Multiplicity Results for a Superlinear Elliptic System with Partial Interference with the Spectrum*

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Abstract

In this work, we consider an elliptic system of two equations in dimension greater than one, with nonlinearities which are linear at $-\infty$ and superlinear at $+\infty$.

We prove, by variational techniques which involve a strongly indefinite functional, the existence of two solutions for suitable forcing terms, under a condition on the linear part which prevents resonance with the eigenvalues of the operator.

Keyword: Elliptic systems, variational methods, linear-superlinear problems, strongly indefinite functionals.
MSC: 35J55 49J35

1 Introduction

In this work we consider the problem

$$\begin{cases} -\Delta u = au + bv + (v^+)^p + f_1 + t\phi_1 & in \quad \Omega \\ -\Delta v = cu + dv + (u^+)^q + f_2 + r\phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases}, \tag{1.1}$$

where $u^+(x) = \max\{0, u(x)\}$, $\phi_1 > 0$ is the first eigenfunction of the Laplacian with Dirichlet boundary conditions and $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$.

The nonlinearities will be assumed both superlinear and subcritical, that is, $1 < p, q < 2^* - 1$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$.

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Our concern with problem (1.1) is mainly motivated by the results in the paper [1], where the scalar equation $-\Delta u = \lambda u + (u^+)^p + f + t\phi_1$ is considered, and by the work in [2], where this result is extended to systems, but with the terms $(v^+)^p$, $(u^+)^q$ appearing in the second and first equation respectively, instead of as they appear in (1.1). This difference, as will be shown in the following, completely changes the variational setting of the problem, forcing us to work with a strongly indefinite functional.

We may write (1.1) in vectorial form as

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} (v^+)^p \\ (u^+)^q \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} t \\ r \end{bmatrix} \phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases},$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; we will assume that A has real eigenvalues $\nu_{1,2} = \frac{a+d}{2} \pm \sqrt{\left(\frac{a-d}{2}\right)^2 + bc}$.

Throughout the paper, we will denote by $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_i \leq ...$ the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and by $\{\phi_i\}_{i\in\mathbb{N}}$ the corresponding eigenfunctions, taken orthogonal and normalized with $\|\phi_i\|_{L^2} = 1$ and $\phi_1 > 0$; by $\sigma(-\Delta)$ we will denote the spectrum of the Laplacian, that is, the set $\{\lambda_i : \ i \in \mathbb{N}\}$.

Our results are

Theorem 1.1. *If A has real eigenvalues $\nu_{1,2} \notin \sigma(-\Delta)$ and $f_{1,2} \in L^s(\Omega)$ with $s > N \geq 2$, then there exists $(t_0, r_0) \in \mathbb{R}^2$ such that if $(t, r)^t = (t_0, r_0)^t + (\lambda_1 I - A)(\tau, \rho)^t$ with $\tau, \rho < 0$, then a negative solution (u_{neg}, v_{neg}) of problem (1.1) exists.*

For negative solution we intend that both $u_{neg}, v_{neg} < 0$ in Ω.

Theorem 1.2. *In the same hypotheses of theorem 1.1, if moreover $a = d$, then for the same vectors $(t, r) \in \mathbb{R}^2$, a second solution exists.*

Remark 1.3. *The assumption $a = d$ in the last theorem is required in order to work with variational techniques; observe that in this case the remaining hypotheses on A may be written as*

$$bc \geq 0 \quad and \quad a \pm \sqrt{bc} \notin \sigma(-\Delta)\,.$$

1.1 Literature and techniques

The scalar counterpart of problem (1.1) is

$$\begin{cases} -\Delta u = \lambda u + (u^+)^p + f + t\phi_1 & in \ \Omega \\ u = 0 & on \ \partial\Omega \end{cases}, \qquad (1.2)$$

and it has been considered in many works.

For $\lambda < \lambda_1$ it is the so called Ambrosetti-Prodi problem (first considered in [3]) and it has zero, at least one or at least two solutions, depending on the forcing term; in particular, it has two solutions for large negative values of t and no solution for large positive values.

For $\lambda > \lambda_1$, we already cited the work [1], where it is proven that for a continuous f there exist at least two solutions for large positive values of t, provided λ is not an eigenvalue. In [4], an analogous result is proven for a larger class of nonlinearities.

It is also interesting to cite the result in [5], where the case in dimension one is considered, and it turns out that if $\lambda \in (\lambda_k, \lambda_{k+1})$, then at least $2k + 2$ solutions exist for $t \geq 0$.

Our theorems 1.1 and 1.2 look to be the equivalent of the results in [1], in the sense that, provided no resonance occurs with the eigenvalues of the operator, we find a region for the parameters (t, r) for which two solutions exist, one being negative.

For the case of a system with the nonlinearities $(v^+)^p$, $(u^+)^q$ appearing in the second and first equation respectively, a result of existence of two solutions for suitable values of the parameters t, r was obtained in [2]; the main condition on the eigenvalues of the matrix A was $\nu_{1,2} < \lambda_1$ or $\nu_{1,2} \in (\lambda_k, \lambda_{k+1})$.

