Liouville-type theorems for nonlinearities with zeros

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Why do we need Liouville type theorems? I

We consider the problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u = \lambda h(x,u); \quad u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

where $\lambda > 0$, Ω bounded smooth domain in $\mathbb{R}^N$, $N > p$.

L. Iturriaga, E. Massa, J. Sánchez, and P. Ubilla, *Positive solutions of the p-Laplacian involving a superlinear nonlinearity with zeros*, J. Differential Equations **248** (2010), no. 2, 309–327.

(H_1) $h : \overline{\Omega} \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous, $h(x,0) = 0$.

(H_2) (positive zero) Exists $a \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ with $-\Delta_p a \geq 0$ and $0 < a_0 \leq a(x) \leq A_0$:

$$\begin{cases} h(x,t) = 0 & \text{if } t = a(x), \\ h(x,t) > 0 & \text{if } t \neq a(x),\ t > 0 \end{cases}$$

Why do we need Liouville type theorems? II

(H_3) (behavior near origin)

$$\lim_{u \longrightarrow 0^+} \frac{h(x,u)}{u^{p-1}} = b(x)$$ uniformly with respect to $x \in \Omega$ with $b \in L^\infty(\Omega)$ and $0 < b_0 \leq b(x) \leq B_0$.

(M_2) there exists a constant $k > 0$ such that $\forall\ x \in \Omega$ the map $s \mapsto h(x,s) + k\, s^{p-1}$ is increasing.

(H_4) (behavior at infinity)

$$\lim_{u \longrightarrow +\infty} \frac{h(x,u)}{u^\sigma} = \rho \text{ uniformly with respect to } x \in \Omega.$$

with $\rho > 0$ e $\sigma \in (p-1, p_* - 1)$,

Existence and Multiplicity

Theorem

In these hypotheses there exists a positive solution for every $\lambda > 0$

Theorem

In the same hypotheses, if one of the following holds,

(a) $p = 2$.

(b) $a(x) \equiv \bar{a}$, *(positive constant)* ; *exists* $C > 0$: $h(x,t) \leq C|\bar{a} - t|^{p-1}$ *for* $t \leq \bar{a}$.

(c) $-\Delta_p a \in L^\infty(\Omega)$ *and* $-\Delta_p a(x) > \varepsilon > 0$ *a.e.* $x \in \Omega$.

(d) $a \in C^1$ *and* $\nabla a \neq 0$ *in* Ω.

there exists a second positive solution for $\lambda > \lambda_1(b)$.

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(c) $-\Delta_p a \in L^\infty(\Omega)$ and $-\Delta_p a(x) > \varepsilon > 0$ a.e. $x \in \Omega$.

(d) $a \in \mathcal{C}^1$ and $\nabla a \neq 0$ in Ω.

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Asymptotical behavior when $\lambda \to \infty$

Theorem

In the above hypotheses, plus:

(H_7) *(behavior above the zero)* *exists* $\gamma > 0$ *such that*
$h(x,t) \geq \gamma |t - a(x)|^\sigma$ *for* $t \geq a(x)$,

then $u_\lambda \to a$ *pointwise in* Ω *when* $\lambda \to +\infty$.

Proof:

(1) get an a-priori estimate for the solutions when λ large (blow-up technique):

- suppose $\lambda_n \to \infty$, $\|u_n\|_\infty = u_n(x_n) \to \infty$;
- For suitable $A_n \to 0$, $w_n(y) := u_n(A_n y + x_n)/u_n(x_n)$ converges to a solution of

$$-\Delta w = w^\sigma, \quad w \geq 0, \qquad in \quad \mathbb{R}^N \quad or \quad \mathbb{R}^N_+;$$

- then (Liouville-type theorem) w=0: contradiction, since $w_n(0) = 1$.

The Liouville-type theorems used here are:

Lemma

a) (Theorem 2.1 of [MP99]) If $u \in \mathcal{C}^1(\mathbb{R}^N)$ satisfies, in the weak sense,

$$-\Delta_p u \geq u^{q-1}, \quad u \geq 0 \quad in \quad \mathbb{R}^N$$

and if $N > p$, $q \in (1, p_*)$, then $u \equiv 0$.

