int_0^1 \frac{\varepsilon |u_n|^{p-1} + C_\varepsilon}{\|u_n\|_{W^{1,p}}^{p-1}} \le \varepsilon C \frac{\|u_n\|_{L^p}^{p-1}}{\|u_n\|_{W^{1,p}}^{p-1}} + \frac{C_\varepsilon}{\|u_n\|_{W^{1,p}}^{p-1}} \ , \tag{9.21}$$

from which there exists $\bar{n}$ such that

$$\int_{u_n \le k} \frac{|g(x,u_n)|}{\|u_n\|_{W^{1,p}}^{p-1}} \le (C+1)\varepsilon \qquad for \ \ n > \bar{n} \ . \tag{9.22}$$

Since $k > s_0$ and using estimate (9.13), one has

$$\int_{u_n > k} \frac{g(x,u_n)}{\|u_n\|_{W^{1,p}}^{p-1}} \le \int_{u_n > k} \frac{g(x,u_n)}{\|u_n\|_{W^{1,p}}^{p-1}} \frac{u_n}{k} \le \int_{u_n > s_0} \frac{g(x,u_n)}{\|u_n\|_{W^{1,p}}^{p-1}} \frac{u_n}{k} \le \frac{A}{k} \le \varepsilon \ . \tag{9.23}$$

Then we conclude that for $n > \bar{n}$

$$\int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{W^{1,p}}^{p-1}} \le (2+C)\varepsilon \ ; \tag{9.24}$$

by the arbitrariness of ε the claim is proved.

□

5. Claim: $z_n \to z_0$ strongly in $W^{1,p}$.

Proof of the claim. consider $\left| \frac{\langle F'(u_n),(z_n - z_0)\rangle}{\|u_n\|_{W^{1,p}}^{p-1}} \right|$:

$$\begin{aligned} &\left| \int_0^1 \psi(z_n')(z_n' - z_0') - \lambda \int_0^1 \psi(z_n)(z_n - z_0) - \int_0^1 \frac{g(x,u_n)(z_n - z_0)}{\|u_n\|_{W^{1,p}}^{p-1}} - \int_0^1 \frac{h(z_n - z_0)}{\|u_n\|_{W^{1,p}}^{p-1}} \right| \le \\ &\le \frac{\varepsilon_n \, \|z_n - z_0\|_{W^{1,p}}}{\|u_n\|_{W^{1,p}}^{p-1}} \ , \end{aligned} \tag{9.25}$$

from which

$$\left| \int_0^1 \psi(z_n')(z_n' - z_0') \right| \le \tag{9.26}$$

$$\le \lambda \int_0^1 |\psi(z_n)||z_n - z_0| + \int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{W^{1,p}}^{p-1}} |z_n - z_0| + \left| \int_0^1 \frac{h(z_n - z_0)}{\|u_n\|_{W^{1,p}}^{p-1}} \right| + \frac{\varepsilon_n \, \|z_n - z_0\|_{W^{1,p}}}{\|u_n\|_{W^{1,p}}^{p-1}} \ ;$$

but now all the terms on the right goes to zero (use equation (9.20) and the strong convergence of z_n in L^p and $\mathcal{C}^0$), and then we conclude that $z_n \to z_0$ strongly in $W^{1,p}$ by the S^+ property of the p-Laplacian. □

6. Claim: under hypothesis (H3-p), $\lambda > 0$ implies $z_0 = 0$.

Proof of the claim. For any $v \in W^{1,p}$ we consider $\left|\frac{\langle F'(u_n), v\rangle}{\|u_n\|^{p-1}_{W^{1,p}}}\right|$:

$$\left|\int_0^1 \psi(z_n')v' - \lambda\int_0^1 \psi(z_n)v - \int_0^1 \frac{g(x,u_n)v}{\|u_n\|^{p-1}_{W^{1,p}}} - \int_0^1 \frac{hv}{\|u_n\|^{p-1}_{W^{1,p}}}\right| \leq \frac{\varepsilon_n \|v\|_{W^{1,p}}}{\|u_n\|^{p-1}_{W^{1,p}}} \ , \tag{9.27}$$

from which

$$\left|\int_0^1 \psi(z_n')v' - \lambda\int_0^1 \psi(z_n)v\right| \leq \int_0^1 \frac{|g(x,u_n)|}{\|u_n\|^{p-1}_{W^{1,p}}}|v| + \left|\int_0^1 \frac{hv}{\|u_n\|^{p-1}_{W^{1,p}}}\right| + \frac{\varepsilon_n \|v\|_{W^{1,p}}}{\|u_n\|^{p-1}_{W^{1,p}}} \ ; \tag{9.28}$$

but now the right hand side goes to zero by equation (9.20) and so, taking the limit and using lemma 8.3, we get

$$\int_0^1 \psi(z_0')v' - \lambda\int_0^1 \psi(z_0)v = 0 \quad for \ \ any \ \ v \in W^{1,p} \ . \tag{9.29}$$

Finally $v = 1$ gives, with $\lambda > 0$, that $\int_0^1 \psi(z_0) = 0$, but for a nonpositive function this implies $z_0 = 0$.

□

7. Claim: u_n is bounded.

   Proof of the claim. Otherwise we get the contradiction $1 = \|z_n\|_{W^{1,p}} \to \|z_0\|_{W^{1,p}} = 0$. □

8. The PS condition follows now with standard calculations from the boundedness of u_n.

   In fact we now take a subsequence such that $u_n \to u$ weakly in $W^{1,p}(0,1)$ and strongly in $L^p(0,1)$ and $\mathcal{C}^0[0,1]$.

