Weighted Trudinger-Moser inequalities and associated Liouville type equations*

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Abstract

We discuss some Trudinger–Moser inequalities with weighted Sobolev norms. Suitable logarithmic weights in these norms allow an improvement in the maximal growth for integrability, when one restricts to radial functions.

The main results concern the application of these inequalities to the existence of solutions for certain mean-field equations of Liouville-type. Sharp critical thresholds are found such that for parameters below these thresholds the corresponding functionals are coercive and hence solutions are obtained as global minima of these functionals. In the critical cases the functionals are no longer coercive and solutions may not exist.

We also discuss a limiting case, in which the allowed growth is of double exponential type. Surprisingly, we are able to show that in this case a local minimum persists to exist for critical and also for slightly supercritical parameters. This allows to obtain the existence of a second (mountain-pass) solution, for almost all slightly supercritical parameters, using the Struwe monotonicity trick. This result is in contrast to the non-weighted case, where positive solutions do not exist (in star-shaped domains) in the critical and supercritical case.

Keywords: Trudinger-Moser inequality, Liouville type equations.

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1 Introduction

The well known *Trudinger-Moser (TM) inequality* provides continuous embeddings into exponential Orlicz spaces in the borderline cases of the standard Sobolev embeddings, when the embeddings into Lebesgue L^p spaces hold for every $p < \infty$ but not for $p = \infty$. Let us recall Moser's result for the case $N = 2$:

Theorem A (Moser [Mos71]). Let $N = 2$; then

$$\sup_{\int_\Omega |\nabla u|^2 dx \le 1} \int_\Omega e^{\alpha u^2} \begin{cases} \le C|\Omega| & \text{if} \quad \alpha \le 4\pi\,, \\ = +\infty & \text{if} \quad \alpha > 4\pi. \end{cases} \tag{1.1}$$

A useful variant of the TM inequality is the following *logarithmic TM inequality*:

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Theorem B (Moser [Mos71]). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then there exists a constant $C > 0$ such that

$$\log \int_\Omega e^u \, dx \leq \frac{1}{16\pi} \int_\Omega |\nabla u|^2 \, dx + C \,, \quad u \in H_0^1(\Omega)\,. \tag{1.2}$$

The value $\dfrac{1}{16\pi}$ is optimal (see e.g. [CLMP92]).

1.1 Weighted Trudinger-Moser inequalities

Recent results concern the influence of weights on such type of inequalities. In [CT05], [AS07], [dFdOdS16], for instance, the authors consider the effect of power weights in the integral term on the maximal growth. On the other hand in [Cal14], [CR15c], [CR15a], [CR15b], [CRS17] the interest is devoted to the impact of weights in the Sobolev norm.

We concentrate our attention on this second type of results. More precisely, let $w \in L^1(\Omega)$ be a non-negative function, and consider the weighted Sobolev space

$$H_0^1(\Omega, w) = cl\Big\{u \in C_0^\infty(\Omega) \;;\; \int_\Omega |\nabla u|^2 w(x) dx < \infty\Big\}\,. \tag{1.3}$$

It turns out that for weighted Sobolev spaces of the form (1.3) logarithmic weights have a particular significance. However, as was observed in [CR15a, Proposition 8], one needs to restrict the attention to radial functions in order to obtain an actual improvement of the embedding inequalities. One is therefore lead to consider problems of the following type: let $B \subset \mathbb{R}^2$ denote the unit ball in $\mathbb{R}^2$, and consider the weighted Sobolev space of radial functions

$$\widetilde{H}_\beta = H^1_{0,rad}(B, w_\beta) := cl\Big\{u \in C^\infty_{0,rad}(B) \;;\; \|u\|_\beta^2 := \int_B |\nabla u|^2 w_\beta(x) dx < \infty\Big\}\,,$$

where

$$w_\beta(x) = \Big(\log \frac{e}{|x|}\Big)^\beta \,, \;\; \beta \geq 0\,. \tag{1.4}$$

The following results were obtained in [CR15a] and will be fundamental in this paper.

Theorem C ([CR15a]). Let $\beta \in [0, 1)$. Then

(*a*) $\displaystyle\int_B e^{|u|^\gamma} dx < +\infty, \quad \text{for all} \;\; u \in \widetilde{H}_\beta \iff \gamma \leq \gamma_\beta := \frac{2}{1-\beta}$

and

(*b*) $\displaystyle\sup_{u\in \widetilde{H}_\beta,\, \|u\|_\beta \leq 1} \int_B e^{\alpha |u|^{\gamma_\beta}} dx < +\infty$

if and only if

$$\alpha \leq \alpha_\beta := 2\,[2\pi(1-\beta)]^{\frac{1}{1-\beta}} \qquad (\textit{critical growth}). \tag{1.5}$$

Remark 1.1. This result extends the Trudinger-Moser inequality (1.1); indeed, for $\beta = 0$ we recover the classical TM inequality where $\gamma_0 = 2$ and $\alpha_0 = 4\pi$.

