On the validity of some classical techniques for the stationary Kirchhoff Equation

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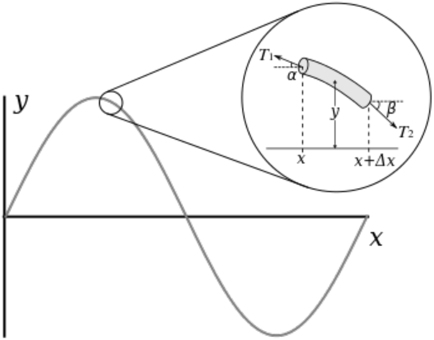
Kirchhoff equation

The vibrating string equation:



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Kirchhoff equation

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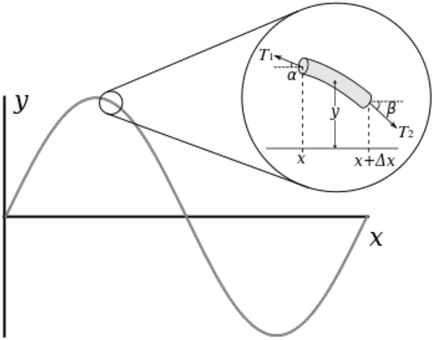

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In (Kirchhoff, 1883) the tension depends on the length:

$$\rho u_{tt} = f + (T_0 + k\Delta L)u_{xx}\,.$$

Approximating the length:

$$L \simeq \int_0^{L_0} \sqrt{1+u_x^2} \simeq \int_0^{L_0} 1 + \frac{1}{2}u_x^2 = L_0 + \frac{1}{2}\int_0^{L_0} u_x^2$$

We get the Kirchhoff equation: (nonlocal equation)

$$\rho u_{tt} = f + \left(T_0 + \frac{k}{2}\int_0^{L_0} u_x^2\right) u_{xx}$$

and the stationary Kirchhoff equation:

$$-\left(T_0 + \frac{k}{2}\int_0^{L_0} u_x^2\right) u_{xx} = f$$

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Stationary Kirchhoff equation

Here we consider the "Stationary Kirchhoff Equation": the following (time independent) generalization of the Kirchhoff vibrating string equation:

$$(K) \qquad \begin{cases} -M(\|u\|_H^2)\Delta u = f(x,u) & in \;\; \Omega\,, \\ u = 0 & on \;\; \partial\Omega\,, \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain,
- M: nonlocal weight function,
- $\|\cdot\|_H$ is the norm in $H_0^1(\Omega)$,
- f is some nonlinearity.

Comparison Principle, Sub- Supersolutions Method

Comparison principle for the Laplacian (weak form)

$$\begin{cases} -\Delta \ell \leq -\Delta w & in \ \Omega\,, \\ \ell \leq w & on \ \partial\Omega\,, \end{cases} \implies \ell \leq w \ in \ \Omega\,. \qquad (1)$$

Question: Does it hold for Kirchhoff operator?

$$\begin{cases} -M(\|\ell\|_H^2)\Delta \ell \leq -M(\|w\|_H^2)\Delta w & in \ \Omega\,, \\ \ell \leq w & on \ \partial\Omega\,, \end{cases} \qquad (2)$$

$\implies \quad \ell \leq w \ in \ \Omega \qquad ?$

Some answers:

- (Alves and Corrêa, 2001) : if $M(t) \geq 0$, $M(t)$ nonincreasing, $M(t^2)t$ increasing, then CP and SSM hold true.
- If $M(t_1^2)t_1 \geq M(t_2^2)t_2$ for some positive $t_1 < t_2$, then CP is false: take $\ell = t_2\phi_1$ and $w = t_1\phi_1$.
- Several papers claiming CP and SSM hold true if $M(t) \geq m_0 > 0$, $M(t)$ nondecreasing.

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Counterexamples:

- (García-Melián and Iturriaga, 2016): if M “increases enough”, then CP and SSM are false.
- (Figueiredo and Suárez, 2018): for certain $M(t) = a + b(t + c)^p$ CP and SSM are false.

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Assume $N \geq 3$ and M is continuous, positive and verifies:

$$\text{(H)} \qquad \text{there exist } R_2 > R_1 > 0 \text{ such that } \frac{M(R_2^{N-2})}{R_2^2} > \frac{M(R_1^{N-2})}{R_1^2}.$$

Then CP and SSM hold false in $\Omega = B \subseteq \mathbb{R}^N$.