The proof of the theorem 1.1 will be relatively simple, since for negative solutions system (1.1) turns out to be a linear problem.

The theorem 1.2 will be proved by finding a critical point of a suitable functional defined in section 3.2.

The techniques we will use are inspired by those in [1]. However, we will need to adapt them to the characteristics of the functional (3.3) and of its variational setting. For this purpose we refer to [6, 7] and we make use of the minimax theorem for strongly indefinite functionals in [8]. Actually, one important characteristic of systems like (1.1) is that, in order to treat them variationally, one is led to work with a strongly indefinite functional, in the sense that there exist two infinite dimensional subspaces of its space of definition such that it is unbounded from above in one and from below in the other (see in the lemma 3.4).

2 The negative solution

In this section, we will look for negative solutions, in the sense that both components are negative: this is relatively simple since in this case the nonlinear term disappears in (1.1).

We will need the following

Lemma 2.1. *If A has real eigenvalues $\nu_{1,2} \notin \sigma(-\Delta)$ and $f_{1,2} \in L^s(\Omega)$ with*

$s > N$, then there exists a unique solution (u_0, v_0) of the problem

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases} . \tag{2.1}$$

Moreover, we have $u_0, v_0 \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ for $\alpha < 1 - N/s$.

Proof. Since A has real eigenvalues it may be reduced to its Jordan form, this means that the equations uncouple sequentially and reduce to scalar equations, which are uniquely solvable provided each eigenvalue of A is not in $\sigma(-\Delta)$.

The hypothesis $f_{1,2} \in L^s(\Omega)$ implies, by regularity theory, that $u_0, v_0 \in W^{2,s}(\Omega) \subseteq \mathcal{C}^{1,\alpha}(\overline{\Omega})$ for $\alpha < 1 - N/s$. □ □

With this result we may obtain the negative solution:

Proof of theorem 1.1. Given $f_{1,2}$, let (u_0, v_0) be the corresponding solution for (2.1) and consider the problem

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} t \\ r \end{bmatrix} \phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases} : \tag{2.2}$$

by looking for a solution of the form $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \phi_1$ one obtains $(\lambda_1 I - A) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} t \\ r \end{bmatrix}$, and then, by superposition principle,

$$(\lambda_1 I - A)^{-1} \begin{bmatrix} t \\ r \end{bmatrix} \phi_1 + \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \tag{2.3}$$

is a solution of (1.1), provided it is non positive.

Since $u_0, v_0 \in \mathcal{C}^{1,\alpha}$, there exist finite

$$\alpha_0 = \sup \{\alpha : \ \alpha\phi_1 + u_0 < 0\}, \qquad \beta_0 = \sup \{\beta : \ \beta\phi_1 + v_0 < 0\} ;$$

so (2.3) is negative provided $\alpha < \alpha_0$ and $\beta < \beta_0$, that is,

$$\begin{bmatrix} t \\ r \end{bmatrix} = (\lambda_1 I - A) \left(\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} + \begin{bmatrix} \alpha - \alpha_0 \\ \beta - \beta_0 \end{bmatrix} \right) :$$

by setting $(t_0, r_0)^t = (\lambda_1 I - A)(\alpha_0, \beta_0)^t$, $\tau = \alpha - \alpha_0 < 0$ and $\rho = \beta - \beta_0 < 0$, we get the conditions in the claim. □ □

3 The second solution

From now on we want to apply variational techniques, for this we will be forced to assume $a = d$ in the matrix A. In this case the condition for having real eigenvalues reads $bc \geq 0$.

In order to simplify the computations of the proofs, we will exploit the following lemma (whose proof is simply a change of unknowns).

Lemma 3.1. *If (U,V) is a solution of*

$$\begin{cases} -\Delta U = aU + bV/\delta + (V^+)^p/\delta + (f_1 + t\phi_1)/\delta & in \quad \Omega \\ -\Delta V = c\delta U + dV + \delta^q (U^+)^q + (f_2 + r\phi_1) & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases} \tag{3.1}$$

with $\delta > 0$, then $(u,v) = (\delta U, V)$ is a solution of (1.1).

In view of this lemma, whenever $b, c \neq 0$, we will consider system (3.1) with the choice $\delta = \sqrt{b/c}$, so that the two out of diagonal coefficients become equal to $\sqrt{bc}$, a rescaling occurs for the forcing term of the first equation, and two new positive constants appear as coefficients of the nonlinear terms; also the case $b = c = 0$ will fit in the following proofs without any need of the lemma 3.1, while in section 3.5 we will consider the remaining case in which one (and only one) of the two coefficients b, c is zero.

Thus, we consider from now on the system

$$\begin{cases} -\Delta u = au + bv + C_1(v^+)^p + f_1 + t\phi_1 & in \quad \Omega \\ -\Delta v = bu + av + C_2(u^+)^q + f_2 + r\phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases}, \tag{3.2}$$

where C_1, C_2 are two positive constants.

