On a variational characterization of the Fučík spectrum of the Laplacian and a superlinear Sturm-Liouville equation

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2003

Abstract

In the first part of this paper a variational characterization of parts of the Fučík spectrum for the Laplacian in a bounded domain Ω is given. The proof uses a linking theorem on sets obtained through a suitable deformation of subspaces of $H^1(\Omega)$.

In the second part a nonlinear Sturm-Liouville equation with Neumann boundary conditions on an interval is considered, where the nonlinearity intersects all but a finite number of eigenvalues. It is proved that under certain conditions this equation is solvable for arbitrary forcing terms. The proof uses a comparison of the minimax levels of the functional associated to this equation with suitable values related to the Fučík spectrum.

1 Introduction

The purpose of this paper is twofold. First we consider the so called Fučík problem for the Laplacian, both with Dirichlet and Neumann boundary conditions:

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & in \ \Omega \\ \left[\begin{array}{l} \frac{\partial u}{\partial n} = 0 \\ \quad or \\ u = 0 \end{array} \right. & in \ \partial\Omega \end{cases}, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $u^{\pm}(x) = \max\{0, \pm u(x)\}$.

The notion of Fučík spectrum was introduced in [9] and [4]; it is defined as the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a non trivial solution of problem (1.1).

To know the Fučík spectrum is important in many applications, for example in the study of problems with "jumping nonlinearities", that is nonlinearities which are asymptotically linear at both $+\infty$ and $-\infty$, but with different slopes: if the slopes correspond to a point (λ^+, λ^-) which is not in the Fučík spectrum, then it is possible to guarantee a priori estimates for the solutions and the PS condition for the associated functional; if moreover the point (λ^+, λ^-) may be connected by a curve which does not intersect the Fučík spectrum to a point of the line $\{\lambda^+ = \lambda^-\}$ (not belonging to the Fučík spectrum), then it is possible to prove existence of solutions.

If one has also a variational characterization of this spectrum, then other interesting results can be obtained, cf. [2], [5], [7] and [3]. However these papers deal only with the first nontrivial curve of the

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Fučík spectrum or with the periodic case on an interval.

In the following we will call H the space $H^1(\Omega)$ when considering the Neumann problem and $H_0^1(\Omega)$ when considering the Dirichlet problem; we will denote with
$0 \le \lambda_1 < \lambda_2 \le \lambda_3 \le ... \le \lambda_k \le ...$ the eigenvalues of $-\Delta$ in H and with $(\phi_k, \ \ k = 1, 2, ..)$ the corresponding eigenfunctions, which will be taken orthogonal and normalized with $\|\phi_k\|_{L^2} = 1$.

First we give a variational characterization of parts of the Fučík spectrum for problem (1.1), in particular we prove:

Theorem 1.1. *Suppose that the point $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ with $\alpha^+ \ge \alpha^-$ is Σ-connected to the diagonal between λ_k and λ_{k+1} in the sense of definition 2.1, then we can find and characterize one intersection of the Fučík spectrum with the halfline*
$\{(\alpha^+ + t, \alpha^- + rt), \ t > 0\}$, for each value of $r \in (0, 1]$.

The cases $\alpha^+ \le \alpha^-$ and $r \in [1, +\infty)$ can be done in a similar way.

The second main theme of the paper is the following Sturm-Liouville equation with Neumann boundary conditions:

$$\begin{cases} -u'' = \lambda u + g(x, u) + h(x) & in \ \ (0, 1) \\ u'(0) = u'(1) = 0 \end{cases} , \tag{1.2}$$

where $g \in \mathcal{C}^0([0, 1] \times \mathbb{R})$ with

$$\lim_{s \to -\infty} \frac{g(x, s)}{s} = 0, \qquad \lim_{s \to +\infty} \frac{g(x, s)}{s} = +\infty \tag{H1}$$

uniformly with respect to $x \in [0, 1]$, and $h \in L^2(0, 1)$.

We will compare it to the Fučík problem

$$\begin{cases} -u'' = \lambda^+ u^+ - \lambda^- u^- & in \ \ (0, 1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{1.3}$$

and, taking advantage of the fact that in the one dimensional case the Fučík spectrum may be exactly calculated, we will prove existence results for problem (1.2). The proof uses the variational characterization above to make a comparison of these minimax levels with those of the functional associated to problem (1.2), in order to prove the existence of a linking structure for this last functional.

Some hypotheses on the growth at infinity of the nonlinearity g will be needed to obtain the PS condition for the functional associated to problem (1.2): defining $G(x, s) = \int_0^s g(x, \xi) d\xi$, we ask

$$\exists \theta \in \left(0, \frac{1}{2}\right), \quad s_0 > 0 \quad s.t. \quad 0 < G(x, s) \le \theta s g(x, s) \quad \forall s > s_0\,; \tag{H2}$$

$$\exists s_1 > 0, \ \ C_0 > 0 \quad s.t. \quad G(x, s) \le \frac{1}{2} s g(x, s) + C_0 \quad \forall s < -s_1\,. \tag{H3}$$

For certain "resonant" values of λ also the following hypothesis will be needed:

$$\exists \rho_0 > 0, \ \ M_0 \in \mathbb{R} \quad s.t. \quad G(x, s) + h(x) s \le M_0 \quad a.e. \ x \in [0, 1], \ \forall s < -\rho_0\,. \tag{HR}$$

The exact statement of the results is:

Theorem 1.2. *Under hypotheses (H1), (H2) and (H3), if $\lambda \in (\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4})$ for some $k \ge 1$, then there exists a solution of problem (1.2) for all $h \in L^2(0, 1)$.*

Theorem 1.3. *Under hypotheses (H1), (H2), (H3) and (HR), if* $\lambda = \frac{\lambda_{k+1}}{4}$ *for some* $k \geq 1$ *and* $h \in L^2(0,1)$*, then there exists a solution of problem (1.2).*

Remark 1.4. *The hypotheses (H1) to (H3) are satisfied for example by the function* $g(x,s) = e^s$*; in this case to satisfy also (HR) we need* $h(x) \geq 0$ *a.e.*

Another example of nonlinearity which satisfies also (HR) and where there is some more freedom on h*, is when* g *behaves at* $-\infty$ *as* $|s|^\delta$ *with* $\delta \in (0,1)$*; then* h *may be chosen arbitrarily in* $L^\infty(0,1)$*.*

1.1 Bibliography

Theorem 1.2 extends the result obtained in [6], where the existence is proved for $\lambda \in (0, \frac{\pi^2}{4})$, that is the case $k = 1$ of theorem 1.2.

Results similar to [6] (with slightly different hypotheses) can be found in [18].

Perera in [13] proved the existence of a solution for $\lambda \in (\frac{\pi^2}{4}, \lambda^*)$, where λ^* is some value in $(\frac{\pi^2}{4}, \frac{\pi^2}{2})$, and so theorem 1.2 extends this result, too.

We also mention that for periodic boundary conditions the equivalent of theorem 1.2 is proved in [7].

Theorem 1.3 deals with some kind of resonance (as will be clear from the proofs in the following); the case $\lambda = \frac{\lambda_2}{4}$ was already discussed in [13], where the existence is proved under different hypotheses, while the case $\lambda = \frac{\lambda_1}{4}$ (that is $\lambda = 0$) is treated in [6].

For what concerns the variational characterization of the Fucik spectrum we cite [5] and [2] where the second curve in any spatial dimension is characterized (in two different ways), and [7], where the whole spectrum for periodic boundary conditions on an interval is characterized.

In the construction of the characterization of the Fučík spectrum we will use a technique derived from a similar one used in [8], which will be discussed in section 2.1.

For other bibliography on problem (1.2), see [6].

1.2 The Fučík spectrum

The notion of Fučík spectrum was introduced in [9] and [4]; it is defined as the set $\Sigma \subseteq \mathbb{R}^2$ of points (λ^+, λ^-) for which there exists a non trivial solution of problem (1.1).