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b) (Theorem 3.1 of [Lor07]) If $u \in \mathcal{C}^1(\mathbb{R}^N_+)$ satisfies, in the weak sense,

$$Cu^{q-1} \geq -\Delta_p u \geq u^{q-1}, \quad u \geq 0 \quad in \quad \mathbb{R}^N_+$$

and if $q \in (p, p_*)$, then $u \equiv 0$.

(2) Blow-up-like argument:

• Fix $x_0 \in \Omega$, suppose $\lambda_n \to \infty$;

• For suitable $A_n \to 0$, $w_n(y) := u_n(A_n y + x_0)$ converges to a solution of

$$-\Delta w = h(x_0, w), \quad w > 0 \qquad in \quad \mathbb{R}^N,$$

• What can we say about w?

−− The previous Liouville Theorem does not apply because the nonlinearity is not strictly positive: $h(x_0, w) \not\geq w^{q-1}$.

−− However, we can prove a Liouville-type Theorem which implies that $h(x_0, w) \equiv 0$, then $w \equiv a(x_0)$:

• As a consequence we prove that $u_n(x_0) \to a(x_0)$: the solution tends pointwise to the function $a(x)$ in Ω, as claimed.

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Our Liouville-type theorem

Theorem

Let $f : [0, \infty) \to [0, \infty)$ be a continuous function satisfying the following four assumptions:

(f_1) *(zero) There exists an $\bar{a} > 0$ such that* $\begin{cases} f(t) = 0 & \text{if } t = 0 \text{ or } t = \bar{a}\,, \\ f(t) > 0 & \text{if } t \neq \bar{a},\ t > 0. \end{cases}$

(f_2) *(above the zero) There exist constants $\gamma > 0$ and $\sigma \in (p-1, p_* - 1)$ such that $f(t) \geq \gamma(t - \bar{a})^\sigma$, for $t > \bar{a}$.*

(f_3) *(near the origin) There exists a constant $\bar{b} > 0$ such that $\lim_{t \longrightarrow 0^+} \frac{f(t)}{t^{p-1}} = \bar{b}$.*

(f_4) *(growth) There exists a constant $\Lambda > 0$ such that $0 \leq f(t) \leq \Lambda(t^\sigma + 1)$, for $t \geq 0$.*

Then any $\mathcal{C}^1$ weak solution of the problem $\begin{cases} -\Delta_p w = f(w) & \text{in } \mathbb{R}^N, \\ w \geq 0\,, \end{cases}$

is either the constant function $w \equiv 0$, or else $w \equiv \bar{a}$.

Proof of the Liouville-type theorem

We will use the following results:
Let w be a $\mathcal{C}^1$ weak solution of the equation

$$-\Delta_p w = f(w) \ \text{ in } \ \mathbb{R}^N ,$$

• Harnack-type inequality from Theorem-V [SZ02]:
Provided $w \geq 0$ and there exists $\delta, \Lambda > 0$ such that, for $w \geq 0$,

$$\delta w^\sigma - w^{p-1} \leq f(w) \leq \Lambda (w^\sigma + 1) , \tag{2.1}$$

then $\forall R > 0$, $\exists c(R)$ such that $\sup_{B_R} w \leq c(R) \inf_{B_R} w$, for any ball B_R of radius R.

• Extension to the p-Laplacian of result due to Redheffer (see Theorem 1–[Red86]).
If f is a continuous nonnegative function, then either $\inf_{\mathbb{R}^N} w = -\infty$, or $\inf_{\mathbb{R}^N} w$ is a zero of f.

Proof of the Liouville-type theorem •

We also use

- Picone´s identity:

Let $u, v \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ be such that $u \geq 0$, $v > 0$, and $\frac{u}{v} \in W^{1,p}_{loc}(\Omega)$. Then

$$\int_\Omega \nabla \left(\frac{u^p}{v^{p-1}} \right) |\nabla v|^{p-2} \nabla v =$$
$$= \int_\Omega p \left(\frac{u}{v} \right)^{p-1} \nabla u |\nabla v|^{p-2} \nabla v - (p-1) \left(\frac{u}{v} \right)^p |\nabla v|^p \leq \int_\Omega |\nabla u|^p . \tag{2.2}$$