We will prove the following proposition:

Proposition 3.2. *Let $a \pm b \notin \sigma(-\Delta)$, $f_{1,2} \in L^s(\Omega)$ with $s > N \geq 2$ and (t,r) as in the theorem 1.1; then there exists a second solution for system (3.2).*

3.1 The minimax theorem

As anticipated in the introduction, we will find the second solution by using a minimax theorem due to Felmer [8]. For sake of completeness, we report here this theorem, in a form adapted to our setting.

Theorem 3.3. *Let $E = X \oplus Y$ be a Hilbert space and $F : E \to \mathbb{R}$ be a $\mathcal{C}^1$ functional satisfying the Palais-Smale (PS) condition and having the structure $F(u) = \langle L\,u, u \rangle_E + b(u)$ where*
1) $L : E \to E$ is a linear, bounded and self-adjoint operator;
2) $b' : E \to E$ is a compact operator;
3) the linear operator $P_X \exp(\mu L) : X \to X$ is invertible for any $\mu > 0$.

Suppose also that we have $M > \rho > 0$, $R > 0$ and $z \in Y$ with $\|z\|_E = 1$ such that
i) $F(u) \geq \xi > 0$ for $u \in S = \{u : \ u \in Y, \ \|u\|_E = \rho\}$;
ii) $F(u) \leq 0$ for $u \in \partial Q$ where
$Q = \{u = w + s\,z : \ w \in X\,, \ \|w\|_E \leq R, \ 0 \leq s \leq M\}$.
Then F possesses a critical point with critical value $e \geq \xi$.

The purpose of the following sections will be to provide the structure required for the application of this theorem.

3.2 The variational structure

We consider the (Hilbert) space $E = H_0^1 \times H_0^1$ equipped with the scalar product

$$\langle\, (u,v),(w,z)\,\rangle_E = \int \nabla u\ \nabla w + \int \nabla v\ \nabla z\,,$$

the related norm $\|(u,v)\|_E$ and the bounded symmetric quadratic form

$$B((u,v),(w,z)) = \int_\Omega \nabla u \nabla z + \nabla v \nabla w - a(uz+vw) - b(uw+vz)\,.$$

Let (t,r) be as in theorem 1.1 and (u_{neg}, v_{neg}) be the corresponding negative solution for (3.2); then it is simple to see that the functional

$$\begin{aligned} F : E \to \mathbb{R} : \mathbf{u} = (u,v) \mapsto F(\mathbf{u}) &= \frac{1}{2}B(\mathbf{u},\mathbf{u}) - H(\mathbf{u}) = \\ &= \int_\Omega \nabla u\ \nabla v - \frac{1}{2}\int_\Omega \big(b(v^2+u^2) + 2auv\big) + \\ &\quad - C_1 \int_\Omega \frac{[(v+v_{neg})^+]^{p+1}}{p+1} - C_2 \int_\Omega \frac{[(u+u_{neg})^+]^{q+1}}{q+1} \end{aligned} \tag{3.3}$$

is $\mathcal{C}^1(E;\mathbb{R})$ and its critical points (u,v) are such that $(u+u_{neg}, v+v_{neg})$ are solutions of (3.2), in particular, the origin is a critical point at level zero and corresponds to the already found negative solution.

In order to find a orthogonal base for E which diagonalizes B, we consider, in a way similar to what was done in [7], the eigenvalue problem

$$(u,v) \in E\ : \qquad B((u,v),(\phi,\psi)) = \mu\langle\, (u,v),(\phi,\psi)\,\rangle_E \quad \forall (\phi,\psi) \in E\ :$$

this gives (use $(0,\phi_i)$ and $(\phi_i,0)$ as test functions and let u_i, v_i be the Fourier's coefficients for u and v)

$$\begin{bmatrix} a-\lambda_i & \mu\lambda_i + b \\ \mu\lambda_i + b & a-\lambda_i \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = 0 \qquad (i \in \mathbb{N})\,, \tag{3.4}$$

so we get nontrivial solutions when μ is such that the determinant of the above matrix is zero for some $i \in \mathbb{N}$. This gives $(\lambda_i - a)^2 - (b+\lambda_i\mu)^2 = 0$ and so

$$\mu_{\pm i} = \frac{-b \pm (\lambda_i - a)}{\lambda_i} \qquad (i \in \mathbb{N})\,;$$

from (3.4) we also get the related eigenvectors $\Psi_{\pm i}$, which we choose such that $\|\Psi_{\pm i}\|_E = 1$:

$$\Psi_{\pm i} = \frac{(\phi_i, \pm\phi_i)}{\sqrt{2\lambda_i}} \qquad (i \in \mathbb{N})\,;$$

observe that in the case $a = \lambda_j$, if it occurs, we get the double eigenvalue $-b/a$ and we may still choose as eigenvectors $\Psi_{\pm j} = \frac{(\phi_j, \pm\phi_j)}{\sqrt{2\lambda_j}}$.

