Multiple solutions for some elliptic equations with a nonlinearity concave in the origin*

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Abstract

In this paper we establish the existence of multiple solutions for the semilinear elliptic problem

$$\begin{array}{lll} -\Delta u = -\lambda |u|^{q-2}u + au + g(u) & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $g : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that $g(0) = g'(0) = 0$, $\lambda > 0$ is real parameter, $a \in \mathbb{R}$, and $1 < q < 2$.

Keywords: multiplicity of solution, variational methods, concave nonlinearity

1 Introduction

Let us consider the problem

$$\begin{array}{lll} -\Delta u = -\lambda |u|^{q-2}u + au + g(u) & \text{in} & \Omega \\ u = 0 & \text{on} & \partial\Omega, \end{array} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary $\partial\Omega$, $a \in \mathbb{R}$, $\lambda > 0$ is a real parameter, $1 < q < 2$ and $g : \mathbb{R} \to \mathbb{R}$ is a function of class C^1.

Elliptic problems with nonlinearities having the concave term $|u|^{q-2}u$, $1 < q < 2$, have been studied by several authors (see [1, 8, 9, 10, 11]). The case $\lambda > 0$ was considered by Perera in [10] and by Wu & Yang in [11]. In [10], the author studied existence of multiple solutions for the coercive case and for the

*This is an Accepted Manuscript of an article published in Nonlinear Analysis, DOI: doi:10.1016/j.na.2006.04.015

†The author was supported by ProDoc-CAPES/Brazil

‡The author was supported by Fapesp Brazil.

Ambrosetti-Prodi like case, while in [11] only existence of solutions is studied, for the asymptotically linear case.

Here we will consider the multiplicity of solutions when the behavior at infinity of the nonlinearity is asymptotically linear or superlinear; in particular we consider the following assumptions:

(g_0) $g(0) = 0$,

(g_1) $g'(0) = 0$ and $a \in [\lambda_k, \lambda_{k+1})$,

(g_2) (i) $G(u) \geq 0$,
(ii) $G(u) \leq C + C|u|^p$ with $2 < p < p^* = (2N)/(N-2)$.

Moreover, let $m(t) = at + g(t)$, $M(t) = \int_0^t m(s)ds = \frac{a}{2}t^2 + G(t)$ and

$$\lim_{s \to \pm\infty} \frac{m(s)}{s} = b^\pm \ :$$

(g_3) $b^\pm \in (\lambda_{k+1}, +\infty]$.

Finally, in order to have the PS condition for the functionals we will work with, one of the following conditions will be assumed:

(g_4) (i) There exist $\bar{t} > 0$ and $\mu < 1/2$ such that $M(t) \leq \mu t m(t)$ for $|t| > \bar{t}$ (this implies $b^\pm = +\infty$);
(ii) $\mu(p-1) < \frac{N+2}{2N}$.

(g_4') $b^\pm \in \mathbb{R}$ but $(b^+, b^-) \notin \Sigma$.

(g_4'') (i) There exist $\bar{t} > 0$ and $\mu < 1/2$ such that $M(t) \leq \mu t m(t)$ for $t > \bar{t}$ (this implies $b^+ = +\infty$);
(ii) $b^- \in \mathbb{R}$ but $b^- \neq \lambda_1$;
(iii) There exists $\alpha \in [0, 1)$ such that $\lim_{s \to -\infty} \frac{m(s) - \lambda|s|^{q-2}s - (b^-)s}{|s|^\alpha} = 0$ and $\mu(p-1) < \min\{\frac{1}{1+\alpha}, \frac{N+2}{2N}\}$.

We are denoting by $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots$ the eigenvalues of $(-\Delta, H_0^1)$, where each λ_j occurs in the sequence as often as its multiplicity, and by Σ the Fučík spectrum of the operator (see for example in [6] or in [4]).

The main results of this work are the following two theorems:

Theorem 1.1 *Assume that g satisfies (g_0), (g_2)(ii), (g_3) with $k \geq 0$ and one of the (g_4)'s; then for all $\lambda > 0$ problem (1) has at least two nontrivial solutions (one positive and one negative).*

Theorem 1.2 *Assume that g satisfies (g_0)...(g_3) with $k \geq 1$ and one of the (g_4)'s; then there exist $\lambda^* > 0$ such that problem (1) has at least three nontrivial solutions for $\lambda \in (0, \lambda^*)$ (of which, one positive and one negative).*

Remark 1.3 The condition (g_4)(ii) will be used only in lemma 3.1, in order to prove the PS condition for $F^{\pm}_{\lambda}$ (see equation (3)): if we remove this condition, the effect is that we may no more distinguish the positive and the negative solution, so that we obtain only one of unknown sign in place of these two solutions.

