Weighted Trudinger-Moser inequalities and associated Liouville type equations

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For $\Omega \subset \mathbb{R}^N$ a bounded smooth domain, some standard **Sobolev embeddings** are

$$W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for} \quad \begin{cases} p < N & \text{and} \quad q \in [1, p^* = \frac{pN}{N-p}] , \\ p = N & \text{and} \quad q \in [1, \infty) . \end{cases}$$

The **Trudinger-Moser inequalities** consider the limiting case $p = N$:

$$\text{actually } W_0^{1,N}(\Omega) \not\hookrightarrow L^\infty(\Omega).$$

Consider $N = 2$ and $u(x) = \log(1 - \log|x|) \in W_0^{1,2}(B_1(0))$.

Then one seeks a **maximal growth function** $f(t)$ such that

$$u \in W_0^{1,N} \Rightarrow \int_\Omega f(u) < \infty.$$

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In particular, let (from now on) $N = 2$:

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Trudinger - Moser inequality (Moser, 1970)

$$(TM) \quad C(\alpha) := \sup_{u \in H_0^1,\ ||u|| \leq 1} \int_\Omega e^{\alpha u^2} \begin{cases} \leq C|\Omega| & \text{if } \alpha \leq 4\pi \\ = \infty & \text{if } \alpha > 4\pi. \end{cases}$$

$(||u||^2 := \int_\Omega |\nabla u|^2)$

A useful consequence is the *logarithmic TM inequality:*

There exists a constant $\widetilde{C} > 0$ *such that*

$$(LogTM) \quad \log \int_\Omega e^{|u|} \leq \frac{1}{16\pi}||u||^2 + \widetilde{C}.$$

$$\left(\int e^{\frac{|u|}{||u||}||u||} \leq \int e^{4\pi\left(\frac{u}{||u||}\right)^2 + \frac{1}{16\pi}||u||^2} \leq C|\Omega| e^{\frac{1}{16\pi}||u||^2} \right)$$

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An application to a mean field equation

Consider the mean field equation of Liouville type (see Caglioti-Lions-Marchioro-Pulvirenti (92), Chanillo-Kiessling (94))

$$\begin{cases} -\Delta u = \lambda \dfrac{e^u}{\int_\Omega e^u}, & \text{in } \Omega \subset\subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The associated functional is

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \log \int_\Omega e^u.$$

Li (99), Chen-Li (10)

If $0 < \lambda < 8\pi$, *the equation has a (positive) solution*, which is a *global minimizer* of J.

Actually, by the Logarithmic TM inequality, the functional is coercive for $\lambda < 8\pi$.

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If $\lambda \geq 8\pi$, Ω starshaped $\Rightarrow$ no solution (via a Pohozaev identity).

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Influence of weights on TM inequalities:

- Weight in the integral: find maximal growth function $f(t)$ such that

$$u \in H_0^1(\Omega) \Rightarrow \int_\Omega f(u) w(x)\, dx < \infty,$$

(Calanchi-Terraneo, Adimurthi-Sandeep, de Oliveira-do Ó, do Ó- de Figueiredo-Dos Santos, mostly for $w(x) = |x|^\alpha$, $\alpha \in \mathbb{R}$)

- Weight in the norm: find maximal growth function $f(t)$ such that

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where H_w is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||_w := \left(\int_B |\nabla u|^2\, w(x)\, dx \right)^{1/2}$$

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Logarithmic radial weights

It turns out that a interesting case is when $\Omega = B = B_1(0) \subseteq \mathbb{R}^2$

$$w_\beta(x) = \left(\log \frac{e}{|x|}\right)^\beta \qquad (\beta \geq 0)$$

and one restricts to radial functions:

$$\widetilde{H}_\beta := cl\left\{u \in C^\infty_{0,rad}(B)\ ;\ \|u\|^2_\beta := \int_B |\nabla u|^2 w_\beta(x) dx < \infty\right\} \ :$$