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We answer in (Iturriaga and M., 2018):[1]

Theorem ((Iturriaga and M., 2018))

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Suppose M is not nonincreasing, that is, there exist positive $t_1 < t_2$ such that $M(t_1) < M(t_2)$. Then the Comparison Principle (both in its weak and strong form) and the Sub and Supersolution Method do not hold in Ω, for the operator

$$-M(\|u\|_H^2)\Delta u\,.$$

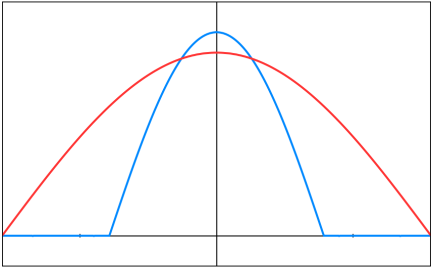
[1] L. Iturriaga and E. M. (2018). "On necessary conditions for the comparison principle and the sub- and supersolution method for the stationary Kirchhoff equation". In: *J. Math. Phys.* 59.1, pp. 011506, 6

(H) there exist $R_2 > R_1 > 0$ such that $\frac{M(R_2^{N-2})}{R_2^2} > \frac{M(R_1^{N-2})}{R_1^2}$.

In B_{R_2} take

$$\ell = \begin{cases} \eta\phi_1\left(x/R_1\right) & |x| \le R_1 \\ 0 & |x| \ge R_1 \end{cases} \qquad w = \phi_1(x/R_2)$$

where ϕ_1 is the first eigenfunction in the unitary ball.



Then (H) allows to select $\eta > 1$ so that

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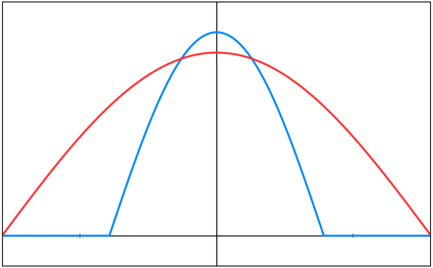

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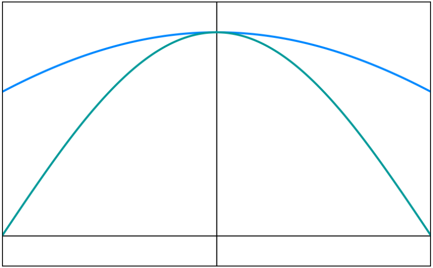
Comparison Principle - Our argument

Idea (dimension 1: $\Omega = (-\pi/2, \pi/2)$): take $\tau > 0$ small and

$$w = \min\left\{\cos\left(\frac{x}{1+\tau}\right), \frac{1}{\varepsilon}\cos(x)\right\},$$



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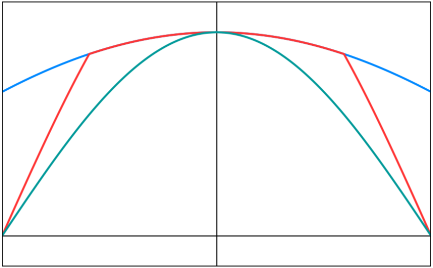

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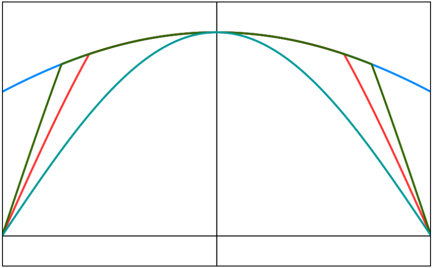

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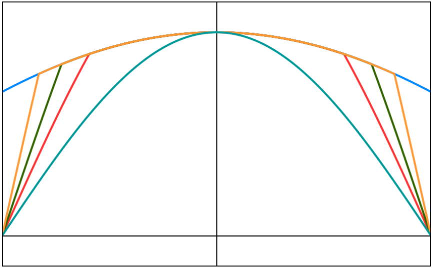

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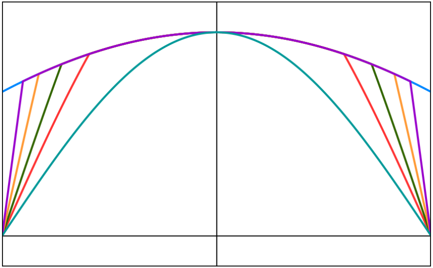