In the case of problem (1.3) the spectrum can be completely calculated, with the corresponding non trivial solutions. It is composed by curves (which we will call Σ_k) in $\mathbb{R}^2$, arising from each point (λ_k, λ_k):

$$\begin{array}{ll} \Sigma_1 : & \{\lambda^+ = \lambda_1\} \cup \{\lambda^- = \lambda_1\} \\ \Sigma_k : & \frac{(k-1)\pi}{2\sqrt{\lambda^+}} + \frac{(k-1)\pi}{2\sqrt{\lambda^-}} = 1 \qquad k = 2, 3, .. \end{array} . \tag{1.4}$$

Note that each curve with $k \geq 2$ is monotone decreasing, has asymptotes at $\lambda^\pm = \frac{\lambda_k}{4}$ and lies completely in the quadrant $\lambda^\pm > \frac{\lambda_k}{4}$.

In the case of higher dimension less is known: Σ is always a closed set, the lines $\{\lambda^+ = \lambda_1\}$ and $\{\lambda^- = \lambda_1\}$ belong to Σ, and Σ does not contain any other point with $\lambda^+ < \lambda_1$ or $\lambda^- < \lambda_1$; moreover we still know (see for example [16], [10] and [11]) that in each square $(\lambda_{k-1}, \lambda_{k+m+1})^2$, where $\lambda_{k-1} < \lambda_k = ... = \lambda_{k+m} < \lambda_{k+m+1}$, from the point (λ_k, λ_k) arises a continuum composed by a lower and a upper curve, both decreasing (may be coincident); other points in $\Sigma \cap (\lambda_{k-1}, \lambda_{k+m+1})^2$ can only lie between the two curves (and hence in the open squares $(\lambda_{k-1}, \lambda_k)^2$ and $(\lambda_{k+m}, \lambda_{k+m+1})^2$ there never are points of Σ). Something more can be said about the lower part of the continuum arising from (λ_2, λ_2): see [5].

In [1] it is proved, under a non-degeneracy condition (which was first introduced in [12] and [14]) that the whole spectrum is composed by curves arising from a point (λ_k, λ_k), never intersecting and going to infinity; this non-degeneracy condition is discussed in [14], where it is proved that it holds for ‘almost all’ (in a suitable sense) domains; however in general it seems not possible to arrive at the same conclusion.

For a larger bibliography about the Fučík spectrum see [17].

1.3 Idea and plan of the paper

If we consider a point $a \in (\lambda_k, \lambda_{k+1})$ and the functional $J_a : H \to \mathbb{R}$

$$J_a(u) = \int_\Omega |\nabla u|^2 - a \int_\Omega u^2 \ , \tag{1.5}$$

we have a natural splitting $H = V \oplus W$, where $V = span\{\phi_1, .., \phi_k\}$.

Taking $\partial B^k_{L^2}$ to be the boundary of the unit ball in L^2 norm in V, one knows that there exists $\mu > 0$ such that

$$J_a(u) \le -\mu < 0 \qquad for\ all\ u \in \partial B^k_{L^2} \ , \tag{1.6}$$

$$J_a(u) \ge \mu \|u\|^2_H \ge 0 \qquad for\ all\ u \in W \ , \tag{1.7}$$

and that the two sets link (for a definition of the concept of linking see for example [15]).
The existence of this structure allows to characterize the eigenvalue λ_{k+1} as

$$\lambda_{k+1} = a + \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(B^k)} J_a(u) \ , \tag{1.8}$$

where the family Γ is defined as

$$\Gamma = \{\gamma \in \mathcal{C}^0(B^k; \partial B_{L^2}) \ s.t.\ \gamma|_{\partial B^k} \ is\ an\ homeomorphism\ onto\ \partial B^k_{L^2}\} ; \tag{1.9}$$

here B_{L^2} denotes the unit ball in L^2-norm in H and $B^k = \{(x_1..., x_k) \in \mathbb{R}^k \ s.t.\ \sum_{i=1}^k x_i^2 \le 1\}$.

In this paper we will build suitable sets to play the same role for the functional

$$J_\alpha(u) = \int_\Omega |\nabla u|^2 - \alpha^+ \int_\Omega (u^+)^2 - \alpha^- \int_\Omega (u^-)^2 \ , \tag{1.10}$$

in order to characterize a point in the Fučík spectrum.

These sets will be obtained in section 2.1 as a deformation of the sets in (1.6) and (1.7), using a technique similar to the one described in [8].

Then the variational characterization will be done in section 2.2.

In section 3.1, a comparison of the obtained minimax levels with those of the functional associated to problem (1.2) will allow to prove the existence of a linking structure for this last functional, and then to prove theorem 1.2 and 1.3.

Finally in section 4 is reported the complete proof of PS condition for the functional associated to problem (1.2).

2 Variational characterization of the Fučík spectrum

2.1 Construction of the linking structure

Consider first the Dirichlet problem (thus here H will be H^1_0 and the norm considered $\|u\|^2_H = \int_\Omega |\nabla u|^2$). Take a point (α^+, α^-), Σ-connected to the diagonal between λ_k and λ_{k+1}, that is:

Definition 2.1. *A point $(\alpha^+, \alpha^-) \notin \Sigma$ is Σ-connected to the diagonal between λ_k and λ_{k+1} if: $\exists a \in (\lambda_k, \lambda_{k+1})$ and a $\mathcal{C}^1$ function $\alpha : [0, 1] \to \mathbb{R}^2$ such that:*

a) $\alpha(0) = (a, a)$, $\alpha(1) = (\alpha^+, \alpha^-)$;

b) $\alpha([0, 1]) \cap \Sigma = \emptyset$.

Remark 2.2. *Since Σ is closed and $\alpha([0, 1])$ is compact, this implies the property* *b') $\exists d > 0$ such that $\xi\alpha([0,1]) \cap \Sigma = \emptyset$ for all $\xi \in [1-d, 1+d]$;*

this property will be used in the following proofs.

Now consider

$$J_{\alpha(t)}(u) = \int_\Omega |\nabla u|^2 - \alpha^+(t) \int_\Omega (u^+)^2 - \alpha^-(t) \int_\Omega (u^-)^2 \ , \tag{2.1}$$

where $\alpha(t) = (\alpha^+(t), \alpha^-(t))$; then splitting as before $H = V \oplus W$, we have

$$J_{\alpha(0)}(u) \le -\mu \|u\|_H^2 \qquad for\ all\ u \in V \tag{2.2}$$

$$J_{\alpha(0)}(u) \ge \mu \|u\|_H^2 \qquad for\ all\ u \in W \tag{2.3}$$

for some $\mu > 0$.

Lemma 2.3 (from lemma 2.3 of [8])**.** *If (α^+, α^-) is as in definition 2.1, we can find $\eta \in (0, \mu)$ and $\delta_\eta > 0$ such that:*
$\forall t \in [0,1], u \in H$ with $\|u\|_H = 1$:

$$if\ \ J_{\alpha(t)}(u) \in [-\eta, \eta], \qquad then\ \ \|\nabla_u J_{\alpha(t)}(u)\|_H^2 - \left\langle \nabla_u J_{\alpha(t)}(u),\, u\right\rangle_H^2 \ge \delta_\eta.$$

Proof. Following [8], take $\eta = \min(\frac{d}{3(d+1)}, \mu)$, suppose by contradiction the existence of a sequence $t_n \subseteq [0,1]$ and $u_n \in H$, $\|u_n\|_H = 1$ such that

$$-\eta \le J_{\alpha(t_n)}(u_n) \le \eta \quad and \quad \|\nabla_u J_{\alpha(t_n)}(u_n)\|_H^2 - \left\langle \nabla_u J_{\alpha(t_n)}(u_n),\, u_n\right\rangle_H^2 \to 0 \tag{2.4}$$

as $n \to +\infty$.