Going to the limiting case $\beta = 1$ in Theorem C, one sees that the exponent γ of u in the integral can take any value, that is, we are again in a borderline case. But again, the embedding does not go into L^∞, in fact, we find a critical growth of *double exponential type*, as described in the following

Theorem D ([CR15a]). Let $\beta = 1$ (i.e. $w_1(x) = \log \frac{e}{|x|}$). Then,

$$(a) \qquad \int_B e^{e^{u^2}} dx < +\infty \quad , \ \forall \ u \in \widetilde{H}_1 = H^1_{0,rad}(B, w_1)$$

and

$$(b) \qquad \sup_{u \in \widetilde{H}_1, \, \|u\|_1 \le 1} \int_B e^{a \, e^{2\pi u^2}} dx < +\infty \iff a \le 2.$$

Finally, in the case $\beta > 1$, one has the following result

Theorem E ([CR15a]). Let $\beta > 1$. Then we have the following embedding:

$$\widetilde{H}_\beta = H^1_{0,rad}(B, w_\beta) \hookrightarrow L^\infty(B)\,.$$

Logarithmic inequalities similar to (1.2) can be obtained also in this setting. We have the following results, which were partially obtained in a previous paper [CR15c], but we recall the proofs for completeness in the Appendix.

Proposition 1.2.
a) For $\beta \in [0, 1)$, there exists a constant $C(\beta)$ such that

$$\log \left(\frac{1}{|B|} \int_B e^{|u|^{\theta_\beta}} dx \right) \ \le \ \frac{1}{2\lambda^*_\beta} \|u\|^2_\beta + C(\beta) \qquad \forall u \in \widetilde{H}_\beta\,, \tag{1.6}$$

where

$$\lambda^*_\beta := \pi(1-\beta)^\beta (2-\beta)^{2-\beta} 2^{1-\beta} \qquad \text{and} \qquad \theta_\beta = \frac{2}{2-\beta}\,. \tag{1.7}$$

b) For $\beta = 1$, there exists a constant C_{MB} such that

$$\log\log \left(\frac{1}{|B|} \int_B e^{e^{|u|}} \, dx \right) \ \le \ \frac{1}{2\pi} \|u\|^2_1 + \log \left(\frac{1}{8} + \frac{\log C_{MB}}{e^{\frac{1}{2\pi}\|u\|^2_1}} \right) \qquad \forall u \in \widetilde{H}_1\,. \tag{1.8}$$

The values $\frac{1}{2\lambda^*_\beta}$ and $\frac{1}{2\pi}$ in (1.6) and (1.8), respectively, are optimal.

Remark 1.3. Notice that in the case $\beta = 0$ inequality (1.6) gives the classical logarithmic TM inequality (1.2), actually, $\lambda^*_0 = 8\pi$ and $\theta_0 = 1$.

Remark 1.4. The optimality of $\frac{1}{2\lambda^*_0}$ can be found in [CLMP92], while the optimality of $\frac{1}{2\lambda^*_\beta}$ and $\frac{1}{2\pi}$ in (1.6) and (1.8), respectively, is new, and it will be a consequence of Theorem 1.5 in this paper.

1.2 Mean field equations of Liouville type

The logarithmic version of the TM inequality is crucial in the study of mean field equations of Liouville type (see [Lio53]) of the form

$$\begin{cases} -\Delta u &= \lambda \dfrac{e^u}{\int_\Omega e^u} \qquad \text{in} \ \ \Omega \subset \mathbb{R}^2\,, \\[2ex] \quad u &= 0 \qquad\qquad \text{on} \ \ \partial\Omega\,. \end{cases} \tag{1.9}$$

Equation (1.9) was derived by Caglioti, Lions, Marchioro and Pulvirenti in their pioneering works [CLMP92, CLMP95] from the mean field limit of point vortices of the Euler flow, see also Chanillo-Kiessling [CK94] and Kiessling [Kie93]. Equation (1.9) occurs also in the study of multiple condensate solutions for the Chern-Simons-Higgs theory, see Tarantello [Tar96], [Tar04].

In particular, it has been shown (see also [Li99], [CL10]) that equation (1.9) has a solution if

$$\lambda < 8\pi, \tag{1.10}$$

while a Pohozhaev identity shows that no solution exists for $\lambda \geq 8\pi$ in starshaped domains (see e.g. [CLMP92]). In view of this, we call the case $\lambda < 8\pi$ *subcritical*, the case $\lambda = 8\pi$ *critical*, and the case $\lambda > 8\pi$ *supercritical*.

The existence of a solution in the *subcritical case* can be proved by using variational methods; in fact, solutions of (1.9) are critical points of the functional

$$J : H_0^1(\Omega) \to \mathbb{R}, \qquad J(u) = \frac{1}{2}\|u\|^2 - \lambda \log \int_\Omega e^u \, dx\,. \tag{1.11}$$

Indeed, for $\lambda < 8\pi$, as a consequence of the logarithmic TM inequality, the functional J is coercive, hence bounded from below, and then admits an absolute minimum. For $\lambda = 8\pi$ the functional J is still bounded below, but no longer coercive, and the infimum is not attained.

1.3 Main results

In this article we concentrate our attention on some functionals similar to (1.11) and related nonlocal equations that generalize (1.9), under the impact of the above mentioned weighted logarithmic inequalities.

We define the following functionals:

i) for $\beta \in [0, 1)$, let

$$J_\lambda : \widetilde{H}_\beta \to \mathbb{R}, \qquad J_\lambda(u) := \frac{1}{2}\|u\|_\beta^2 - \lambda \log \left(\fint_B e^{u^\theta} \, dx \right)\,, \tag{1.12}$$

where $\theta = \theta_\beta$ from Proposition 1.2, and writing $u^\theta := |u|^{\theta-1}u$ and $\fint_B := \frac{1}{|B|}\int_B$;

ii) for $\beta = 1$, let

$$I_\lambda : \ \widetilde{H}_1 \to \mathbb{R}, \qquad I_\lambda(u) := \frac{1}{2}\|u\|_1^2 - \lambda \log\log \left(\fint_B e^{e^u} \, dx \right)\,. \tag{1.13}$$

Our purpose in this paper is to study the geometry of these functionals in dependence of the positive parameter λ and, as a consequence, to obtain existence results for some related nonlocal equations. In particular we will prove the following results.