Calanchi-Ruf (15) - Case $0 \leq \beta < 1$

$$\int_B e^{|u|^\gamma}\, dx < \infty, \forall u \in \widetilde{H}_\beta, \iff \gamma \leq \gamma_\beta := \frac{2}{1-\beta}.$$

$$\sup_{u \in \widetilde{H}_\beta,\ \|u\|_\beta \leq 1} \int_B e^{\alpha |u|^{\gamma_\beta}}\, dx < \infty \iff \alpha \leq \alpha_\beta = 2\left[2\pi(1-\beta)\right]^{\frac{1}{1-\beta}}.$$

(The case $\beta = 0$ is the classical TM: $\gamma_0 = 2,\ \alpha_0 = 4\pi$).

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The maximal growth is now a double exponential:

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$$\int_B e^{e^{u^2}} < +\infty \quad \forall u \in \widetilde{H}_{\beta=1},$$

$$\sup_{u \in \widetilde{H}_{\beta=1} \, \|u\|_{\beta=1} \le 1} \int_B e^{a\, e^{2\pi u^2}}\, dx < \infty \iff a \le 2.$$

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Proof of Weigthed TM inequalities

The classical proof for TM inequality uses *symmetrization*, which doesn't work in the presence of a weight.
In this case one needs a radial Lemma:

Radial Lemma (Calanchi Ruf - 15)

Let $u \in C^1_{0,rad}(B)$. Then

- $$|u(x)| \leq \frac{\left|[\log(e/|x|)]^{1-\beta} - 1\right|^{1/2}}{\sqrt{2\pi|1-\beta|}} \, ||u||_\beta, \quad 0 \leq \beta < 1$$
- $$|u(x)| \leq \sqrt{\frac{\log(\log(e/|x|))}{2\pi}} \, ||u||_\beta, \quad \beta = 1.$$

• If $||u||_{\beta=1} \leq 1$ then

$$e^{ae^{2\pi u^2}} \leq e^{a\log(e/|x|)} = (e/|x|)^a,$$

which is integrable in B for $1 - a > -1$ ($\iff a < 2$)

• if $a > 2$ then $\int_B e^{a\,e^{2\pi u^2}} \to \infty$ along a suitable sequence.

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Logarithmic TM inequality for $\beta \geq 0$

(Calanchi-Ruf-15)

a) $\beta \in [0,1)$*: there exists a constant* $C(\beta)$ *such that*

$$\log\left(\frac{1}{|B|}\int_B e^{|u|^{\theta_\beta}}\,dx\right) \leq \frac{1}{2\lambda_\beta}\|u\|_\beta^2 + C(\beta) \qquad \forall u \in \widetilde{H}_\beta\,,$$

where $\lambda_\beta := \pi(1-\beta)^\beta(2-\beta)^{2-\beta}2^{1-\beta}$ and $\theta_\beta = \dfrac{2}{2-\beta}$.

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Open question from Calanchi-Ruf-15

Are the values $\dfrac{1}{2\lambda_\beta}$ and $\dfrac{1}{2\pi}$ optimal?

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Two associated functional

In view of the LogTM inequality we will consider the following functionals:

i) for $\beta \in [0,1)$, let

$$J_\lambda : \widetilde{H}_\beta \to \mathbb{R}, \qquad J_\lambda(u) := \frac{1}{2}\|u\|_\beta^2 - \lambda \log\left(\fint_B e^{u^{\theta_\beta}}\, dx\right)^* :$$

it is coercive for $\lambda \in [0, \lambda_\beta)$ and it is bounded from below if $\lambda \leq \lambda_\beta$.

