Estimates on the derivatives and analyticity of positive definite functions on $\mathbb{R}^m$

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Abstract

In this paper we obtain some results about derivatives of positive definite functions in $\mathbb{R}^m$, using known properties of positive definite kernels. We prove, by purely algebraic methods, that certain derivatives of such functions are also positive definite and we show that simple conditions on their even order derivatives at the origin strongly determine their global properties. In particular, one can obtain an estimate for f and its derivatives at any point and a condition for real analyticity, using only the value of these derivatives at the origin.

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1. Introduction

Let m be a positive integer and $f : \mathbb{R}^m \to \mathbb{C}$ a positive definite function, that is,

$$\sum_{\mu=1}^{s}\sum_{\nu=1}^{s} c_\mu \overline{c_\nu} f(x_\mu - x_\nu) \geq 0, \tag{1.1}$$

for all positive integers s, complex numbers $c_1, c_2, \dots, c_s$ and points $x_1, x_2, \dots, x_s$ in $\mathbb{R}^m$.

In this paper we obtain some results about derivatives of positive definite functions in $\mathbb{R}^m$, exploiting known properties of positive definite kernels. Actually, given a positive definite function f defined in $\mathbb{R}^m$, the kernel $K(x, y) := f(x - y)$ is a positive definite kernel defined in $\mathbb{R}^m \times \mathbb{R}^m$, that is,

$$\sum_{\mu=1}^{s}\sum_{\nu=1}^{s} c_\mu \overline{c_\nu} K(x_\mu, x_\nu) \geq 0, \tag{1.2}$$

for all positive integers s, complex numbers $c_1, c_2, \dots, c_s$ and points $x_1, x_2, \dots, x_s$ in $\mathbb{R}^m$.

Historically, positive definite functions and kernels have been studied by many authors in various branches of Mathematics, such as Fourier analysis, probability theory, operator theory, complex function-theory, integral equations, boundary-value problems for partial differential equations, approximation theory and others (the reader can see [5, 9, 10, 11, 14, 16, 17, 19] and the references therein). In particular, differentiability of positive definite kernels is related to the decay rates of the eigenvalues and the singular values of integral operators generated by the kernel; in fact, in order to improve the decay rates one usually need to assume the existence and boundedness of certain derivatives of the kernel (see for example in [4, 8]). Differentiability of positive definite functions is also related to the reproducing kernel Hilbert spaces generated by the associated kernels, which appear in many problems from learning theory (see [2, 7, 18]).

Recently, Buescu and Paixão ([3]) considered the case $m = 1$ and proved that if some even order derivative at the origin of a positive definite function (defined in $\mathbb{R}$) vanishes, then the function is constant; moreover, if all even order derivatives at the origin are non-zero and satisfy a certain natural growth condition, then the function is real-analytic and it extends holomorphically to a stripe of the complex plane.

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Our main purpose in this paper is to obtain the same kind of result as in [3], but in the general case of positive definite functions defined in $\mathbb{R}^m$. In particular, by using results about positive definite kernels, we prove that certain derivatives of f are also positive (or negative) definite and we obtain a condition on even order derivatives of f at the origin that implies that f is constant. Furthermore, we obtain sufficient conditions for real-analyticity of positive definite functions. In fact, our results show that the global behavior of a smooth positive definite function is strongly determined by certain even order derivatives at the origin.

2. Statement of the main results

In this section we introduce the basic notation, we give a brief introduction to the matter and we state our main results, along with some remarks.

From now on, points in $\mathbb{R}^m$ will be written as $x = (x_1, \dots, x_m)$. Multi-index notation will be used throughout the paper, namely, if $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$, then $|\alpha| := \alpha_1 + \cdots + \alpha_m$, $\alpha! := \alpha_1! \dots \alpha_m!$ and $x^\alpha := x_1^{\alpha_1} \dots x_m^{\alpha_m}$; we will denote by e_j, $j = 1, 2, \dots, m$, the multi-index with j-th component equal to 1 and all the others equal to 0. The following relation ([12, p. 55]) about multi-indexes will be used:

$$\alpha! \le |\alpha|! \le m^{|\alpha|}\alpha! \,. \tag{2.1}$$

If O denotes an open subset of $\mathbb{R}^m$, then $C^{2n}(O \times O)$ is the classic set of the kernels $K : O \times O \to \mathbb{C}$ for which all the derivatives

$$D^{\alpha,\beta}_{x,y} K(x, y) := \frac{\partial^{|\alpha+\beta|}K}{\partial x^\alpha \partial y^\beta}(x, y) = \frac{\partial^{|\alpha+\beta|}K}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m} \partial y_1^{\beta_1} \dots \partial y_m^{\beta_m}}(x, y), \tag{2.2}$$

$|\alpha|, |\beta| \le n$, exist and are continuous in $O \times O$. Also, we say that $f : O \to \mathbb{C}$ belongs to the class $C^{2n}(O)$ (resp. $C^\infty(O)$) when it is $2n$ times (resp. infinite times) continuously differentiable in O.

