Multiplicity of solutions for the p-Laplacian involving a nonlinearity with zeros

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The first problem (Iturriaga Sanchez Ubilla)

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u = \lambda h(x,u); \quad u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

where $\lambda > 0$, Ω bounded smooth domain in $\mathbb{R}^N$,

(H_1) $h : \overline{\Omega} \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous, $h(x,0) = 0$.

(H_2) Exists $a \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ with $-\Delta_p a \geq 0$ and $0 < a_0 \leq a(x) \leq A_0$

$$\begin{cases} h(x,t) = 0 & \text{if } t = a(x), \\ h(x,t) > 0 & \text{if } t \neq a(x),\ t > 0 \end{cases}$$

(H_3) $\displaystyle\lim_{u \longrightarrow 0^+} \frac{h(x,u)}{u^{p-1}} = b(x)$ uniformly with respect to $x \in \Omega$ with $b \in L^\infty(\Omega)$ and $0 < b_0 \leq b(x) \leq B_0$.

(M_2) there exists a constant $k > 0$ such that $\forall\ x \in \Omega$ the map $s \mapsto h(x,s) + k\, s^{p-1}$ is increasing.

Existence

(H_4)

$$\lim_{u \longrightarrow +\infty} \frac{h(x,u)}{u^\sigma} = \rho \text{ uniformly with respect to } x \in \Omega.$$

with $\rho > 0$ e $\sigma \in (p-1, p_* - 1)$,

Theorem

In these hypotheses exists a solution for every $\lambda > 0$

- For $\lambda > \lambda_1(b)$ exists a subsolution $(\varepsilon\phi_1(b))$; a is supersolution
- For $\lambda < \lambda_1(b)$ origin is a minimum, superlinearity implies mountain pass geometry
- For $\lambda = \lambda_1(b)$ taking limit of the above, with $\lambda < \lambda_1(b)$

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Multiplicity

Theorem

In the same hypotheses, if one of the following holds,

(a) $p = 2$.

(b) $a(x) \equiv \bar{a}$, *(positive constant)* ;
exists $C > 0$: $h(x, t) \leq C|\bar{a} - t|^{p-1}$ for $t \leq \bar{a}$.

(c) $-\Delta_p\, a \in L^\infty(\Omega)$ *and* $-\Delta_p\, a(x) > \varepsilon > 0$ *a.e.* $x \in \Omega$.

(d) $a \in \mathcal{C}^1$ *and* $\nabla\, a \neq 0$ *in*Ω.

the there exist a second positive solution for $\lambda > \lambda_1(b)$.

- using a,b,c,d, one shows that first solution satisfies $u < a$.
- then it is a local minimum (De Figueiredo Gossez Ubilla)
- second solution via mountain pass.

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Asymptotical behavior when $\lambda \to \infty$

Theorem

In the above hypotheses, plus:

(H_7) *exists* $\gamma > 0$ *such that* $h(x,t) \geq \gamma |t - a(x)|^\sigma$ *for* $t \geq a(x)$,

then $u_\lambda \to a$ *poointwise in* Ω *when* $\lambda \to +\infty$.

Proof:

- get an a-priori estimate for the solutions when λ large (blow-up technique):
  suppose $\lambda_n \to \infty$, $\|u_n\|_\infty = u_n(x_n) \to \infty$;
  $w_n(y) := u_n(A_n y + x_n)/S_n$ converges to a solution of
  $-\Delta w = w^\sigma$, $w \geq 0$ in $\mathbb{R}^N$ or half space;
  then (Liouville-type theorem) w=0: contradiction.
- similar blow-up argument:
  fix $x_0 \in \Omega$, suppose $\lambda_n \to \infty$;
  $w_n(y) := u_n(A_n y + x_0)$ converges to a solution of
  $-\Delta w = h(x_0, w)$, $w > 0$ in $\mathbb{R}^N$,
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Remark: if we put $w_n(y) := u_n(A_n y + x_n)$ **we would get a bound from above, then we could truncate. However we do not know if the limit problem is in** R^N **and we have no Liouville-type theorem in the half space.**

We need a moving plane type result!

(Damascelli) Ω **convexo,** $h(x,u) = f(u)$ **locally Lipschitz in** $(0,\infty)$**,** $f(u) > 0$ **for** $u > 0$**,**

second problem (Iturriaga Lorca)

$$(\Pi_\lambda) \qquad \begin{cases} -\Delta_p u = \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{em} \quad \partial\Omega, \end{cases}$$

Ω bounded smooth convex domain in $\mathbb{R}^N$, $\lambda > 0$.

(F_1) $f : [0, +\infty) \to [0, +\infty)$ continuous and locally Lipschitz in $(0, \infty)$; $f(0) = f(1) = 0$ and $f(x) > 0$ for $x \notin \{0; 1\}$.

(F_2) $\liminf_{s \to 0^+} \dfrac{f(s)}{s^{p-1}} \geq 1$.

(F_3) $\lim_{t \to 1} \dfrac{f(t)}{|t-1|^\sigma} = \gamma$, with $\gamma > 0$ e $\sigma \in (p-1, p_* - 1)$

(F_4) Exist $k > 0$ and $T > 1$ such that $t \mapsto f(t) + kt^{p-1}$ is increasing for $t \in [0, T]$.

(no restriction on the growth at infinity)
example: $f(u) = u^{p-1} e^u |1-u|^\sigma$ with $\sigma \in (p-1, p_* - 1)$.

Theorem

There exists $\lambda^* > 0$ such that the problem (Π_λ) has at least two positive solutions $u_{1,\lambda}$, $u_{2,\lambda}$, for $\lambda > \lambda^*$.
moreover $\|u_{1,\lambda}\|_\infty \to 1^-$ and $\|u_{2,\lambda}\|_\infty \to 1^+$, when $\lambda \to \infty$.

- truncate the nonlinearity above $T > 1$
- let $\tau > 0$ and consider the positive nonlinearity $f_T(u) + \tau(u^+)^{p-1}$
- 1 is no more supersolution, but one find a family of supersolutions near 1
- first solution via sub-supersolutions (strictly below 1)
- second solution via degree argument
- By Damascelli the maxima are far from the boundary
- taking limit $\tau \to 0$ we get solutions with $\tau = 0$ and still maxima are far from the boundary
- then $\|u\|_\infty \to 1$ when $\lambda \to \infty$
- then for λ large solution below T (original problem).

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