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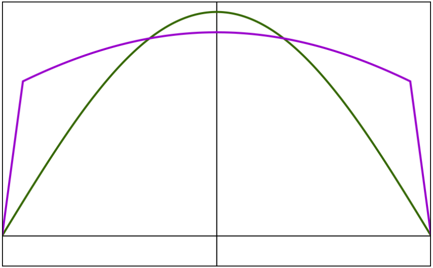

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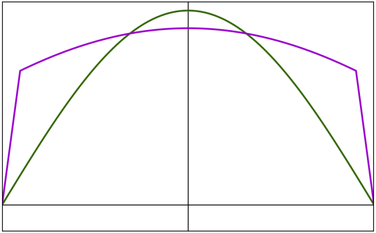

finally choose ε, τ, rescale and find parameters so that

$$\lambda^\tau M(t_2) > \lambda_1 M(t_1), \quad \|w\|_H^2 = t_2 > t_1 = \|\ell\|_H^2, \qquad \eta > 1,$$

$$-M(\|w\|_H^2)\Delta w \geq M(t_2)\lambda^\tau w \geq M(t_1)\lambda_1 \ell = -M(\|\ell\|_H^2)\Delta\ell$$

Higher dimension and general domain:

Same idea:

$$\boxed{w} = \min\left\{\phi_\tau, \frac{1}{\varepsilon}\phi_1\right\}, \qquad \boxed{\ell} = \eta\phi_1,$$

where

- ϕ_1 is the first eigenfunction in Ω,
- ϕ_τ is the first eigenfunction in

$$\Omega_\tau = \left\{x \in \mathbb{R}^N : \ d(x, \Omega) < \tau\right\}.$$

Some remarks:

- counterexamples for Strong Comparison Principle and the Sub and Supersolution method are obtained in a similar way,
- the same argument works for p-Laplacian,
- versions of CP and SSM which work for a wider range of M exist, but always require additional hypotheses (Alves and Corrêa, 2015; Figueiredo and Suárez, 2018).

Kirchhoff equation: variational approach

$$\begin{cases} -M(\|u\|_W^p)\Delta_p u = f(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3}$$

$$J(u) = \frac{1}{p}\widehat{M}(\|u\|_W^p) - \int_\Omega F(x,u)\,, \qquad u \in W_0^{1,p}(\Omega)\,. \tag{4}$$

Here T4

- $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain,
- $\|\cdot\|_W$ is the norm in $W_0^{1,p}(\Omega)$, $p > 1$,
- $\widehat{M}(t) = \int_0^t M(s)\,ds$ and $F(x,v) = \int_0^v f(x,s)\,ds$.

Several authors:
Alves, Ambrosetti, Anello, Arcoya, Cheng, Colasuonno, Corrêa, Figueiredo, Liu, Ma, Madeira, Nunes, Pucci, Santos J., Siciliano, Song, Tang, Wu.

Hölder versus Sobolev minimizers

Theorem ((Brezis and Nirenberg, 1993))

Let $J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u), \qquad u \in H_0^1(\Omega) \qquad (..)$

If $J(u_0) \leq J(u_0 + v)$ *for* $v \in \mathcal{C}_0^1(\Omega)$ *with* $\|v\|_{\mathcal{C}^1}$ *small*

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Analogous in $W_0^{1,p}$ by (García Azorero, Peral Alonso, and Manfredi, 2000; Guo and Zhang, 2003; Brock, Iturriaga, and Ubilla, 2008).

Question: What happens for

$J(u) = \frac{1}{p}\widehat{M}(\|u\|_W^p) - \int_\Omega F(x,u), \qquad u \in W_0^{1,p}(\Omega)$?

- If $M(t) \geq m_0 > 0$ (non degenerate case) then an analogous holds true. (Fan, 2010).

We study the degenerate case, in particular we take

- $M \geq 0$, M continuous, $M(0) = 0$.

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Model problem: let $p^* > \omega > q \geq 1$ and $r > p$:

$$\mathcal{J}(u) = \frac{1}{r}\|u\|_W^r + \frac{1}{q}\|u\|_q^q - \frac{\lambda}{\omega}\|u\|_\omega^\omega \,, \tag{5}$$

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First result: a negative answer:

Theorem ((Iturriaga and M., 2019)[1])

If $r > p^*$ then

- $\mathcal{J}(0) \leq \mathcal{J}(v)$ *for* $v \in L^\infty \cap W_0^{1,p}$ *with* $\|v\|_{L^\infty}$ *small*
- *there exists a sequence* u_n *in* $W_0^{1,p}(\Omega)$ *with* $\|u_n\|_W \to 0$, *such that* $\mathcal{J}(u_n) < 0$.

