Multiplicity Results for a Superlinear Elliptic System with Partial Interference with the Spectrum

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Problem and results

In this work we consider the problem

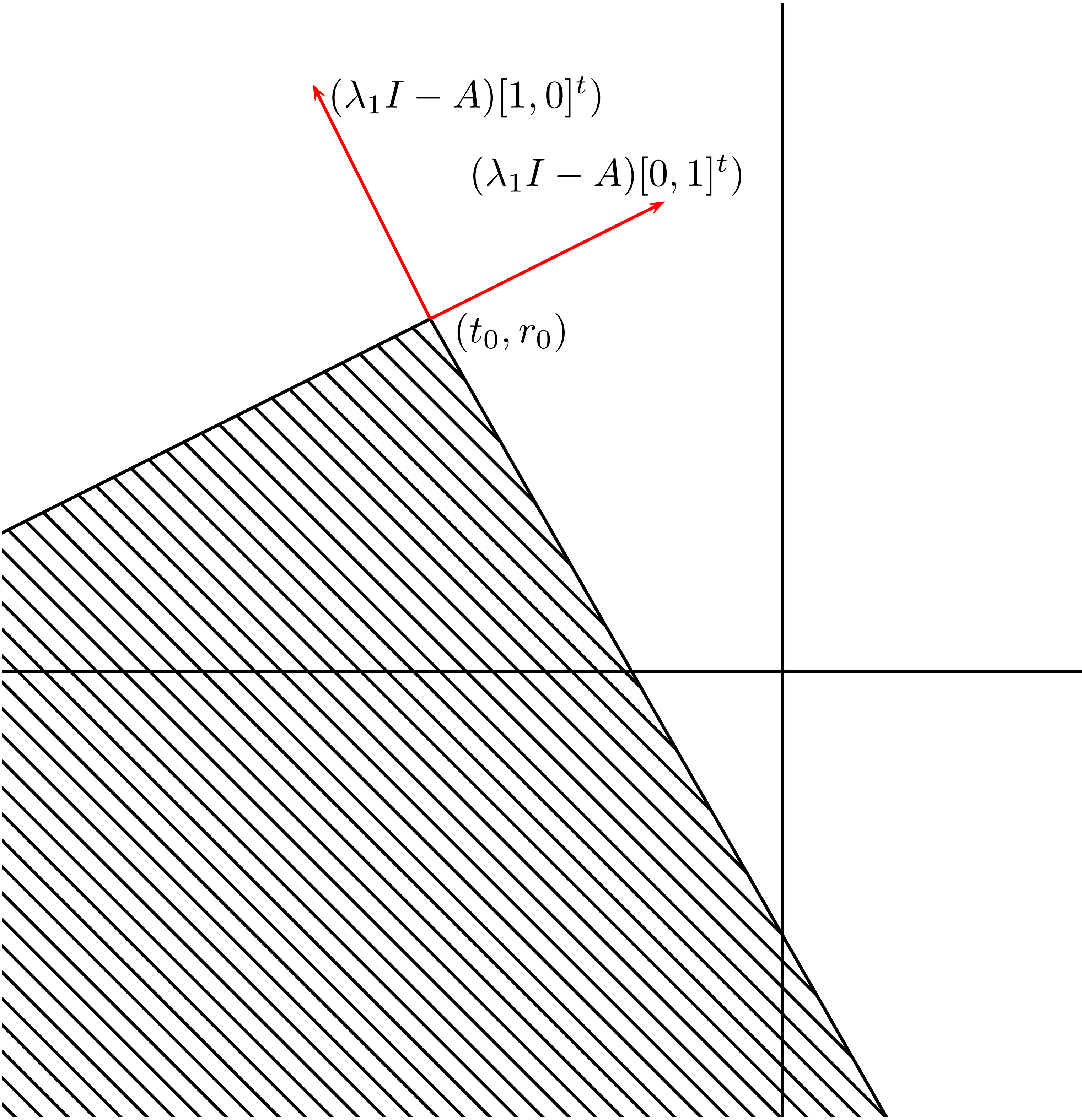
$$\begin{cases} -\Delta u = au + bv + (v^+)^p + f_1 + t\phi_1 & in \quad \Omega \\ -\Delta v = cu + dv + (u^+)^q + f_2 + r\phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases} \qquad \text{(S)}$$

where:

