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# Semilinear elliptic problems near resonance with a non-principal eigenvalue[1]

Francisco Odair de Paiva[2]
UNICAMP
Eugenio Massa[3]
ICMC-USP

Workshop in Nonlinear Differential Equations
PUC-RIO

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[3]The author was supported by Fapesp/Brazil
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# The problem

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In this work we consider the problem

$$\begin{cases} -\Delta u = \lambda u \pm f(x,u) + h(x) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1±}$$

where:

- $|f(x,t)| \leq C(1+|t|^{q-1})$ with $q \in (1,2)$,
- $h \in L^2(\Omega)$,
- $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain.
- ...more hypotheses on $f$...

---

Observe that if $\lambda \notin \sigma(-\Delta)$ at least one solution exists, moreover if $f = h = 0$ then the solution is unique (the trivial one)

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**Question:** which hypotheses to guarantee at least two solutions for $\lambda$ near to an eigenvalue $\lambda_k$? **(almost resonant problem)**
(in particular, we want conditions on $f$ only at infinity)

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# Existing literature for almost resonant problems

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- $\lambda_1$, *ODE*, bifurcation and degree.
  - J. Mawhin, K. Schmitt, (1990)
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We study the case

$\lambda_k$, *PDE*, variational techniques.

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# Some notation



$J^{\pm} : H^1_0(\Omega) \to \mathbb{R}$ :
$J(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) dx \mp \int_\Omega F(x,u)\,dx - \int_\Omega h\,u\,dx$

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$V = span\{\phi_1, \dots, \phi_{k-1}\}$ ,
$Z = span\{\phi_k, \dots, \phi_{k+m-1}\} = H_{\lambda_k}$ ,
$W = (V \oplus Z)^{\perp}$ ,

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$S_V$, $S_{VZ}$, $S_{ZW}$, the unit spheres in $V$, $V \oplus Z$, $Z \oplus W$
$B_V$, $B_{VZ}$, $B_{ZW}$, the unit balls.

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If $\lambda \notin \sigma(-\Delta)$ there exists a solution from Saddle Point Theorem.
however, a suitable behaviour of $f$ may give rise to a further solution.

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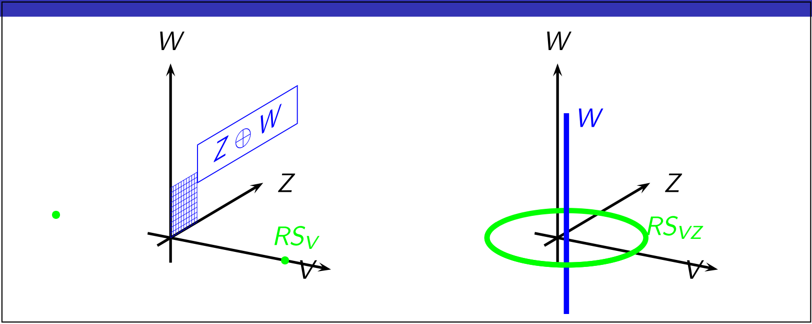
# One solution




$$(\lambda < \lambda_k) \quad c_{k-1} = \inf_{\gamma \in \Gamma_{k-1}} \sup_{v \in RB_V} J(\gamma(v))\,.$$

$$\Gamma_{k-1} = \{\gamma \in \mathcal{C}^0(RB_V; H_0^1)\ \ s.t.\ \ \gamma|_{RS_V} = Id\}\,,$$

$$(\lambda > \lambda_k) \quad c_k = \inf_{\gamma \in \Gamma_k} \sup_{v \in RB_{VZ}} J(\gamma(v))\,.$$

$$\Gamma_k = \{\gamma \in \mathcal{C}^0(RB_{VZ}; H_0^1)\ \ s.t.\ \ \gamma|_{RS_{VZ}} = Id\}\,,$$

# The main theorem: hypotheses

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**(H)** $|f(x,t)| \leq C(1+|t|^{q-1})$ with $q \in (1,2)$,
$h \in L^2(\Omega)$,
$\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain.

either **(H1)** **(f2)** : $\lim_{t\to\pm\infty} f(x,t) = \pm\infty$ uniformly $x \in \Omega$;

or **(H2)** **(f3)** : $\lim_{|t|\to\infty} F(x,t) = +\infty$ uniformly $x \in \Omega$,
**(f4)** : $F(x,t) \geq -C_F$,
**(h1)** : $\int_\Omega h\,\phi\,dx = 0 \quad \forall\,\phi \in H_{\lambda_k}$.