Moreover, if hypothesis (g_4'') is satisfied without condition (iii), then F_λ does not satisfies the PS conditions but F^{-}_{λ} does, and then we still have at least the negative solution; in fact, in this case the nonlinearity $m(s)$ satisfies the hypothesis (g_4') for $s \leq 0$, then it is obvious that the negative solution we find in that case is still a solution in this one.

2 Proofs of the main Theorems

The classical solutions of the problem (1) correspond to critical points of the functional F_λ, defined, on $H^1_0 = H^1_0(\Omega)$, by

$$F_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{a}{2}\int_\Omega u^2 dx - \int_\Omega G(u)\, dx, \quad u \in H^1_0. \quad (2)$$

Under the above assumptions F_λ is a functional of class C^1.

Moreover, define, for $u \in H^1_0$,

$$F^{\pm}_{\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda}{q}\int_\Omega |u^{\pm}|^q dx - \frac{a}{2}\int_\Omega (u^{\pm})^2 dx - \int_\Omega G(u^{\pm})\, dx, \quad (3)$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$: by virtue of condition (g_0), $F^{\pm}_{\lambda} \in C^1$ and the critical points $u_\pm$ of $F^{\pm}_{\lambda}$ satisfy $\pm u_\pm \geq 0$ and so are critical points of F_λ too: actually, $(F^{\pm}_{\lambda})'(u_\pm)[(u_\pm)^{\mp}] = \int_\Omega |\nabla (u_\pm)^{\mp}|^2 dx = 0$.

2.1 Preliminary Lemmata

We first give some useful estimates based on the given hypotheses:

Lemma 2.1 *(a) If (g_0), (g_1) and (g_2)(ii) are satisfied, then for any $\varepsilon > 0$ there exists D_ε such that $G(s) \leq \frac{\varepsilon}{2}s^2 + D_\varepsilon |s|^p$.*

(b) If (g_0) and (g_2)(ii) are satisfied, then there exists D such that $G(s) \leq Ds^2 + D|s|^p$.

(c) If (g_3) is satisfied, then for $\varepsilon \geq 0$ small enough, there exists C_ε such that $M(s) \geq \frac{\lambda_{k+1}+\varepsilon}{2}s^2 - C_\varepsilon$.

Proof: By (g_0), (g_1) we have that there exists a $\delta \in (0,1)$ such that $G(s) \leq \frac{\varepsilon}{2}s^2$ for $|s| < \delta$; then the claim follows since by (g_2)(ii) we have $G(s) \leq C + C|s|^p \leq \frac{2C}{\delta^p}|s|^p$ for any $|s| \geq \delta$.

Estimate (b) is analogous, since (g_0) implies $G(s) \leq Ds^2$ for $|s| < \delta$ and suitable $D, \delta > 0$.

Finally, estimate (c) is trivial. □

Now, we give the following three theorems, which will provide the structure we need to obtain the solutions claimed in theorems 1.1 and 1.2.

Lemma 2.2 *If g satisfies (g_0) and $(g_2)(ii)$, then $u \equiv 0$ is a local minimizer for F_λ and $F_\lambda^\pm$, for any $\lambda > 0$.*

Proof: It suffices to show that 0 is a local minimizer of F_λ in the C^1 topology (see [2]). Then, for $u \in C_0^1(\overline{\Omega})$ (we use estimate (b) in lemma 2.1),

$$\begin{aligned} F_\lambda(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{a}{2}\int_\Omega u^2 dx - \int_\Omega G(u)\, dx \\ &\geq \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{a}{2}\int_\Omega u^2 dx - \int_\Omega G(u)\, dx \\ &\geq \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{a}{2}\int_\Omega u^2 dx - D\int_\Omega |u|^2 dx - D\int_\Omega |u|^p dx \\ &\geq \left(\frac{\lambda}{q} - \frac{a-2D}{2}|u|_{C^0}^{2-q} - D|u|_{C^0}^{p-q}\right)\int_\Omega |u|^q dx \geq 0 \end{aligned}$$

if $\frac{|a-2D|}{2}|u|_{C_0^1}^{2-q} + C|u|_{C_0^1}^{p-q} \leq \frac{\lambda}{q}$. The same argument works for $F_\lambda^\pm$. □