   Then consider $|\langle F'(u_n),(u_n - u)\rangle|$:

$$\begin{aligned}&\left|\int_0^1 \psi(u_n')(u_n' - u') - \lambda\int_0^1 \psi(u_n)(u_n-u) - \int_0^1 g(x,u_n)(u_n-u) - \int_0^1 h(u_n-u)\right| \\ &\quad \leq \varepsilon_n \|u_n - u\|_{W^{1,p}} \ ,\end{aligned} \tag{9.30}$$

   from which

$$\begin{aligned}&\left|\int_0^1 \psi(u_n')(u_n' - u')\right| \\ &\leq \lambda\int_0^1 |\psi(u_n)||u_n-u| + \int_0^1 |g(x,u_n)||u_n-u| + \left|\int_0^1 h(u_n-u)\right| + \varepsilon_n \|z_n - z_0\|_{W^{1,p}} \ ;\end{aligned} \tag{9.31}$$

   where now the right hand side goes to zero by the uniform boundedness of u_n and its L^p convergence, and then again the property S^+ for the p-Laplacian implies $u_n \to u$ strongly in $W^{1,p}$.

Remark 9.1. *The above proof may easily be adapted to the multidimensional Neumann problem under the hypothesis $p > N$ that guarantees the compact inclusion $W^{1,p}(\Omega) \subseteq \mathcal{C}^0(\bar{\Omega})$.*

9.1 PS condition for the multi-Laplacian

Now we want to show that the PS condition may be extended also to the functional (7.6), that is with the operator $(-\Delta)^m$ under the boundary conditions considered in section 7.

We have to change a little bit the proof of the claim $z_0 \leq 0$, since in equation (9.5) we tested against z_0^+ which may not be in H^m; however if we consider $\Omega^+ = \{x \in \Omega : \quad z_0(x) > 0\}$ and $v \in \mathcal{C}_0^\infty(\Omega^+)$ with $v \geq 0$ then $z_0^+ v \in H_*^m(\Omega)$ and using it as test function with all possible choices of v, we get the same results.

After this, in the case (N) with hypothesis (HN) the extension is straightforward provided $\|\nabla^m u\|_{L^2}^2 + \|u\|_{L^2}^2$ is a norm and one verifies the property S^+; in the case (D) one arrives at equation (9.29) and there one deduces that for $\lambda > \lambda_1^m$ and test function ϕ_1 it implies $\int_\Omega z_0 \phi_1 = 0$ and so again $z_0 = 0$.

We remark that for this proof it is just required $H_D^m(\Omega) \subseteq \mathcal{C}^0(\bar{\Omega})$ and so we may assume hypothesis (HN) instead of (HD) also in the case (D).

Then the reult is:

Lemma. 7.33. *For Ω of class $\mathcal{C}^m$, under hypotheses (HN), (H1-m), (H2-m) and (H3-m) with $h \in L^2(\Omega)$, the functional (7.6) satisfies the PS condition in $H_N^m(\Omega)$ (resp. in $H_D^m(\Omega)$) for any $\lambda > \lambda_1^m$.*

Proof of the property S^+ for the multi-Laplacian.
The property S^+ in this case is simple since we work in an Hilbert space: in fact $u_n \rightharpoonup u$ in H^m implies strong convergence in L^2 and $\limsup_{n\to+\infty} \int_\Omega \nabla^m(u_n - u)\nabla^m(u_n - u) \leq 0$ simply reads $\|\nabla^m u_n - \nabla^m u\|_{L^2} \to 0$ and so the convergence is strong too.

□

9.2 PS condition below the value γ

In [dFR91] the PS condition was proven without hypothesis (H3), but only for $\lambda \in [0, \gamma)$ (see the definition of γ on page 73).

That proof may be useful to avoid (H3-m) for $\lambda \in [\lambda_1^m, \gamma)$ in the problem with the multi-Laplacian treated in section 7.

The result is

Lemma. 7.15. *For Ω of class $\mathcal{C}^m$, under hypotheses (HN) (resp. (HD)), (H1-m) and (H2-m), with $h \in L^2(\Omega)$, the functional (7.6) defined in $H_N^m(\Omega)$ (resp. in $H_D^m(\Omega)$) satisfies the PS condition for $\lambda \in (\lambda_1^m, \gamma)$.*

Moreover under hypothesis (HR0-m) and $\int_\Omega h\phi_1 < 0$ it satisfies the PS condition also for $\lambda = \lambda_1^m$.

We outline here the proof: one starts with a PS sequence $\{u_n\} \subseteq H_*^m(\Omega)$, i.e. there exist $T > 0$ and $\varepsilon_n \to 0^+$ such that

$$|F(u_n)| = \left| \frac{1}{2} \int_\Omega |\nabla^m u|^2 - \frac{\lambda}{2} \int_\Omega |u_n|^2 - \int_\Omega G(x, u_n) - \int_\Omega h u_n \right| \leq T \ , \tag{9.32}$$

$$\begin{aligned} |\langle F'(u_n), v\rangle| = \left| \int_\Omega \nabla^m u \nabla^m v - \lambda \int_\Omega u_n v - \int_\Omega g(x, u_n) v - \int_\Omega h v \right| \leq & \\ \leq \varepsilon_n \|v\|_{H^m} \ , \qquad\qquad & \forall v \in H_*^m \ . \end{aligned} \tag{9.33}$$

The PS condition follows as before if we prove that u_n is bounded, then supposing $1 \leq \|u_n\|_{H^m} \to +\infty$ define $z_n = \frac{u_n}{\|u_n\|_{H^m}}$, and extract a subsequence $z_n \to z_0$ weakly in $H^m_*(\Omega)$ and strongly in $L^2(\Omega)$ and $\mathcal{C}^0(\bar{\Omega})$ (resp. $\mathcal{C}^1(\bar{\Omega})$).