# The main theorem: statement



## Theorem

*Let $\lambda_k$ $(k \geq 2)$ be an eigenvalue of multiplicity $m$ and $h \in L^2(\Omega)$. Under the hypotheses (H) plus one of the sets of hypotheses (H1) or (H2), one gets:*

a) *there exists $\varepsilon_0 > 0$ such that for $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$ there exist two solutions of (1+);*

b) *there exists $\varepsilon_1 > 0$ such that for $\lambda \in (\lambda_k, \lambda_k + \varepsilon_1)$ there exist two solutions of (1–).*

---

equation $(1\pm)$: $-\Delta u = \lambda u \pm f(x,u) + h(x)$
model **(H1)**: $f(x,u) = a(x)|u|^{q-2}u$
model **(H2)**: $f(x,u) = a(x)\arctan(u)$

$$f(x,u) \sim a(x)\frac{1}{u}$$

(may be plus a lower order perturbation)

$(0 < \delta < a(x) < M$ and $q \in (1,2))$

# The case $\lambda < \lambda_k$



## Proposition

*If one of the sets of hypotheses (H1) or (H2) is satisfied, then:*

$$\exists\ D_W : \quad J^+(u) \geq D_W \qquad \text{for} \quad u \in W\,; \tag{4.1}$$

*there exist $R^+, \varepsilon_0 > 0$ such that, for any $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$*

$$J^+(u) < D_W \qquad \text{for} \quad u \in R^+ S_{VZ}\,, \tag{4.2}$$

$$\text{for} \quad u \in V\,,\ \|u\| \geq R^+\,; \tag{4.3}$$

▸ D

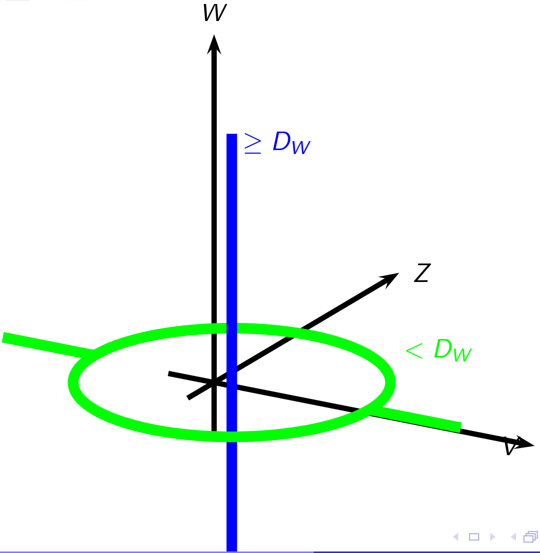
*if now we fix $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$ then*

$$\exists\ D_\lambda : \quad J^+(u) \geq D_\lambda \qquad \text{for} \quad u \in Z \oplus W\,, \tag{4.4}$$

$$\exists\ \rho_\lambda^+ > R^+ : \quad J^+(u) < D_\lambda \qquad \text{for} \quad u \in \rho_\lambda^+ S_V\,. \tag{4.5}$$

▸ D

▸ E ▸ E




We have

$$c_k \geq D_W\ ,$$

$$c_{k-1} \geq D_\lambda\ ,$$

but also

$$c_{k-1} < D_W\ ,$$

then the solutions are distinct.