With this structure we have

$$\langle \Psi_i, \Psi_j \rangle_E = \delta_{i,j}\,, \quad B(\Psi_i, \Psi_j) = \mu_i \delta_{i,j}\,, \quad \langle \Psi_i, \Psi_j \rangle_{[L^2]^2} = \lambda_i^{-1}\delta_{i,j} \quad (i,j \in \mathbb{Z}^0)\,, \tag{3.5}$$

so if we write $(u,v) = \sum_{i\in\mathbb{Z}^0} c_i \Psi_i$, we get

$$\|(u,v)\|_E^2 = \sum_{i\in\mathbb{Z}^0} c_i^2\,, \quad B((u,v),(u,v)) = \sum_{i\in\mathbb{Z}^0} \mu_i c_i^2\,, \quad \|(u,v)\|_{[L^2]^2}^2 = \sum_{i\in\mathbb{Z}^0} \lambda_i^{-1} c_i^2\,. \tag{3.6}$$

In view of this structure we may define

$$\begin{aligned} E^+ &= \overline{span\,\{\Psi_i : \ \mu_i > 0,\ i \in \mathbb{Z}^0\}}\,, \\ E^- &= \overline{span\,\{\Psi_i : \ \mu_i < 0,\ i \in \mathbb{Z}^0\}}\,, \\ E^0 &= span\left\{\Psi_i : \ \mu_i = 0,\ i \in \mathbb{Z}^0\right\}, \end{aligned}$$

and we have

Lemma 3.4. *There exists $\xi^* > 0$ such that*

$$B(\mathbf{u},\mathbf{u}) \geq 2\xi^* \|\mathbf{u}\|_E^2 \qquad \text{for } \mathbf{u} \in E^+\,, \tag{3.7}$$

$$B(\mathbf{u},\mathbf{u}) \leq -2\xi^* \|\mathbf{u}\|_E^2 \qquad \text{for } \mathbf{u} \in E^-\,. \tag{3.8}$$

Moreover, if $a \pm b \notin \sigma(-\Delta)$, then $E^0 = \{0\}$.

Proof. The claim is satisfied by setting

$$2\xi^* := \inf\left\{|\mu_i| : \quad |\mu_i| > 0,\ i \in \mathbb{Z}^0\right\} \ :$$

actually this is strictly positive since $\lim_{i\to\pm\infty} \mu_i = \pm 1$.
The condition $a \pm b \notin \sigma(-\Delta)$ implies $\mu_i \neq 0$ for any $i \in \mathbb{Z}^0$. □ □

For later use, we also define $\widetilde{n}$ such that for $i \geq \widetilde{n}$ we have $\lambda_i - a > |b|$ and

$$E_h = \overline{span\,\{\Psi_i : \ |i| \geq \widetilde{n},\ i \in \mathbb{Z}^0\}}\,, \qquad E_l = span\left\{\Psi_i : \ |i| < \widetilde{n},\ i \in \mathbb{Z}^0\right\} \ : \tag{3.9}$$

we have the following

Lemma 3.5. *$(u,v) \in E^+ \cap E_h$ implies $u = v$ and $(u,v) \in E^- \cap E_h$ implies $u = -v$.*

Proof. It follows readily from the fact that for $i \geq \widetilde{n}$ we have $\pm\mu_{\pm i} > 0$ and that $\Psi_{\pm i} = \frac{(\phi_i, \pm\phi_i)}{\sqrt{2\lambda_i}}$. □ □

3.3 Estimates for the linking structure

In this section we will prove the estimates we need in order to apply the minimax theorem 3.3.

Lemma 3.6. *There exist $\rho, \xi > 0$ such that*

$$F(\mathbf{u}) \geq \xi \quad \textit{for } \mathbf{u} = (u, v) \in E^+ \textit{ and } \|\mathbf{u}\|_E = \rho\,.$$

Proof. Let $\mathbf{u}$ be as above: by (3.7) and the continuous embedding of H_0^1 in L^{q+1} and L^{p+1} we get

$$\begin{aligned} F(\mathbf{u}) &= \frac{1}{2}B(\mathbf{u},\mathbf{u}) - C_1 \int_\Omega \frac{[(v+v_{neg})^+]^{p+1}}{p+1} - C_2 \int_\Omega \frac{[(u+u_{neg})^+]^{q+1}}{q+1} \\ &\geq \xi^* \|\mathbf{u}\|_E^2 - C_1 \int_\Omega \frac{(|v|)^{p+1}}{p+1} - C_2 \int_\Omega \frac{(|u|)^{q+1}}{q+1} \\ &\geq \xi^*(\|u\|_{H_0^1}^2 + \|v\|_{H_0^1}^2) - C(\|v\|_{H_0^1}^{p+1} + \|u\|_{H_0^1}^{q+1}) \\ &\geq \|u\|_{H_0^1}^2 (\xi^* - C\rho^{q-1}) + \|v\|_{H_0^1}^2 (\xi^* - C\rho^{p-1}) \end{aligned} \tag{3.10}$$

where C is a positive constant. Since $p, q > 1$, for $\rho, \xi > 0$ small enough we obtain the claim. □ □