Lemma 2.3 *If g satisfies $(g_2)(ii)$ and (g_3) with $k \geq 0$, then there exists $t_0 > 0$ such that $F_\lambda^\pm(\pm t_0\varphi_1) \leq 0$, for all λ in a limited set.*

Proof: Using estimate (c) in lemma 2.1 and denoting by φ_1 the eigenfunction associated to λ_1, we have, for $t > 0$,

$$\begin{aligned} F_\lambda^\pm(\pm t\varphi_1) &= \frac{t^2}{2}\int_\Omega |\nabla\varphi_1|^2 dx + \frac{t^q\lambda}{q}\int_\Omega |\varphi_1|^q dx - \int_\Omega M(\pm t\varphi_1)\, dx \\ &\leq \frac{1}{2}t^2(\lambda_1 - \lambda_{k+1} - \varepsilon)\int_\Omega \varphi_1^2 dx + \frac{t^q\lambda}{q}\int_\Omega |\varphi_1|^q dx + C_\varepsilon|\Omega| \end{aligned}$$

and, since $\lambda_1 \leq \lambda_{k+1}$ and $q < 2$, there exists a choice of $\varepsilon > 0$ which proves the lemma. □

Lemma 2.4 *Assume that g satisfies $(g_0 ... g_3)$ with $k \geq 1$; then there exist positive numbers $r, \rho, \bar{R}$ and $\eta = \eta(\lambda)$, such that*

(i) *for all $u \in H_k := \oplus_{j=1}^k \ker(-\Delta - \lambda_j I)$ we have*

$$F_\lambda(u) \leq \eta(\lambda) \qquad \text{and} \qquad \lim_{\lambda\to 0^+}\eta(\lambda) = 0\,;$$

(ii) *for all $u \in S_r = \{u \in H_k^\perp$ with $||u|| = r\}$, we have*

$$F_\lambda(u) \geq \rho > 0 \qquad \forall\lambda > 0;$$

(iii) *for all* $u \in H_{k+1} := \oplus_{j=1}^{k+1} \ker(-\Delta - \lambda_j I)$ *and* $\|u\| \geq \bar{R}$ *we have*

$$F_\lambda(u) < 0\,.$$

Proof: (i): Let $u \in H_k$: we have, since $a \geq \lambda_k$,

$$\begin{aligned} F_\lambda(u) &= \frac{\lambda_k - a}{2}\int_\Omega u^2 dx + \frac{\lambda}{q}\int_\Omega |u|^q dx - \int_\Omega G(u)\,dx \\ &\leq \frac{\lambda}{q} K_q ||u||^q - \int_\Omega G(u)\,dx, \end{aligned} \tag{4}$$

where we denoted by K_q the constant of the continuous embedding of H_1^0 in L^q.

From estimate (c) in lemma 2.1 (with $\varepsilon = 0$), we see that we may find s_0 such that $G(s) \geq \frac{\lambda_{k+1}-a}{2} s^2 - C_0 \geq 0$ for $|s| > s_0$; then (using also $(g_2)(i)$)

$$\int_\Omega G(u)\,dx \quad \geq \quad \int_{|u|\geq s_0} \left(\frac{\lambda_{k+1} - a}{2} u^2 - C_0 \right) dx\,;$$

now, for any $z \in H_k$ with $\|z\| = 1$,

$$F_\lambda(tz) \quad \leq \quad \frac{\lambda}{q} K_q ||tz||^q - \int_{|tz|\geq s_0} \left(\frac{\lambda_{k+1} - a}{2} (tz)^2 - C_0 \right) dx\,.$$