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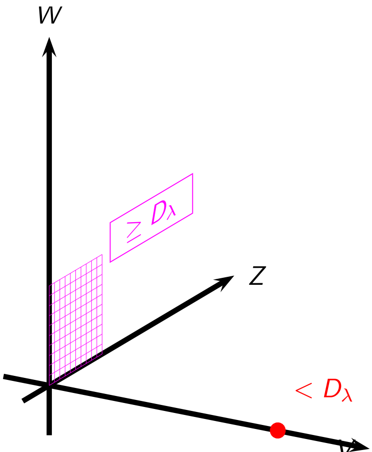


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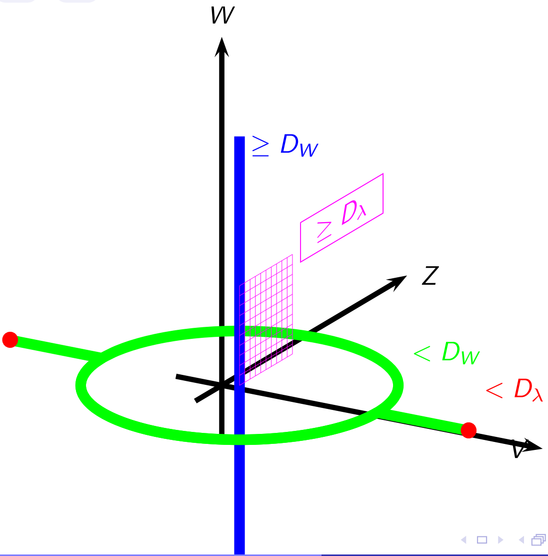


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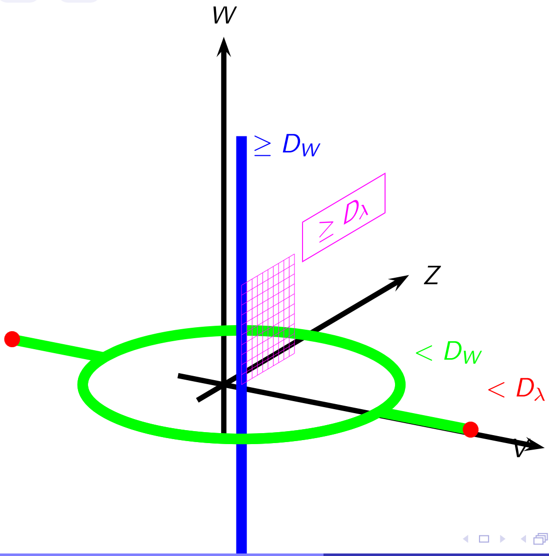


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# The case $\lambda > \lambda_k$



## Proposition

*If one of the sets of hypotheses (H1) or (H2) is satisfied, then*

$$\exists\, K_V : \quad J^-(u) < K_V \qquad \text{for} \quad u \in V\,; \tag{4.6}$$

*there exists* $R^-, \varepsilon_1 > 0$ *such that, for any* $\lambda \in (\lambda_k, \lambda_k + \varepsilon_1)$

$$J^-(u) \geq K_V \qquad \text{for} \quad u \in R^- S_{ZW}\,, \tag{4.7}$$

$$\text{for} \quad u \in W, \quad \|u\| \geq R^-\,; \tag{4.8}$$

*also* ▸ D

$$\exists\, E : \quad J^-(u) > E \qquad \text{for} \quad u \in R^- B_{ZW}\,, \tag{4.9}$$

$$\exists\, \xi : \quad J^-(u) < E \qquad \text{for} \quad u \in \xi S_V\,; \tag{4.10}$$

**Proposition (continuation)**

*if now we fix* $\lambda \in (\lambda_k, \lambda_k + \varepsilon_1)$ *then* D

$$\exists\ K_\lambda : \quad J^-(u) \geq K_\lambda \qquad for \quad u \in W\,, \tag{4.11}$$

$$\exists\ \rho_\lambda^- > R^- : \quad J^-(u) < K_\lambda \qquad for \quad u \in \rho_\lambda^- S_{VZ}\,. \tag{4.12}$$

**Theorem (From theorem 8.1 of A. Marino, A. M. Micheletti, A. Pistoia, (1994) - see also M. Frigon (1999) )**

*With the geometry given by (4.6),(4.7),(4.9),(4.10),* D
*there exists a critical point* $u_0$ *such that* $J^-(u_0) \in [E, K_V)$.

