Blow-up of solutions of semilinear heat equations with almost Hénon-critical exponent[1]

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(joint work with S. Alarcon[b] and L. Iturriaga[b])

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The problem

We study the parabolic problem

$$\begin{cases} u_t - \Delta u = |x|^\alpha |u|^{p-1} u & \text{in } B_1 \times (0, T) \\ u = 0 & \text{on } \partial B_1 \times (0, T) \\ u = u_0 & \text{in } B_1 \times \{0\}, \end{cases} \qquad (P_\alpha)$$

where

- B_1 is the unit ball in $\mathbb{R}^N$, $N \geq 3$;
- $T = T_{max}(u_0) \in (0, +\infty]$: the maximal existence time for the (classical) solution;
- $\alpha > 0$, $p > 1$;
- $u_0 \in C_0(B_1) := \{v \in C(\overline{B_1}) : v = 0 \text{ on } \partial B_1\}$.

Background: case $\alpha = 0$

Consider the problem

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where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $p > 1$,
$u_0 \in C_0(\Omega) = \{v \in C(\overline{\Omega}) : v(x) = 0 \text{ for } x \in \partial\Omega\}$.

- The **well-posedness** and several useful results (**comparison, regularity**..) for this equation can be found for example in (Quittner and Souplet, 2007).
- Our concern will be the **study of the sets of initial data that produce global solutions or blow-up solutions**.

$$\mathcal{G} = \{u_0 \in C_0(\Omega) : \; T_{\max}(u_0) = \infty\}$$
$$\mathcal{F} = \{u_0 \in C_0(\Omega) : \; T_{\max}(u_0) < +\infty\}.$$
$$\mathcal{G}^+ = \{u_0 \in C_0(\Omega) : \; u_0 \geq 0, \; T_{\max}(u_0) = \infty\}$$
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Known properties of (P_0)

- Let $w \in C_0(\Omega)$, $w \not\equiv 0$ and consider $u_0 = \lambda w$, $\lambda \geq 0$:
  - if λ is small enough then $u_0 \in \mathcal{G}$
  - if λ is large enough then $u_0 \in \mathcal{F}$

  Moreover, if $w \geq 0$, there exists $\bar{\lambda} > 0$ such that
  - if $0 < \lambda < \bar{\lambda}$ then $u_0 \in \mathcal{G}$
  - if $\lambda > \bar{\lambda}$ then $u_0 \in \mathcal{F}$
  - if $\lambda = \bar{\lambda}$ both cases can occur
- Thus, $\mathcal{G}^+$ **is star-shaped** (in fact convex) with respect to 0.
- When the initial value changes sign, the situation is different. ¿Is $\mathcal{G}$ star-shaped?

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$\mathcal{G}$ is not starshaped - 1

(Cazenave, Dickstein, and Weissler, 2009) consider radial solutions in $\Omega = B_1$:

Theorem

There $\exists p^* < p_0 := \frac{N+2}{N-2}$ *such that if* $p^* < p < p_0$ *and* ψ_p *is a radial sign changing stationary solution of* (P_0)*, that is,*

$$\begin{cases} -\Delta \psi_p = |\psi_p|^{p-1}\psi_p & \text{in } B_1 \\ \psi_p = 0 & \text{on } \partial B_1, \end{cases}$$

then there exists $\eta > 0$ *such that*

$$0 < |1 - \lambda| < \eta \Rightarrow u_0 = \lambda \psi_p \in \mathcal{F}$$

- i.e. the solution of (P_0), with $\Omega = B_1$ and $u_0 = \lambda\psi_p$, blows up in finite time both for λ slightly greater than 1 and λ slightly less than 1. Hence $\mathcal{G}$ **is not star-shaped with respect to the origin.**

$\mathcal{G}$ is not starshaped - 2

(Marino, Pacella, and Sciunzi, 2015) extended the previous result by considering a general bounded smooth domain $\Omega \subset \mathbb{R}^N$