- Claim: $z_0 \leq 0$.

Proof of the claim. As before, with $z_0^+ v$: $v \in \mathcal{C}_0^\infty(\{x \in \Omega : \quad z_0(x) > 0\}$ in place $of z_0^+$, as observed in section 9.1. □

- Claim:

$$\limsup_{n\to+\infty} \int_\Omega \frac{g(x,u_n)z_n}{\|u_n\|_{H^m}} \leq 0 \; . \tag{9.34}$$

Proof of the claim. We make the same calculations as before until equation (9.15), then we use estimate (9.2) (once integrated and once multiplied by u_n) to get a constant D_ε such that

$$\int_{u_n \leq s_0} 2G(x,u_n) - g(x,u_n)u_n \leq \int_\Omega (\varepsilon u_n^2 + \tilde{D}_\varepsilon |u_n|) \leq \varepsilon \|u_n\|_{L^2}^2 + D_\varepsilon \|u_n\|_{L^2} \; ; \tag{9.35}$$

so in place of equation (9.19) we get

$$\int_{u_n > s_0} g(x,u_n)u_n \leq \frac{1}{1-2\theta}(A_\varepsilon \|u_n\|_{H^m} + \varepsilon \|u_n\|_{L^2}^2 + 2T) \; . \tag{9.36}$$

Then we have to estimate also (again using (9.2))

$$\int_{u_n \leq s_0} g(x,u_n)u_n \leq \varepsilon \|u_n\|_{L^2}^2 + E_\varepsilon \|u_n\|_{L^2} \; ; \tag{9.37}$$

Finally joining the previous two equations and dividing by $\|u_n\|_{H^m}^2$ we get (redefining the constants)

$$\int_\Omega \frac{g(x,u_n)z_n}{\|u_n\|_{H^m}} \leq C \left(\varepsilon \frac{\|u_n\|_{L^2}^2}{\|u_n\|_{H^m}^2} + \frac{A_\varepsilon}{\|u_n\|_{H^m}} + \frac{T}{\|u_n\|_{H^m}^2} \right) \; ; \tag{9.38}$$

taking the lim sup one concludes that

$$\limsup_{n\to+\infty} \int_\Omega \frac{g(x,u_n)z_n}{\|u_n\|_{H^m}} \leq C\varepsilon \tag{9.39}$$

for any choice of $\varepsilon > 0$, from which follows the claim (9.34). □

- Claim:

$$\limsup_{n\to+\infty} \int_\Omega |\nabla^m z_n|^2 \leq \lim_{n\to+\infty} \lambda \int_\Omega z_n^2 = \lambda \int_\Omega z_0^2 \, . \tag{9.40}$$

Proof of the claim. Consider $\left|\frac{\langle F'(u_n),z_n\rangle}{\|u_n\|_{H^m}}\right|$:

$$\left|\int_\Omega |\nabla^m z_n|^2 - \lambda\int_\Omega z_n^2 - \int_\Omega \frac{g(x,u_n)z_n}{\|u_n\|_{H^m}} - \int_\Omega \frac{hz_n}{\|u_n\|_{H^m}}\right| \quad \leq \quad \frac{\varepsilon_n\,\|z_n\|_{H^m}}{\|u_n\|_{H^m}}\ , \tag{9.41}$$

from which

$$\int_\Omega |\nabla^m z_n|^2 \quad\leq\quad \lambda\int_\Omega z_n^2 + \int_\Omega \frac{g(x,u_n)z_n}{\|u_n\|_{H^m}} + \left|\int_\Omega \frac{hz_n}{\|u_n\|_{H^m}}\right| + \frac{\varepsilon_n\,\|z_n\|_{H^m}}{\|u_n\|_{H^m}} \tag{9.42}$$

where, taking the lim sup and using equation (9.34), all the terms in the right hand side go to zero except the first which converges to $\lambda\,\|z_0\|^2_{L^2}$.

□

- Claim: if $\lambda \in (\lambda_1^m, \gamma)$ then $z_0 = 0$.

Proof of the claim. We will first prove that $z_0 \in S_0$ (see the definition of S_0 on page 73). Suppose by contradiction that $sup_{x\in\Omega}\frac{z_0(x)}{\phi_1(x)} < 0$.

Since $z_n \to z_0$ in $\mathcal{C}^0(\bar\Omega)$ in the case (N) and in $\mathcal{C}^1(\bar\Omega)$ in the case (D) we have that $\frac{z_n}{\phi_1} < \frac{1}{2}\frac{z_0}{\phi_1} < 0$ for $n > \bar n$ and then $u_n < 0$ in Ω for $n > \bar n$.