Theorem 1.5.
i) For $\beta \in [0, 1)$, the functional J_λ is coercive for $\lambda \in [0, \lambda_\beta^*)$ and it is bounded from below if and only if $\lambda \leq \lambda_\beta^*$ (see expression (1.7)).

ii) For $\beta = 1$, the functional I_λ is coercive for $\lambda \in [0, \pi)$ and it is bounded from below if and only if $\lambda \leq \pi$.

Both results have a natural application to some weighted mean field equations of Liouville type. As for equation (1.9) we distinguish the *subcritical* and *critical* case.

Theorem 1.6 (Subcritical case)**.**
i) Let $\beta \in [0,1)$*, and* $\theta = \theta_\beta = \frac{2}{2-\beta}$*. Then the equation*

$$\begin{cases} -\mathrm{div}\big(w_\beta(x)\nabla u\big) &= \lambda \dfrac{\theta |u|^{\theta-1} e^{u^\theta}}{\int_B e^{u^\theta}} & \text{in } B\,, \\ u &= 0 & \text{on } \partial B\,, \end{cases} \tag{1.14}$$

has a positive weak radial solution, which is a global minimizer for J_λ*, for every value* $\lambda \in (0, \lambda^*_\beta)$*.*
ii) The equation

$$\begin{cases} -\mathrm{div}\big(w_1(x)\,\nabla u\big) &= \lambda \dfrac{e^u}{\log \fint_B e^{e^u}} \dfrac{e^{e^u}}{\int_B e^{e^u}} & \text{in } B\,, \\ u &= 0 & \text{on } \partial B\,, \end{cases} \tag{1.15}$$

has a positive weak radial solution, which is a global minimizer for I_λ*, for every* $\lambda \in (0,\pi)$*.*

In contrast to the situation for equation (1.9), and somewhat surprisingly, for problem (1.15) with the double exponential nonlinearity we can also prove an existence result for the critical and slightly supercritical case:

Theorem 1.7 (Critical and supercritical case)**.** *There exists* $\varepsilon_0 > 0$ *such that equation (1.15) has a positive weak radial solution, which is a local minimizer for* I_λ*, also for* $\lambda \in [\pi, \pi + \varepsilon_0)$*. When* $\lambda = \pi$ *the minimum is global.*

Remark 1.8. The nonlinearities in the problems above are always nonnegative, and so only trivial or positive solutions may exist. In fact, the trivial solution exists for problem (1.14) if $\beta \in (0,1)$, while for $\beta = 0$ (problem (1.9)) and for $\beta = 1$ (problem (1.15)) $u = 0$ is not a solution.

We observe that in the supercritical situation, that is for $\lambda \in (\pi, \pi + \varepsilon_0)$, the functional I_λ has a mountainpass-structure, since we have a local minimum, and directions along which the functional tends to $-\infty$. A direct application of the Mountain-pass Theorem by Ambrosetti-Rabinowitz [AR73] seems difficult due to loss of compactness. However, we can apply the so-called "monotonicity trick" by Struwe [Str88] (see also [ST98, Jea99]) to obtain

Theorem 1.9. *Let* ε_0 *as in Theorem 1.7. Then for a.e.* $\lambda \in (\pi, \pi + \varepsilon_0)$ *equation* (1.15) *has a second positive radial solution which is of mountain-pass type.*

2 Proofs

We first prove the result concerning the geometry of the functionals J_λ and I_λ.

Proof of Theorem 1.5. Coercivity for $\lambda < \lambda^*_\beta$ (resp. $\lambda < \pi$) is an immediate consequence of (1.6) and (1.8), actually (with $\lambda \geq 0$)

$$J_\lambda(u) \geq \Big(\frac{1}{2} - \frac{\lambda}{2\lambda^*_\beta}\Big)\|u\|^2_\beta - \lambda C(\beta)$$

and

$$I_\lambda(u) \geq \left(\frac{1}{2} - \frac{\lambda}{2\pi}\right)\|u\|_1^2 - \lambda \log\left(\frac{1}{8} + \frac{\log C_{MB}}{e^{\frac{\|u\|_1^2}{2\pi}}}\right) \geq \left(\frac{1}{2} - \frac{\lambda}{2\pi}\right)\|u\|_1^2 - \lambda \log\left(\frac{1}{8} + \log C_{MB}\right).$$

The above estimates also show that the functionals J_λ and I_λ are bounded from below when $\lambda \leq \lambda_\beta^*$ (resp. $\lambda \leq \pi$).

Sharpness is much more delicate. When λ exceeds those critical values, the functionals are not bounded from below: we will produce a sequence along which they tend to $-\infty$.