Theorem

There $\exists p^* < p_0 := \frac{N+2}{N-2}$ such that if $p^* < p < p_0$ and ψ_p is a sign changing stationary solution of (P_0) in Ω, satisfying

$$\int_\Omega |\nabla \psi_p|^2 \to 2S_0^{\frac{N}{2}} \quad \text{as } p \to p_0 \tag{2.1}$$

$$\frac{\max \psi_p}{\min \psi_p} \to -\infty \quad \text{as } p \to p_0. \tag{2.2}$$

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- Existence of solutions as above were proved in [(Pistoia and Weth, 2007; Musso and Pistoia, 2010)]

Sketch of the argument

The argument for both results is in three steps:
let ψ_p be a sign-changing stationary solution of (P_0):

- **step 3** (proved in (Gazzola and Weth, 2005)):
  if $\exists t \geq 0$: $u(\cdot,t) \geq\not\equiv \psi_p$ then u blows-up (positively).
  if $\exists t \geq 0$: $u(\cdot,t) \leq\not\equiv \psi_p$ then it blows-up (negatively).
- **step 2**: (proved in (Cazenave, Dickstein, and Weissler, 2009)):
  Proposition. let $\varphi_{1,p}$ be a first eigenfunction of the linearized problem around ψ_p:

$$\begin{cases} -\Delta\varphi - p|\psi_p|^{p-1}\varphi = \lambda\varphi \text{ in } \Omega \\ \varphi = 0 \text{ on } \partial\Omega, \end{cases}$$

  and assume that

$$\int_\Omega \psi_p \varphi_{1,p} > 0.$$

  Then
  - for $\lambda > 1$ near 1, the solution u_p^λ of (P_0) with initial value $u_0 = \lambda\psi_p$, satisfy $u_p^\lambda(\cdot,t) \geq\not\equiv \psi_p$ for t large enough.
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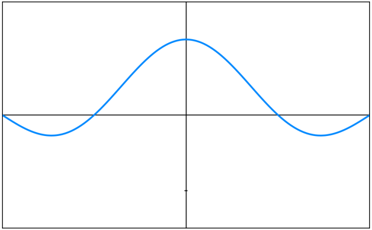
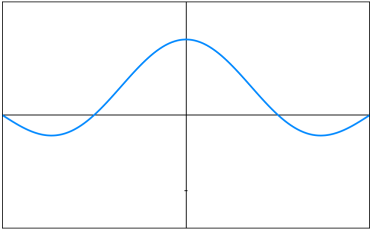
Blow-up mechanism

Stationary sign changing solution ψ_p with $\int_\Omega \psi_p \varphi_{1,p} > 0$.





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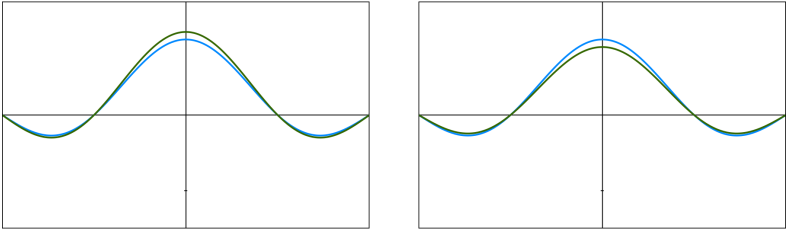

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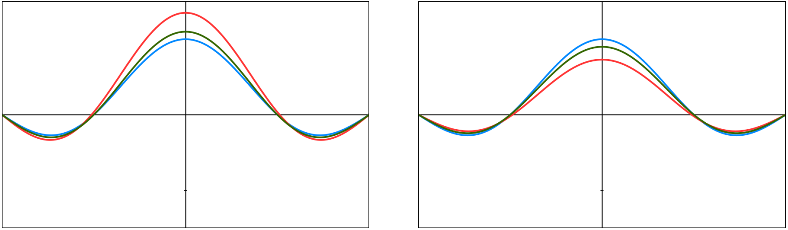