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Coercivity

Coercivity for $\lambda < \lambda_\beta$ (resp. $\lambda < \pi$) is an immediate consequence of the LogTM inequality: for $\lambda \geq 0$

$$J_\lambda(u) \geq \left(\frac{1}{2} - \frac{\lambda}{2\lambda_\beta}\right)\|u\|_\beta^2 - \lambda C(\beta)$$

and

$$I_\lambda(u) \geq \left(\frac{1}{2} - \frac{\lambda}{2\pi}\right)\|u\|_1^2 - \lambda \log\left(\frac{1}{8} + \frac{\log C_{MB}}{e^{\frac{\|u\|_1^2}{2\pi}}}\right)$$
$$\geq \left(\frac{1}{2} - \frac{\lambda}{2\pi}\right)\|u\|_1^2 - \lambda \log\left(\frac{1}{8} + \log C_{MB}\right) .$$

Moreover J_λ and I_λ are still bounded from below when $\lambda \leq \lambda_\beta$ (resp. $\lambda \leq \pi$).

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Coercivity

Coercivity for $\lambda < \lambda_\beta$ (resp. $\lambda < \pi$) is an immediate consequence of the LogTM inequality: for $\lambda \geq 0$

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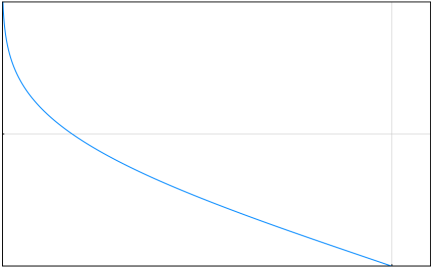
Sharpness ($\beta = 1$) [CMR18]

Sharpness (case $\beta = 1$): we produce a sequence u_k such that $I_{\pi+\varepsilon}(u_k) \to -\infty$: take a generalized Moser sequence:

$$u_k(x) = \begin{cases} \log\log\left(\frac{e}{|x|}\right) & \text{for } e^{-k} \leq |x| < 1\,, \\ \log(1+k) & \text{for } |x| < e^{-k}\,. \end{cases} \qquad (1)$$



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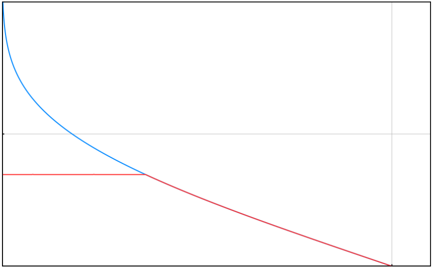

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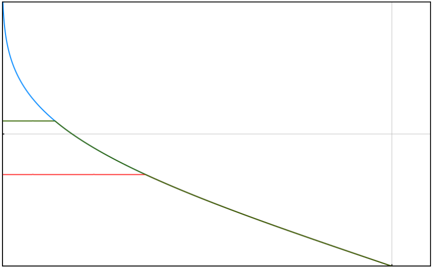

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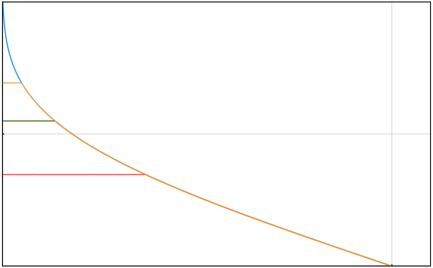

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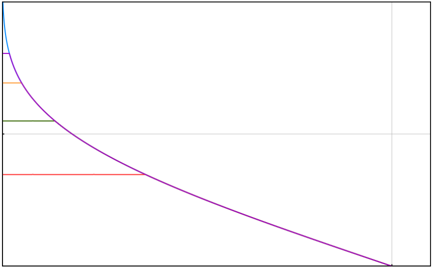

Then $\|u_k\|_{\beta=1} = 2\pi \log(1+k) \to \infty$ and

$$\begin{aligned} I_\lambda(Cu_k) &= \frac{1}{2}\|Cu_k\|_1^2 - \lambda \log\log\left(\fint_B e^{e^{Cu_k}}\, dx \right) \\ &\leq \frac{1}{2}\|Cu_k\|_1^2 - \lambda \log\log\left(\frac{1}{|B|} \int_{|x|<e^{-k}} e^{e^{C\log(1+k)}}\, dx \right) \\ &\leq C^2\pi \log(1+k) - \lambda\left[\log\left((1+k)^C - 2k\right)\right]. \end{aligned}$$