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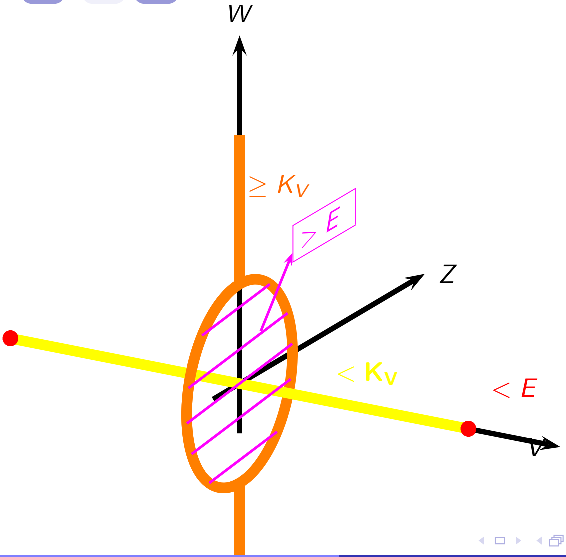


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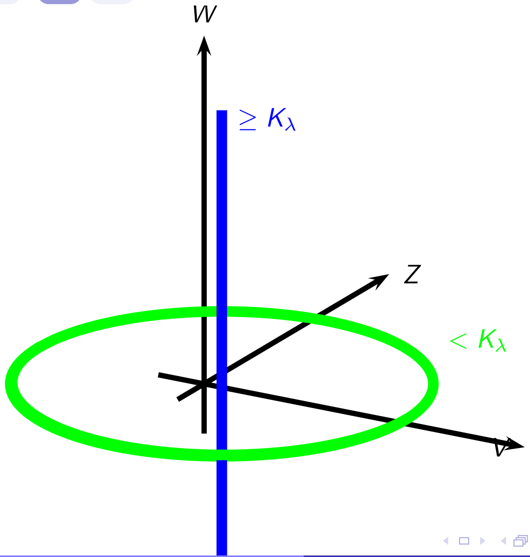


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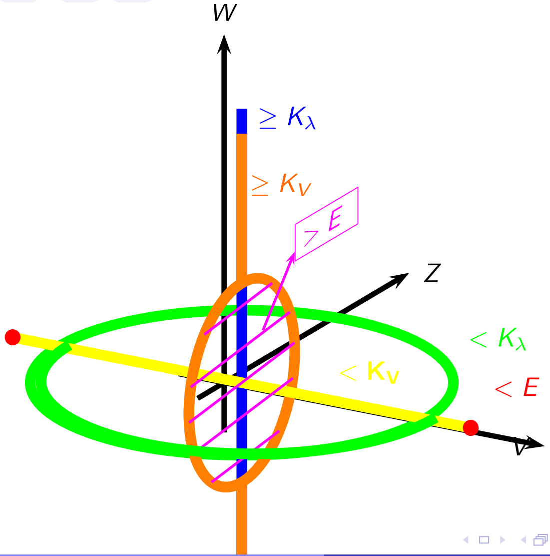


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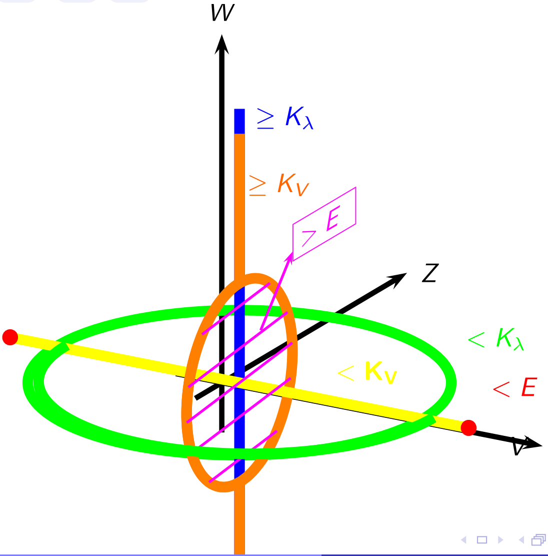


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# The main claim in the case $\lambda < \lambda_k$

## Claim

*Given $D_W \in \mathbb{R}$, there exist $R^+, \varepsilon_0 > 0$ such that, for any $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$*

$$J^+(u) < D_W \qquad \text{for} \quad u \in R^+ S_{VZ}$$

case H1

Let $2\tau = 1 - \frac{\lambda}{\lambda_k} > 0$;