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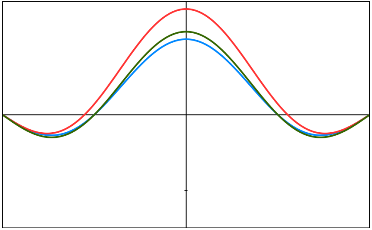

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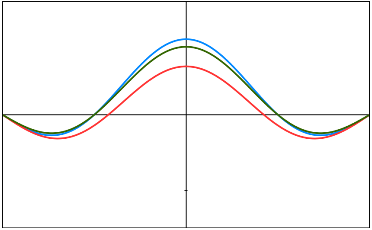

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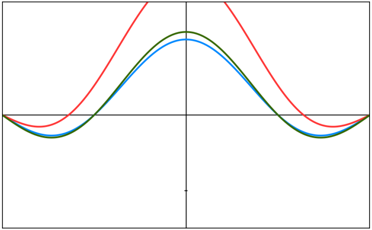

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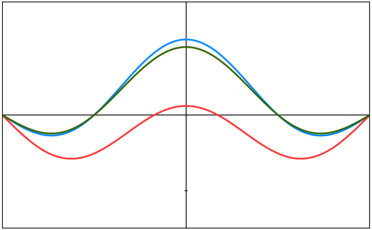

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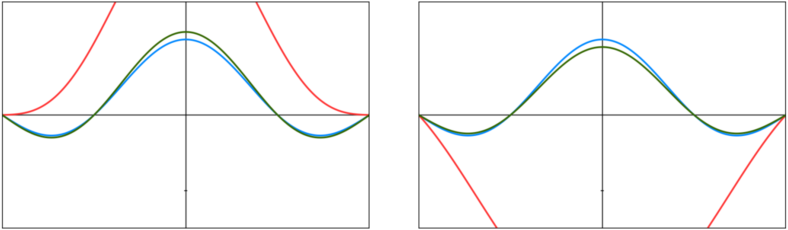

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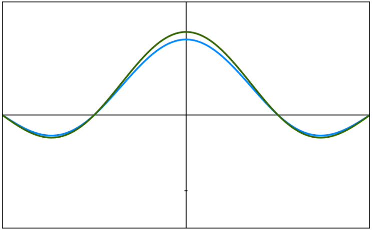

blew in finite time

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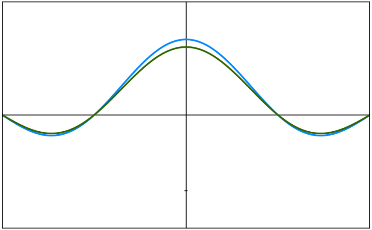

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The problem with $\alpha > 0$

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where B_1 is the unit ball in $\mathbb{R}^N$, $N \geq 3$, $p > 1$, $\alpha > 0$.

- We restrict to B_1 and radial solutions because we need to work near the "relevant" critical exponent: $p_\alpha = \frac{N+2+2\alpha}{N-2}$

  Actually,

  $$H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad \text{for } 2 \leq p \leq p_0 + 1$$

  but

  $$H^1(\mathbb{R}^N) \not\hookrightarrow L^{p_\alpha+1}(\mathbb{R}^N), \quad H^1(B_1) \not\hookrightarrow L^{p_\alpha+1}(B_1, |x|^\alpha)$$

$$H^1_{rad}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, |x|^\alpha), \quad \text{for } 2 + \frac{\alpha}{N-1} \leq p \leq p_\alpha + 1$$

- Problem (P_α) is **well posed** for $u_0 \in C_0(B_1)$ (see (Wang, 1993)).
- Classical results (**comparison, energy, ..**) still hold or can be adapted.
- **Step 3** was based on these methods then can be adapted.
- **Step 2** ...see below...
- So we need to prove **step 1**:

Proposition

Given $\alpha > 0$ there exists $p^* > 0$ such that for each $p \in (p^*, p_\alpha)$ the exists a radial sign-changing solution $\psi_p \in C_0(B_1)$ of the elliptic problem

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{p-1} u & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{3.1}$$

such that, if $\varphi_{1,p}$ is a first eigenfunction of the linearized problem around ψ_p:

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then

$$\int_{B_1} \psi_p \varphi_{1,p} > 0. \tag{3.2}$$

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Step 2

- Let u_p^λ be the solution of (P_α) with $u_0 = \lambda\psi_p$ and define

$$z_p^\lambda(\cdot,t) = \frac{u_p^\lambda(\cdot,t) - \psi_p}{\lambda - 1} \qquad \text{in } B_1 \times (0,T).$$

- By continuous dependence, given $0 < \tau < T < \infty$, for $|1-\lambda| > 0$ small enough, u_p^λ is well defined on $[0,T]$ and

$$z_p^\lambda \to z_p \quad \text{in } \mathcal{C}([\tau,T],\mathcal{C}^1(\overline{B_1})) \ \text{ as } \lambda \to 1\ ,$$

with z_p being a solution of the limiting problem

$$\begin{cases} (z_p)_t = \Delta z_p + p_\alpha |x|^\alpha |\psi_p|^{p_\alpha - 1} z_p & \text{in } B_1 \times (0,T) \\ z_p = 0 & \text{on } \partial B_1 \times (0,T) \\ z_p = \psi_p & \text{in } B_1 \times \{0\}. \end{cases}$$

- But

$$\int_{B_1} \psi_p \varphi_{1,p} > 0$$

then at some $t_0 > 0$

$$z_p^\lambda(\cdot,t_0) > 0 \quad \text{for } |\lambda - 1| \leq \delta.$$

Step 2

- Let u_p^λ be the solution of (P_α) with $u_0 = \lambda\psi_p$ and define

$$z_p^\lambda(\cdot, t) = \frac{u_p^\lambda(\cdot, t) - \psi_p}{\lambda - 1} \qquad \text{in } B_1 \times (0, T).$$

- By continuous dependence, given $0 < \tau < T < \infty$, for $|1 - \lambda| > 0$ small enough, u_p^λ is well defined on $[0, T]$ and

$$z_p^\lambda \to z_p \quad \text{in } \mathcal{C}([\tau, T], \mathcal{C}^1(\overline{B_1})) \;\; \text{as } \lambda \to 1 \; ,$$

with z_p being a solution of the limiting problem

$$\begin{cases} (z_p)_t = \Delta z_p + p_\alpha |x|^\alpha |\psi_p|^{p_\alpha - 1} z_p & \text{in } B_1 \times (0, T) \\ z_p = 0 & \text{on } \partial B_1 \times (0, T) \\ z_p = \psi_p & \text{in } B_1 \times \{0\}. \end{cases}$$

- But

$$\int_{B_1} \psi_p \varphi_{1,p} > 0$$

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Step 1 - The sign changing solutions

The radial sign-changing solutions ψ_p were found in (Alarcón, 2017) in the form

$$\psi_p(x) = +PU_{M_1\varepsilon^{3/(N-2)},\alpha} - PU_{M_2\varepsilon^{1/(N-2)},\alpha} + \sigma_p(x) \qquad x \in B_1,$$

where

- $$U_{\lambda,\alpha}(x) = \gamma_{N,\alpha}\left(\frac{\lambda^{\frac{2+\alpha}{2}}}{\lambda^{2+\alpha} + |x|^{2+\alpha}}\right)^{\frac{N-2}{2+\alpha}}$$

  are the "bubbles of order α": all the radial classical solutions of the problem

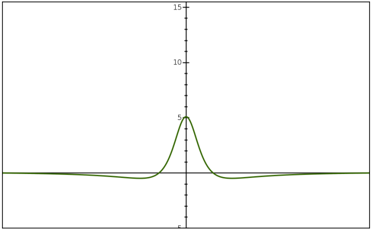
  $$\begin{cases} -\Delta U = |x|^\alpha U^{p_\alpha} & \text{in } \mathbb{R}^N \\ U > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{3.3}$$

- P is the projection on $H^1(B_1)$;
- $\varepsilon = p_\alpha - p$, M_1, M_2 are positive constants depending only on N and α, and σ_p is a function which is of a lower order than the other terms (in $\mathcal{C}^1$ norm) as $p \nearrow p_\alpha$.