It follows immediately by the definition (1.1) that a positive definite function f satisfies the following properties (see [5]):

$$f(0) \ \ge \ 0 \,, \tag{2.3}$$

$$f(-x) \ = \ \overline{f(x)} \,, \qquad x \in \mathbb{R}^m \,, \tag{2.4}$$

$$|f(x)| \ \le \ f(0) \,, \qquad x \in \mathbb{R}^m \,. \tag{2.5}$$

Another remarkable property of positive definite functions is related to their smoothness:

Proposition 2.1. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be a positive definite function. If f is of class C^{2n} for some n nonnegative integer (resp. C^∞), in some neighborhood of the origin, then $f \in C^{2n}(\mathbb{R}^m)$ (resp. $C^\infty(\mathbb{R}^m)$).*

This property is a consequence of Bochner's characterization and can be found, for example, in [6, p.186] or [17, p.77].

Inequality (2.5) implies that if $f(0) = 0$ then f vanishes identically. Our first result extends this property by showing that if a positive definite function has certain even order derivatives null at the origin then it is constant:

Theorem 2.2. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be a positive definite function of class C^{2n} in some neighborhood of the origin, for some positive integer n. Assume that there exist numbers $l_j \in \{1, \dots, n\}$ such that $D^{2l_j e_j} f(0) = 0$, for $j = 1, \dots, m$. Then f is constant in $\mathbb{R}^m$.*

In the case $m = 1$ this theorem coincides with Theorem 3.1 in [3]. We remark that no mixed derivative appears in the statement, but only even order derivatives with respect to the same variable x_j, $j = 1, \dots, m$, not necessarily of the same order. This theorem is proved in Section 3.

In Section 4, we provide a condition that guarantees real-analyticity of a positive definite function. In order to guarantee that a general function of class C^∞ is real-analytic one usually needs to estimate all its derivatives in a whole open set (see equation (4.1)); however, in the case of positive definite functions, it turns out that real-analyticity can be guaranteed by the following condition, involving only even order derivatives with respect to one single variable, and only at the origin:

Hypothesis $\mathbf{H_{ra}}$: there exist positive constants d and $\widetilde{R}$ such that

$$|D^{2le_j} f(0)| \leq d \frac{(2l)!}{\widetilde{R}^{2l}}, \tag{2.6}$$

for every $l \geq 0$ and $j = 1, \ldots, m$.

Theorem 2.3. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be a positive definite function of class C^∞ in some neighborhood of the origin, satisfying Hypothesis H_{ra}. Then, f is real-analytic in $\mathbb{R}^m$. In fact, Hypothesis H_{ra} implies that there exist constants M and r such that*

$$|D^\alpha f(x)| \leq M \frac{\alpha!}{r^{|\alpha|}}, \quad x \in \mathbb{R}^m,$$

for all multi-indexes $\alpha \in \mathbb{Z}_+^m$.

Again, this theorem reduces to Theorem 4.3-(i) in [3] when $m = 1$.

The fundamental tool for proving all the above results is a property similar to (2.5), but involving derivatives of f, and is obtained in Proposition 3.2 in Section 3. Roughly speaking, it says that any derivative of a smooth positive definite function f can be estimated in terms of the even order derivatives at the origin. This estimate is a consequence of known results about positive definite kernels in $\mathbb{R}^m \times \mathbb{R}^m$ (see Proposition 3.1).

3. Differentiability

The purpose of this section is to transpose some results about positive definite kernels from [1, 7, 15], to the case of positive definite functions, and then to prove Theorem 2.2. In particular, we make use of the following Proposition, which extends to $\mathbb{R}^m$ a result from [2]:

Proposition 3.1. *If $K \in C^{2n}(O \times O)$ is a positive definite kernel, then:*

(i) ([15]) *$D^{\alpha,\alpha} K$ is a positive definite kernel of class $C^{2(n-|\alpha|)}(O \times O)$, whenever $|\alpha| \leq n$;*

(ii) ([1]) *the following inequality holds:*

$$|D^{\alpha,\beta}_{x,y} K(x,y)|^2 \leq D^{\alpha,\alpha}_{x,y} K(x,x) D^{\beta,\beta}_{x,y} K(y,y), \quad x, y \in O\,, \tag{3.1}$$

whenever $|\alpha|, |\beta| \leq n$.