Define $j_n = \left\langle \nabla_u J_{\alpha(t_n)}(u_n),\, u_n\right\rangle_H = 2J_{\alpha(t_n)}(u_n) \in [-2\eta, 2\eta]$; from Pythagoras' theorem deduce that

$$\|\nabla_u J_{\alpha(t_n)}(u_n)\|_H^2 - \left\langle \nabla_u J_{\alpha(t_n)}(u_n),\, u_n\right\rangle_H^2 = \|\nabla_u J_{\alpha(t_n)}(u_n) - j_n u_n\|_H^2 \ , \tag{2.5}$$

then evaluating the norm considering the points in H as operators on H one concludes that

$$(1 - j_n) \int_\Omega \nabla u_n \nabla v_n - \alpha^+(t_n) \int_\Omega u_n^+ v_n + \alpha^-(t_n) \int_\Omega u_n^- v_n \to 0 \tag{2.6}$$

for any bounded sequence $v_n \subseteq H$.

Up to a subsequence we may say that $j_n \to j \in [-2\eta, 2\eta]$, $t_n \to t_0 \in [0,1]$ and $u_n \rightharpoonup u \in H$ (strongly in L^2); taking the limit of (2.6) with $v_n = u_n$ gives

$$1 - j = \alpha^+(t_0) \int_\Omega (u^+)^2 + \alpha^-(t_0) \int_\Omega (u^-)^2 \ , \tag{2.7}$$

where $j \le 2\eta < 1$ and then u is not trivial.

From equation (2.6) with arbitrary test function and using the weak convergence of u_n, we get that u is a solution of the Fučík problem with coefficient $(\frac{\alpha^+(t_0)}{1-j}, \frac{\alpha^-(t_0)}{1-j})$, but the choice of η and remark 2.2 imply that this is not possible, since $\frac{1}{1-j} \in [1-d, 1+d]$, $\forall j \in [-2\eta, 2\eta]$.

□

Then as in [8] (see there for the proofs and the details) define a continuous flow $\sigma_t(u) : [0,1] \times H \to H$:

$$\begin{cases} \frac{d}{dt}\sigma_t(u) = M\, F_t(\sigma_t(u)) \\ \sigma_0(u) = u \end{cases} , \tag{2.8}$$

where

- M is a suitable positive constant, defined as $M = 2KS^2/\delta_\eta$, with

– $K = \sup_{t\in[0,1]}(|\alpha^+(t)'| + |\alpha^-(t)'|)$,

– $S = \lambda_1^{-\frac{1}{2}} = \sup_{u\in H} \frac{\|u\|_{L^2}}{\|u\|_H}$;

- $F_t : H \to H$ is defined such that it is locally Lipschitz and

$$\begin{cases} F_t(u) = \nabla_u J_{\alpha(t)}(u) & \textit{where} \quad \frac{J_{\alpha(t)}(u)}{\|u\|_H^2} \geq \eta/2 \\ F_t(u) = -\nabla_u J_{\alpha(t)}(u) & \textit{where} \quad \frac{J_{\alpha(t)}(u)}{\|u\|_H^2} \leq -\eta/2 \end{cases} . \tag{2.9}$$

Then $\sigma_t(u)$ has the properties:

- $\sigma_t(0) = 0$ and $\sigma_t(u) \neq 0$ $\forall u \neq 0$,
- $\forall t$, $\sigma_t : H \to H$ is an homeomorphism.

Moreover

Lemma 2.4 (from lemma 2.6 of [8]). *Defining* $\Theta_t(u) = \frac{J_{\alpha(t)}(\sigma_t(u))}{\|\sigma_t(u)\|_H^2}$, *we have that, fixing* u, $\Theta_t(u)$ *is increasing (resp. decreasing) in the variable* t *in any interval* $[t_1, t_2]$ *such that*

$\eta/2 \leq \Theta_t(u) \leq \eta$, $\forall t \in [t_1, t_2]$
(resp. $-\eta \leq \Theta_t(u) \leq -\eta/2$, $\forall t \in [t_1, t_2]$*).*

Proof. Consider first the case $\eta/2 \leq \Theta_t(u) \leq \eta$: then the flow is defined by

$$\frac{d}{dt}\sigma_t(u) = M\nabla_u J_{\alpha(t)}(\sigma_t(u)) \tag{2.10}$$

for all $t \in [t_1, t_2]$.

Then we have (we will omit the dependence from u in the notation)

$$\begin{aligned} \frac{d\Theta_t}{dt} &= \frac{1}{\|\sigma_t\|_H^2}\left[\frac{\partial J_{\alpha(t)}(\sigma_t)}{\partial t} + \langle \nabla_u J_{\alpha(t)}(\sigma_t), \frac{d}{dt}\sigma_t\rangle_H\right] + J_{\alpha(t)}(\sigma_t)\frac{d}{dt}\left(\frac{1}{\|\sigma_t\|_H^2}\right) \\ &= \frac{1}{\|\sigma_t\|_H^2}\left[-\alpha^+(t)'\int_\Omega (\sigma_t^+)^2 - \alpha^-(t)'\int_\Omega (\sigma_t^-)^2 + \right. \\ &\quad \left. + \langle \nabla_u J_{\alpha(t)}(\sigma_t), M\nabla_u J_{\alpha(t)}(\sigma_t)\rangle_H\right] + \\ &\quad + \frac{\langle \nabla_u J_{\alpha(t)}(\sigma_t), \sigma_t\rangle_H}{2}\left(-\frac{2}{\|\sigma_t\|_H^4}\langle \sigma_t, \frac{d}{dt}\sigma_t\rangle_H\right) \\ &\geq -KS^2 + M\left(\frac{\|\nabla_u J_{\alpha(t)}(\sigma_t)\|_H^2}{\|\sigma_t\|_H^2} - \frac{\langle \nabla_u J_{\alpha(t)}(\sigma_t), \sigma_t\rangle_H^2}{\|\sigma_t\|_H^4}\right) \geq -KS^2 + M\delta_\eta . \end{aligned}$$

By the choice made above $M > KS^2/\delta_\eta$ and then the proof of the first part is complete.

In the case $-\eta \leq \Theta_t(u) \leq -\eta/2$ the proof follows the same ideas. □

Finally denote $\sigma_1(u)$ with $\tau_{\alpha,\eta}(u)$ (to remember its dependence from α and η), to obtain

Lemma 2.5 (from equation (2.9) and lemma (2.7) of [8]).

$$J_\alpha(\tau_{\alpha,\eta}(u)) \leq -\eta\|\tau_{\alpha,\eta}(u)\|_H^2 \quad \textit{for all } u \in V \ ; \tag{2.11}$$

$$J_\alpha(\tau_{\alpha,\eta}(u)) \geq \eta\|\tau_{\alpha,\eta}(u)\|_H^2 \quad \textit{for all } u \in W \ ; \tag{2.12}$$

$\forall R>0$, $\tau_{\alpha,\eta}(W)$ links with $R\,\tau_{\alpha,\eta}(\partial B^k_V)$ where B^k_V is the unit ball, in the H-norm, of V.

Proof. Equations (2.11) and (2.12) follow easily from lemma 2.4.

For the linking property we need to prove that:
$\forall\gamma\in\Gamma=\{\gamma\in\mathcal{C}^0(R\tau_{\alpha,\eta}(B^k_V);H)$ *and s.t.* $\gamma(u)=u$ *for* $u\in R\tau_{\alpha,\eta}(\partial B^k_V)\}$, there exists a point $\bar u\in\gamma(R\tau_{\alpha,\eta}(B^k_V))\cap\tau_{\alpha,\eta}(W)$.

We start by proving that

$$\xi\tau_{\alpha,\eta}(u)\neq\tau_{\alpha,\eta}(v) \tag{2.13}$$

for any $u\in\partial B^k_V$, $v\in W$ and $\xi>0$: actually if it were not so, from equations (2.11) and (2.12) we would get $\eta\|\tau_{\alpha,\eta}(v)\|^2_H\leq J_\alpha(\tau_{\alpha,\eta}(v))=J_\alpha(\xi\tau_{\alpha,\eta}(u))=\xi^2J_\alpha(\tau_{\alpha,\eta}(u))\leq-\eta\xi^2\|\tau_{\alpha,\eta}(u)\|^2_H$ which implies (using also the uniqueness of the Cauchy problem) $u=v=0$: contradiction since $u\in\partial B^k_V$.