This allows one to use the estimate (9.2) to obtain that

$$\left|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\right| \leq \varepsilon|z_n(x)| + \frac{C_\varepsilon}{\|u_n\|_{H^m}}\ ; \tag{9.43}$$

taking the lim sup we get

$$\limsup_{n\to+\infty}\left|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\right| \leq \varepsilon|z_0(x)| \tag{9.44}$$

for any choice of $\varepsilon > 0$, and then

$$\lim_{n\to+\infty}\left|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\right| = 0\ . \tag{9.45}$$

Now consider $\left|\frac{\langle F'(u_n),\phi_1\rangle}{\|u_n\|_{H^m}}\right|$:

$$\left|\int_\Omega \nabla^m z_n\nabla^m\phi_1 - \lambda\!\int_\Omega z_n\phi_1 - \!\int_\Omega \frac{g(x,u_n)\phi_1}{\|u_n\|_{H^m}} - \!\int_\Omega \frac{h\phi_1}{\|u_n\|_{H^m}}\right| \quad\leq\quad \frac{\varepsilon_n\,\|\phi_1\|_{H^m}}{\|u_n\|_{H^m}}\ , \tag{9.46}$$

from which

$$\left|(\lambda_1^m - \lambda)\int_\Omega z_n\phi_1\right| \quad\leq\quad \left|\int_\Omega \frac{g(x,u_n)\phi_1}{\|u_n\|_{H^m}}\right| + \left|\int_\Omega \frac{h\phi_1}{\|u_n\|_{H^m}}\right| + \frac{\varepsilon_n\,\|\phi_1\|_{H^m}}{\|u_n\|_{H^m}}\ . \tag{9.47}$$

Since equation (9.43) also tells us that the functions in the sequence are dominated (for $n > \bar n$) by $\max_{x\in\bar\Omega}|z_0| + 1 + C_{\varepsilon=1}$, we can use dominated convergence to assert that

$$\lim_{n\to+\infty}\left|\int_\Omega \frac{g(x,u_n)\phi_1}{\|u_n\|_{H^m}}\right| \leq \lim_{n\to+\infty}\int_\Omega \left|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\right|\phi_1 = \int_\Omega \lim_{n\to+\infty}\left|\frac{g(x,u_n)}{\|u_n\|_{H^m}}\right|\phi_1 = 0\ . \tag{9.48}$$

Now we may take the limit in equation (9.47), to get

$$(\lambda_1^m - \lambda) \int_\Omega z_0 \phi_1 = 0 \,. \tag{9.49}$$

This, with $\lambda \neq \lambda_1^m$, gives $\int_\Omega z_0 \phi_1 = 0$ which, since $z_0 \leq 0$, would imply $z_0 \equiv 0$: we conclude that $z_0 \in S_0$ as claimed.

Finally this implies $\int_\Omega |\nabla^m z_0|^2 \geq \gamma \int_\Omega z_0^2$ by the definition of γ, which contradicts equation (9.40) unless $z_0 = 0$ since otherwise, by the weak convergence,

$$\int_\Omega |\nabla^m z_0|^2 \leq \liminf_{n \to +\infty} \int_\Omega |\nabla^m z_n|^2 \leq \lambda \int_\Omega z_0^2 < \gamma \int_\Omega z_0^2 \,. \tag{9.50}$$

□

- Claim: if $\lambda = \lambda_1^m$, $\int_\Omega h\phi_1 < 0$ and hypothesis (HR0-m) holds, then $z_0 = 0$.

Proof of the claim. Equation (9.40) and the weak convergence of z_n to z_0 imply

$$\int_\Omega |\nabla^m z_0|^2 \leq \liminf_{n \to +\infty} \int_\Omega |\nabla^m z_n|^2 \leq \lambda_1^m \int_\Omega z_0^2 \,, \tag{9.51}$$

which implies that $z_0 \in span(\phi_1)$, that is $z_0 = -\rho\phi_1$ for some $\rho \geq 0$.

Now consider $|\langle F'(u_n), \phi_1 \rangle|$:

$$\left| \int_\Omega \nabla^m u_n \nabla^m \phi_1 - \lambda_1^m \int_\Omega u_n \phi_1 - \int_\Omega g(x, u_n)\phi_1 - \int_\Omega h\phi_1 \right| \quad \leq \quad \varepsilon_n \left\| \phi_1 \right\|_{H^m} \ , \tag{9.52}$$

from which

$$\left| \int_\Omega g(x, u_n)\phi_1 + \int_\Omega h\phi_1 \right| \leq \varepsilon_n \left\| \phi_1 \right\|_{H^m} \ . \tag{9.53}$$

Taking the lim sup we get

$$\limsup_{n \to +\infty} \int_\Omega g(x, u_n)\phi_1 = -\int_\Omega h\phi_1 > 0 \,, \tag{9.54}$$

but this implies $z_0 = -\rho\phi 1 \equiv 0$ since otherwise

$$u_n(x) = z_n(x) \left\| u_n \right\|_{H^m} \leq -\frac{\rho}{2}\phi_1(x) \left\| u_n \right\|_{H^m} \to -\infty \quad \forall x \in \Omega \tag{9.55}$$

and so by hypothesis (HR0-m) the limit in the left hand side would be zero. □

- Claim: u_n is bounded.

Proof of the claim. Equation (9.40) now implies $1 = \|\nabla^m z_n\|_{L^2}^2 + \|z_n\|_{L^2}^2 \to 0$, a contradiction.

□

A Appendix

We summarize in this appendix some basic definitions and results used throughout the work.

A.1 Sobolev Spaces

Let $\Omega \subseteq \mathbb{R}^N$ be an open set (bounded or unbounded) and $\partial\Omega$ its boundary.