- These solutions are called **Bubble towers**: the superposition of two bubbles that, as $p \nearrow p_\alpha$, concentrate at the origin at different speeds.
- They are obtained via the **Lyapunov-Schmidt finite dimensional reduction**.
- They satisfy $\psi_p(0) > 0$,

$$\int_{B_1} |\nabla \psi_p|^2 \to 2S_\alpha^{\frac{N+\alpha}{2+\alpha}} \quad \text{as } p \nearrow p_\alpha,$$

$$\max \psi_p, \ -\min \psi_p \to +\infty, \qquad \frac{\max \psi_p}{\min \psi_p} \to -\infty \quad \text{as } p \nearrow p_\alpha,$$



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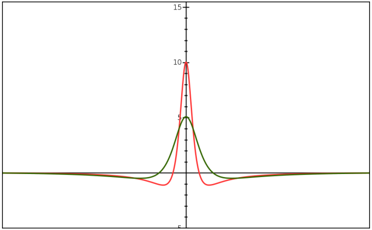

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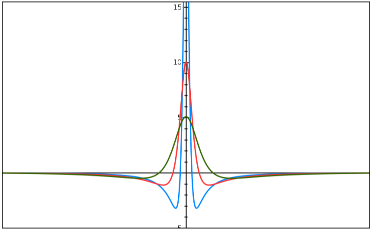


Step 1

We compare these four problems:

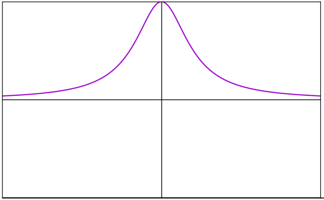
ψ_p radial solution of $\begin{cases} -\Delta u = |x|^\alpha |u|^{p-1} u & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1, \end{cases}$

$U \in \mathcal{D}^{1,2}_{rad}(\mathbb{R}^N)$ the radial positive solution of $\begin{cases} -\Delta U = |x|^\alpha U^{p_\alpha} & \text{in } \mathbb{R}^N \\ U(0) = 1, \end{cases}$

$\varphi_{1,p}$ is a radial first eigenfunction of $\begin{cases} -\Delta \varphi - p|x|^\alpha |\psi_p|^{p-1} \varphi = \lambda \varphi & \text{in } B_1 \\ \varphi = 0 & \text{on } \partial B_1, \end{cases}$

$\varphi_1^* \in H^1_{rad}(\mathbb{R}^N)$ is a first eigenfunction of $-\Delta \varphi - p|x|^\alpha U^{p-1} \varphi = \lambda \varphi$ in $\mathbb{R}^N$,

The rescaling



Set

$$M_p := \psi_p(0) = \|\psi_p\|_{L^\infty}$$

$$\widetilde{B}_p = M_p^{\frac{p-1}{2+\alpha}} B_1$$

$$\widetilde{\psi}_p(x) := \frac{1}{M_p} \psi_p \left(\frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in } \widetilde{B}_p$$

$$\widetilde{\varphi}_{1,p}(x) = \left(\frac{1}{M_p^{\frac{p-1}{2+\alpha}}} \right)^{\frac{N}{2}} \varphi_{1,p} \left(\frac{x}{M_p^{\frac{p-1}{2+\alpha}}} \right) \quad \text{in } \widetilde{B}_p$$

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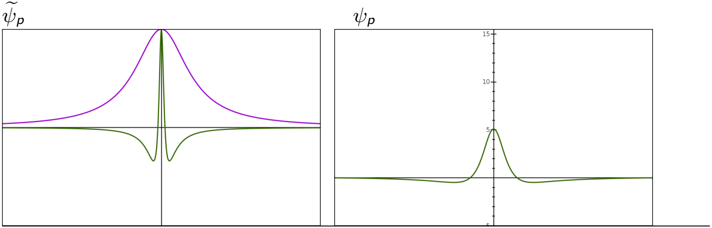