- $x^+ = \max\{0, x\}$,
- $\phi_1 > 0$ is the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$,
- $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain with $N \geq 2$.
- $1 < p, q < 2^* - 1$

Theorem 1. *In the above hypotheses, if* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *has real eigenvalues* $\nu_{1,2} \notin \sigma(-\Delta)$ *and* $f_{1,2} \in L^s(\Omega)$ *with* $s > N \geq 2$*, then there exists* $(t_0, r_0) \in \mathbb{R}^2$ *such that if* $(t, r)^t = (t_0, r_0)^t + (\lambda_1 I - A)(\tau, \rho)^t$ *with* $\tau, \rho < 0$*, then a negative solution* (u_{neg}, v_{neg}) *of problem (S) exists.*

Theorem 2. *In the same hypotheses of theorem 1, if moreover* $a = d$*, then for the same vectors* $(t, r) \in \mathbb{R}^2$*, a second solution exists.*


$(\lambda_1 I - A)[1,0]^t)$
$(\lambda_1 I - A)[0,1]^t)$
(t_0, r_0)

Motivations: scalar problem

The scalar counterpart of problem (S):

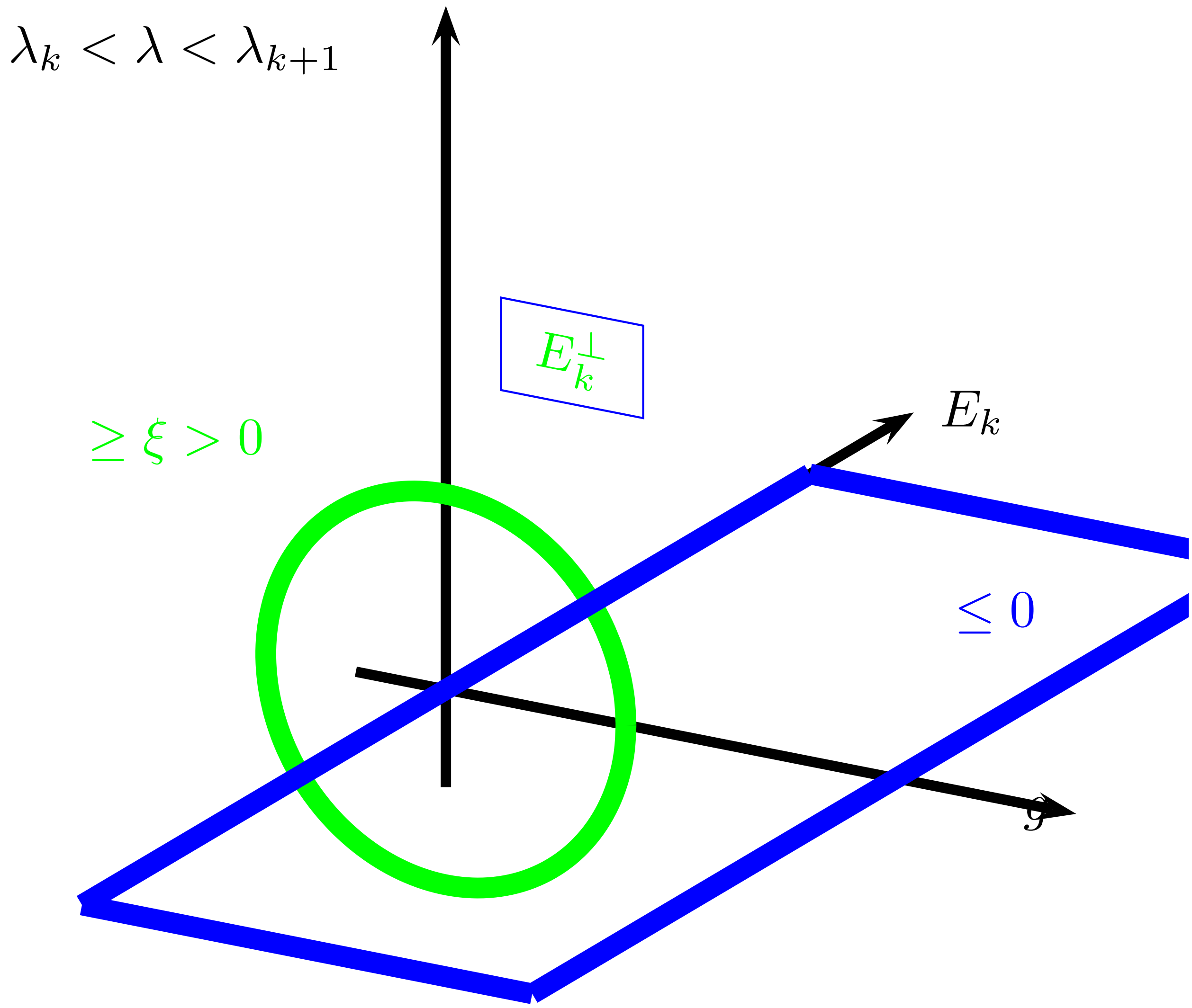
$$\begin{cases} -\Delta u = \lambda u + (u^+)^p + f + t\phi_1 & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}.$$