The inequality in (ii) was proved in [7] for the case where the kernel admits a Mercer-like expansion. However, it was proved in more general contexts in [1, 2], relying on positive definiteness only.

For the case of a positive definite function, we obtain the following result, exploiting also Proposition 2.1.

Proposition 3.2. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be a positive definite function. Assume that f is of class C^{2n} in some neighborhood of the origin, for some positive integer n. Then*

(i) *each function*

$$f_\alpha := (-1)^{|\alpha|} D^{2\alpha} f, \quad |\alpha| \leq n, \tag{3.2}$$

is positive definite of class $C^{2(n-|\alpha|)}(\mathbb{R}^m)$;

(ii) *the following inequality holds:*

$$|D^{\alpha+\beta} f(x)|^2 \leq (-1)^{|\alpha+\beta|} D^{2\alpha} f(0)\, D^{2\beta} f(0), \quad x \in \mathbb{R}^m, \tag{3.3}$$

whenever $|\alpha|, |\beta| \leq n$.

Proof. By Proposition 2.1, $f \in C^{2n}(\mathbb{R}^m)$ and then $K(x,y) := f(x-y)$ is a positive definite kernel of class $C^{2n}(\mathbb{R}^m \times \mathbb{R}^m)$. Using the chain rule it is easy to see that

$$D^{\alpha,\beta}_{x,y} K(x,y) = D^{0,\beta}_{x,y} [(D^\alpha f)(x-y)] = (-1)^{|\beta|} D^{\alpha+\beta} f(x-y), \quad x, y \in \mathbb{R}^m. \tag{3.4}$$

In particular, for $|\alpha| \leq n$,

$$(-1)^{|\alpha|} D^{2\alpha} f(x-y) = D^{\alpha,\alpha}_{x,y} K(x,y), \quad x, y \in \mathbb{R}^m. \tag{3.5}$$

Since $D^{\alpha,\alpha}_{x,y}K$ is a positive definite kernel of class $C^{2(n-|\alpha|)}(\mathbb{R}^m\times\mathbb{R}^m)$ (by Proposition 3.1), equation (3.5) implies that f_α is a positive definite function in $\mathbb{R}^m$. Finally, from (3.1) and (3.4), it follows that

$$\begin{aligned} |D^{\alpha+\beta}f(x)|^2 &= |D^{\alpha,\beta}_{x,y}K(x,0)|^2 \\ &\leq D^{\alpha,\alpha}_{x,y}K(x,x)D^{\beta,\beta}_{x,y}K(0,0) = (-1)^{|\alpha|+|\beta|}D^{2\alpha}f(0)D^{2\beta}f(0), \quad x\in\mathbb{R}^m, \end{aligned}$$

whenever $|\alpha|,|\beta|\leq n$. ■

Observe that the right hand side in (3.3) is nonnegative because both functions $(-1)^{|\alpha|}D^{2\alpha}f$ and $(-1)^{|\beta|}D^{2\beta}f$ are positive definite and hence they are nonnegative at the origin. For the same reason, Hypothesis H_{ra} can be written as

$$0\leq(-1)^lD^{2le_j}f(0)\leq d\frac{(2l)!}{\widetilde{R}^{2l}},$$

for every $l\geq 0$ and $j=1,\ldots,m$. Finally, observe that claim (ii) in the above proposition holds true also for $n=0$, where it reduces to the known relation (2.5).

We are now in the position to give the

Proof of Theorem 2.2. Fix $j\in\{1,\ldots,m\}$ and suppose that $D^{2l_je_j}f(0)=0$. We will show that $D^{2e_j}f(0)=0$.
If $l_j=1$, there is nothing to prove. So, we assume $l_j>1$ and we define the non-increasing sequence $\{k^j_p\}$ of even numbers by setting $k^j_1=2l_j$ and

$$k^j_{p+1}=\begin{cases} k^j_p/2 & \text{if } k^j_p/2 \text{ is even} \\ k^j_p/2+1 & \text{if } k^j_p/2 \text{ is odd} \end{cases}, \quad p=1,2,\ldots.$$