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Proof (simplified): Let ψ_ε be compact support approximations of the generalized Talenti functions

$$\psi_\varepsilon(x) = \left(C_{N,p} \frac{\epsilon^{\frac{1}{p-1}}}{\epsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}} \right)^{\frac{N-p}{p}} \qquad \text{with } \epsilon > 0, \qquad (7)$$

take $u_n = \varepsilon_n^\sigma \psi_{\varepsilon_n}$ where $\varepsilon_n \to 0$ and

$$\frac{N}{p^*} > \sigma > \frac{N}{p^*} \frac{p^* - \omega}{r - \omega} \geq 0,$$

then

- u_n is unbounded in L^∞,
- the last term in $\mathcal{J}$ dominates then $\mathcal{J}(u_n) < 0$ as $n \to \infty$.

$J(u_n) < 0$ holds true under more general hypotheses:

- $\frac{1}{p}\widehat{M}(s^p) \leq C_1 s^r, \qquad$ for s small,
- $F(x, v) \geq C_2 v^\omega - C_3 v^q, \qquad$ for $x \in \Omega$, $v \geq 0$.

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- $\frac{1}{p}\widehat{M}(s^p) \leq C_1 s^r, \qquad$ for s small,
- $F(x, v) \geq C_2 v^\omega - C_3 v^q, \qquad$ for $x \in \Omega,\ v \geq 0$.

Now a positive answer

Theorem ((Iturriaga and M., 2019))

Suppose in (3-4)

- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *is continuous and there exist constants* $D > 0$ *and* $\ell \in [p, p^*)$ *such that*

$$f(x, v) sgn(v) \leq D|v|^{\ell-1}, \qquad \forall\, (x, v) \in \Omega \times \mathbb{R}.$$

- $M(t) \geq 0$ *for every* $t \geq 0$ *and there exist constants* $a_1, \delta > 0$ *and* $r \in (p, p^*)$ *such that*

$$M(s^p) \geq \frac{r\, a_1}{p} s^{r-p} \; (\Rightarrow \widehat{M}(s^p) \geq Cs^r), \qquad \text{for } 0 \leq s^p < \delta.$$

Then, If the origin is a local minimum for J with respect to the L^∞ *norm, then it is also a local minimum with respect to the* $W_0^{1,p}$ *norm.*

Steps of the proof:

- Suppose the origin is not a local minimum.
- Then there exist a sequence v_n of minimizers in sets $B_n = \{\int_\Omega u^\ell \leq \frac{1}{n}\}$, with $J(v_n) < 0$, which satisfy the equation (may be with an additional term due to a Lagrange multiplier), moreover $\|v_n\|_W \to 0$.
- By Moser's iterations, $f(x,v)sgn(v) \leq D|v|^{\ell-1}$ implies

$$\|u\|_\infty \leq C_1(\ell, p, \Omega) D^{\frac{1}{p^*-\ell}} \|u\|_{p^*}^{\frac{p^*-p}{p^*-\ell}}$$

  for weak solutions of $-\Delta_p u = f(x,u)$.
- For weak solutions of $-M(\|u\|_W^p)\Delta_p u = f(x,u)$, using $M(\|u\|_W^p)^{-1} \leq \frac{p}{ra_1} \|u\|_W^{p-r}$, we get

$$\begin{aligned}\|v_n\|_\infty &\leq C_1(....) \|v_n\|_W^{\frac{p-r}{p^*-\ell}} \|v_n\|_{p^*}^{\frac{p^*-p}{p^*-\ell}} \\ &\leq C(....) \|v_n\|_W^{\frac{p^*-r}{p^*-\ell}}.\end{aligned}$$

- Since $\|v_n\|_W \to 0$, then $\|v_n\|_\infty \to 0$.
- Then the origin is not a minimum w.r.t. to L^∞ norm either.

For the more classical kind of result involving the $\mathcal{C}^1$ norm, one needs more restrictions, in particular a balance between r and ℓ:

$$(\ell - 1) > (r - p)\,\frac{p^* - 1}{p^* - p}\,.$$

Steps of the proof:

- Obtain the estimate for $\|v_n\|_\infty$ as before,
- obtain an estimate for $M(\|v_n\|_W^p)^{-1}\,\|f(x, v_n)\|_\infty$,
- bootstrap to a uniform estimate for the $\mathcal{C}^{1,\alpha}$ norm (via (Lieberman, 1988)),
- apply Ascoli-Arzela Theorem to get a subsequence converging in $\mathcal{C}^1$,
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Applications

Consider the problem T5 (10) T6

$$\begin{cases} -\|u\|_W^{r-p}\Delta_p u = -|u|^{q-2}u + \lambda|u|^{\omega-2}u & \text{in } \Omega\,, \\ u \not\equiv\geq 0, & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{8}$$

with $1 < q < \omega < p^*$, $\lambda > 0$.

