Fučík spectrum for systems of PDEs and ODEs

Bernhard Ruf
Università degli Studi di Milano (Italy)
Eugenio Massa[1]
ICMC - USP, São Carlos (SP)

U. Chile
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The classical Fučík spectrum

Given the problem

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & in\ \Omega \\ Bu = 0 & in\ \partial\Omega \end{cases} . \qquad \text{(PF)}$$

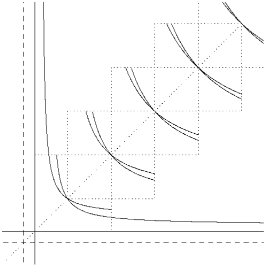
The Fučík spectrum (First introduced by Fučík and Dancer in 1976-77):

$$\Sigma_{eq} = \left\{ (\lambda^+, \lambda^-) \in \mathbb{R}^2 \quad \text{such that (PF) has nontrivial solutions} \right\} .$$

(Here Ω is a bounded domain in $\mathbb{R}^n$, $u^\pm(x) = \max\{0, \pm u(x)\}$ and $Bu = 0$ represents Dirichlet or Neumann boundary conditions).

Fučík spectrum: PDE case

bk1 bk2



Known parts:

1. • trivial part $\lambda^\pm = \lambda_1$,
   • nontrivial part in $\lambda^\pm > \lambda_1$;
2. near the diagonal:
   in $(\lambda_{k-1}, \lambda_{k+1})^2$, two curves through (λ_k, λ_k),
   in $(\lambda_{k-1}, \lambda_k)^2$ and $(\lambda_k, \lambda_{k+1})^2$, empty;
3. first nontrivial curve (obtained variationally).

[Gallouët-Kavian (81), Ruf (81), Magalhães (90), de Figueiredo-Gossez (94), Cuesta——- (99)]

Fučík spectrum: ODE case

bk1 bk2

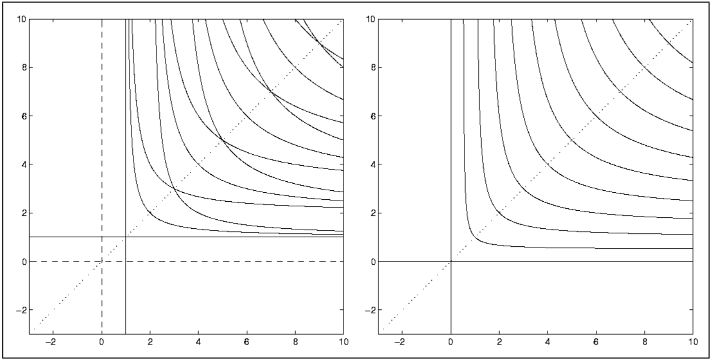


Figure: ODE Dirichlet and Neumann Fučík spectrum

Some classical applications

Fučík spectrum for systems
Eugenio Massa
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More properties
–degree
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bk

- The LS-degree of $u-(-\Delta)^{-1}(\lambda^+u^+ + \lambda^-u^-)$ is constant in the connected components of $\mathbb{R}^2 \setminus \Sigma$.
- this degree is nonzero in the connected components containing a piece of diagonal:
  - $\Rightarrow$ solvability of equation $-\Delta u - \lambda^+u^+ + \lambda^-u^- = h \in L^2$ in these regions (jumping nonlinearities - asymptotically linear problems)
  - $\simeq\Rightarrow$ Also, solvability of linear-superlinear problems if there is a gap between the asymptotes of subsequent curves (Neumann - Periodic problem in $[0,1]$)

[Fučík (77), Dancer (78...), de Figueiredo-Ruf (93), myself (04)]

A Fučík spectrum for the system

We will consider here the following generalization to the case of coupled systems:

$$\begin{cases} -\Delta u = \lambda^+ v^+ - \lambda^- v^- & in \ \Omega \\ -\Delta v = \mu^+ u^+ - \mu^- u^- & in \ \Omega \\ Bu = Bv = 0 & in \ \partial\Omega \end{cases} \qquad \text{(PFS)}$$

$$\Sigma = \left\{(\lambda^+, \lambda^-, \mu^+, \mu^-) \in \mathbb{R}^4 \quad \text{such that (PFS) has nontrivial solutions}\right\}$$

Simple properties

Lemma

Let $(\lambda^+, \lambda^-, \mu^+, \mu^-) \in \Sigma$ and (u, v) be a corresponding solution, then

1. *Both u and v change sign or none of the two.*
2. *If both u and v change sign then all the coefficients have the same sign (and no one is zero);*
3. *If u and v do not change sign then they are both non zero multiples of ϕ_1*

$\rightarrow$ Then we concentrate on the "non trivial" part: when both u and v change sign.