$\boxed{\lambda < \lambda_1}$ Ambrosetti-Prodi, *On the inversion of some differentiable mappings with singularities between Banach spaces*, Ann. Mat. Pura Appl. (4) **93** (1972), 231–246. and many others.

Two solutions for $t << 0$.
(first is negative and a minimum, then mountain pass theorem).

$\boxed{\lambda > \lambda_1}$ B. Ruf, P. N. Srikanth, Multiplicity results for superlinear elliptic problems with partial interference with the spectrum, J. Math. Anal. Appl. 118 (1) (1986) 15–23.

Two solutions for $\lambda \notin \sigma(-\Delta)$ and $t >> 0$.
(first is negative but not a minimum, then linking theorem).


$\lambda_k < \lambda < \lambda_{k+1}$
$E_k^{\perp}$
$\geq \xi > 0$
E_k
≤ 0
g

Motivations: gradient system

For the system

$$\begin{cases} -\Delta u = au + bv + (u^+)^p + f_1 + t\phi_1 & in \quad \Omega \\ -\Delta v = cu + dv + (v^+)^q + f_2 + r\phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases}$$

F.R. Pereira, *Problemas do tipo Ambrosetti-Prodi para sistemas envolvendo expoentes subcrítico e crítico*, Ph.D. thesis, UNICAMP, Brasil (2005).

two solutions for suitable values of the parameters t, r.

If $\nu_{1,2} < \lambda_1$, then one negative solution which is a minimum and second through MP theorem.
If $\nu_{1,2} \in (\lambda_k, \lambda_{k+1})$, then one negative solution and second through linking theorem.

Observe that for this system one uses a functional with principal part $\int_\Omega |\nabla u|^2 + |\nabla v|^2$

With system (S) the principal part is $\int_\Omega \nabla u \nabla v$: **strongly indefinite**.

Then we will never have minimum: we will need a **linking theorem with infinite dimensional negative space**.

The negative solution

Consider

$$\begin{cases} -\Delta \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} t \\ r \end{bmatrix} \phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases}$$

If A has real eigenvalues $\nu_{1,2} \notin \sigma(-\Delta)$ then there exists a $\boxed{\text{unique solution } (u_0, v_0)}$ for the case $t = r = 0$.

Since $f_{1,2} \in L^s(\Omega)$ with $s > N$, we have $\boxed{u_0, v_0 \in \mathcal{C}^{1,\alpha}(\overline{\Omega})}$ for $\alpha < 1 - N/s$.

Then by superposition

$$(\lambda_1 I - A)^{-1} \begin{bmatrix} t \\ r \end{bmatrix} \phi_1 + \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}$$

is a solution of (S), provided it is non positive.

This results in the condition in theorem 1:

$$\begin{bmatrix} t \\ r \end{bmatrix} = \begin{bmatrix} t_0 \\ r_0 \end{bmatrix} + (\lambda_1 I - A) \begin{bmatrix} \tau \\ \rho \end{bmatrix} \text{ with } \tau, \rho < 0$$

An equivalent problem

Now let $a = d$ and $bc > 0$.