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$$\log((1+k)^{1+2\delta} - 2k) \geq \log((1+k)^{1+\delta}) = (1+\delta)\log(1+k).$$

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Mean field equations with weight

Consider the following problem, $w(x) = \log^{\beta}\left(\frac{e}{|x|}\right)$, $\beta > 0$

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As we have seen, for $\beta = 0$, J is

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Increasing the growth

The "interesting" problems are the ones associated with the above functionals:

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Theorem (subcritical case)[CMR18]

The problem (2) has a weak radial positive solution for every $\lambda < \lambda_\beta$.
The problem (3) has a weak radial positive solution for every $\lambda < \pi$.

Both solutions correspond to a global minimum of the (coercive) associated functional.
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Existence of a solution for $\beta = 1$ and $\lambda = \pi$

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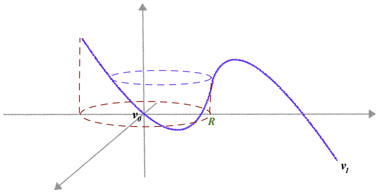

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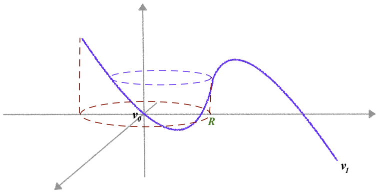

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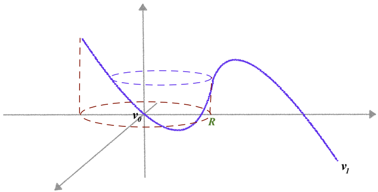
Finally, since $I_{\pi+\varepsilon}$ is unbounded from below, for $\pi < \lambda < \pi + \varepsilon$ the functional has a mountain-pass structure.



This suggests the possibility of a second solution.

Existence of a solution for $\beta = 1$ and $\lambda > \pi$

For $\lambda = \pi + \varepsilon$ the minimum persists if $\varepsilon > 0$ is small. (now it is only a local minimum).
Actually, using

$$I_{\pi+\varepsilon}(u) \geq \left(\frac{1}{2} - \frac{\pi+\varepsilon}{2\pi}\right) ||u||_\beta^2 - (\pi+\varepsilon)\log\left(\frac{1}{8} + \frac{C}{e^{\frac{1}{2\pi}||u||_\beta^2}}\right),$$

for a suitable $R > 0$ and small enough $\varepsilon > 0$ one has, for $||u||_\beta = R$.

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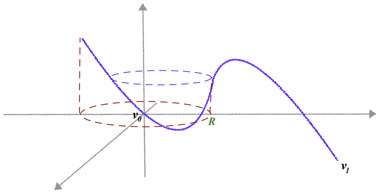

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The second solution.

Now the problem is that we could not prove the (PS) condition!!!

We used a generalization of a result by L. Jeanjean, based on the so called *monotonicity trick* by Struwe.

This shows that for almost every $\lambda \in [\pi, \pi + \varepsilon_0)$, there exists a bounded PS-sequence for I_λ at the Mountain pass level.

Summing up

Theorem (Critical and supercritical case) [CMR18]

There exists $\varepsilon > 0$ such that the equation

$$\begin{cases} -div\left(\log \frac{e}{|x|} \nabla u\right) = \lambda \dfrac{e^u}{\log \fint_B e^{e^u}} \dfrac{e^{e^u}}{\int_B e^{e^u}} & in \;\; B\,, \\ u = 0 & on \;\; \partial B\,, \end{cases} \tag{4}$$

has a positive radial solution, which is a local minimizer for I_λ, $\lambda \in [\pi, \pi + \varepsilon)$. Moreover for a.e. $\lambda \in (\pi, \pi + \varepsilon)$, there is a second positive radial solution which is of mountain-pass type.

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