Case $\beta \in (0,1)$**.** We evaluate the functional along a generalized Moser sequence (see [CR15a]): let $u_k(x) = \frac{\psi_k(t)}{\sqrt{2\pi(1-\beta)}}$ with $|x| = e^{-t}$, where

$$\psi_k(t) = \begin{cases} \dfrac{(1+t)^{1-\beta} - 1}{\sqrt{(1+k)^{1-\beta} - 1}} & \text{for } t \leq k\,, \\[2ex] \sqrt{(1+k)^{1-\beta} - 1} & \text{for } t > k\,. \end{cases} \tag{2.1}$$

With this definition one has $\|u_k\|_\beta = 1$.

We set $\alpha_k = C\sqrt{2\pi(1-\beta)}\left(\sqrt{(1+k)^{1-\beta} - 1}\right)^{1/(1-\beta)}$, where C will be fixed later, and evaluate the functional (1.12) along the sequence $\{\alpha_k u_k\}$: using the new variable t the functional reads as

$$J_\lambda(\alpha_k u_k) \;\; = \frac{\alpha_k^2}{2} - \lambda \log\left[2\int_0^\infty \exp\left(\left|\frac{\alpha_k}{\sqrt{2\pi(1-\beta)}}\,\psi_k(t)\right|^{\theta_\beta} - 2t\right)dt\right].$$

We estimate

$$\begin{aligned}
\int_0^\infty \exp\left(\left|\frac{\alpha_k}{\sqrt{2\pi(1-\beta)}}\,\psi_k(t)\right|^{\theta_\beta} -2t\right)dt &= \int_0^\infty \exp\left(\left|C\left(\sqrt{(1+k)^{1-\beta} - 1}\right)^{\frac{1}{(1-\beta)}}\psi_k(t)\right|^{\theta_\beta} - 2t\right)dt \\
&\geq \int_k^\infty \exp\left(\left|C\left(\sqrt{(1+k)^{1-\beta} - 1}\right)^{\frac{1}{(1-\beta)}+1}\right|^{\theta_\beta} - 2t\right)dt \\
&= \frac{1}{2}\exp\left[\left|C\left(\sqrt{(1+k)^{1-\beta} - 1}\right)^{(2-\beta)/(1-\beta)}\right|^{2/(2-\beta)} - 2k\right] \\
&= \frac{1}{2}\exp\left[C^{2/(2-\beta)}\left[(1+k)^{1-\beta} - 1\right]^{1/(1-\beta)} - 2k\right].
\end{aligned}$$

Therefore

$$J_\lambda(\alpha_k \psi_k) \leq C^2\left[(1+k)^{1-\beta} - 1\right]^{\frac{1}{1-\beta}}\pi(1-\beta) - \lambda\left[C^{2/(2-\beta)}\left[(1+k)^{1-\beta} - 1\right]^{1/(1-\beta)} - 2k\right].$$

We now set $C^{2/(2-\beta)} = 2\,\dfrac{2-\beta}{1-\beta} + 2\delta$, for some $\delta > 0$.

Since $(2+\delta)\left[(1+k)^{1-\beta}-1\right]^{1/(1-\beta)}-2k\to\infty$ when $k\to\infty$, we estimate, for k large,

$$\begin{aligned}C^{2/(2-\beta)}\Big[(1+k)^{1-\beta}-1\Big]^{\frac{1}{1-\beta}}-2k &\geq \Big[2\tfrac{2-\beta}{1-\beta}-2+\delta\Big]\left[(1+k)^{1-\beta}-1\right]^{\frac{1}{1-\beta}}\\ &= \Big[\frac{2}{1-\beta}+\delta\Big]\left[(1+k)^{1-\beta}-1\right]^{\frac{1}{1-\beta}}.\end{aligned}$$

Then

$$J(\alpha_k\psi_k)\leq\Big[(1+k)^{1-\beta}-1\Big]^{\frac{1}{1-\beta}}\;\Bigg[\Big(\Big[2\frac{2-\beta}{1-\beta}\Big]+2\delta\Big)^{2-\beta}\pi(1-\beta)-\lambda\Big[\frac{2}{1-\beta}+\delta\Big]\Bigg]. \tag{2.2}$$

Let $\lambda=(1+\varepsilon)\lambda^*_\beta=(1+\varepsilon)\left(\left[2\frac{2-\beta}{1-\beta}\right]\right)^{2-\beta}\pi(1-\beta)^2/2$, for some $\varepsilon>0$. Then (2.2) can be rewritten as

$$J_\lambda(\alpha_k\psi_k)\leq\Big[(1+k)^{1-\beta}-1\Big]^{\frac{1}{1-\beta}}\;\Big(2\,\frac{2-\beta}{1-\beta}\Big)^{2-\beta}\pi(1-\beta)\,\{\,\dots\},$$

where

$$\{\,\dots\}=\left\{\Big(1+\frac{\delta(1-\beta)}{2-\beta}\Big)^{2-\beta}-(1+\varepsilon)\Big(1+\delta\frac{1-\beta}{2}\Big)\right\}.$$

Since this term tends to $-\varepsilon$ as $\delta\to 0$, the expression in braces is negative for $\delta>0$ small enough, and then $J\to-\infty$ along this sequence.