Lemma. *If* (U, V) *is a solution of*

$$\begin{cases} -\Delta U = aU + bV/\delta + (V^+)^p/\delta + (f_1 + t\phi_1)/\delta & in \quad \Omega \\ -\Delta V = c\delta U + dV + \delta^q (U^+)^q + (f_2 + r\phi_1) & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases}$$

with $\delta > 0$, *then* $(u, v) = (\delta U, V)$ *is a solution of (S).*

Thus, we consider from now on the system

$$\begin{cases} -\Delta u = au + bv + C_1 (v^+)^p + f_1 + t\phi_1 & in \quad \Omega \\ -\Delta v = bu + av + C_2 (u^+)^q + f_2 + r\phi_1 & in \quad \Omega \\ u = v = 0 & on \quad \partial\Omega \end{cases} \qquad \text{(S*)}$$

Then the eigenvalues of A are $\boxed{\nu_{1,2} = a \pm b}$

Remark 3. $a = d$ *by hypothesis in theorem 2.*

$bc \geq 0$ *since* A *has real eigenvalues.*

The case $c = 0$ *or* $b = 0$ *is just slightly different.*

The functional

We will work in the Hilbert space $\boxed{E = H_0^1(\Omega) \times H_0^1(\Omega)}$, equipped with the norm $\|(u,v)\|_E^2 = \int |\nabla u|^2 + |\nabla v|^2$

Let (u_{neg}, v_{neg}) be the negative solution, and consider

$$F : E \to \mathbb{R} : \mathbf{u} = (u,v) \mapsto F(\mathbf{u}) =$$

$$= \int_\Omega \nabla u \; \nabla v - \frac{1}{2} \int_\Omega \big(b(v^2 + u^2) + 2auv \big)$$

$$- C_1 \int_\Omega \frac{\left[(v + v_{neg})^+\right]^{p+1}}{p+1} - C_2 \int_\Omega \frac{\left[(u + u_{neg})^+\right]^{q+1}}{q+1}$$

• F is $\mathcal{C}^1(E;\mathbb{R})$

• (u,v) <u>critical</u> for F implies $\boxed{(u + u_{neg}, v + v_{neg}) \text{ solution}}$ of (S*).

• $(0,0)$ is critical at level 0, and corresponds to the negative solution.

The minimax theorem of Felmer

We will use the following minimax theorem:

Theorem 4. *(From P. L. Felmer, Periodic solutions of "superquadratic" Hamiltonian systems, J. Differential Equations 102 (1) (1993) 188–207.)*
Let $E = X \oplus Y$ be a Hilbert space;
$F : E \to \mathbb{R}$ such that
1) F is a $\mathcal{C}^1$ functional,
2) F satisfies the Palais-Smale (PS) condition,
3) F have the structure $F(u) = \langle\, L\,u, u \,\rangle_E + b(u)$ where:
3a) $L : E \to E$ is linear, bounded and self-adjoint,
3b) $b' : E \to E$ is compact,
3c) $P_X \exp(\mu L) : X \to X$ is invertible for any $\mu > 0$.

Moreover, exist $R, \Theta, \rho > 0$ with $R\Theta > \rho$ and $g \in Y$ with $\|g\|_E = 1$ such that:
i) $F(u) \geq \xi > 0$ for $u \in S = \{u :\ u \in Y,\ \|u\|_E = \rho\}$;
ii) $F(u) \leq 0$ for $u \in \partial Q$ where
$Q = \{u = w + s\,g :\ w \in X\,,\ \|w\|_E \leq R,\ 0 \leq s \leq R\Theta\}$.

Then $\boxed{\exists u \in E\text{: } F'(u) = 0 \text{ and } F(u) \geq \xi}$

The variational setting

Let

$$B((u,v),(w,z)) = \int_\Omega \nabla u \nabla z + \nabla v \nabla w - a(uz+vw) - b(uw+vz)$$

We look for a E-orthogonal base which diagonalizes B, so we consider

$$(u,v) \in E \ : \quad B((u,v),(\phi,\psi)) = \mu \langle\, (u,v),(\phi,\psi)\,\rangle_E \quad \forall (\phi,\psi) \in E$$

and we obtain two sequences:

$$\mu_{+i} = \frac{-b + (\lambda_i - a)}{\lambda_i} \ \ (\to +1) \qquad \Psi_{+i} = \frac{(+\phi_i, +\phi_i)}{\sqrt{2\lambda_i}} \quad (i \in \mathbb{N})\,;$$

$$\mu_{-i} = \frac{-b - (\lambda_i - a)}{\lambda_i} \ \ (\to -1) \qquad \Psi_{-i} = \frac{(+\phi_i, -\phi_i)}{\sqrt{2\lambda_i}} \quad (i \in \mathbb{N})\,;$$