A.1.1 The spaces L^p

For $p \in [1, +\infty)$, let

$$\tilde{L}^p(\Omega) = \{u : \Omega \to \mathbb{R} \;\; such \;\; that \;\; \int_\Omega |u|^p < +\infty\}\,, \tag{A.1}$$

and define $L^p(\Omega)$ as the set of the equivalence classes of the elements in $\tilde{L}^p(\Omega)$ coinciding almost everywhere, equipped with the norm $\|u\|_{L^p} = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}$.

Moreover let

$$\tilde{L}^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \;\; measurable \;\; and \;\; such \;\; that \;\; \exists C : \;\; |u| \le C \;\; a.e.\}\,, \tag{A.2}$$

and define as above $L^\infty(\Omega)$ equipped with the norm $\|u\|_{L^\infty} = \inf\{C\colon\ |u| \le C \text{ a.e.}\}$.

It is known (see for example [Bre83]) that

- $L^p(\Omega)$ is a Banach space for $p \in [1, +\infty]$, moreover it is a Hilbert space for $p = 2$ with the scalar product $(u, v)_{L^2} = \int_\Omega uv$.
- $L^p(\Omega)$ is reflexive and separable for $p \in (1, +\infty)$ and its dual is $L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$, where the duality is given by $\langle u, v\rangle = \int_\Omega uv$.
- $L^1(\Omega)$ is separable, while $L^\infty(\Omega)$ is not, and the dual of $L^1(\Omega)$ is $L^\infty(\Omega)$ while the dual of $L^\infty(\Omega)$ contains $L^1(\Omega)$ but is larger, so that both are not reflexive.

A.1.2 The spaces $W^{k,p}$

Let $L^1_{loc}(\Omega) = \{u : \Omega \to \mathbb{R}$ such that $u \in L^1(\omega)$ for all compact sets $\omega \subseteq \Omega\}$; then, given a function $f \in L^1_{loc}(\Omega)$, we say that $g \in L^p_{loc}(\Omega)$ is the **distributional derivative** of f with respect to the variable x_i if:
$\int_\Omega g\psi = -\int_\Omega f\frac{\partial\psi}{\partial x_i}$ for all $\psi \in \mathcal{C}^\infty_0(\Omega) = \{\psi \in \mathcal{C}^\infty(\Omega)$ with compact support contained in $\Omega\}$.

Then we can define:

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \text{ s.t. all distributional derivatives of } u \text{ up to order } k \text{ in } L^p(\Omega)\},$$

equipped with the norm given by the sum of the L^p norms of all the derivatives from order 0 to k (or, in an equivalent way, the norm at the p^{th} power may be defined as the sum of the p^{th} power of the L^p norms of all the derivatives).

We will usually denote $H^k(\Omega) = W^{k,2}(\Omega)$.

Just as in the case of the L^p spaces we have

- $W^{k,p}(\Omega)$ is a Banach space for $p \in [1, +\infty]$, is a Hilbert space for $p = 2$ (that is the case H^k) with the scalar product $(u, v)_{H^k}$ given by the sum of the L^2 scalar products of each couple of corresponding derivatives.
- $W^{k,p}(\Omega)$ is reflexive and separable for $p \in (1, +\infty)$, but merely separable for $p = 1$.

A.1.3 The spaces $W_0^{1,p}$

$W_0^{1,p}(\Omega)$ may be defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$, equipped with this norm.

One important property of these spaces is the Poincaré inequality:

Theorem A.1. *If Ω is bounded, then there exists a constant C only depending on Ω such that $\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}$ for all $u \in W_0^{1,p}(\Omega)$.*

This implies that in these spaces $\|\nabla u\|_{L^p}$ is a norm equivalent to the usual one.

Finally one has for these spaces the same conclusions for what concerns the reflexivity and separability, and for the case $p = 2$, where the space is Hilbert and is usually denoted by $H_0^1(\Omega)$.

The Poincaré inequality has a useful version for the space $W^{1,p}$:

Theorem A.2. *If Ω is bounded and $\partial\Omega$ is Lipschitz, then there exists a constant C only depending on Ω such that $\|u - \bar{u}\|_{L^p} \leq C \|\nabla u\|_{L^p}$ for all $u \in W^{1,p}(\Omega)$, where $\bar{u} = |\Omega|^{-1} \int_\Omega u$: the mean value of u.*

A.1.4 Other results

We give here some important properties of the above defined spaces.

Approximation by smooth functions

Theorem A.3. *Provided $\partial\Omega$ is $\mathcal{C}^1$, given a function $u \in W^{1,p}(\Omega)$ (resp. $u \in W_0^{1,p}(\Omega)$), there exits a sequence $\{u_n\} \subseteq \mathcal{C}_0^\infty(\mathbb{R}^N)$ (resp. $\{u_n\} \subseteq \mathcal{C}_0^\infty(\Omega)$) converging to u in the $W^{1,p}$ norm.*

Embeddings

Define the Banach space $\mathcal{C}^{k,\alpha}(\bar{\Omega}) = \{\phi \in \mathcal{C}^k(\bar{\Omega})$ such that $\sup_{x \neq y \in \Omega} \frac{|\psi(x) - \psi(y)|}{|x-y|^\alpha} < +\infty$ for each ψ derivative of ϕ up to order $k\}$, normed with the sum of the $\mathcal{C}^k$-norms and the given above suprema.