Lemma 3.7. *There exists $\mathbf{g} = (g, g) \in E^+ \bigcap E_h$ with $\|\mathbf{g}\|_E = 1$ and $\|g^+\|_{L^\infty} = +\infty$.*

Proof. Since H_0^1 is not embedded in L^∞ (here is where we need the condition $N \geq 2$), there exists $u \in H_0^1$ such that $\|u^+\|_{L^\infty} = +\infty$; by removing the components of u in the directions of the eigenvectors ϕ_i with $i < \widetilde{n}$ we maintain this property since we simply subtract a finite linear combination of regular functions, so we may assume that such components are zero.

Now we have that $(u, u) \in E^+ \bigcap E_h$ since, for $i \geq \widetilde{n}$, $\Psi_i = (\phi_i, \phi_i)/\sqrt{2\lambda_i}$ and $\mu_i > 0$.

Finally, we obtain $\|(g, g)\|_E = 1$ by a suitable rescaling of (u, u). □ □

Lemma 3.8. *Let $\mathbf{g} = (g, g)$ as in the lemma above.*

Then there exist $R, \theta > 0$ with $R\theta > \rho$ such that $F(\mathbf{u}) \leq 0$ for:

a) $(u, v) \in E^-$,

b) $(u, v) = \mathbf{u} = \mathbf{w} + s\mathbf{g}$*:* $\mathbf{w} \in E^-$, $\|\mathbf{w}\|_E = R$, $0 \leq s \leq \theta R$,

c) $(u, v) = \mathbf{u} = \mathbf{w} + s\mathbf{g}$*:* $\mathbf{w} \in E^-$, $\|\mathbf{w}\|_E \leq R$, $s = \theta R$,

Proof.

a) Let $\mathbf{u} \in E^-$: by equation (3.8)

$$\begin{aligned} F(\mathbf{u}) &= \frac{1}{2}B(\mathbf{u},\mathbf{u}) - C_1 \int_\Omega \frac{[(v+v_{neg})^+]^{p+1}}{p+1} - C_2 \int_\Omega \frac{[(u+u_{neg})^+]^{q+1}}{q+1} \\ &\leq -\xi^* \|\mathbf{u}\|_E^2 \leq 0\,. \end{aligned} \tag{3.11}$$

b) Let $\mathbf{w} = (w, z) \in E^-$ with $\|\mathbf{w}\|_E = R$ and $0 \le s \le \theta R$: observe that $\mathbf{g}$ is orthogonal to $\mathbf{w}$, that is, $\langle \mathbf{w}, \mathbf{g} \rangle_E = 0 = B(\mathbf{w}, \mathbf{g})$; then we estimate, by using again (3.8),

$$\begin{aligned} F(\mathbf{u}) &= \frac{1}{2} B(\mathbf{u}, \mathbf{u}) - C_1 \int_\Omega \frac{[(v + v_{neg})^+]^{p+1}}{p+1} - C_2 \int_\Omega \frac{[(u + u_{neg})^+]^{q+1}}{q+1} \\ &\le \frac{1}{2} B(\mathbf{w} + s\mathbf{g}, \mathbf{w} + s\mathbf{g}) \quad = \quad \frac{1}{2}(B(\mathbf{w}, \mathbf{w}) + s^2 B(\mathbf{g}, \mathbf{g})) \\ &\le -\xi^* \|\mathbf{w}\|_E^2 + \frac{1}{2} s^2 B(\mathbf{g}, \mathbf{g}) \,. \end{aligned} \tag{3.12}$$

From this we may conclude (let $B_g := \frac{1}{2} B(\mathbf{g}, \mathbf{g})$ and observe that it is positive by equation (3.7))

$$F(\mathbf{u}) \quad \le \quad R^2 (-\xi^* + \theta^2 B_g) \,. \tag{3.13}$$

By fixing $\theta < \sqrt{\xi^*/B_g}$, such that the last term is negative, the claim b) is proved.

c) Consider now $\|\mathbf{w}\|_E \le R$, $s = \theta R$, and let

$$P_l \mathbf{w} = (\sigma_1, \sigma_2) \,, \qquad P_h \mathbf{w} = (\delta_1, \delta_2) \,,$$

where P_l, P_h are the orthogonal projections onto the spaces E_l and E_h respectively (see the definition in (3.9)). In this way, $P_h \mathbf{w} \in E^- \bigcap E_h$ and then it is of the form $P_h \mathbf{w} = (\delta_1, -\delta_1)$, by lemma 3.5.