(normalization $\|\Psi_{\pm i}\|_E = 1$)

[see Hulshof, van der Vorst; de Figueiredo, Felmer]

$$\mu_{+i} = \frac{-b + (\lambda_i - a)}{\lambda_i} \ (\to +1) \qquad \Psi_{+i} = \frac{(+\phi_i, +\phi_i)}{\sqrt{2\lambda_i}} \quad (i \in \mathbb{N})\,;$$

$$\mu_{-i} = \frac{-b - (\lambda_i - a)}{\lambda_i} \ (\to -1) \qquad \Psi_{-i} = \frac{(+\phi_i, -\phi_i)}{\sqrt{2\lambda_i}} \quad (i \in \mathbb{N})\,;$$

(normalization $\|\Psi_{\pm i}\|_E = 1$)

We observe:

- for i large, $\boxed{\mu_{+i} > 0}$, $\boxed{\mu_{-i} < 0}$;
- for any $i \in \mathbb{N}$,

$\boxed{\Psi_{+i} \text{ equal components}}$, $\boxed{\Psi_{-i} \text{ opposite components}}$;

- for $a \pm b \notin \sigma(-\Delta)$, $\boxed{|\mu_{\pm i}| \geq 2\xi^* > 0}$.

We define two different splits of E:

Let $\widetilde{n}$ be such that if $i \geq \widetilde{n}$ then $\lambda_i - a > |b|$ and

$$\begin{aligned} E_h &= \overline{span\left\{\Psi_i : \ |i| \geq \widetilde{n}, \ i \in \mathbb{Z}^0\right\}}, \\ E_l &= span\left\{\Psi_i : \ |i| < \widetilde{n}, \ i \in \mathbb{Z}^0\right\}. \end{aligned}$$

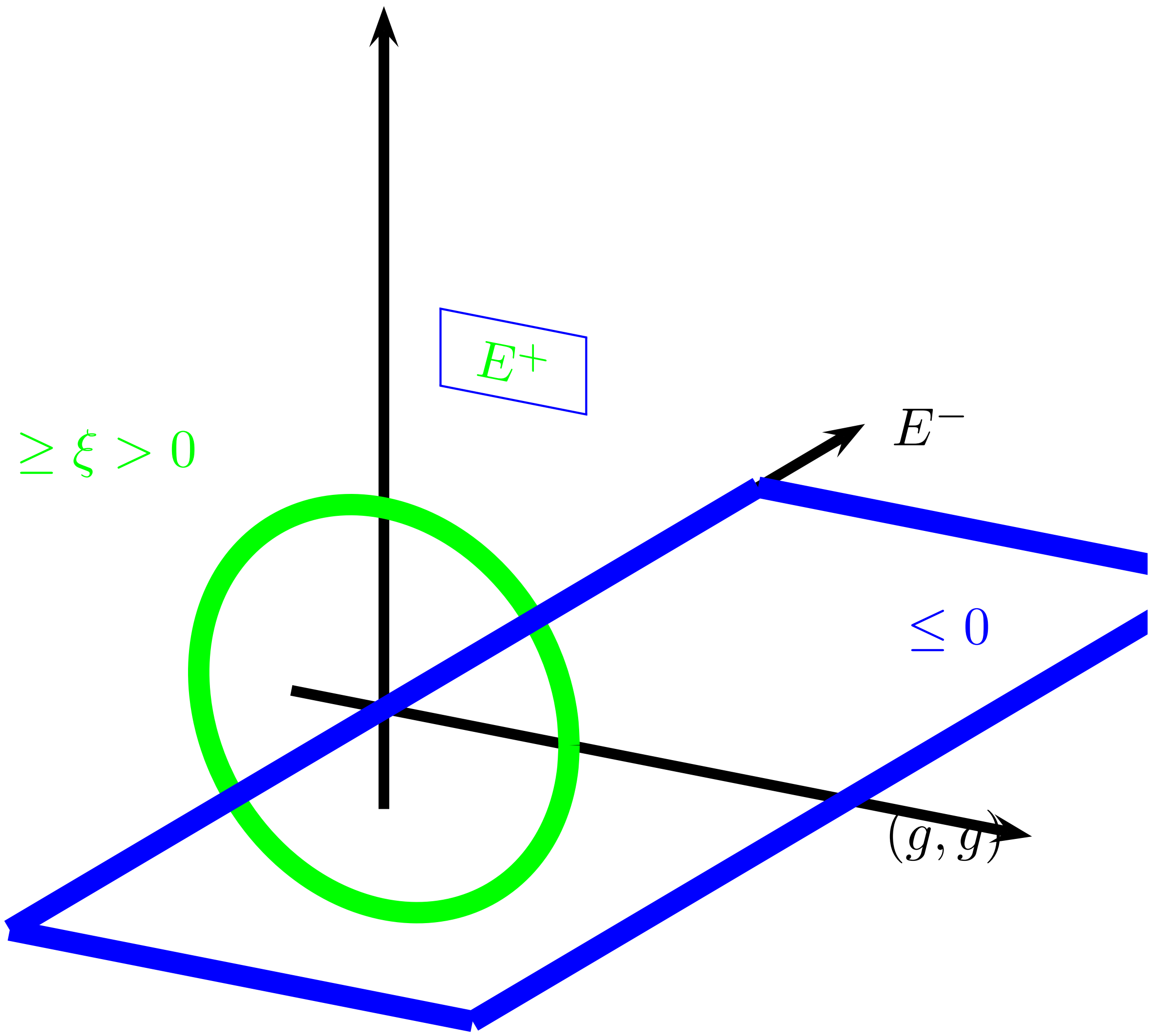
$$\begin{aligned} E^+ &= \overline{span\left\{\Psi_i : \ \mu_i > 0, \ i \in \mathbb{Z}^0\right\}}, \\ E^- &= \overline{span\left\{\Psi_i : \ \mu_i < 0, \ i \in \mathbb{Z}^0\right\}}, \end{aligned}$$