Case $\beta=1$**.** Again we prove that the value $\lambda=\pi$ is sharp by considering a generalized Moser sequence: let $u_k(x)=\frac{\psi_k(t)}{\sqrt{2\pi}}$ with $|x|=e^{-t}$, where now we use the sequence

$$\psi_k(t)=\begin{cases}\dfrac{\log(1+t)}{\sqrt{\log(1+k)}} & \text{for } t\leq k\,,\\[2ex] \sqrt{\log(1+k)} & \text{for } t>k\,.\end{cases} \tag{2.3}$$

Then $\|u_k\|_1=1$ and evaluating I_λ along the sequence $\{\alpha_k u_k\}$, with $\alpha_k=C\sqrt{2\pi\log(1+k)}$, we obtain

$$I_\lambda(\alpha_k u_k)=\frac{\alpha_k^2}{2}\;-\;\lambda\log\log 2\int_0^\infty exp\left[e^{(\alpha_k\psi_k/\sqrt{2\pi})}-2t\right]dt\,.$$

We estimate

$$\int_0^\infty exp\left[e^{(\alpha_k\psi_k/\sqrt{2\pi})}-2t\right]dt\geq\int_k^\infty exp\left[e^{(C\sqrt{2\pi\log(1+k)}\sqrt{\log(1+k)}/\sqrt{2\pi})}-2t\right]dt$$

$$=\int_k^\infty exp\left[e^{(C\log(1+k)}-2t\right]=\frac{1}{2}\,exp\big[(1+k)^C-2k\big]dt,$$

and then

$$I_\lambda(\alpha_k u_k)\leq C^2\pi\log(1+k)-\lambda\big[\log\big((1+k)^C-2k\big)\big]\,.$$

For $\lambda=\pi+\varepsilon$ we choose $C=1+2\delta(\varepsilon)$ and for k large we can estimate

$$\log((1+k)^{1+2\delta}-2k)\geq\log((1+k)^{1+\delta}),$$

and then

$$I_\lambda(\alpha u_k) \le (1+2\delta)^2\pi \log(1+k) - (\pi+\varepsilon)(1+\delta)\log(1+k)\ ;$$

since for $\delta > 0$ small $(1+2\delta)^2\pi < (\pi+\varepsilon)(1+\delta)$, we have proved that, if $\lambda > \pi$, there exists a sequence along which $I_\lambda \to -\infty$. □

In order to prove Theorem 1.6 we need a compactness result. The following Lemma due to de Figueiredo-Miyagaki-Ruf (Lemma 2.1 in [dFMR95]) will be needed:

Lemma F ([dFMR95]). Let (u_n) be a sequence of functions in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $F : \mathbb{R} \to \mathbb{R}$ is measurable, and that $F(u_n(x))$ and $F(u(x))$ are also L^1 functions. If

$$\int_\Omega \big|F(u_n(x))\, u_n(x)\big| dx \le C$$

then $F(u_n(x))$ converges to $F(u(x))$ in L^1.

The compactness result is in the following Lemma:

Lemma 2.1. *Let $\beta \in [0,1)$ and $\theta \in (0,\gamma_\beta)$ or $\beta \ge 1$ and $\theta > 0$.*
Let (u_n) be a bounded sequence in $\widetilde{H}_\beta$. Then there exists $u \in \widetilde{H}_\beta$ such that (up to a subsequence)

$$\log \fint_B e^{u_n^\theta} dx \to \log \fint_B e^{u^\theta} dx \quad \text{as } n \to +\infty$$

and, if $\beta \ge 1$,

$$\log\log \fint_B e^{e^{u_n}} dx \to \log\log \fint_B e^{e^{u}} dx \quad \text{as } n \to +\infty.$$

Proof. Let $\|u_n\|_\beta \le C$. Then there exists $u \in \widetilde{H}_\beta$, such that (up to a subsequence)

$$u_n \rightharpoonup u \ \text{ in } \ \widetilde{H}_\beta\ ,\ u_n \to u \ \text{ in } \ L^1(B)\ ,\ u_n \to u \ \text{ a.e. }\ ,\ \text{ as } \ n \to +\infty\,.$$

Observe that the nonlinearity e^{u^θ} is subcritical with respect to the maximal growth γ_β given by Theorem C, and that there exists a constant C_1 (depending on C, θ and β) such that

$$|te^{t^\theta}| \le C_1 e^{\alpha_\beta\left(\frac{|t|}{C}\right)^{\gamma_\beta}}\ ,\ \forall\, t \in \mathbb{R}\ . \tag{2.4}$$

From Theorem C and this estimate we have that

$$e^{|u_n|^\theta},\, e^{|u|^\theta} \in L^1 \qquad \text{and} \qquad \int_B \big|u_n e^{|u_n|^\theta}\big| dx \le C_2\,.$$

We now apply Lemma F using $F(t) = e^{t^\theta}$.

For the case $\beta \ge 1$ one proceeds in the same way using the inequalities

$$|te^{t^\theta}|,\, |te^{e^t}| \le C_1 e^{2e^{2\pi\left(\frac{|t|}{C}\right)^2}}\ ,\ \forall\, t \in \mathbb{R}\ ,$$

Theorem D or E, and applying Lemma F using $F(t) = e^{t^\theta}$ and $F(t) = e^{e^t}$.

□

We are now able to prove our first existence result.

Proof of Theorem 1.6. Since $\lambda < \lambda^*_\beta$ (resp $\lambda < \pi$), the functional J_λ (resp. I_λ) is bounded from below by Theorem 1.5, and one can take a minimizing sequence (u_n), i.e.

$$\lim_{n\to+\infty,} J_\lambda(u_n) = m = \inf_{i\in\widetilde{H}_\beta} J_\lambda(u),$$

which is trivially bounded in $\widetilde{H}_\beta$ by coercivity. Therefore there exists $u \in \widetilde{H}_\beta$, such that (up to a subsequence)

$$u_n \rightharpoonup u \ \text{ in } \ \widetilde{H}_\beta \ , \ u_n \to u \ \text{ in } \ L^1(B) \ , \ u_n \to u \ \text{ a.e. } \ , \ \text{ as } \ n \to +\infty$$

By Lemma 2.1 and the weak lower semicontinuity of the norm

$$m \le J_\lambda(u) \le \liminf_{n\to+\infty} J_\lambda(u_n) = m.$$

Then u is a global minimizer and therefore a solution of Problem (1.14). The case $\beta = 1$ is analogous.