Write now

$$\int_\Omega \left[(z + \theta R\, g + v_{neg})^+\right]^{p+1} = R^{p+1} \int_\Omega \left[\left(\frac{\sigma_2 - \delta_1 + v_{neg}}{R} + \theta g \right)^+ \right]^{p+1} \tag{3.14}$$

$$\int_\Omega \left[(w + \theta R\, g + u_{neg})^+\right]^{q+1} = R^{q+1} \int_\Omega \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g \right)^+ \right]^{q+1} \tag{3.15}$$

since u_{neg}, v_{neg} are fixed and bounded, and σ_1, σ_2 are a linear combination of a finite number of eigenvectors of the Laplacian, there exists a constant C such that

$$|u_{neg}|, |v_{neg}| < C/2 \qquad \text{and} \qquad |\sigma_1|, |\sigma_2| < C \|w\|_E /2 \le CR/2 \,;$$

so, for $R > 1$,

$$\frac{|\sigma_1 + u_{neg}|}{R}, \frac{|\sigma_2 + v_{neg}|}{R} < C \,.$$

Moreover, since $\mathbf{g}$ and θ have already been fixed and $\|g^+\|_{L^\infty} = \infty$, we know that

$$\Omega^* = \{x \in \Omega : \theta g > C + 1\}$$

has positive measure; we observe that $\theta g > C+1$ implies

$$\max\{\theta g \pm \delta_1/R\} > C+1$$

for any function δ_1 and any $R \in \mathbb{R}$: then $\Omega^* \subseteq \Omega^*_+ \cup \Omega^*_-$, where

$$\Omega^*_\pm = \{x \in \Omega : \theta g \pm \delta_1/R > C+1\}$$

(observe that both $\Omega^*_\pm$ depend on $\mathbf{w}$ and R, but Ω^* does not).
Then either $|\Omega^*_-| \geq |\Omega^*|/2$ or $|\Omega^*_+| \geq |\Omega^*|/2$ and, as a consequence, for any $\mathbf{w}$ as assumed and $R > 1$, one of the following two inequalities hold:

$$\int_\Omega \left[\left(\frac{\sigma_2 - \delta_1 + v_{neg}}{R} + \theta g\right)^+\right]^{p+1} \geq |\Omega^*|/2 \tag{3.16}$$

$$\int_\Omega \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g\right)^+\right]^{q+1} \geq |\Omega^*|/2\,. \tag{3.17}$$

We conclude from (3.14-3.15) and (3.16-3.17) that

$$-C_1\int_\Omega \frac{[(v+v_{neg})^+]^{p+1}}{p+1} - C_2\int_\Omega \frac{[(u+u_{neg})^+]^{q+1}}{q+1} \leq -\widetilde{C}\, R^{\min\{p,q\}+1}\,,$$

where now $\widetilde{C} > 0$ does not depend on $R, \mathbf{w}$.
Finally, by estimating the first terms as in point b), we get

$$\begin{aligned} F(\mathbf{u}) &\leq \frac{1}{2}B(\mathbf{w}+\theta R\mathbf{g}, \mathbf{w}+\theta R\mathbf{g}) - \widetilde{C}\, R^{\min\{p,q\}+1} \\ &\leq -\xi^*\,\|\mathbf{w}\|_E^2 + \frac{1}{2}\theta^2 R^2 B(\mathbf{g},\mathbf{g}) - \widetilde{C}\, R^{\min\{p,q\}+1} \\ &\leq R^2\left(\theta^2 B_g - \widetilde{C} R^{\min\{p,q\}-1}\right)\ : \end{aligned} \tag{3.18}$$

since $p, q > 1$, we may choose $R > 1$ (and also $R > \rho/\theta$) large enough to make the last expression negative; this concludes the proof of the claim c).

□ □

3.4 The second solution through the minimax theorem

Now, we may prove

Proposition 3.9. *There exists a critical point $\mathbf{u} \in E$ for the functional F, with $F(\mathbf{u}) \geq \xi > 0$ (and then $\mathbf{u} \neq (0,0)$, so that it is a second solution).*

Proof. It is a consequence of the minimax theorem 3.3, by using the estimates in the lemmas 3.6 and 3.8 and the PS condition in the lemma 4.1.

Actually, we set

$$S = \left\{\mathbf{u} : \mathbf{u} \in E^+, \|\mathbf{u}\|_E = \rho\right\} ,$$

$$Q = \left\{\mathbf{u} = \mathbf{w} + s\mathbf{g} : \ \mathbf{w} \in E^- , \ \|\mathbf{w}\|_E \le R, \ 0 \le s \le \theta R\right\} ,$$

and then we just need to check that also the hypotheses on the form of the functional that are required in [8] are satisfied. This is the case if we set $L(u, v) = \frac{1}{2}(v, u)$: this is linear, bounded, self-adjoint and $P^- \exp(\mu L) : E^- \to E^-$ is an invertible linear operator for any $\mu > 0$: actually, it is simple to check that L is diagonal with respect to the base we are considering, in fact,

$$L\,\Psi_{\pm i} = \pm\frac{1}{2}\Psi_{\pm i} \qquad (i \in \mathbb{N})\,;$$

then $\exp(\mu L)$ is diagonal too and takes the form

$$\exp(\mu L)\Psi_{\pm i} = \exp(\pm\mu/2)\Psi_{\pm i} \qquad (i \in \mathbb{N})\,,$$

which shows that it maps E^- onto itself and is invertible on it. □ □

Now, proposition 3.9 implies proposition 3.2.