Lemma

Let $(\lambda^+, \lambda^-, \mu^+, \mu^-) \in \Sigma$ *and* (u, v) *be a corresponding solution, then:*

a) $\left(\frac{\lambda^+}{\delta}, \frac{\lambda^-}{\delta}, \delta\mu^+, \delta\mu^-\right) \in \Sigma$ *for any* $\delta > 0$, *with corresp. sol.* $(u, \delta v)$,

b) $(-\lambda^-, -\lambda^+, -\mu^+, -\mu^-) \in \Sigma$, *with corresp. sol.* $(u, -v)$

c) $(\mu^+, \mu^-, \lambda^+, \lambda^-) \in \Sigma$, *with corresp. sol.* (v, u)

d) $(\lambda^-, \lambda^+, \mu^-, \mu^+) \in \Sigma$, *with corresp. sol.* $(-u, -v)$

Symmetry b) links points with all negative coefficients to points with all positive coefficients.

Symmetry a) implies four parameters are redundant: we may make a change of the unknown functions: obtain one single point that represents the whole curve generated by this symmetry for $\delta \in \mathbb{R}^+$.

Reformulation

Thus we may do the following simplifications:

- Consider only sign changing solutions.
- Consider positive coefficients
- Choose δ such that $\delta\mu^+ = \lambda^+/\delta$:

Reformulation

$$\begin{cases} -\Delta u = \lambda^+ v^+ - \lambda^- v^- & in\ \Omega \\ -\Delta v = \lambda^+ u^+ - \mu^- u^- & in\ \Omega \\ Bu = Bv = 0 & on\ \partial\Omega \end{cases} . \qquad \text{(PFS*)}$$

and $\widehat{\Sigma}_{nt}$ (the non trivial part)

$$\widehat{\Sigma}_{nt} = \{(\lambda^+, \lambda^-, \mu^-) \in \mathbb{R}^3 \text{ such that:} \lambda^\pm, \mu^- > 0 \text{ and (PFS*) has nontrivial solutions which (both) change sign}\} .$$

More properties

JMP1 bk3

- Exist points in $\widehat{\Sigma}_{nt}$?
  If $(\lambda^+,\lambda^-)\in\Sigma_{eq}$ with $\lambda^\pm>\lambda_1$, then $(\lambda^+,\lambda^-,\lambda^-)\in\widehat{\Sigma}_{nt}$ and $u=v$ (in particular $(\lambda_k,\lambda_k,\lambda_k)\in\widehat{\Sigma}_{nt}$).
- Where?
  - $(\lambda^+,\lambda^-,\mu^-)\in\widehat{\Sigma}_{nt}$ implies $\lambda^+>\lambda_1$ and $\sqrt{\lambda^-\mu^-}>\lambda_1$.
  - If $\lambda^\pm,\mu^->0$ are such that $\lambda_k<\frac{\lambda^+}{\delta},\frac{\lambda^-}{\delta},\delta\lambda^+,\delta\mu^-<\lambda_{k+1}$ for some $\delta>0$, $k\geq 1$, then $(\lambda^+,\lambda^-,\mu^-)\notin\widehat{\Sigma}_{nt}$.
- Characteristics of the corresp. solutions?
  Let $(\lambda^+,\lambda^-,\mu^-)\in\widehat{\Sigma}_{nt}$ and (u,v) be a corresponding nontrivial solution: then $u^+v^+\not\equiv 0$ and $u^-v^-\not\equiv 0$.

JMP

Proposition

The LS-degree of

$$(u,v) - diag(-\Delta)^{-1}\left(\lambda^+v^+ + \lambda^-v^-, \lambda^+u^+ + \mu^-u^-\right)$$

is constant in the connected components of $\{\lambda^+,\lambda^-,\mu^- > \lambda_1\} \setminus \widehat{\Sigma}_{nt}$, *and is nonzero in those which contain a piece of the diagonal.*

Useful to study the solvability of asymptotically linear problems as

$$\begin{cases} -\Delta u = \lambda^+v^+ - \lambda^-v^- + g_1(x,u,v) + h_1(x) & in\ \Omega \\ -\Delta v = \mu^+u^+ - \mu^-u^- + g_2(x,u,v) + h_2(x) & in\ \Omega \\ Bu = Bv = 0 & in\ \partial\Omega \end{cases},$$

where $h_{1,2} \in L^2(\Omega)$, $g_{1,2}$ sublinear in u, v.

Existence of surfaces in $\widehat{\Sigma}_{nt}$

Local surfaces near λ_k

Through each point $(\lambda_k, \lambda_k, \lambda_k) \in \widehat{\Sigma}_{nt}$ (λ_k simple) there pass two (maybe coincident) "Fučík surfaces", in $\widehat{\Sigma}_{nt}$, parameterized by $\lambda^+ = \lambda_{k+}(\lambda^-, \mu^-)$ and $\lambda^+ = \lambda_{k-}(\lambda^-, \mu^-)$ with $|\lambda^- - \lambda_k|, |\mu^- - \lambda_k|$ small enough.
Between the two surfaces the degree is zero (Ambrosetti-Prodi results)

(topological degree and Lyapunov-Schmidt reduction),

A global surface near λ_2

We can find and characterize variationally a surface in $\widehat{\Sigma}_{nt}$ of the form

$$(\lambda_1 + d(r,s), \lambda_1 + s\, d(r,s), \lambda_1 + r\, d(r,s)) : \quad r, s \in (0, +\infty),$$

which pass through $(\lambda_2, \lambda_2, \lambda_2)$

(variational techniques, Galerkin approximation) JMP2

JMP3 QUESTION: exist points in $\widehat{\Sigma}_{nt}$ not related to any point in Σ_{eq}?