Now define P to be the orthogonal projection of H onto V and consider the map $H_t=P\circ\tau^{-1}_{\alpha,\eta}\circ(1+(R-1)t)\tau_{\alpha,\eta}$: property (2.13) implies that $H_t\neq0$ on ∂B^k_V for any $t\in[0,1]$ and then $deg(H_1,B^k_V,0)=deg(H_0,B^k_V,0)=deg(Id,B^k_V,0)=1$.

Now for any $\gamma\in\Gamma$, $deg(P\circ\tau^{-1}_{\alpha,\eta}\circ\gamma\circ R\tau_{\alpha,\eta},B^k_V,0)=1$ since on ∂B^k_V the function is exactly H_1, and then there is a point $p\in B^k_V$ such that $\gamma(R\tau_{\alpha,\eta}(p))\in\tau_{\alpha,\eta}(W)$.

□

For the Neumann problem, as shown in the proof of theorem 3.4 of [8] for the periodic case, one can get the same conclusions, working with the operator $-\Delta u+u$ to avoid the problems arising since the first eigenvalue is 0.

Finally we prove one more property that we will need later:

Lemma 2.6. *If $u\in V$ or $u\in W$, and $\xi>0$ then $\tau_{\alpha,\eta}(\xi u)=\xi\tau_{\alpha,\eta}(u)$.*

Proof. From lemma 2.4 and 2.5 and equations (2.8) and (2.9) we have that in these two cases the equation just contains the gradient of $J_{\alpha(t)}$.

If we take $u\in V$, then the flow is defined by

$$\begin{cases}\frac{d}{dt}\sigma_t(u)=-M\nabla_uJ_{\alpha(t)}(\sigma_t(u))\\ \sigma_0(u)=u\in V\end{cases}. \tag{2.14}$$

Consider then the change of variable $\sigma=k\pi$ with $k>0$: equation (2.14) becomes

$$\begin{cases}k\frac{d}{dt}\pi_t(u)=-M\nabla_uJ_{\alpha(t)}(k\pi_t(u))\\ k\pi_0(u)=u\in V\end{cases} \tag{2.15}$$

and considering the linear positive homogeneity of $\nabla_uJ_{\alpha(t)}$ it can be simplified to obtain

$$\begin{cases}\frac{d}{dt}\pi_t(u)=-M\nabla_uJ_{\alpha(t)}(\pi_t(u))\\ \pi_0(u)=u/k\in V\end{cases}, \tag{2.16}$$

which is the same equation as (2.14) with a different initial condition: then
$\sigma_t(u)=k\pi_t(u)=k\sigma_t(u/k)$.

The case $u\in W$ is treated in the same way. □

2.2 Construction of the variational characterization

Now we use the results of section 2.1 to obtain a variational characterization of some parts of the Fučík spectrum (problem (1.1)).

The result is the one stated in theorem 1.1.

Note that in the one dimensional case, since the spectrum is known, (α^+, α^-) may be taken anywhere between the continuous curves arising from a point (λ_k, λ_k) and the ones arising from $(\lambda_{k+1}, \lambda_{k+1})$; in the multi dimensional case one has to be more careful, but Σ-connection may be assured at least for (α^+, α^-) in the square $(\lambda_{k-1}, \lambda_{k+m+1})^2$ (being $\lambda_{k-1} < \lambda_k = ... = \lambda_{k+m} < \lambda_{k+m+1}$) when it is not between (or on) the lower and the upper curve arising from (λ_k, λ_k).

We will now imitate the characterization of λ_{k+1} described in (1.8).

We fix a point (α^+, α^-) Σ-connected to the diagonal between λ_k and λ_{k+1} and with $\alpha^+ \geq \alpha^-$, then we apply the results of section 2.1, obtaining the deformation $\tau_{\alpha,\eta}$, we choose $r \in (0, 1]$, we split again $H = V \oplus W$ with $V = span\{\phi_1, .., \phi_k\}$ and we consider:

- The set
$$Q_r = \left\{ u \in H \ \ s.t. \ \ \int_\Omega (u^+)^2 + r(u^-)^2 = 1 \right\} . \tag{2.17}$$
- The radial projection on Q_r of the set obtained in section 2.1 by the deformation of ∂B^k_V, that is
$$L_{\alpha,r} = P^r(\tau_{\alpha,\eta}(\partial B^k_V)) , \tag{2.18}$$
where $P^r : u \to \frac{u}{\sqrt{\int_\Omega (u^+)^2 + r \int_\Omega (u^-)^2}}$.
- The class of maps
$$\Gamma_{\alpha,r} = \{\gamma \in \mathcal{C}^0(B^k; Q_r) \ \ s.t. \ \ \gamma|_{\partial B^k} \ \ is \ \ an \ \ homeomorphism \ \ onto \ \ L_{\alpha,r}\} , \tag{2.19}$$
where $B^k = \{(x_1..., x_k) \in \mathbb{R}^k \ s.t. \ \sum_{i=1}^k x_i^2 \leq 1\}$.
- The functional
$$J_\alpha(u) = \int_\Omega (\nabla u)^2 - \alpha^+ \int_\Omega (u^+)^2 - \alpha^- \int_\Omega (u^-)^2 \ . \tag{2.20}$$

The idea now is to consider
$$d_{\alpha,r} = \inf_{\gamma \in \Gamma_{\alpha,r}} \sup_{u \in \gamma(B^k)} J_\alpha(u) \tag{2.21}$$
and to prove that this leads to a nontrivial solution of the Fučík problem (1.1), that is to a point in Σ.

We first prove that the above definitions are well posed and derive some properties of the defined sets:

Lemma 2.7. *For $u \in Q_r$ we have that $1 \leq \int_\Omega u^2 \leq 1/r$.*

Proof. $1 = \int_\Omega (u^+)^2 + r(u^-)^2 \leq \int_\Omega (u^+)^2 + (u^-)^2 = \int_\Omega u^2 \leq (\int_\Omega (u^+)^2 + r(u^-)^2)/r = 1/r$. □

Lemma 2.8.

(i) The set $L_{\alpha,r}$ is homeomorphic to ∂B^k.

(ii) $L_{\alpha,r} \subseteq \tau_{\alpha,\eta}(V)$.

Proof. (i) Since ∂B^k_V is homeomorphic to ∂B^k and $\tau_{\alpha,\eta}$ is an homeomorphism, we just need to prove that P^r is an homeomorphism when restricted to $\tau_{\alpha,\eta}(\partial B^k_V)$.

$\tau_{\alpha,\eta}$ on ∂B^k_V has the property (see lemma 2.6) that $\forall \xi > 0$, $\tau_{\alpha,\eta}(\xi u) = \xi \tau_{\alpha,\eta}(u)$, then P^r is one to one on $\tau_{\alpha,\eta}(\partial B^k_V)$ and so can be inverted.

Finally P^r is continuous together with its inverse because, since ∂B^k_V is a compact set which does not contain the origin, $\int_\Omega (u^+)^2 + r \int_\Omega (u^-)^2$ is continuous, bounded and bounded away from zero on it.

(ii) The second point is a trivial consequence of lemma 2.6.

□

Lemma 2.9. *$\tau_{\alpha,\eta}(W)$ links with $L_{\alpha,r}$.*

Proof. From lemma 2.5 $\tau_{\alpha,\eta}(W)$ links with $\tau_{\alpha,\eta}(\partial B^k_V)$.

Then the claim could be false only if for some $u \in L_{\alpha,r}$, $\xi > 0$, and $v \in \tau_{\alpha,\eta}(W)$ we had $\xi u = v$. But by the homogeneity property of $\tau_{\alpha,\eta}$ in V and W (lemma 2.6) this would imply $\xi(\tau_{\alpha,\eta})^{-1}(u) = (\tau_{\alpha,\eta})^{-1}(v)$ and then $u = v = 0$, which is impossible since $u \in P^r(\tau_{\alpha,\eta}(\partial B^k_V))$. □

In the next three lemmas we verify the conditions for the "Linking Theorem" which will be used to prove the criticality of $d_{\alpha,r}$.