we have the following

Lemma. *There exists $\xi^* > 0$ such that*

$$\begin{aligned} B(\mathbf{u},\mathbf{u}) \geq 2\xi^* \|\mathbf{u}\|_E^2 \qquad & \text{for } \mathbf{u} \in E^+ , \\ B(\mathbf{u},\mathbf{u}) \leq -2\xi^* \|\mathbf{u}\|_E^2 \qquad & \text{for } \mathbf{u} \in E^- . \end{aligned}$$

- $(u,v) \in E^+ \cap E_h$ *implies* $\boxed{u = v}$
- $(u,v) \in E^- \cap E_h$ *implies* $\boxed{u = -v}$
- E_l *is* $\boxed{\textit{finite dimensional}}$



E^+
$\geq \xi > 0$
E^-
≤ 0
(g, g)

Lemma. *There exists* $\boxed{\mathbf{g} = (g, g) \in E^+ \bigcap E_h}$ *with* $\boxed{\|\mathbf{g}\|_E = 1}$ *and* $\boxed{\|g^+\|_{L^\infty} = +\infty}$.

Lemma. *There exist* $\rho, \xi > 0$ *such that*
1) $\boxed{F(\mathbf{u}) \geq \xi}$ *for* $\mathbf{u} = (u, v) \in E^+$ *and* $\|\mathbf{u}\|_E = \rho$ *.*

Use $\mathbf{u} \in E^+$ *and* $H_0^1 \hookrightarrow L^{p+1}, L^{q+1}$ *where* $p, q > 1$*.*

Lemma. *Let* $\mathbf{g} = (g, g)$ *as in the lemma above.*

Then there exist $R, \theta > 0$ *with* $R\theta > \rho$ *such that* $\boxed{F(\mathbf{u}) \leq 0}$ *for:*

a) $(u, v) \in E^-$,
Just use $(u, v) \in E^-$.

b) $(u, v) = \mathbf{u} = \mathbf{w} + s\mathbf{g}$*:* $\mathbf{w} \in E^-$, $\|\mathbf{w}\|_E = R$, $0 \leq s \leq \theta R$,
Use $B(\mathbf{w}, \mathbf{g}) = 0$ and
$F(u, v) \leq \frac{1}{2}(B(\mathbf{w}, \mathbf{w}) + s^2 B(\mathbf{g}, \mathbf{g})) \leq$
$\leq R^2(-\xi^* + \theta^2 \frac{1}{2} B(\mathbf{g}, \mathbf{g}))$: then $\boxed{\text{fix } \Theta \text{ small}}$.