When $\beta = 0$ or $\beta = 1$ the solution obtained is positive by Remark 1.8. For $\beta \in (0,1)$ we still have to show that the obtained solution is not trivial. This is the case since the origin is not a minimizer. Indeed, let $v \in \widetilde{H}_\beta$, $v \not\equiv 0$, $0 \le v \le 1$, $t \in (0,1)$: then $e^{(tv)^\theta} \ge 1 + (tv)^\theta$ and

$$⨍_B e^{(tv)^\theta} dx \ge 1 + ⨍_B (tv)^\theta dx\,.$$

Since $⨍_B (tv)^\theta dx \le 1$ we can use the estimate $\log(1+\tau) \ge \frac{1}{2}\tau$ for $\tau \in (0,1)$ to conclude

$$\log ⨍_B e^{(tv)^\theta} dx \ge \frac{1}{2} ⨍_B (tv)^\theta dx\,.$$

With this we get

$$J_\lambda(tv) = \frac{t^2}{2}\,\|v\|^2_\beta - \lambda \log ⨍_B e^{(tv)^\theta} dx \le \frac{t^2}{2}\,\|v\|^2_\beta - \frac{\lambda}{2}\, t^\theta ⨍_B v^\theta dx\,;$$

since $\theta \in (1,2)$, the above expression is negative for t small and then $m < 0 = J_\lambda(0)$.

□

In the next proof we consider problem (1.15) when $\lambda \ge \pi$.

Proof of Theorem 1.7. Beyond the threshold $\lambda = \pi$.

The functional I_π is still bounded from below by Theorem 1.5. We need to prove that minimizing sequences are still bounded, despite that in this case coercivity does not hold. However, the particular form of the logarithmic TM inequality will help in this direction.

Let (u_n) be a minimizing sequence, that is

$$I_\pi(u_n) \to m = \inf_{u\in\widetilde{H}_1} I_\pi(u).$$

We observe first that the infimum cannot be positive, since $m \leq I_\pi(0) = 0$. On the other hand, from inequality (1.8) we have

$$I_\pi(u_n) = \frac{1}{2}\|u_n\|_1^2 - \pi \log\log\left(\fint_B e^{e^{u_n}}\ dx\right) \geq -\pi \log\left(\frac{1}{8} + \frac{\log C_{MB}}{e^{\frac{\|u_n\|_1^2}{2\pi}}}\right).$$

If $\|u_n\|_1 \to \infty$ we would have

$$0 \geq m = \lim_{n\to+\infty} I_\pi(u_n) \geq \liminf_{n\to+\infty}\left[-\pi \log\left(\frac{1}{8} + \frac{\log C_{MB}}{e^{\frac{\|u_n\|_1^2}{2\pi}}}\right)\right] = \pi \log 8 > 0,$$

a contradiction. Then (u_n) is bounded and we are done (as in the proof of Theorem 1.6).

Now we prove that a minimum (now only local) exists also for $\lambda = \pi + \varepsilon > \pi$, ε small. Let $R > 0$ be such that

$$\frac{\log C_{MB}}{e^{\frac{\|u\|_1^2}{2\pi}}} < \frac{1}{8} \quad \text{for} \quad \|u\|_1 \geq R\,, \tag{2.5}$$

then for $\|u\|_1 = R$ and every $\varepsilon > 0$, one has

$$-(\pi+\varepsilon)\log\left(\frac{1}{8} + \frac{\log C_{MB}}{e^{\frac{\|u\|_1^2}{2\pi}}}\right) \geq -(\pi+\varepsilon)\log 1/4 = (\pi+\varepsilon)\log 4 \geq \pi \log 4; \tag{2.6}$$

as a consequence, for $\|u\|_1 = R$ and $\varepsilon > 0$ small enough,

$$\begin{aligned} I_{\pi+\varepsilon}(u) &\geq \left(\frac{1}{2} - \frac{\pi+\varepsilon}{2\pi}\right) R^2 + \pi \log 4 \\ &\geq -\frac{\varepsilon}{2\pi} R^2 + \pi \log 4 > \pi \log 2 > 0\,. \end{aligned}$$

Let then $B_R = \{u \in \widetilde{H}_1 :\ \|u\|_1 < R\}$. Since

$$\inf_{u \in \overline{B_R}} I_{\pi+\varepsilon}(u) \leq I_{\pi+\varepsilon}(0) = 0 < \inf_{u \in \partial B_R} I_{\pi+\varepsilon}(u)\,, \tag{2.7}$$

we conclude (up to a compactness argument as above) that the infimum is attained at a local minimum in B_R, which then yields a nontrivial positive solution.

We observe that the limiting value for ε_0 can be estimated in term of C_{MB}: in the argument above (but a finer estimate could be obtained) $R > 2\pi \log(8 \log C_{MB})$ and then $\varepsilon_0 < \frac{\log 2}{2\log^2(8\log C_{MB})}$.