Since the functions $z \in H_k$ with $\|z\| = 1$ are smooth, they are uniformly bounded and then (since their L^2 norm is at least $\lambda_k^{-1/2}$) there exists a common $\varepsilon > 0$ such that the sets $\Omega_z = \{x \in \Omega : |z(x)| > \varepsilon\}$ have measure $|\Omega_z| > \varepsilon$; by using this property we get, for $t > s_0/\varepsilon$ so that $\Omega_z \subseteq \{x \in \Omega : |tz(x)| > s_0\}$,

$$\int_{|tz|\geq s_0} \left(\frac{\lambda_{k+1} - a}{2} (tz)^2 - C_0 \right) dx \geq \varepsilon \frac{\lambda_{k+1} - a}{2} t^2 \varepsilon^2 - C_0 |\Omega|\,;$$

then

$$F_\lambda(tz) \quad \leq \quad t^q \frac{\lambda}{q} K_q + C_0 |\Omega| - t^2 \varepsilon^3 \frac{\lambda_{k+1} - a}{2}\,.$$

This implies that, fixed a $\bar{\lambda} > 0$, the set $T = \{u \in H^k : F_\lambda(u) \geq 0$ for some $\lambda \in (0, \bar{\lambda}]\}$ is bounded in the H_0^1 norm, that is, $\|u\| \leq C_T$ in T; then from equation (4) we get, since $G \geq 0$, that

$$F_\lambda(u) \quad \leq \quad \lambda \frac{K_q\, C_T^q}{q} \to 0 \mathrm{for} \lambda \to 0^+\,,$$

which proves the claim.

(ii): Let $u \in H_k^\perp$: we have, by estimate (a) in lemma 2.1,

$$\begin{aligned} F_\lambda(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\lambda}{q}\int_\Omega |u|^q dx - \frac{a}{2}\int_\Omega u^2 dx - \frac{\varepsilon}{2}\int_\Omega u^2 dx - C_\varepsilon \int_\Omega |u|^p dx \\ &\geq \frac{1}{2}\Big(1 - \frac{a + \varepsilon}{\lambda_{k+1}}\Big) ||u||^2 - C_\varepsilon K_p ||u||^p\,; \end{aligned}$$

since $2 < p$ and $a < \lambda_{k+1}$, we can choose r and ρ as in the lemma.

(iii): By estimate (c) in lemma 2.1, we get, for any $u \in H_{k+1}$,

$$\begin{aligned} F_\lambda(u) &\le \frac{\lambda_{k+1}}{2}\int_\Omega u^2 dx + \frac{\lambda}{q}\int_\Omega |u|^q dx - \int_\Omega M(u)\,dx \\ &\le -\frac{\varepsilon}{2}\int_\Omega u^2 dx + C_\varepsilon|\Omega| + \frac{\lambda}{q}K_q\|u\|^q\,; \end{aligned}$$

since $2 > q$ the claim is proved. □

Proof of Theorem 1.1

By lemma 2.2 and 2.3, we may apply the Relaxed Mountain Pass Theorem in [7] (Corollary 5.11) to obtain that the following levels are critical for the functional F_λ and correspond to nontrivial solutions of problem (1):

$$c_\lambda^\pm = \inf_{\gamma\in\Gamma^\pm}\sup_{t\in[0,1]} F_\lambda^\pm(\gamma(t))\,,$$

where

$$\Gamma^\pm = \{\gamma \in \mathcal{C}([0,1]) : \quad \gamma(0) = 0, \quad \gamma(1) = \pm t_0\varphi_1\}\ .$$

Moreover, as observed before, the solution corresponding to c_λ^+ is positive while that corresponding to c_λ^- is negative, and so they are distinct.

As observed in remark 1.3, if only hypothesis (g_4)(i) is satisfied, then we loose the PS condition for $F_\lambda^\pm$, so that we may guarantee only one mountain pass critical point for F_λ, of unknown sign. □

Proof of Theorem 1.2

In view of lemma 2.4 we define the set

$$Q = \left\{u + v : u \in H_k, \quad v = t\varphi_{k+1}, \quad t \ge 0, \quad \|u+v\| \le \bar{R}\right\}\ :$$

we have

- $F_\lambda(u) \le \eta(\lambda)$ for $u \in \partial Q$, by (i) and (iii) in lemma 2.4;
- $F_\lambda(u) \ge \rho$ for $u \in S_r$, by (ii) in lemma 2.4;
- ∂Q links with S_r, since by comparing (i) and (iii) in lemma 2.4 it is clear that $\bar{R} > r$.