c) $(u, v) = \mathbf{u} = \mathbf{w} + s\mathbf{g}$*:* $\mathbf{w} \in E^-$, $\|\mathbf{w}\|_E \leq R$, $s = \theta R$,
more difficult...

$$\mathbf{u} = (u,v) = (w,z) + R\Theta(g,g)\,, \qquad \|w,z\|_E \le R$$

$$E^- \ni \mathbf{w} = (w,z) = (\sigma_1,\sigma_2) + (\delta_1,-\delta_1) :$$
$$(\sigma_1,\sigma_2) \in E^- \cap E_l\,,\ (\delta_1,-\delta_1) \in E^- \cap E_h$$

So

$$\int_\Omega \left[(z+\theta R\,g+v_{neg})^+\right]^{p+1} = R^{p+1}\int_\Omega \left[\left(\frac{\sigma_2-\delta_1+v_{neg}}{R}+\theta g\right)^+\right]^{p+1},$$
$$\int_\Omega \left[(w+\theta R\,g+u_{neg})^+\right]^{q+1} = R^{q+1}\int_\Omega \left[\left(\frac{\sigma_1+\delta_1+u_{neg}}{R}+\theta g\right)^+\right]^{q+1};$$

$$|u_{neg}|, |v_{neg}| < C/2 \quad \text{and} \quad |\sigma_1|, |\sigma_2| < C\,\|\mathbf{w}\|_E\,/2 \le CR/2\,;$$

so, for $R > 1$,

$$\frac{|\sigma_1+u_{neg}|}{R}, \frac{|\sigma_2+v_{neg}|}{R} < C\,.$$

$$\Omega^* = \{x \in \Omega : \theta g > C+1\} \quad \text{has positive measure}\,;$$

$\theta g > C{+}1 \ \Rightarrow\ \max\{\theta g \pm \delta_1/R\} > C{+}1 \quad \forall\, \delta_1 \in H^1_0,\ R \in \mathbb{R}$

then $\Omega^* \subseteq \Omega^*_+ \cup \Omega^*_-$, where

$$\Omega^*_\pm = \{x \in \Omega : \theta g \pm \delta_1/R > C+1\}$$

Then either $|\Omega^*_-| \ge |\Omega^*|/2$ or $|\Omega^*_+| \ge |\Omega^*|/2$

Then one of the following two inequalities hold:

$$\int_\Omega \left[\left(\frac{\sigma_2 - \delta_1 + v_{neg}}{R} + \theta g\right)^+\right]^{p+1} \geq |\Omega^*|/2$$

$$\int_\Omega \left[\left(\frac{\sigma_1 + \delta_1 + u_{neg}}{R} + \theta g\right)^+\right]^{q+1} \geq |\Omega^*|/2\,.$$

We conclude that

$$- C_1 \int_\Omega \frac{[(v + v_{neg})^+]^{p+1}}{p+1} - C_2 \int_\Omega \frac{[(u + u_{neg})^+]^{q+1}}{q+1} \leq -\widetilde{C}\, R^{\min\{p,q\}+1}\,,$$

where now $\widetilde{C} > 0$ does not depend on $R, \mathbf{w}$.

Finally,

$$\begin{aligned} F(\mathbf{u}) &\leq \frac{1}{2} B(\mathbf{w} + \theta R \mathbf{g}, \mathbf{w} + \theta R \mathbf{g}) - \widetilde{C}\, R^{\min\{p,q\}+1} \\ &\leq -\xi^* \|\mathbf{w}\|_E^2 + \frac{1}{2}\theta^2 R^2 B(\mathbf{g}, \mathbf{g}) - \widetilde{C}\, R^{\min\{p,q\}+1} \\ &\leq R^2 \left(\theta^2 B_g - \widetilde{C} R^{\min\{p,q\}-1}\right) \end{aligned}$$

Then $\boxed{\text{fix } R \text{ large}}$.

Interesting questions

- Can we do it for more general nonlinearities?
- Can we do it with p, q below the critical hyperbola?
- Can we do it in the critical case?