- Nonnegative solutions are critical points of

$$\mathcal{J}^+(u) = \frac{1}{r}\|u\|_W^r + \frac{1}{q}\|u^+\|_q^q - \frac{\lambda}{\omega}\|u^+\|_\omega^\omega\,. \tag{9}$$

- They are positive if $q \geq p$.
- PS condition holds for $\omega \neq r > p$.

First case: If $r > \omega > q$ then $\mathcal{J}^+$ is coercive.

Theorem

Let $1 < q < \omega < p^*$, $r > \omega$ *and* $\lambda > 0$. (8)

- *If* $r \in (\omega, p^*)$, *then*
  - *no solution (even sign changing) for* $\lambda << 1$
  - *at least two nonnegative nontrivial solutions for* $\lambda >> 1$.
- *If* $r = p^*$, *then*
  - *no solution (even sign changing) for* $\lambda << 1$
  - *at least one nonnegative nontrivial solution for* $\lambda >> 1$.
- *If* $r > p^*$, *then there exists at least one nonnegative nontrivial solution for every* $\lambda > 0$.

Remark

- If $r = p$, similar (more precise) result in (Anello, 2012), using sub and supersolution method.
- if $r \in (p, p^*)$, the $0 - 2$ solution situation is maintained
- ir $r > p^*$, things change!

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Proof:

- $r > \omega > q$ then $\mathcal{J}^+$ is coercive. (8)
- if $r > p^*$ then $inf \mathcal{J}^+ < 0$. Then at least one solution
- if $r < p^*$ then the origin is a local minimum.
  - With some estimates

$$\|u\|_\omega^\omega \leq \frac{r-\omega}{r-q}\|u\|_q^q + \frac{\omega-q}{r-q}(C\|u\|_W)^r.$$

Then a necessary condition is

$$\begin{aligned} 0 &= \|u\|_W^r + \|u\|_q^q - \lambda\|u\|_\omega^\omega \\ &\geq \left(1-\lambda\frac{\omega-q}{r-q}C^r\right)\|u\|_W^r + \left(1-\lambda\frac{r-\omega}{r-q}\right)\|u\|_q^q : \end{aligned}$$

No nontrivial solution for λ small

  - $inf \mathcal{J}^+ < 0$ for λ large enough, then global minimum $+$ Mountain pass solution.
- if $r = p^*$ then for λ large enough we still have the global minimum .

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Second case: If $r < \omega$ then $\mathcal{J}^+$ is not coercive.

Theorem

Suppose $1 < q < \omega < p^*$ *and* $r \in [p, \omega)$. *(8)*
Then at least one nonnegative nontrivial solution for all $\lambda > 0$.

Proof:

- PS holds true,
- $\mathcal{J}^+(tu) \to -\infty$ if $t \to \infty$ and $u > 0$,
- the origin is a minimum.

$\Longrightarrow$ Mountain pass solution.

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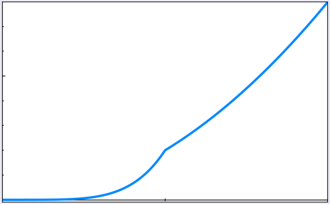
Third case: M not a pure power.

Theorem



Let $1 < q < \omega < p^*$ *and*
$M(s^p) = min\{s^{r_0 - p};\ s^{r_\infty - p}\}$
with $r_\infty \in (p, \omega)$,

Then the problem

$$\begin{cases} -M(\|u\|_W^p)\Delta_p u = -|u|^{q-2}u + \lambda|u|^{\omega-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

has at least one nonnegative nontrivial solution for $\lambda > 0$ *small enough. Moreover,*

1. *if* $r_0 < p^*$, *then the nonnegative nontrivial solution exists for every* $\lambda > 0$,
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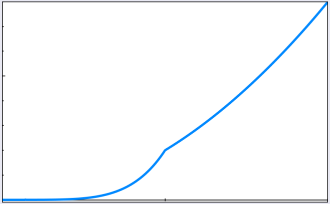

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Proof:

- PS holds true for the associated $\mathcal{J}^+$ ($r_\infty < \omega$)
- $\mathcal{J}^+(tu) \to -\infty$ if $t \to \infty$ and $u > 0$, (ω largest power)
- Since $\mathcal{J}^+(u) \geq \frac{1}{p}\widehat{M}(\|u\|_W^p) - \lambda C \|u\|_W^\omega$, there exist $\Lambda, S, \rho > 0$ such that

$$\mathcal{J}^+(u) \geq S > 0 \qquad \text{for } \|u\|_W = \rho \text{ and } \lambda \in [0, \Lambda).$$

$\Longrightarrow$ Mountain pass solution for $\lambda \in [0, \Lambda)$.