□

In order to prove Theorem 1.9, we will use the following generalization (whose proof is given in the appendix) of a result by Jeanjean [Jea99] which is based on the so-called *monotonicity trick* by Struwe, see [Str88, ST98].

Theorem 2.2. *Let X be a Banach space equipped with the norm $\|\cdot\|$, and let $\mu : X \to X$ be a continuous map. We consider a family $(I_\lambda)_{\lambda\in J}$ ($J \subset \mathbb{R}^+$ is an open interval) of C^1- functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in J,$$

and suppose that

i) $B(u) \geq 0$ for all $u \in \mu(X)$;

ii) $I_\lambda(\mu(u)) \leq I_\lambda(u)$, for all $u \in X$ and $\lambda \in J$;

iii) either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$.

Assume that there are two fixed points v_0 and v_1 of μ such that for the family of maps

$$\Gamma = \{\gamma \in C([0,1], X),\ \gamma(0) = v_0,\ \gamma(1) = v_1\}, \tag{2.8}$$

it holds

$$c_\lambda := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1),\ I_\lambda(v_0)\}\ ,\quad \text{for all } \lambda \in J\ . \tag{2.9}$$

Then for almost every $\lambda \in J$ there exists a bounded PS-sequence for I_λ at level c_λ, i.e. there is $\{u_n\}_n \subset X$ with

a) $\{u_n\}_n$ is bounded,

b) $I_\lambda(u_n) \to c_\lambda$,

c) $I'_\lambda(u_n) \to 0$ in the dual space X' of X.

Remark 2.3. When μ is the identity, this is exactly [Jea99, Theorem 1.1].

Proof of Theorem 1.9. We apply Theorem 2.2 to our functional $I_{\pi+\varepsilon}$, $\varepsilon \in (0, \varepsilon_0)$, with $X = \widetilde{H}_1$, and $\eta(u) = |u|$. In view of (2.7) and Theorem 1.5-*ii*, for all $\varepsilon_1 \in (0, \varepsilon_0)$ there exists $v_1 \geq 0$ such that $I_{\pi+\varepsilon}$ with $\varepsilon \in (\varepsilon_1, \varepsilon_0)$ satisfies condition (2.9) with $v_0 = 0$. Hence, we find for a.e. $\varepsilon \in (\varepsilon_1, \varepsilon_0)$ a sequence $\{v_n\}$ satisfying a), b) and c). Due to the arbitrariness of ε_1 this is true for a.e. $\varepsilon \in (0, \varepsilon_0)$.

Since $\{v_n\}$ is bounded, we find a subsequence converging weakly and a.e. to $v \in \widetilde{H}_1$. Then the proof is easily concluded: by Lemma 2.1 we have $⨍_B e^{e^{v_n}} dx \to ⨍_B e^{e^v} dx > 1$, and similarly one also obtains $\int_B e^{v_n} e^{e^{v_n}} \varphi dx \to \int_B e^v e^{e^v} \varphi\, dx$, for all $\varphi \in \widetilde{H}_1$. Thus, from

$$0 \leftarrow I'_{\pi+\varepsilon}(v_n)(v_n - v) = \int_B \nabla v_n \nabla(v_n - v) w_1 dx - (\pi + \varepsilon) \frac{\int_B e^{v_n} e^{e^{v_n}} (v_n - v) dx}{\log ⨍_B e^{e^{v_n}} dx \int_B e^{e^{v_n}} dx},$$

we conclude that $\int_B \nabla v_n \nabla(v_n - v) w_1 dx \to 0$. As a consequence $v_n \to v$ strongly and v is a weak solution of equation (1.15). It is different from the first solution since $I_{\pi+\varepsilon}(v) > 0$, and positive by Remark 1.8.

□

3 Appendix

In this appendix we give, for sake of completeness, the proof of the logarithmic TM inequalities, that were already proved in [CR15c], and the proof of Theorem 2.2.

Proof of Proposition 1.2 . Let $\beta \in (0, 1)$. By Young inequality, if $\delta,\ \delta'$ are two conjugate exponents, then for every $s,\ t \geq 0$ one has

$$st \leq \frac{(\tau s)^{\delta'}}{\delta'} + \frac{t^\delta}{\delta \tau^\delta}, \quad \forall \tau > 0\,. \tag{3.1}$$

We need δ, δ' to be conjugate exponents and satisfy

$$\begin{cases} \theta\delta = \gamma_\beta = \frac{2}{1-\beta} \\ \theta\delta' = 2\,. \end{cases}$$

This implies

$$\theta = \tfrac{2}{2-\beta}, \quad \delta = \tfrac{2-\beta}{1-\beta}, \quad \delta' = 2-\beta\,.$$

Then one has, by taking $s = \|u\|_\beta^\theta$ and $t = \left(\frac{|u|}{\|u\|_\beta}\right)^\theta$ in (3.1), and selecting τ so that $\delta\tau^\delta = \frac{1}{\alpha_\beta}$ (see equation (1.5))

$$|u|^\theta \leq \frac{\|u\|_\beta^2}{2\lambda_\beta^*} + \alpha_\beta \left(\frac{|u|}{\|u\|_\beta}\right)^{\gamma_\beta}, \tag{3.2}$$

where λ_β^* is given in (1.7). Let now

$$C(\beta) = \log\left(\sup_{u\in\widetilde{H}_\beta\setminus\{0\}} \fint_B e^{\alpha_\beta\left(\frac{|u|}{\|u\|_\beta}\right)^{\gamma_\beta}}\,dx\right),$$

which is finite by Theorem C. Then we conclude

$$\log\fint_B e^{|u|^\theta}dx \leq \log\left(e^{\frac{\|u\|_\beta^2}{2\lambda_\beta^*}}\fint e^{\alpha_\beta\left(\frac{|u|}{\|u\|_\beta}\right)^{\gamma_\beta}}\,dx\right) \leq \frac{\|u\|_\beta^2}{2\lambda_\beta^*} + C(\beta)\,.$$