3.5 The case $b = 0$ (or $c = 0$)

In order to complete the proof of theorem 1.2, we still need to consider the case in which exactly one of the diagonal terms of the matrix A in system (1.1) is zero (that is, when its eigenvalues coincide and are equal to a, but A is not diagonal).

In this case lemma 3.1 allows us to choose the nonzero parameter (say b) as small as desired. We will first forget about the term $\int_\Omega bv^2$, that is we consider

$$B((u, v), (w, z)) = \int_\Omega \nabla u \nabla z + \nabla v \nabla w - a(uz + vw) \tag{3.19}$$

and we make the same construction as in section 3.2 in order to diagonalize B; then we obtain the corresponding value ξ^* of lemma 3.4 and, by virtue of lemma 3.1, we may consider the equivalent system

$$\begin{cases} -\Delta u = au + bv + C_1(v^+)^p + f_1 + t\phi_1 & in \quad \Omega \\ -\Delta v = 0 + av + C_2(u^+)^q + f_2 + r\phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases} , \tag{3.20}$$

where we choose the rescaling parameter δ such that $b = \xi^*\lambda_1$.

We will prove

Proposition 3.10. *Let $a \notin \sigma(-\Delta)$, b as fixed above, $f_{1,2} \in L^s(\Omega)$ with $s > N \ge 2$, and (t, r) as in the theorem 1.1, then there exists a second solution for system (3.20).*

Proof. Actually, a solution of (3.20) will be a critical point for the functional

$$F : E \to \mathbb{R} : \mathbf{u} = (u, v) \mapsto F(\mathbf{u}) = \frac{1}{2}B(\mathbf{u}, \mathbf{u}) - \frac{1}{2}\int_\Omega bv^2 - H(\mathbf{u}) =$$
$$= \int_\Omega \nabla u \ \nabla v - \frac{1}{2}\int_\Omega \left(bv^2 + 2auv\right) +$$
$$- C_1 \int_\Omega \frac{[(v + v_{neg})^+]^{p+1}}{p+1} - C_2 \int_\Omega \frac{[(u + u_{neg})^+]^{q+1}}{q+1} \,, \quad (3.21)$$

where the term $\frac{1}{2}\int_\Omega bv^2$ will be considered as a (small) perturbation which we estimate as

$$\left|\int_\Omega bv^2\right| \le \xi^* \lambda_1 \|v\|^2_{L^2} \le \xi^* \|v\|^2_{H^1_0} \le \xi^* \|(u, v)\|^2_E \ :$$

then we obtain the new version of the estimates (3.7-3.8)

$$B(\mathbf{u}, \mathbf{u}) - \int_\Omega bv^2 \ge \xi^* \|\mathbf{u}\|^2_E \qquad \text{for } \mathbf{u} = (u, v) \in E^+ \,, \qquad (3.22)$$

$$B(\mathbf{u}, \mathbf{u}) - \int_\Omega bv^2 \le -\xi^* \|\mathbf{u}\|^2_E \qquad \text{for } \mathbf{u} = (u, v) \in E^- \,, \qquad (3.23)$$

which allow us to use the same arguments as in section 3.3 for this case, and then to obtain a solution for system (3.20) through the minimax theorem 3.3.
□ □

Finally, we may conclude the proof of the theorem 1.2:

Proof of theorem 1.2. The propositions 3.2 and 3.10 imply the theorem 1.2 by virtue of the lemma 3.1. □ □

4 The PS conditions

In this section we will prove the PS conditions, which was required for the application of theorem 3.3.

Lemma 4.1 (PS condition)**.** *Under the considered hypotheses, the functional F satisfies the PS condition, that is, let ε_n be a sequence of positive reals converging to zero and $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subseteq E$ be such that*

$$|F(\mathbf{u}_n)| \le T \,, \qquad (4.1)$$

$$|F'(\mathbf{u}_n)[\phi, \psi]| \le \varepsilon_n \|(\phi, \psi)\|_E \quad \forall (\phi, \psi) \in E \ : \qquad (4.2)$$

then $\{\mathbf{u}_n\}$ admits a convergent subsequence.