Lemma 2.10. *The functional $J_\alpha(u)$ constrained to Q_r satisfies the PS condition.*

Proof. Consider the sequences $\{u_n\} \subseteq Q_r$, $\{\beta_n\} \subseteq \mathbb{R}$ (Lagrange multipliers) and $\varepsilon_n \to 0^+$ such that

$$\left| \int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2 \right| \le C \tag{2.22}$$

$$\begin{aligned} \left| \int_\Omega \nabla u_n \nabla v - \alpha^+ \int_\Omega (u_n^+) v + \alpha^- \int_\Omega (u_n^-) v + \right. & \\ \left. + \beta_n \left(\int_\Omega u_n^+ v - r u_n^- v \right) \right| & \le \varepsilon_n \|v\|_H, \quad \forall v \in H \; . \end{aligned} \tag{2.23}$$

Since $\{u_n\} \subseteq Q_r$, it is a bounded sequence in L^2, and then equation (2.22) implies that it is also a bounded sequence in H. Then there is a subsequence converging weakly in H and strongly in L^2 to some u.

The L^2 convergence implies that $u \in Q_r$.

Taking $v = u_n$ we get that

$$\beta_n + \left(\int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2 \right) \to 0 \, . \tag{2.24}$$

Then, with $v = u_n - u$ we have

$$\begin{aligned} & \int_\Omega \nabla u_n \nabla (u_n - u) - \alpha^+ \int_\Omega (u_n^+)(u_n - u) + \alpha^- \int_\Omega (u_n^-)(u_n - u) + \\ & - \left(\int_\Omega (\nabla u_n)^2 - \alpha^+ \int_\Omega (u_n^+)^2 - \alpha^- \int_\Omega (u_n^-)^2 \right) \left(\int_\Omega (u_n^+ - r u_n^-)(u_n - u) \right) \to 0, \end{aligned}$$

where all terms except the first go to zero. Then we conclude that
$\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}$ and then $u_n \to u$ strongly in H. □

Lemma 2.11. $\sup_{u \in \gamma(\partial B^k)} J_\alpha(u) \le 0 \quad \forall \gamma \in \Gamma_{\alpha,r}$.

Proof. By lemma 2.5, since $\gamma(\partial B^k) = L_{\alpha,r} \subseteq \tau_{\alpha,\eta}(V)$ and then
$J_\alpha(u) \le -\eta \|u\|^2_H < 0$. □

Lemma 2.12. $+\infty > \sup_{u \in \gamma(B^k)} J_\alpha(u) \ge \eta > 0$ *for each* $\gamma \in \Gamma_{\alpha,r}$

Proof. By lemma 2.9 there is always a point $u \in \gamma(B^k) \cap \tau_{\alpha,\eta}(W)$, and by lemma 2.5 we have in that point $J_\alpha(u) \ge \eta \|u\|^2_H$; considering lemma 2.7 and that $u \in Q_r$, this becomes $\ge \eta$.

Finally it is less than $+\infty$ since each $\gamma(B^k)$ is a compact set.

□

At this point we can state the following standard "Linking Theorem" (see for example [15]):

Proposition 2.13. *The level $d_{\alpha,r} \ge \eta > 0$ is a critical value for $J_\alpha(u)$ constrained to Q_r.*

The importance of the criticality of the level $d_{\alpha,r}$ is clarified in the following proposition:

Proposition 2.14. *The critical points associated to the critical value $d_{\alpha,r}$ are non trivial solutions of the Fučík problem (1.1) with coefficients (λ^+, λ^-), where $\lambda^+ - \alpha^+ = d_{\alpha,r}$ and $\lambda^- - \alpha^- = rd_{\alpha,r}$.*

Proof. Criticality of u implies that there exists a Lagrange multiplier $\beta \in \mathbb{R}$ such that

$$\int_\Omega \nabla u \nabla v - \alpha^+ \int_\Omega (u^+)v + \alpha^- \int_\Omega (u^-)v + \beta \left(\int_\Omega u^+v - ru^-v \right) = 0 \quad \forall v \in H \ , \tag{2.25}$$

but testing against u we get $\beta = -d_{\alpha,r}$ and so u solves

$$-\Delta u = \alpha^+u^+ - \alpha^-u^- + d_{\alpha,r}u^+ - d_{\alpha,r}ru^- = (\alpha^+ + d_{\alpha,r})u^+ - (\alpha^- + rd_{\alpha,r})u^- \tag{2.26}$$

in Ω, with the considered boundary conditions.

Finally u is not trivial since it is in Q_r. □

Proposition 2.13 and 2.14 imply that the point $(\alpha^+ + d_{\alpha,r},\ \alpha^- + rd_{\alpha,r})$ belongs to the halfline $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$ (since $d_{\alpha,r} > 0$) and also to the Fučík spectrum; thus theorem 1.1 is proved.

Remark 2.15. *We did not prove that this solution corresponds to the first intersection (that is the one with smallest t) of the halfline with Σ.*

Thus, even in the one dimensional case (that is when the spectrum is known), we cannot assert that the point belongs to the continuum arising from $(\lambda_{k+1}, \lambda_{k+1})$: what we can say (since $d_{\alpha,r} > 0$) is just that it belongs to the continuum arising from (λ_h, λ_h) for some $h \geq k + 1$.

3 The superlinear problem

3.1 Proof of theorem 1.2

Consider now the superlinear problem (1.2): the idea here is to prove the existence of a non constrained critical point of the functional

$$F(u) = \frac{1}{2} \int_0^1 (u')^2 - \frac{\lambda}{2} \int_0^1 u^2 - \int_0^1 G(x, u) - \int_0^1 hu \ , \tag{3.1}$$

which corresponds to a solution of the problem.

We will follow a strategy inspired by [7].

Note that $H^1(0, 1) \subseteq \mathcal{C}^0([0, 1])$ with compact inclusion and recall that in this case the asymptotes of each Σ_k with $k \geq 2$ are at $\lambda^- = \frac{\lambda_k}{4}$ and that Σ_k lies entirely in $\{\lambda^- > \frac{\lambda_k}{4}\}$.

This structure of Σ implies that, fixed $\lambda \in (\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4})$, $k \geq 1$, it is always possible to find:

- a point (α^+, α^-) Σ-connected to the diagonal between λ_k and λ_{k+1} and such that $\alpha^- < \lambda$,
- a $\delta > 0$ such that $\alpha^- < \lambda - \delta$ and $\lambda + \delta < \frac{\lambda_{k+1}}{4}$.

Now, using the notation of section 2.2, we define, for $R > 0$, the family of maps

$$\Gamma^R_{\alpha,\bar{r}} = \{\gamma^* \in \mathcal{C}^0(B^k; H) \ \textit{s.t.}\ \gamma^*|_{\partial B^k} \ \textit{is an homeomorphism onto}\ RL_{\alpha,\bar{r}}\} \ . \tag{3.2}$$

We want to prove that, for a suitable $R > 0$, the level

$$f = \inf_{\gamma^* \in \Gamma^R_{\alpha,\bar{r}}} \sup_{u \in \gamma^*(B^k)} F(u) \tag{3.3}$$

is a critical value for the functional F.