Recall that we say that $A \subseteq B$,

- with **continuous inclusion** if:
  there exists a constant C such that $\|u\|_B \leq C \|u\|_A$,
- with **compact inclusion** if:
  if $u_n \rightharpoonup u$ in A then, up to a subsequence, $u_n \to u$ in B.

Then we have (with some hypotheses on the set Ω to avoid cusps on the boundary, for example a sufficient condition is to have a Lipschitz boundary):

- For $\frac{1}{p} - \frac{m}{N} > 0$, be $p^* = \left(\frac{1}{p} - \frac{m}{N}\right)^{-1}$:
  $W^{m,p}(\Omega) \subseteq L^q(\Omega)$

- for all $q \in [p, p^*]$ with continuous inclusion, if Ω is unbounded,
- for all $q \in [1, p^*]$ with continuous inclusion and also compact except for the limiting case $q = p^*$, if Ω is bounded.

- For $\frac{1}{p} - \frac{m}{N} = 0$:
  $W^{m,p}(\Omega) \subseteq L^q(\Omega)$
  - for all $q \in [p, +\infty)$ with continuous inclusion, if Ω is unbounded,
  - for all $q \in [1, +\infty)$ with continuous and compact inclusion, if Ω is bounded.
- for $\frac{1}{p} - \frac{m}{N} < 0$:
  $W^{m,p}(\Omega) \subseteq L^q(\Omega)$
  - for all $q \in [p, +\infty]$ with continuous inclusion, if Ω is unbounded,
  - for all $q \in [1, +\infty]$ with continuous and compact inclusion, if Ω is bounded.

  Moreover be k and α respectively the integral and decimal part of $m - \frac{N}{p}$, then one has:

  - if $\alpha \neq 0$ then $W^{m,p}(\Omega) \subseteq \mathcal{C}^{k,\alpha}(\bar{\Omega})$ with continuous inclusion (also compact if Ω is bounded),
  - if $\alpha = 0$ then $W^{m,p}(\Omega) \subseteq \mathcal{C}^{k-1,1}(\bar{\Omega})$ with continuous inclusion, (also compact if Ω is bounded).

A.2 Trace operators

Theorem A.4. *Let Ω be of class $\mathcal{C}^1$ and $p \in [1, +\infty)$.*
Then there exists a bounded linear operator $T_r : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that if $u \in W^{1,p}(\Omega) \cap \mathcal{C}^0(\bar{\Omega})$ then $T_r u = u|_{\partial\Omega}$.

$T_r u$ is usually called the **trace** of u on $\partial\Omega$ and is sometimes denoted as $u|_{\partial\Omega}$.
Since a derivative of order $k < m$ of a function in $W^{m,p}$ belongs to $W^{1,p}$ we also get that:

Corollary A.5. *Let Ω be of class $\mathcal{C}^1$, $m = \{1, 2, ...\}$, $p \in [1, +\infty)$ and D_* an operator of derivation of order $k < m$.*
Then there exists a bounded linear operator $T_{r*} : W^{m,p}(\Omega) \to L^p(\partial\Omega)$ such that if $u \in W^{m,p}(\Omega) \cap \mathcal{C}^k(\bar{\Omega})$ then $T_{r*} u = D_* u|_{\partial\Omega}$.

Then one may define such an operator in $W^{m,p}(\Omega)$ for any derivative up to order $m - 1$.
This allows one to define linear closed subspaces of $W^{m,p}(\Omega)$ of the form $W^{m,p}_{BC}(\Omega) = \{u \in^{m,p} (\Omega)$ such that $T_{rBC} u = 0\}$, being T_{rBC} an operator of the above type that maps u to a vector of traces on $\partial\Omega$ of derivatives of u of order strictly lower than m.

A.3 Fréchet derivative

Given a functional $I : E \to \mathbb{R}$ with E a Banach space, we say that
I is **Fréchet differentiable** in $u_0 \in E$ if
there exists $I'(u_0) \in E^*$ such that $I(u_0 + v) - I(u_0) = \langle I'(u_0), v \rangle_E + o(\|v\|_E)$ for $\|v\|_E \to 0$.
Then we say $I \in \mathcal{C}^1(E, \mathbb{R})$ if the map $E \to E^* : u \mapsto I'(u)$ is continuous.

A.4 Green's function method

Consider a solution $u(x) \in \mathcal{C}^2(0,1) \cap \mathcal{C}^1([0,1])$ of the problem

$$\begin{cases} -u''(x) = f(x) & in \ (0,1) \\ \left[\begin{array}{l} u'(0) = u'(1) = 0 \\ \quad or \\ u(0) = u(1) = 0 \end{array} \right. \end{cases} . \tag{A.3}$$

Let

$$G(x,y) = \frac{-|x-y|}{2} + a(x)y + b(x) \ : \tag{A.4}$$

if we integrate over an interval $(a,b) \subseteq [0,1]$ with $x \notin (a,b)$ we have:

$$\begin{aligned} 0 &= \int_{(a,b)} \left[G_y(x,y)u'(y) - G_y(x,y)u'(y) \right] dy = \\ &= \left[G_y(x,y)u(y) - G(x,y)u'(y) \right]_{y=a}^{b} - \int_{(a,b)} \left[G_{yy}(x,y)u(y) - G(x,y)u''(y) \right] dy \, . \end{aligned} \tag{A.5}$$