Observe that there exists an index $p(l_j)$ depending on l_j such that $k^j_p=2$, for any $p\geq p(l_j)$. We now prove, by induction on p, that

$$D^{k^j_pe_j}f(0)=0, \text{ for all positive integers } p. \tag{3.6}$$

For $p=1$, we have $k^j_1e_j=2l_je_j$ and then (3.6) is true by hypothesis. Suppose now that $D^{k^j_pe_j}f(0)=0$ for some positive integer p. Using the Proposition 3.2-(ii), with $\alpha=0$ and $\beta=(k^j_p/2)e_j$, we obtain

$$|D^{(k^j_p/2)e_j}f(x)|^2\leq(-1)^{k^j_p/2}f(0)D^{k^j_pe_j}f(0)=0, \quad x\in\mathbb{R}^m. \tag{3.7}$$

Thus, for every $x\in\mathbb{R}^m$, one has $D^{(k^j_p/2)e_j}f(x)=0$, as a consequence $D^{((k^j_p/2)+1)e_j}f(x)=0$, and then

$$D^{k^j_{p+1}e_j}f(x)=0, \quad x\in\mathbb{R}^m, \tag{3.8}$$

in particular, $D^{k^j_{p+1}e_j}f(0)=0$. Therefore, $D^{k^j_pe_j}f(0)=0$ for all $p=1,2,\ldots$. When $p=p(l_j)$, we have $k_{p(l_j)}e_j=2e_j$.

In order to finalize the proof, we apply again Proposition 3.2-(ii), now with $\alpha=e_j$ and $\beta=0$:

$$|D^{e_j}f(x)|^2\leq -f(0)D^{2e_j}f(0)=0, \quad x\in\mathbb{R}^m. \tag{3.9}$$

Therefore, $D^{e_j}f\equiv 0$, for every $j=1,2,\ldots,m$, which implies that f is constant in $\mathbb{R}^m$. ■

Remark 3.3. *No analogous to Theorem 2.2 holds for odd order derivatives, actually, the positive definite function (*[5, p. 104]*)*

$$f(x)=\exp(-||x||^2), \quad x\in\mathbb{R}^m, \tag{3.10}$$

is a simple example where $D^{(2l+1)e_j}f(0)=0$*, for every* $j=1,2,\ldots,m$ *and* $l=0,1,\ldots$*, but* f *is not constant.*

4. Real-Analyticity

We recall that if $f \in C^\infty(O)$ is a complex-valued function defined in the open set $O \subseteq \mathbb{R}^m$, then f is real-analytic if and only if, for every $y \in O$, there exist an open ball U, with $y \in U \subset O$, and positive constants M and r such that

$$|D^\alpha f(x)| \leq M \frac{\alpha!}{r^{|\alpha|}}, \quad x \in U, \tag{4.1}$$

for all multi-indexes $\alpha \in \mathbb{Z}_+^m$ (see for example [13, p. 34]).

In this section we prove Theorem 2.3, which asserts that, in the case of positive definite functions, real-analyticity can be guaranteed by an estimate similar to (4.1), but involving only even order derivatives with respect to one single variable, and only at the origin: Hypothesis H_{ra}.

It is worth noting that, for general functions, an estimate on the derivatives in one single point is not enough to guarantee analyticity: a classic example is given by

$$g(x) = \begin{cases} 1 - e^{-1/||x||^2} & for\ x \neq 0\,, \\ 1 & for\ x = 0\,, \end{cases}$$

which has all derivatives at the origin equal to zero, but is not analytic in any neighborhood of the origin. As a consequence, g cannot be positive definite in $\mathbb{R}^m$, despite it satisfies (2.3-2.5) and it has a qualitative graph similar to (3.10). In fact, if g were positive definite, then Theorem 2.2 would imply that it is constant.

First, we need the following Lemma, which shows that Hypothesis H_{ra} allows to estimate every derivative at the origin, whose multi-index has all even entries:

Lemma 4.1. *Let $f : \mathbb{R}^m \to \mathbb{C}$ be a positive definite function. If Hypothesis H_{ra} holds, then*

$$|D^{2\alpha} f(0)| \leq d \frac{(2\alpha)!}{R^{|2\alpha|}}, \tag{4.2}$$

for every multi-index $\alpha \in \mathbb{Z}_+^m$, where $R = 2^{-t}\widetilde{R}$ and t is a integer number such that $2^t \geq m$.

Proof. We will need the estimate

$$(2\alpha)! \leq (2^{|\alpha|}\alpha!)^2\,, \tag{4.3}$$

which is a consequence of $(2n)! = |(n,n)|! \leq 2^{2n}(n,n)!$, when n is a nonnegative integer.

We will prove by induction that

$$|D^{2\alpha} f(0)| \leq d \frac{(2\alpha)!}{(\widetilde{R}/2^t)^{|2\alpha|}}$$

for every α such that at most 2^t indexes are positive. This holds true for $t = 0$ by Hypothesis H_{ra}.