Proof of the Liouville-type theorem • •

• if $w \geq \bar{a}$ then change of unknown $v = w - \bar{a}$:

$$\begin{cases} -\Delta_p v = f(v + \bar{a}) \ \text{ in } \ \mathbb{R}^N , \\ v \geq 0 . \end{cases} \tag{2.3}$$

By hypothesis (f_2), we have $f(v + \bar{a}) \geq \gamma v^\sigma$.
It then follows from ▸ L-T1 that $v \equiv 0$, that is, $w \equiv \bar{a}$.
• Then suppose $\inf_{\mathbb{R}^N} w < \bar{a}$, and so ▸ Redheffer implies that $\inf_{\mathbb{R}^N} w = 0$.

Proof of the Liouville-type theorem • • •

- Let $\varepsilon \in (0, \overline{a})$ such that $f(t)/\left(t^{p-1}\right) > \overline{b}/2$,for $t \in (0, \varepsilon)$.
- Let $R > 0$ be such that $\lambda_1(B_R) < \overline{b}/4$·
- Since $\inf_{\mathbb{R}^N} w = 0$, $\exists\, x_R$ such that $B_R(x_R)$ satisfies $\inf_{B_R(x_R)} < \dfrac{\varepsilon}{c(R)}$.
- Then, by Harnack, $w < \varepsilon < \overline{a}$ in $B_R(x_R)$.

Let now Φ_1 be the first eigenfunction in B_R, suppose $\inf_{B_R} w > 0$. Then $\frac{\Phi_1^p}{w^{p-1}}$ is in $W^{1,p}(B_R)$ and by Picone

$$\int_{B_R} \nabla\left(\frac{\Phi_1^p}{w^{p-1}}\right) |\nabla w|^{p-2}\nabla w \leq \int_{B_R} |\nabla\Phi_1|^p = \lambda_1(B_R)\int_{B_R}\Phi_1^p$$

On the other hand,

$$\int_{B_R} \nabla\left(\frac{\Phi_1^p}{w^{p-1}}\right) |\nabla w|^{p-2}\nabla w = \int_{B_R} f(w)\frac{\Phi_1^p}{w^{p-1}} \geq \int_{B_R}\frac{\overline{b}}{2}\Phi_1^p$$

which is impossible because $\lambda_1(B_R) < \overline{b}/4$. Thus $\inf_{B_R} w = \sup_{B_R} w = 0$, that is, $w \equiv 0$.

Extension of the Redheffer result

Proposition

Let w be a $\mathcal{C}^1$ weak solution of the equation

$$-\Delta_p w = f(w) \quad in \quad \mathbb{R}^N,$$

where f is a continuous nonnegative function. Then either $\inf_{\mathbb{R}^N} w = -\infty$, or $\inf_{\mathbb{R}^N} w$ is a zero of f.

•Proof similar to Redheffer´s for the Laplacian:
– Roughly speaking, if $\inf_{\mathbb{R}^N} w = M \in \mathbb{R}$ with $f(M) > 0$, then one finds a set where $M \leq w \leq M + \varepsilon$, $-\Delta_p w = f(w) > \alpha > 0$, and then, considering $W = w + \delta |x|^{p/(p-1)}$ with δ small, one gets a contradiction with the maximum principles (a set where W is p-superharmonic and has an interior minimum).
– For the p-Laplacian one has to find a suitable maximum principle and cannot use the linearity of the operator.

Any growth at infinity

Remark: if we put $w_n(y) := u_n(A_n y + x_n)$ in blow-up we would get, for λ large, a bound from above ($\|u_n\|_\infty \leq \|a\|_\infty + \varepsilon$), then we could truncate and remove the growth condition.
However we do not know if the limit problem is in $\mathbb{R}^N$ and we have no Liouville-type theorem in the $\mathbb{R}^N_+$.

In

L. Iturriaga, S. Lorca, and E. Massa, *Positive solutions for the p-laplacian involving critical and supercritical nonlinearities with zeros*, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010), no. 2, 763–771,

we used a moving plane type result for the p-Laplacian to guarantee that the limit problem is in $\mathbb{R}^N$, but we needed several restrictions.
A better result could be obtained if we could prove a Liouville-type theorem in $\mathbb{R}^N_+$ for a nonlinearity with zeros.

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