Remark 3.1. *In the definition of* $\Gamma^R_{\alpha,\bar r}$ *the choice of* $\bar r \in (0,1]$ *has no importance: it can be chosen arbitrarily.*

Using that $h \in L^2$ and hypothesis (H1), we can find constants C_1, C_2 and C_3 as follows:

- $C_1(\delta,h)$ such that
$$\left|\int_0^1 hu\right| \le \frac{\delta}{4}\|u\|^2_{L^2} + C_1(\delta,h)\,; \tag{3.4}$$
- $C_2(\delta,g)$ such that
$$\left|\int_0^1 G(x,-u^-)\right| \le \frac{\delta}{4}\|u\|^2_{L^2} + C_2(\delta,g)\,; \tag{3.5}$$
- for any M, $C_3(M,g)$ such that
$$\int_0^1 G(x,u^+) \ge \frac{M}{2}\|u^+\|^2_{L^2} - C_3(M,g)\,. \tag{3.6}$$

To find a Generalized Mountain Pass structure we first need

Lemma 3.2. $\forall C \in \mathbb{R}$ *we can find* $R>0$ *such that*
$$\sup_{u\in\gamma^*(\partial B^k)} F(u) < C \quad \forall \gamma^* \in \Gamma^R_{\alpha,\bar r}\,. \tag{3.7}$$

Proof. We evaluate, for $u \in L_{\alpha,\bar r}$ and $\rho > 0$,

$$\begin{aligned}
\frac{F(\rho u)}{\rho^2} &= \frac12\int_0^1 (u')^2 - \frac{\lambda}{2}\int_0^1 u^2 - \frac{\int_0^1 G(x,\rho u)}{\rho^2} - \frac{\int_0^1 h\rho u}{\rho^2}\\
&\le \frac12\int_0^1 (u')^2 - \frac{\lambda}{2}\int_0^1 u^2 + \frac{|\int_0^1 G(x,-\rho u^-)|}{\rho^2} - \frac{\int_0^1 G(x,\rho u^+)}{\rho^2} + \frac{|\int_0^1 h\rho u|}{\rho^2}\\
&\le \frac12\int_0^1 (u')^2 - \frac{\lambda}{2}\int_0^1 u^2 + \left(\frac{\delta}{4}\int_0^1 u^2 + \frac{C_2(\delta,g)}{\rho^2}\right)\\
&\quad -\left(\frac{M}{2}\int_0^1 (u^+)^2 - \frac{C_3(M,g)}{\rho^2}\right) + \left(\frac{\delta}{4}\int_0^1 u^2 + \frac{C_1(\delta,h)}{\rho^2}\right)\\
&\le \frac12\int_0^1 (u')^2 - \frac{\lambda-\delta}{2}\int_0^1 u^2 - \frac{M}{2}\int_0^1 (u^+)^2 +\\
&\quad +\frac{C_1(\delta,h)+C_2(\delta,g)+C_3(M,g)}{\rho^2}\\
&= \frac12 J_\alpha(u) - \frac{\lambda-\delta+M-\alpha^+}{2}\int_0^1 (u^+)^2 - \frac{\lambda-\delta-\alpha^-}{2}\int_0^1 (u^-)^2\\
&\quad +\frac{C_1+C_2+C_3(M,g)}{\rho^2}\,.
\end{aligned}$$

Now if we fix $M = \alpha^+ - \alpha^-$ and consider that $J_\alpha(u) \le 0$ and $\int_0^1 u^2 \ge 1$ on $L_{\alpha,\bar r}$, we get
$$\frac{F(\rho u)}{\rho^2} \le -\frac{\lambda-\delta-\alpha^-}{2} + \frac{\tilde C(\delta,\alpha,g,h)}{\rho^2}\ , \tag{3.8}$$

where the first part is negative by the choice made for δ and then we can find the required R, namely $R > \sqrt{\frac{2(\tilde{C}(\delta,\alpha,g,h)-C)}{\lambda-\delta-\alpha^-}}$. $\square$

Next we need

Lemma 3.3.
$$\sup_{u\in\gamma^*(B^k)} F(u) \geq -C_1(\delta,h) - C_2(\delta,g) - 1 \quad \forall \gamma^* \in \Gamma^R_{\alpha,\bar{r}}\,. \tag{3.9}$$

Proof. Fix a $\gamma^* \in \Gamma^R_{\alpha,\bar{r}}$.

Since $\gamma^*(B^k)$ is a compact set in a space of continuous functions, we can find
$$b(\gamma^*) = \max\{|u(x)| : x \in [0,1], u \in \gamma^*(B^k)\} \tag{3.10}$$
and then there exists $\mu_{\gamma^*} > 0$ such that
$$\frac{\mu_{\gamma^*}}{2} s^2 \geq G(x,s) - 1 \quad for\ \ all \quad s \in [0, b(\gamma^*)]\ . \tag{3.11}$$

Then
$$\begin{aligned}\int_0^1 G(x,u) + \int_0^1 hu \quad &\leq \quad \frac{\delta}{4}\int_0^1 u^2 + C_1(\delta,h) + \\ &+\frac{\delta}{4}\int_0^1 u^2 + C_2(\delta,g) + \frac{\mu_{\gamma^*}}{2}\int_0^1 (u^+)^2 + \int_0^1 1\ ,\end{aligned} \tag{3.12}$$
and so
$$\begin{aligned}\sup_{u\in\gamma^*(B^k)} F(u) \quad \geq \quad & \frac{1}{2}\sup_{u\in\gamma^*(B^k)} \left(\int_0^1 (u')^2 - (\lambda+\delta)\int_0^1 u^2 - \mu_{\gamma^*}\int_0^1 (u^+)^2\right) + \\ & -C_1(\delta,h) - C_2(\delta,g) - 1\ .\end{aligned} \tag{3.13}$$

Now if $0 \in \gamma^*(B^k)$ the sup on the right is clearly nonnegative.

Otherwise we can rearrange the terms in the sup on the right, adding and subtracting $\alpha^+\int_0^1 (u^+)^2 + \alpha^-\int_0^1 (u^-)^2$, defining $r_{\gamma^*} = \frac{\lambda+\delta-\alpha^-}{\lambda+\delta+\mu_{\gamma^*}-\alpha^+}$ and collecting $\int_0^1 (u^+)^2 + r_{\gamma^*}\int_0^1 (u^-)^2 > 0$, obtaining
$$\begin{aligned}\sup_{u\in\gamma^*(B^k)} \quad & \left[\left(\frac{J_\alpha(u)}{\int_0^1 (u^+)^2 + r_{\gamma^*}\int_0^1 (u^-)^2} - (\lambda+\delta+\mu_{\gamma^*}-\alpha^+)\right)\right. \\ & \left.\left(\int_0^1 (u^+)^2 + r_{\gamma^*}\int_0^1 (u^-)^2\right)\right]\ .\end{aligned} \tag{3.14}$$

Now if the sup of the first part is nonnegative, then so is all the sup.

But $\sup_{u\in\gamma^*(B^k)} \frac{J_\alpha(u)}{\int_0^1 (u^+)^2 + r_{\gamma^*}\int_0^1 (u^-)^2}$ is equivalent to $\sup_{u\in\gamma(B^k)} J_\alpha(u)$ for some $\gamma \in \Gamma_{\alpha,r_{\gamma^*}}$ (compare equation (2.19) and (3.2), considering the definition (2.18)); then it is not lower than the value $d_{\alpha,r_{\gamma^*}}$ obtained in proposition 2.13; this means that by proposition 2.14 and remark 2.15
$$\sup_{u\in\gamma(B^k)} J_\alpha(u) \geq \lambda^+_{\gamma^*} - \alpha^+ \tag{3.15}$$
where $(\lambda^+_{\gamma^*}, \lambda^-_{\gamma^*}) \in \Sigma_h$ with $h \geq k+1$ and $\frac{\lambda^-_{\gamma^*}-\alpha^-}{\lambda^+_{\gamma^*}-\alpha^+} = r_{\gamma^*}$.

Then remains the calculation
$$(\lambda^+_{\gamma^*} - \alpha^+) - (\lambda+\delta+\mu_{\gamma^*}-\alpha^+) = \frac{(\lambda^-_{\gamma^*}-\alpha^-) - (\lambda+\delta-\alpha^-)}{r_{\gamma^*}} = \frac{\lambda^-_{\gamma^*} - (\lambda+\delta)}{r_{\gamma^*}}\ , \tag{3.16}$$

which is positive for the choice made for δ, since the curves Σ_h with $h \geq k+1$ have all points with $\lambda^- > \frac{\lambda_{k+1}}{4}$.

To conclude note that in this way we eliminated the dependence from γ^* (and from the values which depended upon it: r_{γ^*}, $\lambda^+_{\gamma^*}$ and $\lambda^-_{\gamma^*}$) in the estimates, hence the lemma is proved. □

The PS condition for F was proved (using hypothesis (H2)) in [6] for $\lambda \in (0, \frac{\pi^2}{4})$, and in [7] (using also (H3)) for any $\lambda > 0$, in the case of periodic boundary conditions, but the proof can be extended to the Neumann case. The complete proof is given, for sake of completeness, in section 4.