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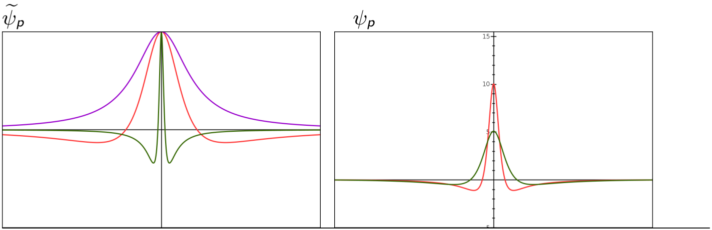

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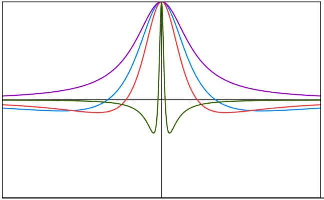
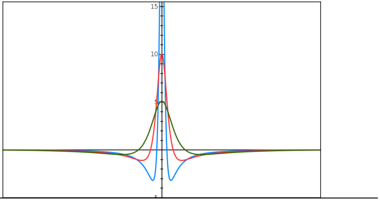
The rescaling

$\widetilde{\psi}_p$



ψ_p



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Estimates on ψ_p and $\widetilde{\psi}_p$

One needs some estimates on the solutions ψ_p:

$$\max_{B_1} |x|^\alpha (\psi_p(x)^+)^{p-1} = O(\varepsilon^{-\frac{6}{N-2}})$$

$$\max_{B_1} |x|^\alpha (\psi_p(x)^-)^{p-1} = O(\varepsilon^{-\frac{2}{N-2}})$$

as $\varepsilon = p_\alpha - p \to 0$.

For the rescaled $\widetilde{\psi}_p$ this implies

$$\max_{\widetilde{B}_p} |x|^\alpha (\widetilde{\psi}_p(x)^+)^{p-1} = O(1)$$

$$\max_{\widetilde{B}_p} |x|^\alpha (\widetilde{\psi}_p(x)^-)^{p-1} = O(\varepsilon^{4/(N-2)})$$

as $\varepsilon = p_\alpha - p \to 0$.

about φ_1^*

Consider the Rayleigh functional

$$\mathcal{R}^*(v) = \int_{\mathbb{R}^N} (|\nabla v|^2 - p_\alpha |x|^\alpha |U|^{p_\alpha - 1} v^2)$$

and define

$$\lambda_1^* := \inf_{\substack{v \in H^1_{\mathrm{rad}}(\mathbb{R}^N) \\ \|v\|_{L^2(\mathbb{R}^N)} = 1}} \mathcal{R}^*(v). \tag{3.4}$$

Then

- $-\infty < \lambda_1^* < 0$
- There exists a unique positive minimizer φ_1^* associated to λ_1^*.

about $\varphi_{1,p}$

Consider the Rayleigh functional

$$\mathcal{R}(v) = \int_{B_1} \left(|\nabla v|^2 - p|x|^\alpha |\psi_p|^{p-1} v^2 \right)$$

and define

$$\lambda_{1,p} := \inf_{\substack{v \in H^1_{0,\mathrm{rad}}(B_1) \\ \|v\|_{L^2(B_1)} = 1}} \mathcal{R}(v). \tag{3.5}$$

Then

- $-\infty < \lambda_{1,p} < 0$
- There exists a unique positive minimizer $\varphi_{1,p}$ associated to $\lambda_{1,p}$.