Then, let $\lambda^* = \sup\{t > 0 : \quad \eta(\lambda) < \rho \text{ for all } \lambda \in (0,t)\}$: for $\lambda \in (0, \lambda^*)$ we may apply standard linking theorem, and then we find a new critical level $c_1 \ge \rho$.

To conclude the proof, we just need to show that the level c_1 corresponds to a new solution, in fact the two paths $\gamma^\pm : [0,1] \to H_0^1 : t \mapsto \pm t t_0\varphi_1$ belong to $\Gamma^\pm$ respectively, and since their image is contained in H_k one gets, from (i) in lemma 2.4, that $c_\lambda^\pm \le \eta(\lambda) < \rho \le c_1$. □

3 PS condition

Lemma 3.1 *If one of the three hypotheses* (g_4), (g_4'), (g_4'') *is satisfied, then the functionals* F_λ *and* $F_\lambda^\pm$ *satisfy the PS condition for any* $\lambda > 0$.

Proof: The claim for F_λ follows from [3] (Lemma 2.2 in chapter III) under the hypothesis (g_4)(i), and from [5] in the case (g_4''); for the case (g_4') we do as follows.

CASE (g_4'): Suppose we have a sequence $\{u_n\} \subseteq H_0^1$ such that $|F(u_n)| \le K$ and $|F'(u_n)[v]| \le \varepsilon_n \|v\|$ for any $v \in H_0^1$: it is a classical result that the PS condition follows if we prove that $\{u_n\}$ is bounded, so suppose $1 \le \|u_n\| \to \infty$ and define $U_n = \frac{u_n}{\|u_n\|}$ so that, up to a subsequence, $U_n \to U$ weakly in H_0^1 and strongly in L^2.

By (g_4') we may estimate $|m(s) - b^+ s^+ - b^- s^-| \le \varepsilon |s| + C_\varepsilon$ and so

$$\int_\Omega \frac{|m(u_n) - b^+ u_n^+ - b^- u_n^-|}{\|u_n\|} dx \le \varepsilon \int_\Omega |U_n|\, dx + \frac{C_\varepsilon}{\|u_n\|}\,,$$

for any $\varepsilon > 0$, and then tends to zero.

Now consider, for any $v \in H_0^1$:

$$\left|\frac{F'(u_n)[v]}{\|u_n\|}\right| = \left|\int_\Omega \nabla U_n \nabla v dx + \lambda \int_\Omega \frac{|u_n|^{q-2} u_n v}{\|u_n\|} dx - \int_\Omega \frac{m(u_n) v}{\|u_n\|} dx\right| \le \varepsilon_n \frac{\|v\|}{\|u_n\|} \;:$$

summing and subtracting $\int_\Omega \frac{b^+ u_n^+ + b^- u_n^-}{\|u_n\|} v\, dx$ and since $q < 2$ gives

$$\int_\Omega \nabla U_n \nabla v dx - b^+ \int_\Omega U_n^+ v dx - b^- \int_\Omega U_n^- v dx \to 0 \tag{5}$$

and taking weak limit gives

$$\int_\Omega \nabla U \nabla v dx - b^+ \int_\Omega U^+ v dx - b^- \int_\Omega U^- v dx = 0\,,$$

which implies $U = 0$ since $(b^+, b^-) \notin \Sigma$; this gives contradiction since once that $U_n \to 0$ in L^2, by choosing $v = U_n$ in (5) one would deduce $1 = \int_\Omega |\nabla U_n|^2 \to 0$.

For F_λ^+ one has that in the cases (g_4) (with (ii)) and (g_4'') it satisfies the hypotheses in [5], while in the case (g_4') it still satisfies (g_4') with $b^- = 0$: actually $(b^+, 0) \notin \Sigma$ since otherwise this would imply $b^+ = \lambda_1$ but then also $(b^+, b^-) \in \Sigma$, which was excluded by hypothesis.

Finally, for F_λ^- one reduces again to the case (g_4'), now with $b^+ = 0$, both in case (g_4') and (g_4''), while the case (g_4) reduces to the hypotheses of [5] by simply changing the sign of the variable in the nonlinearity.

Note that, as observed in remark 1.3, hypothesis (g_4)(ii) is used just to prove the PS condition for $F_\lambda^\pm$: it is not necessary for F_λ; on the other side hypothesis (g_4'')(iii) is needed for the PS condition of F_λ but not for F_λ^-. □

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