Using lemma 3.2 with $C < -C_1(\delta, h) - C_2(\delta, g) - 1$, lemma 3.3 and the PS condition, we are in the conditions to apply a linking theorem that proves the criticality of the level f defined in equation (3.3), and then theorem 1.2 is proved.

3.2 Proof of theorem 1.3

For the values $\lambda = \frac{\lambda_{k+1}}{4}$ one has a kind of resonance which creates difficulties for some of the estimates; actually the proof of lemma 3.2 can be done in the same way, choosing $\delta > 0$ such that $\alpha^- < \lambda - \delta$, but for lemma 3.3 we cannot conclude with the same estimates since no choice of $\delta > 0$ would allow to infer that the expression in (3.16) is not negative.

Thus in this case we need to impose also the hypothesis (HR) and we proceed using the following estimates:

$$\int_{u<-\rho_0} G(x,u) + hu \leq M_0 \int_0^1 1\,,$$

$$\int_{u\in[-\rho_0,0]} G(x,u) + hu \leq \sup_{u\in[-\rho_0,0],x\in[0,1]} G(x,u) \int_0^1 1 + \rho_0 \int_0^1 |h| = C_4(h,g)\,,$$

$$\int_{u>0} G(x,u) + hu \leq \frac{\mu_{\gamma^*}}{2} \int_0^1 (u^+)^2 + \int_0^1 1 + \frac{1}{2} \int_0^1 (u^+)^2 + \frac{1}{2} \int_0^1 h^2\,;$$

then we get, in place of (3.12), that

$$\int_0^1 G(x,u) + \int_0^1 hu \quad \leq \quad \frac{\mu_{\gamma^*}+1}{2} \int_0^1 (u^+)^2 + M_0 + C_4(h,g) + 1 + \frac{1}{2} \int_0^1 h^2$$

and then we can estimate the sup like in (3.13) as

$$\begin{aligned} \sup_{u\in\gamma^*(B^k)} F(u) \quad &\geq \quad \frac{1}{2} \sup_{u\in\gamma^*(B^k)} \left(\int_0^1 (u')^2 - \lambda \int_0^1 u^2 - (\mu_{\gamma^*}+1) \int_0^1 (u^+)^2 \right) + \\ &\quad -M_0 - 1 - \frac{1}{2} \int_0^1 h^2 - C_4(h,g) \ . \end{aligned} \tag{3.17}$$

After this we make the same calculations we did before, now with $r_{\gamma^*} = \frac{\lambda-\alpha^-}{\lambda+\mu_{\gamma^*}+1-\alpha^+}$, to conclude that there is a point $(\lambda^+_{\gamma^*}, \lambda^-_{\gamma^*}) \in \Sigma_h$ with $h \geq k+1$ and $\frac{\lambda^-_{\gamma^*}-\alpha^-}{\lambda^+_{\gamma^*}-\alpha^+} = r_{\gamma^*}$ such that the sup is not negative if the following expression is not negative too:

$$(\lambda^+_{\gamma^*} - \alpha^+) - (\lambda + \mu_{\gamma^*} + 1 - \alpha^+) = \frac{(\lambda^-_{\gamma^*} - \alpha^-) - (\lambda - \alpha^-)}{r_{\gamma^*}} = \frac{\lambda^-_{\gamma^*} - \lambda}{r_{\gamma^*}}\ ; \tag{3.18}$$

but this is actually positive since all points in Σ_h with $h \geq k+1$ have $\lambda^- > \lambda$.

4 Proof of the PS condition

Proposition 4.1. *Under hypotheses (H1), (H2) and (H3), the functional (3.1) satisfies the PS condition for any $\lambda > 0$.*

First note that from hypothesis (H1) one can always make the estimates:
for any $\varepsilon > 0$, $\bar{s} \in \mathbb{R}$ and $M \in R$, there exist C_M, $C_\varepsilon \in \mathbb{R}$ (of course depending also on $\bar{s}$) such that

$$g(x,s) \geq Ms - C_M \quad for \quad s > \bar{s} \ , \tag{4.1}$$
$$|g(x,s)| \leq \varepsilon(-s) + C_\varepsilon \quad for \quad s \leq \bar{s} \ . \tag{4.2}$$

Let now $\{u_n\} \subseteq H^1(0,1)$ be a PS sequence, i.e. there exist $T > 0$ and $\varepsilon_n \to 0^+$ such that

$$|F(u_n)| = \left| \frac{1}{2}\int_0^1 |u_n'|^2 - \frac{\lambda}{2}\int_0^1 u_n^2 - \int_0^1 G(x,u_n) - \int_0^1 hu_n \right| \leq T \ , \tag{4.3}$$

$$|\langle F'(u_n), v\rangle| = \left| \int_0^1 u_n'v' - \lambda\int_0^1 u_n v - \int_0^1 g(x,u_n)v - \int_0^1 hv \right| \leq \varepsilon_n \|v\|_{H^1}, \ \ \forall v \in H^1. \tag{4.4}$$

1. Suppose u_n is not bounded, then we can assume $\|u_n\|_{H^1} \geq 1$, $\|u_n\|_{H^1} \to +\infty$ and define $z_n = \frac{u_n}{\|u_n\|_{H^1}}$, so that z_n is a bounded sequence in H^1 and we can select a subsequence such that $z_n \to z_0$ weakly in H^1 and strongly in $L^2(0,1)$ and $\mathcal{C}^0[0,1]$.

2. Claim: $z_0 \leq 0$.

   Proof of the claim. Consider $\left|\frac{\langle F'(u_n), z_0^+\rangle}{\|u_n\|_{H^1}}\right|$:

$$\left| \int_0^1 z_n'(z_0^+)' - \lambda\int_0^1 z_n z_0^+ - \int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|_{H^1}} - \int_0^1 \frac{hz_0^+}{\|u_n\|_{H^1}} \right| \leq \frac{\varepsilon_n \|z_0^+\|_{H^1}}{\|u_n\|_{H^1}} \ , \tag{4.5}$$

   from which

$$\int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|_{H^1}} \leq \left|\int_0^1 z_n'(z_0^+)'\right| + \lambda\left|\int_0^1 z_n z_0^+\right| + \left|\int_0^1 \frac{hz_0^+}{\|u_n\|_{H^1}}\right| + \frac{\varepsilon_n \|z_0^+\|_{H^1}}{\|u_n\|_{H^1}} \ . \tag{4.6}$$

   Now for any $\bar{x}$ such that $z_0^+(\bar{x}) > 0$, we have that $u_n(\bar{x}) > 0$ for n large enough and then we can use the estimate (4.1) to obtain

$$\frac{g(\bar{x},u_n)}{\|u_n\|_{H^1}} \geq Mz_n(\bar{x}) - \frac{C_M}{\|u_n\|_{H^1}} \ ; \tag{4.7}$$

   taking lim inf we get

$$\liminf_{n\to+\infty} \frac{g(\bar{x},u_n)}{\|u_n\|_{H^1}} \geq Mz_0(\bar{x}) \tag{4.8}$$

   for any choice of M and then

$$\lim_{n\to+\infty} \frac{g(\bar{x},u_n)}{\|u_n\|_{H^1}} = +\infty \ . \tag{4.9}$$

   Joining equations (4.1) and (4.2) with $\bar{s} = 0$ and divided by $\|u_n\|_{H^1}$ we get

$$\frac{g(x,u_n)}{\|u_n\|_{H^1}} \begin{array}{ll} \geq Mz_n - \frac{C_M}{\|u_n\|_{H^1}} & where\ z_n > 0 \\ \geq -\varepsilon(-z_n) - \frac{C_\varepsilon}{\|u_n\|_{H^1}} & where\ z_n \leq 0 \end{array}$$

and so

$$\frac{g(x,u_n)}{\|u_n\|_{H^1}} \geq -\varepsilon|z_n| - \frac{C_{M,\varepsilon}}{\|u_n\|_{H^1}}\,. \tag{4.10}$$