---

(H1): $\lim_{t \to \pm\infty} f(x,t) = \pm\infty$
$\Rightarrow \int F(x,u) \geq M \|u\| - C_M$,

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then for $u \in V \oplus Z$, $\|u\| = R$
$J^+(u) \leq \tau R^2 - (M - \|h\|) R + C_M$

bck

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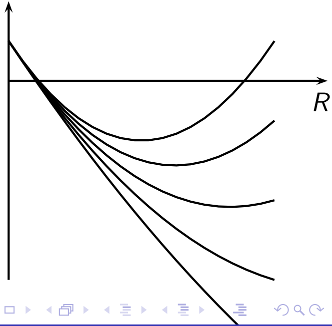

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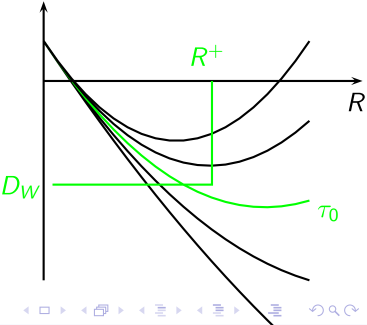

case H2: $\lim_{|t|\to\infty} F(x,t) = +\infty\,, \quad F(x,t) \geq -C_F\,,$
$\int_\Omega h\,\phi\,dx = 0 \quad \forall\,\phi \in H_{\lambda_k}.$

## Lemma

*There exists a nondecreasing function* $D : (0,+\infty) \to \mathbb{R}$ *such that* $\lim_{R\to+\infty} D(R) = +\infty$ *and* $\inf_{u\in RS_{VZ}} \int_\Omega F(x,u)dx > D(R)$

---

for $u = v + z \in V \oplus Z$, $\|u\| = R$ one gets
$J^+(u) \leq \varepsilon \|z\|^2 - \tau \|v\|^2 - \int hv - \int F(x,u)$
$\leq \varepsilon R^2 - D(R) + C$ jmp

case H2: $\lim_{|t|\to\infty} F(x,t) = +\infty\,, \quad F(x,t) \geq -C_F\,,$
$\int_\Omega h\,\phi\,dx = 0 \quad \forall\,\phi \in H_{\lambda_k}.$

### Lemma

*There exists a nondecreasing function $D:(0,+\infty)\to\mathbb{R}$ such that $\lim_{R\to+\infty} D(R) = +\infty$ and $\inf_{u\in RS_{VZ}} \int_\Omega F(x,u)dx > D(R)$*

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for $u = v + z \in V \oplus Z$, $\|u\| = R$ one gets
$J^+(u) \leq \varepsilon\,\|z\|^2 - \tau\,\|v\|^2 - \int hv - \int F(x,u)$
$\leq \varepsilon R^2 - D(R) + C$ jmp

# The main claim in the case $\lambda > \lambda_k$

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**Claim**

*Given $K_V \in \mathbb{R}$, there exist $R^-, \varepsilon_1 > 0$ such that, for any $\lambda \in (\lambda_k, \lambda_k + \varepsilon_1)$*

$$J^-(u) \geq K_V \qquad \text{for} \quad u \in R^- S_{ZW}$$

Suppose that for any two sequences $R_n > 0$ and $\varepsilon_n \to 0^+$ there exist $u_n \in Z \oplus W$ with $\|u_n\| = R_n$ such that

$$J^-_{\lambda_k+\varepsilon_n}(u_n) < K_V \tag{5.1}$$

With no loss of generality let $R_n \to +\infty$ and $\varepsilon_n R_n^2 \to 0$.
write $u_n = z_n + w_n$, divide (5.1) by $R_n^2$,
....
obtain $\frac{\|w_n\|^2}{R_n^2} \to 0$, and deduce that $\|z_n\| \to R_n$.
this implies that exists $\delta > 0$: for $n$ large $\{\left|\{x \in \Omega : |u_n(x)| > \delta R_n\}\right| > \delta$

for case H1) $\int F(x, u_n) \geq MR_n - C_M$
for case H2) $\int F(x, u_n) \to +\infty$

and the result follows....

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