We suppose it holds true for some $t \geq 0$ and we will prove that then it holds true also for $t+1$. Actually, if γ has at most 2^{t+1} positive indexes, we write $\gamma = \alpha + \beta$ in such a way that α, β both have at most 2^t positive indexes, and no index is positive in both. Then, by (3.3) and using (4.3),

$$|D^{2\alpha+2\beta} f(0)|^2 \leq |D^{4\alpha} f(0)||D^{4\beta} f(0)| \leq d^2 \frac{(4\alpha)!(4\beta)!}{(\widetilde{R}/2^t)^{|4\alpha+4\beta|}} \leq d^2 \frac{[2^{|2\alpha+2\beta|}(2\alpha)!(2\beta)!]^2}{(\widetilde{R}/2^t)^{|4\alpha+4\beta|}}\,,$$

that is,

$$|D^{2\gamma} f(0)| = |D^{2\alpha+2\beta} f(0)| \leq d \frac{2^{|2\alpha+2\beta|}(2\alpha)!(2\beta)!}{(\widetilde{R}/2^t)^{|2\alpha+2\beta|}} = d \frac{(2\alpha)!(2\beta)!}{(\widetilde{R}/2^{t+1})^{|2\alpha+2\beta|)}} = d \frac{(2\gamma)!}{(\widetilde{R}/2^{t+1})^{|2\gamma|}}\,.$$

■

Proof of Theorem 2.3. Since f is of class C^∞ in some neighborhood of the origin, by Proposition 2.1, $f \in C^\infty(\mathbb{R}^m)$. We will show that there exist $M, r > 0$ for which (4.1) holds true for all $x \in \mathbb{R}^m$ and all multi-indexes α.

Fix $\alpha \in \mathbb{Z}_+^m$ and consider γ and β multi-indexes such that

$$\alpha = \gamma + \beta, \tag{4.4}$$

where $|\gamma| = |\beta|$ if $|\alpha|$ is even and $|\gamma| = |\beta| + 1$ if $|\alpha|$ is odd.
Using the Proposition 3.2-(ii) we obtain

$$|D^\alpha f(x)|^2 = |D^{\gamma+\beta} f(x)|^2 \le |D^{2\gamma} f(0)||D^{2\beta} f(0)|, \quad x \in \mathbb{R}^m. \tag{4.5}$$

Then, (2.1) and (4.2) imply that

$$|D^\alpha f(x)|^2 \le d^2 \frac{(2\gamma)!}{R^{|2\gamma|}} \frac{(2\beta)!}{R^{|2\beta|}} \le d^2 \frac{|2\gamma|!}{R^{|2\gamma|}} \frac{|2\beta|!}{R^{|2\beta|}} = \frac{d^2}{R^{2|\alpha|}} |2\gamma|!|2\beta|!, \quad x \in \mathbb{R}^m\,. \tag{4.6}$$

Now we have

$$\begin{cases} |2\gamma| = |2\beta| = |\alpha| & \text{if } |\alpha| \text{ is even,} \\ |2\gamma| = |\alpha| + 1, \quad |2\beta| = |\alpha| - 1 & \text{if } |\alpha| \text{ is odd;} \end{cases}$$

then

$$|2\gamma|!|2\beta|! = \begin{cases} (|\alpha|!)^2 & \text{if } |\alpha| \text{ even,} \\ (|\alpha|!)^2 \frac{|\alpha|+1}{|\alpha|} & \text{if } |\alpha| \text{ odd,} \end{cases}$$

and so $|2\gamma|!|2\beta|! \le 2(|\alpha|!)^2$. Then (4.6) becomes

$$|D^\alpha f(x)|^2 \le 2d^2 \frac{(|\alpha|!)^2}{R^{2|\alpha|}}, \quad x \in \mathbb{R}^m. \tag{4.7}$$

Using (2.1),

$$|D^\alpha f(x)| \le \sqrt{2}\, d \frac{|\alpha|!}{R^{|\alpha|}} \le \sqrt{2}\, d\, m^{|\alpha|} \frac{\alpha!}{R^{|\alpha|}} = \sqrt{2}\, d \frac{\alpha!}{(R/m)^{|\alpha|}}, \quad x \in \mathbb{R}^m. \tag{4.8}$$

Writing $r = R/m$ and $M = \sqrt{2}\, d$, the estimative (4.1) is proved for every multi-index α, therefore f is real-analytic in $\mathbb{R}^m$. ■

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