Since z_n is uniformly bounded (by its $\mathcal{C}^0$ convergence) and $\|u_n\|_{H^1} \geq 1$, this implies that the functions $\frac{g(x,u_n)}{\|u_n\|_{H^1}}$ are bounded below uniformly so that we can use Fatou's Lemma and get from (4.6) and supposing $z_0^+ \not\equiv 0$

$$\begin{aligned} +\infty &= \int_0^1 \lim_{n\to+\infty} \frac{g(x,u_n)z_0^+}{\|u_n\|_{H^1}} \leq \liminf_{n\to+\infty} \int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|_{H^1}} \\ &\leq \liminf_{n\to+\infty} \left(\left|\int_0^1 z_n'(z_0^+)'\right| + \lambda\left|\int_0^1 z_n z_0^+\right| + \left|\int_0^1 \frac{hz_0^+}{\|u_n\|_{H^1}}\right| + \frac{\varepsilon_n\|z_0^+\|_{H^1}}{\|u_n\|_{H^1}} \right)\,. \end{aligned} \tag{4.11}$$

The right hand side can be estimated since the first two terms are bounded by $(1+\lambda)\|z_n\|_{H^1}\|z_0^+\|_{H^1} \leq 1+\lambda$ and the last two clearly go to zero; then equation (4.11) gives rise to a contradiction unless $z_0 \leq 0$. □

3. Claim: Using hypotheses (H2) and (H3) we obtain a constant A such that

$$\int_{u_n>s_0} g(x,u_n)u_n \leq A\|u_n\|_{H^1} \tag{4.12}$$

at least for n big enough.

Proof of the claim. Consider first $|2F(u_n) - \langle F'(u_n), u_n\rangle|$:

$$\left|\int_0^1 -2G(x,u_n) + g(x,u_n)u_n + (1-2)\int_0^1 hu_n\right| \leq 2T + \varepsilon_n\|u_n\|_{H^1}\ , \tag{4.13}$$

from which

$$\begin{aligned} \int_{u_n>s_0} g(x,u_n)u_n - 2G(x,u_n) \quad &\leq \quad \int_{u_n\leq s_0} 2G(x,u_n) - g(x,u_n)u_n + \\ &\quad + \left|\int_0^1 hu_n\right| + 2T + \varepsilon_n\|u_n\|_{H^1}\,. \end{aligned} \tag{4.14}$$

The right hand side may be estimated as follows:

- $$\int_{-s_1\leq u_n\leq s_0} 2G(x,u_n) - g(x,u_n)u_n \leq \sup_{x\in[0,1],\ s\in[-s_1,s_0]} (2G(x,s) - g(x,s)s)\ , \tag{4.15}$$
- using hypothesis (H3)
$$\int_{u_n\leq -s_1} 2G(x,u_n) - g(x,u_n)u_n \leq 2C_0\ , \tag{4.16}$$
- $\left|\int_0^1 hu_n\right| \leq \|h\|_{L^2}\|u_n\|_{L^2} \leq \|h\|_{L^2}\|u_n\|_{H^1}$.

For the left hand side we use hypothesis (H2) to obtain

$$(1-2\theta)\int_{u_n>s_0} g(x,u_n)u_n \leq \int_{u_n>s_0} g(x,u_n)u_n - 2G(x,u_n) \tag{4.17}$$

and then, since $(1-2\theta)>0$, joining all estimates from (4.14) to (4.17), we get

$$\int_{u_n>s_0} g(x,u_n)u_n \le \frac{A}{2}\|u_n\|_{H^1} + \frac{A}{2} \le A\|u_n\|_{H^1} \tag{4.18}$$

for some constant A.

□

4. Claim:

$$\int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{H^1}} \to 0 \ . \tag{4.19}$$

Proof of the claim. Fix $\varepsilon>0$ and k such that $\frac{A}{k}\le\varepsilon$ and $k>s_0$.
Estimate (4.2) shows that

$$\int_{u_n\le k} \frac{|g(x,u_n)|}{\|u_n\|_{H^1}} \le \int_0^1 \frac{\varepsilon|u_n|+C_\varepsilon}{\|u_n\|_{H^1}} \le \varepsilon C\frac{\|u_n\|_{L^2}}{\|u_n\|_{H^1}} + \frac{C_\varepsilon}{\|u_n\|_{H^1}} \ , \tag{4.20}$$

from which there exists $\bar n$ such that

$$\int_{u_n\le k} \frac{|g(x,u_n)|}{\|u_n\|_{H^1}} \le (C+1)\varepsilon \qquad for\ \ n>\bar n \ . \tag{4.21}$$

Since $k>s_0$ and using estimate (4.12), one has

$$\int_{u_n>k} \frac{g(x,u_n)}{\|u_n\|_{H^1}} \le \int_{u_n>k} \frac{g(x,u_n)}{\|u_n\|_{H^1}}\frac{u_n}{k} \le \int_{u_n>s_0} \frac{g(x,u_n)}{\|u_n\|_{H^1}}\frac{u_n}{k} \le \frac{A}{k} \le \varepsilon \ . \tag{4.22}$$

Then we conclude that for $n>\bar n$

$$\int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{H^1}} \le (2+C)\varepsilon \ ; \tag{4.23}$$

by the arbitrariness of ε the claim is proved.

□

5. Claim: $\lambda>0$ implies $z_0=0$.

Proof of the claim. For any $v\in H^1$ we consider $\left|\frac{\langle F'(u_n),v\rangle}{\|u_n\|_{H^1}}\right|$:

$$\left|\int_0^1 z_n'v' - \lambda\int_0^1 z_n v - \int_0^1 \frac{g(x,u_n)v}{\|u_n\|_{H^1}} - \int_0^1 \frac{hv}{\|u_n\|_{H^1}}\right| \le \frac{\varepsilon_n\|v\|_{H^1}}{\|u_n\|_{H^1}} \ , \tag{4.24}$$

from which

$$\left|\int_0^1 z_n'v' - \lambda\int_0^1 z_n v\right| \le \int_0^1 \frac{|g(x,u_n)|}{\|u_n\|_{H^1}}|v| + \left|\int_0^1 \frac{hv}{\|u_n\|_{H^1}}\right| + \frac{\varepsilon_n\|v\|_{H^1}}{\|u_n\|_{H^1}} \ ; \tag{4.25}$$

but now the right hand side goes to zero by equation (4.19), and then we get, taking limit and using weak convergence of z_n, that

$$\int_0^1 z_0'v' - \lambda\int_0^1 z_0 v = 0 \quad for\ \ any\ \ v\in H^1 \ . \tag{4.26}$$

Since all eigenfunctions with $\lambda>0$ of the Neumann problem change sign (while $z_0\le 0$), this implies $z_0=0$. □

6. Claim: u_n is bounded.

Proof of the claim. Consider $\left|\frac{\langle F'(u_n),z_n\rangle}{\|u_n\|_{H^1}}\right|$:

$$\left|\int_0^1 (z_n')^2 - \lambda\int_0^1 z_n^2 - \int_0^1 \frac{g(x,u_n)z_n}{\|u_n\|_{H^1}} - \int_0^1 \frac{hz_n}{\|u_n\|_{H^1}}\right| \leq \frac{\varepsilon_n\|z_n\|_{H^1}}{\|u_n\|_{H^1}} \ , \tag{4.27}$$

from which

$$\int_0^1 (z_n')^2 \leq \lambda\int_0^1 z_n^2 + \int_0^1 \frac{|g(x,u_n)||z_n|}{\|u_n\|_{H^1}} + \int_0^1 \frac{hz_n}{\|u_n\|_{H^1}} + \frac{\varepsilon_n\|z_n\|_{H^1}}{\|u_n\|_{H^1}} \ . \tag{4.28}$$

Now using (4.19) and the fact that $z_n \to 0$ in L^2, (4.28) becomes

$$\int_0^1 (z_n')^2 \to 0 \ , \tag{4.29}$$

which gives contradiction since one would get $1 = \int_0^1 (z_n')^2 + \int_0^1 z_n^2 \to 0$.

□

7. The PS condition follows now with standard calculations from the boundedness of u_n.

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