Proof. Equations (4.1-4.2) read

$$|F(\mathbf{u}_n)| = \\ = \left| \frac{1}{2}\widehat{B}(\mathbf{u}_n,\mathbf{u}_n) - C_1 \int_\Omega \frac{[(v_n+v_{neg})^+]^{p+1}}{p+1} - C_2 \int_\Omega \frac{[(u_n+u_{neg})^+]^{q+1}}{q+1} \right| \le T\,, \tag{4.3}$$

$$F'(\mathbf{u}_n)[\phi,\psi] = \widehat{B}(\mathbf{u}_n,(\phi,\psi)) + -C_1 \int_\Omega \left[(v_n+v_{neg})^+\right]^p \psi + \\ - C_2 \int_\Omega \left[(u_n+u_{neg})^+\right]^q \phi \le \varepsilon_n \left\|(\phi,\psi)\right\|_E \quad \forall(\phi,\psi)\in E\ , \tag{4.4}$$

where we are denoting by $\widehat{B}$ the form B plus, in the case of functional (3.21), the term $\int_\Omega bv\psi$; also, since this does not affect the proof at all, we will assume $C_1 = C_2 = 1$.

First, we want to prove that $\|\mathbf{u}_n\|_E$ is bounded; so we consider for sake of contradiction a subsequence such that $\|\mathbf{u}_n\|_E \to \infty$ and we define $(U_n,V_n) = \frac{(u_n,v_n)}{\|\mathbf{u}_n\|_E}$, so that (up to a further subsequence) $(U_n,V_n) \to (U,V)$ weakly in E and strongly in $[L^r]^2$ for $r < 2^*$.

Now observe that

$$\int_\Omega \left[(v_n+v_{neg})^+\right]^p v_n = \int_\Omega \left[(v_n+v_{neg})^+\right]^{p+1} + \left[(v_n+v_{neg})^+\right]^p (-v_{neg})$$

(and an analogous relation holds for the term in u_n); then, by considering $F(\mathbf{u}_n) - \frac{1}{2}F'(\mathbf{u}_n)[\mathbf{u}_n]$, we get

$$\left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega \left[(v_n+v_{neg})^+\right]^{p+1} + \\ + \left(\frac{1}{2} - \frac{1}{q+1}\right)\int_\Omega \left[(u_n+u_{neg})^+\right]^{q+1} + \frac{1}{2}\int_\Omega \left[(v_n+v_{neg})^+\right]^p (-v_{neg}) + \\ + \frac{1}{2}\int_\Omega \left[(u_n+u_{neg})^+\right]^q (-u_{neg}) \le T + \varepsilon_n \left\|\mathbf{u}_n\right\|_E\ ; \tag{4.5}$$

by observing that each term in the expression above is nonnegative, we conclude that the estimate from above holds for each of them, and then

$$\frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega \left[(v_n+v_{neg})^+\right]^{p+1} \to 0\ , \quad \frac{1}{\|\mathbf{u}_n\|_E}\int_\Omega \left[(u_n+u_{neg})^+\right]^{q+1} \to 0\,. \tag{4.6}$$

For any $(\phi,\psi)\in E$ we get, by considering $\frac{F'(\mathbf{u}_n)[\phi,\psi]}{\|\mathbf{u}_n\|_E}$,

$$\widehat{B}((U_n,V_n),(\phi,\psi)) - \int_\Omega \frac{[(v_n+v_{neg})^+]^p}{\|\mathbf{u}_n\|_E}\psi - \int_\Omega \frac{[(u_n+u_{neg})^+]^q}{\|\mathbf{u}_n\|_E}\phi \to 0 \tag{4.7}$$

which, by using the weak convergence of (U_n, V_n) and (4.6), implies that

$$\widehat{B}((U,V),(\phi,\psi)) = 0\,; \tag{4.8}$$

this means that (U, V) is a solution of $-\Delta(U,V)^t = A(U,V)^t$, but then it is zero by our assumptions on the eigenvalues of the matrix A (see the lemma 2.1).

Now consider $\frac{F'(u_n,v_n)[v_n,u_n]}{\|\mathbf{u}_n\|_E^2}$:

$$\widehat{B}((U_n,V_n),(V_n,U_n)) - \int_\Omega \frac{[(v_n+v_{neg})^+]^p}{\|\mathbf{u}_n\|_E} U_n - \int_\Omega \frac{[(u_n+u_{neg})^+]^q}{\|\mathbf{u}_n\|_E} V_n \to 0\,,$$

which implies $\widehat{B}((U_n,V_n),(V_n,U_n)) \to 0$ and then $\int_\Omega |\nabla U_n|^2 + |\nabla V_n|^2 \to 0$; but this gives rise to a contradiction since by definition we have $\int_\Omega |\nabla U_n|^2 + |\nabla V_n|^2 = \|(u,v)\|_E^2 = 1$.

We conclude that $\|\mathbf{u}_n\|_E$ is bounded.

It is now simple to see that $\mathbf{u}_n$ admits a convergent subsequence. In fact, up to a subsequence, $(u_n, v_n) \to (u, v)$ weakly in E and strongly in $[L^r]^2$ for $r < 2^*$, then we may consider $F'(u_n, v_n)[v_n - v, u_n - u]$ to obtain

$$\int_\Omega \nabla u_n \nabla(u_n - u) + \nabla v_n \nabla(v_n - v) \to 0\,, \tag{4.9}$$

which implies that the convergence is in fact strong. □ □

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