Now let $x \in (0,1)$, $0 < \varepsilon < d(x,\{0;1\})$ and $U_{\varepsilon,x} = (0, x-\varepsilon) \cup (x+\varepsilon, 1)$: making the same integration over $U_{\varepsilon,x}$ we get:

(1) $\int_{U_{\varepsilon,x}} [G_{yy}(x,y)u(y) - G(x,y)u''(y)] \, dy = \int_{U_{\varepsilon,x}} G(x,y)[-u''(y)]dy$ since $G_{yy}(x,y) = 0$ in $[0,1] \times U_{\varepsilon,x}$;

(2) the boundary term is: $[G_y(x,y)u(y) - G(x,y)u'(y)]_{y=0}^{1} - [G_y(x,y)u(y) - G(x,y)u'(y)]_{x-\varepsilon}^{x+\varepsilon}$;

taking limit for $\varepsilon \to 0$:

(1) becomes $\int_0^1 G(x,y)[-u''(y)]dy$

(2) becomes $[G_y(x,y)u(y) - G(x,y)u'(y)]_{y=0}^{1} - u(x)\,[G_y(x,x^+) - G_y(x,x^-)]$, where the last term is simply $u(x)$ since $[G_y(x,x^+) - G_y(x,x^-)] = -1$.

So we get

$$u(x) = -\left[G_y(x,y)u(y) - G(x,y)u'(y) \right]_{y=0}^{1} + \int_0^1 G(x,y)[-u''(y)]dy \quad x \in (0,1) \, . \tag{A.6}$$

Now observe that the first term contains the values of u and u' in 0 and 1; then we distinguish between the two kinds of boundary conditions.

- In the Dirichlet case $u(0) = u(1) = 0$, then of the boundary terms would remain only $G(x,1)u'(1) - G(x,0)u'(0)$: if we choose $a(x)$ and $b(x)$ such that $G(x,0) = G(x,1) = 0$ for all $x \in (0,1)$ the equation simplifies giving

$$u(x) = \int_0^1 G(x,y)[-u''(y)]dy \, . \tag{A.7}$$

Moreover we remark that

$$|G(x,y)| \leq d(y,\{0;1\}) \tag{A.8}$$

and

$$|G(x,y)| \leq d(x,\{0;1\})\,; \tag{A.9}$$

actually with the above choices of $a(x)$ and $b(x)$ we have, for any fixed $x \in (0,1)$, that $G(x,y)$ is increasing in $[0,x)$ with derivative smaller than 1 and decreasing in $(x,1]$ with derivative larger than -1.

- In the Neumann case we can not eliminate all the boundary terms since no choice of $a(x)$ and $b(x)$ would give $G_y(x,0) = G_y(x,1) = 0$, however we may for example choose to impose $G(x,0) = G_y(x,0) = 0$ (obtaining then $G_y(x,1) = -1$) and so get

$$u(x) = u(1) + \int_0^1 G(x,y)[-u''(y)]dy \tag{A.10}$$

and the estimate

$$|G(x,y)| \leq 1\,, \tag{A.11}$$

since for any fixed $x \in (0,1)$, $G(x,y)$ is zero in $[0,x)$ and then decreasing with derivative -1 in $(x,1]$.

B Bibliography

References

[AC95] M. Arias and J. Campos, *Radial Fučik spectrum of the Laplace operator*, J. Math. Anal. Appl. **190** (1995), no. 3, 654–666.

[Ana87] A. Anane, *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), no. 16, 725–728.

[AP72] A. Ambrosetti and G. Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. Pura Appl. (4) **93** (1972), 231–246.

[AR73] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.

[AV95] D. Arcoya and S. Villegas, *Nontrivial solutions for a Neumann problem with a nonlinear term asymptotically linear at $-\infty$ and superlinear at $+\infty$*, Math. Z. **219** (1995), no. 4, 499–513.

[BNFS01] A. K. Ben-Naoum, C. Fabry, and D. Smets, *Structure of the Fučík spectrum and existence of solutions for equations with asymmetric nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), no. 2, 241–265.

[Bon93] A. Bonnet, *A deformation lemma on a C^1 manifold*, Manuscripta Math. **81** (1993), no. 3-4, 339–359.

[Bre83] H. Brezis, *Analyse fonctionnelle*, Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Masson, Paris, 1983, Théorie et applications. [Theory and applications].

[CD01] J. Campos and E. N. Dancer, *On the resonance set in a fourth-order equation with jumping nonlinearity*, Differential Integral Equations **14** (2001), no. 3, 257–272.

[CdFG99] M. Cuesta, D. de Figueiredo, and J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations **159** (1999), no. 1, 212–238.

[CG92] M. Cuesta and J.-P. Gossez, *A variational approach to nonresonance with respect to the Fučik spectrum*, Nonlinear Anal. **19** (1992), no. 5, 487–500.

[CS85] A. Castro and R. Shivaji, *Multiple solutions for a Dirichlet problem with jumping nonlinearities*, Trends in the theory and practice of nonlinear analysis (Arlington, Tex., 1984), North-Holland Math. Stud., vol. 110, North-Holland, Amsterdam, 1985, pp. 97–101.

[Dan78] E. N. Dancer, *On the ranges of certain weakly nonlinear elliptic partial differential equations*, J. Math. Pures Appl. (9) **57** (1978), no. 4, 351–366.

[Dan77] E. N. Dancer, *On the Dirichlet problem for weakly non-linear elliptic partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **76** (1976/77), no. 4, 283–300.