After rescaling:

- $\|\widetilde{\varphi}_{1,p}\|_{L^2(\mathbb{R}^N)} = 1$
- $\widetilde{\varphi}_{1,p}$ is the first eigenfunction of the following linearized problem:

$$\begin{cases} -\Delta \widetilde{\varphi}_{1,p} - p|x|^\alpha |\widetilde{\psi}_p|^{p-1} = \widetilde{\lambda}_{1,p} \widetilde{\varphi}_{1,p} & \text{in } \widetilde{B}_p \\ \widetilde{\varphi}_{1,p} = 0 & \text{on } \partial \widetilde{B}_p \,, \end{cases} \tag{3.6}$$

with $\widetilde{\lambda}_{1,p} = \frac{\lambda_{1,p}}{M_p^{\frac{2(p-1)}{2+\alpha}}}$.

about $\varphi_{1,p}$

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final argument

In order to conclude one has to prove:

- $\widetilde{\psi}_p \to U$ in $\mathcal{C}^2_{loc}(\mathbb{R}^N)$ (U is the unique solution of the limiting problem).
- $\widetilde{\lambda}_{1,p} \to \lambda_1^*$ (Several computations comparing the two minimization problems and using the properties of ψ_p)
- $\widetilde{\varphi}_{1,p} \to \varphi_1^*$ in $L^2(R^N)$ (follows from the previous, considering the minimizing sequence $\widetilde{\varphi}_{1,p_n}$)

Finally,

- $\int_{B_1} \psi_p \varphi_{1,p}$, has the same sign as $\int_{B_1} |x|^\alpha |\psi_p|^{p-1} \psi_p \varphi_{1,p}$
- $\int_{B_1} |x|^\alpha |\psi_p|^{p-1} \psi_p \varphi_{1,p} = \int_{\widetilde{B}_p} |x|^\alpha |\widetilde{\psi}_p|^{p-1} \widetilde{\psi}_p \widetilde{\varphi}_{1,p} \to \int_{\mathbb{R}^N} |x|^\alpha U^{p_\alpha} \varphi_1^* > 0$

THEN

$$\int_{\mathbf{B_1}} \psi_\mathbf{p} \varphi_\mathbf{1,p} > \mathbf{0}\,.$$

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THEN

$$\int_{\mathbf{B_1}} \psi_{\mathbf{p}} \varphi_{\mathbf{1,p}} > \mathbf{0}\,.$$

Conclusion

Theorem

There exists $p^* < p_\alpha = \frac{N+2+2\alpha}{N-2}$ with the following property:

If $p^* < p < p_\alpha$, then

$$\exists \textit{sign-changing radial stationary solution } \psi_p \textit{ of } (P_\alpha) \textit{ and } \delta_p > 0$$

such that:

If $0 < |\lambda - 1| < \delta_p$, then the classical solution u of (P_α) with initial value $u_0 = \lambda\psi_p$ blows up in finite time.
That is,

$$0 < |1 - \lambda| < \delta_p \Rightarrow u_0 = \lambda\psi_p \in \mathcal{F}$$

Then **also for $(\mathbf{P}_\alpha)$ the set $\mathcal{G}$ is not starshaped with respect to the origin.**

Thank you very much for your attention.

Main references I

- Alarcón, S. (2017). "Multiple Sign changing solutions at the almost Hénon critical exponent". In: *Preprint*.
- Cazenave, T., F. Dickstein, and F. B. Weissler (2009). "Sign-changing stationary solutions and blowup for the nonlinear heat equation in a ball". In: *Math. Ann.* 344.2, pp. 431–449.
- Gazzola, F. and T. Weth (2005). "Finite time blow-up and global solutions for semilinear parabolic equations with initial data at high energy level". In: *Differential Integral Equations* 18.9, pp. 961–990.
- Marino, V., F. Pacella, and B. Sciunzi (2015). "Blow up of solutions of semilinear heat equations in general domains". In: *Commun. Contemp. Math.* 17.2, pp. 1350042, 17.
- Musso, Monica and Angela Pistoia (2010). "Tower of bubbles for almost critical problems in general domains". In: *J. Math. Pures Appl. (9)* 93.1, pp. 1–40.

Main references II

Pistoia, A. and T. Weth (2007). "Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24.2, pp. 325–340.

Quittner, P. and P. Souplet (2007). *Superlinear parabolic problems*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Blow-up, global existence and steady states. Birkhäuser Verlag, Basel, pp. xii+584.

Wang, X. (1993). "On the Cauchy problem for reaction-diffusion equations". In: *Trans. Amer. Math. Soc.* 337.2, pp. 549–590.