- If $r_0 < p^*$ the origin is a local minimum,
  $\Longrightarrow$ MP solution $\forall\, \lambda > 0$.
- If $r_0 > p^*$ the origin is NOT a local minimum,
  $\Longrightarrow$ for $\lambda \in [0, \Lambda)$, MP solution + local minimum in B_ρ.

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A-priori estimates

Consider the nonlocal problem (P_a)

$$\begin{cases} -\|u\|_W^{r-2}\,\Delta u = g_a(u) = -au^{q-1} + u^{\omega-1} & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \qquad (P_a)$$

with parameter $a \in (0, A]$ and suitable $1 < q < \omega < 2$.

If $r = 2$, there exists $\Lambda > 0$ such that
$|g_a(s)| \leq \Lambda$ for every $s \in [-D, D]$, $a \in (0, A]$.
Then by (Lieberman, 1988:Theorem 1) there exist $\beta(\Lambda, N) \in (0, 1)$ and $C(\Lambda, D, N, \Omega) > 0$, such that

$$\|u\|_{C^{1,\beta}} \leq C$$

for any weak solution satisfying $\|u\|_\infty < D$.

Question: does the same hold with $r > 2$?

Writing the nonlocal problem as

$$-\Delta u = \|u\|_W^{2-r} g_a(u)$$

the RHS is not bounded if $\|u\|_W \to 0$. Then one cannot directly apply (Lieberman, 1988) result.

Actually,

Proposition ((Iturriaga and M., 2019))

If $r \in (2 + \frac{2}{N}, 2^*)$, $N \geq 3$, then there exists a family of functions, satisfying problem (P_a) with $a \in (0, 1]$, which is bounded in L^∞ but unbounded in $\mathcal{C}^1$.

Idea of the proof: By (Il'yasov and Egorov, 2010), for suitable $1 < q < \omega$, b_0, there exists a compact support solution Φ for

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Then consider for $\lambda \geq 1$, $\mu \in (0, 1]$, the family of functions

$$\Phi_{\lambda,\mu}(x) := \begin{cases} \mu\Phi(\lambda x) & \textit{in } B_{1/\lambda}\,, \\ 0 & \textit{in } \Omega \setminus B_{1/\lambda}\,, \end{cases}$$

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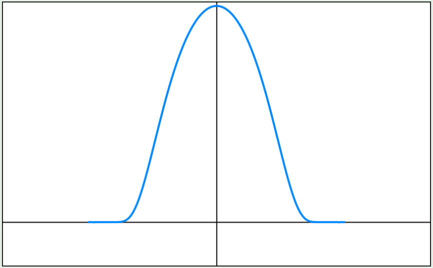

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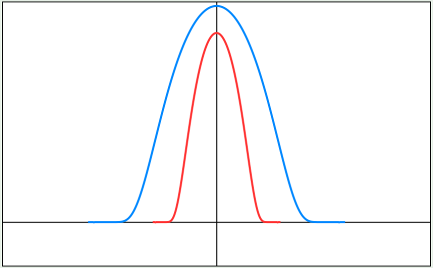

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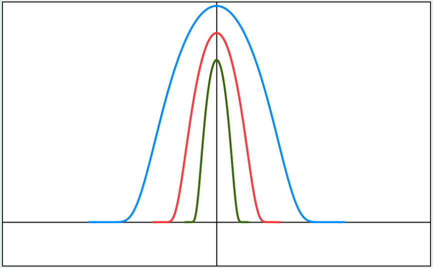

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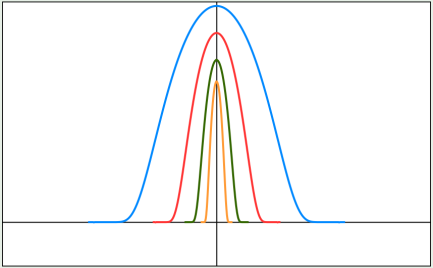

THE END!

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