[dF84] D. G. de Figueiredo, *On the superlinear Ambrosetti-Prodi problem*, Nonlinear Anal. **8** (1984), no. 6, 655–665.

[dF88] D. G. de Figueiredo, *On superlinear elliptic problems with nonlinearities interacting only with higher eigenvalues*, Rocky Mountain J. Math. **18** (1988), no. 2, 287–303, Nonlinear Partial Differential Equations Conference (Salt Lake City, UT, 1986).

[dFG94] D. G. de Figueiredo and J.-P. Gossez, *On the first curve of the Fučik spectrum of an elliptic operator*, Differential Integral Equations **7** (1994), no. 5-6, 1285–1302.

[dFR91] D. G. de Figueiredo and B. Ruf, *On a superlinear Sturm-Liouville equation and a related bouncing problem*, J. Reine Angew. Math. **421** (1991), 1–22.

[dFR93] D. G. de Figueiredo and B. Ruf, *On the periodic Fučik spectrum and a superlinear Sturm-Liouville equation*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), no. 1, 95–107.

[dFS84] D. G. de Figueiredo and S. Solimini, *A variational approach to superlinear elliptic problems*, Comm. Partial Differential Equations **9** (1984), no. 7, 699–717.

[dP89] J. C. N. de Pádua, *Multiplicity results for a superlinear Dirichlet problem*, J. Differential Equations **82** (1989), no. 2, 356–371.

[DR98] A. R. Domingos and M. Ramos, *On the solvability of a resonant elliptic equation with asymmetric nonlinearity*, Topol. Methods Nonlinear Anal. **11** (1998), no. 1, 45–57.

[Drá92] P. Drábek, *Solvability and bifurcations of nonlinear equations*, Pitman Research Notes in Mathematics Series, vol. 264, Longman Scientific & Technical, Harlow, 1992.

[Eva98] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[Fuč76] S. Fučík, *Boundary value problems with jumping nonlinearities*, Časopis Pěst. Mat. **101** (1976), no. 1, 69–87.

[GK81] T. Gallouët and O. Kavian, *Résultats d'existence et de non-existence pour certains problèmes demi-linéaires à l'infini*, Ann. Fac. Sci. Toulouse Math. (5) **3** (1981), no. 3-4, 201–246 (1982).

[Lin90] P. Lindqvist, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc. **109** (1990), no. 1, 157–164.

[LL70] E. M. Landesman and A. C. Lazer, *Nonlinear perturbations of linear elliptic boundary value problems at resonance*, J. Math. Mech. **19** (1969/1970), 609–623.

[Ma90] C. A. Magalhães, *Semilinear elliptic problem with crossing of multiple eigenvalues*, Comm. Partial Differential Equations **15** (1990), no. 9, 1265–1292.

[Mic94] A. M. Micheletti, *A remark on the resonance set for a semilinear elliptic equation*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), no. 4, 803–809.

[MP01] A. M. Micheletti and A. Pistoia, *On the Fučík spectrum for the p-Laplacian*, Differential Integral Equations **14** (2001), no. 7, 867–882.

[MW89] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, vol. 74, Springer-Verlag, New York, 1989.

[Neč83] J. Nečas, *Introduction to the theory of nonlinear elliptic equations*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 52, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1983, With German, French and Russian summaries.

[Pal63] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340.

[Per00] K. Perera, *Existence and multiplicity results for a Sturm-Liouville equation asymptotically linear at* $-\infty$ *and superlinear at* $+\infty$, Nonlinear Anal. **39** (2000), no. 6, Ser. A: Theory Methods, 669–684.

[Pis97] A. Pistoia, *A generic property of the resonance set of an elliptic operator with respect to the domain*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 6, 1301–1310.

[PS64] R. S. Palais and S. Smale, *A generalized Morse theory*, Bull. Amer. Math. Soc. **70** (1964), 165–172.

[Rab86] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

[RS86a] B. Ruf and S. Solimini, *On a class of superlinear Sturm-Liouville problems with arbitrarily many solutions*, SIAM J. Math. Anal. **17** (1986), no. 4, 761–771.

[RS86b] B. Ruf and P. N. Srikanth, *Multiplicity results for ODEs with nonlinearities crossing all but a finite number of eigenvalues*, Nonlinear Anal. **10** (1986), no. 2, 157–163.

[Ruf81] B. Ruf, *On nonlinear elliptic problems with jumping nonlinearities*, Ann. Mat. Pura Appl. (4) **128** (1981), 133–151.

[RW99] W. Reichel and W. Walter, *Sturm-Liouville type problems for the p-Laplacian under asymptotic non-resonance conditions*, J. Differential Equations **156** (1999), no. 1, 50–70.

[Sch00] M. Schechter, *Resonance problems with respect to the Fučík spectrum*, Trans. Amer. Math. Soc. **352** (2000), no. 9, 4195–4205 (electronic).

[Sma64] S. Smale, *Morse theory and a non-linear generalization of the Dirichlet problem*, Ann. of Math. (2) **80** (1964), 382–396.

[Str96] M. Struwe, *Variational methods*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 34, Springer-Verlag, Berlin, 1996, Applications to nonlinear partial differential equations and Hamiltonian systems.

[Vil98] S. Villegas, *A Neumann problem with asymmetric nonlinearity and a related minimizing problem*, J. Differential Equations **145** (1998), no. 1, 145–155.

[Wil96] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser Boston Inc., Boston, MA, 1996.