Consider now the case $\beta = 1$. Taking $a = \|u\|_1$, $b = \frac{|u|}{\|u\|_1}$ and $\varepsilon^2 = \pi$ in

$$ab \leq \frac{a^2}{4\varepsilon^2} + \varepsilon^2 b^2 \tag{3.3}$$

we get

$$|u| \leq \frac{1}{4\pi}\|u\|_1^2 + \pi\left(\frac{u}{\|u\|_1}\right)^2, \tag{3.4}$$

so that

$$\left(\fint_B e^{e^{|u|}}\,dx\right) \;\leq\; \fint_B exp\left(e^{\frac{1}{4\pi}\|u\|_1^2+\pi\left(\frac{u}{\|u\|_1}\right)^2}\right)dx \leq \fint_B exp\left(e^{\frac{1}{4\pi}\|u\|_1^2}\,e^{\pi\left(\frac{u}{\|u\|_1}\right)^2}\right)dx\,.$$

Let

$$C_{MB} = \sup_{u\in\widetilde{H}_1\setminus\{0\}}\fint_{B_1(0)} exp\left(2e^{2\pi\left(\frac{|u|}{\|u\|_1}\right)^2}\right)dx$$

(which is finite in virtue of Theorem D).

Now taking $a = e^{\frac{1}{4\pi}\|u\|_1^2}$, $b = e^{\pi\left(\frac{u}{\|u\|_1}\right)^2}$ and $\varepsilon^2 = 2$ in (3.3), one gets

$$\begin{aligned}
\log\log\left(\fint_B e^{e^u}\,dx\right) &\le \log\log \fint_B exp\left(\frac{1}{8}e^{\frac{1}{2\pi}\|u\|_1^2} + 2e^{2\pi\left(\frac{u}{\|u\|_1}\right)^2}\right)dx \\
&= \log\left[\frac{1}{8}e^{\frac{1}{2\pi}\|u\|_1^2} + \log \fint_B e^{2e^{2\pi\left(\frac{u}{\|u\|_1}\right)^2}}\,dx\right] \\
&\le \log\left(\frac{1}{8}e^{\frac{1}{2\pi}\|u\|_1^2} + \log C_{MB}\right) \\
&\le \frac{1}{2\pi}\|u\|_1^2 + \log\left(\frac{1}{8} + \frac{\log C_{MB}}{e^{\frac{\|u\|_1^2}{2\pi}}}\right).
\end{aligned}$$

□

Proof of Theorem 2.2. It suffices to show that for every $\lambda \in J$

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\mu\circ\gamma(t)) =: c_\lambda^\mu \tag{3.5}$$

Indeed, observe first that given a path $\gamma \in \Gamma$, also the path $\mu\circ\gamma \in \Gamma$, since μ is continuous and v_0, v_1 are fixed points of μ. Hence, $c_\lambda^\mu \ge c_\lambda$ because every path $\mu\circ\gamma$ on the right also appears on the left. Condition (ii) gives the reversed inequality. This, together with (i), implies that the map $\lambda \mapsto c_\lambda$ is non-increasing and then c'_λ exists for almost every $\lambda \in J$.

Then the proof can be completed as in [Jea99], Theorem 1.1, proceeding by the following steps:

1) Given $\lambda \in J$ at which c'_λ exists and a sequence $\{\lambda_n\} \subseteq J$ with $\lambda_n \nearrow \lambda$, there exist a constant $K = K(\lambda) > 0$ and a sequence of paths $\gamma_n \in \Gamma$ such that

a) $\max_{t\in[0,1]} I_\lambda(\gamma_n(t)) \le c_\lambda + (-c'_\lambda + 2)(\lambda - \lambda_n)$.

Moreover, if $\gamma_n(t)$ satisfies $I_\lambda(\gamma_n(t)) \ge c_\lambda - (\lambda - \lambda_n)$, then $\|\gamma_n(t)\| \le K$.

In the proof of this step, it is important to observe that, in view of (3.5), the paths γ_n can be chosen so that they have image in $\mu(X)$, which allows to use condition (i).

2) For $\alpha > 0$ let $F_\alpha = \{u \in X :\ \|u\| \le K + 1 \text{ and } |I_\lambda(u) - c_\lambda| \le \alpha\}$, where K is the constant of the previous step. Then

$$\inf\{\|I'_\lambda(u)\| \ :\ u \in F_\alpha\} = 0\ , \quad \text{for every}\ \ \alpha > 0\ . \tag{3.6}$$

Then, choosing $\alpha = \varepsilon_n \to 0$, we obtain by 2) a $u_n \in F_{\varepsilon_n}$ such that $\|I'_\lambda(u_n)\| \le \varepsilon_n$, which satisfies $\|u_n\| \le K + 1$ and $|I_\lambda(u_n) - c_\lambda| \le \varepsilon_n$.

□

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