(For the linear spectrum the answer would be NO: solutions of

$$\begin{cases} -\Delta u = \lambda v & \text{in } \Omega \\ -\Delta v = \lambda u & \text{in } \Omega \\ Bu = Bv = 0 & \text{on } \partial\Omega \end{cases}$$

are only $(u, v) = (\phi_k, \pm\phi_k)$ with $\lambda = \pm\lambda_k$).

Proposition

($\partial\Omega$ sufficiently regular) If $(\lambda^+, \mu^-, \lambda^-) \in \widehat{\Sigma}_{nt}$ with $\mu^- \neq \lambda^-$ then for the corresponding nontrivial solutions u, v at least three of the products u^+v^+, u^+v^-, u^-v^+, u^-v^- are not identically zero.

This (along with the previous result) means that the Fučík problem for the system is much richer than that for one equation.

The ODEs system

Now we consider the ODE system:

$$\begin{cases} -u'' = \lambda^+ v^+ - \lambda^- v^- & in\ (0,1) \\ -v'' = \lambda^+ u^+ - \mu^- u^- & in\ (0,1) \\ Bu = Bv = 0 & in\ \{0;1\} \end{cases},$$

We will obtain:

- Local description through the implicit function theorem [similar to Campos-Dancer (2001)]
- Global description of each surface by continuation
- Global description of $\widehat{\Sigma}_{nt}$ using the knowledge of the linear spectrum for the system and of the Fučík spectrum for one equation Σ_{eq}

First step: derive some qualitative properties of the nontrivial solutions:

Theorem

If $(\lambda^+,\lambda^-,\mu^-)\in\widehat{\Sigma}_{nt}$ and (u,v) is a corresponding nontrivial solution then u and v have only simple zeros, in the same number, and have the same sign both in a neighborhood of 0 and of 1.

Local study

We consider the IVP

$$\begin{cases} -u'' = \lambda^+ v^+ - \lambda^- v^- \\ -v'' = \lambda^+ u^+ - \mu^- u^- \\ (u, v, u', v')(0) = (0, 0, \pm 1, s) \end{cases}$$

and we define, (Dirichlet case)

$$\widetilde{\Sigma}^\pm = \left\{ \begin{array}{c} (\lambda^+, \lambda^-, \mu^-, s) \in (\mathbb{R}^+)^3 \times \mathbb{R} \ : \ \text{the solutions} \ \ (u, v) \\ \text{change sign and satisfy} \ \ u(1) = v(1) = 0 \end{array} \right\}$$

We denote by $(u, v)\,[\lambda^+, \lambda^-, \mu^-, s]\,(x)$ the solution of the IVP and we apply the implicit function theorem to the system

$$(u, v)\left[\lambda^+, \lambda^-, \mu^-, s\right](1) = (0, 0)\,, \tag{3.1}$$

in a point of $\widetilde{\Sigma}^\pm$, in order to obtain (locally) λ^+ and s as a function of λ^- and μ^-.

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in a point of $\widetilde{\Sigma}^\pm$, in order to obtain (locally) λ^+ and s as a function of λ^- and μ^-.

Lemma

- $\widetilde{\Sigma}^{\pm}$ *is locally of the form* $(\lambda^+(\lambda^-,\mu^-),\lambda^-,\mu^-,s(\lambda^-,\mu^-))$, *where* λ^+, s *are* $\mathcal{C}^1$ *functions;*
- *The partial derivatives are*

$$\frac{\partial\lambda^+}{\partial\lambda^-}(\bar\lambda^-,\bar\mu^-) = \frac{-\int_0^1(\bar v^-)^2}{\int_0^1(\bar u^+)^2+(\bar v^+)^2} < 0\,,$$
$$\frac{\partial\lambda^+}{\partial\mu^-}(\bar\lambda^-,\bar\mu^-) = \frac{-\int_0^1(\bar u^-)^2}{\int_0^1(\bar u^+)^2+(\bar v^+)^2} < 0\,.$$

- *In this region, the related nontrivial solutions maintain the number of zeros and the sign in a neighborhood of* 0 *and of* 1.

Global study of the surfaces

- Each connected component of $\widetilde{\Sigma}^{\pm}$ is characterized by initial sign and number of zeros of its solutions
- Each connected component eventually crosses the diagonal $\lambda^+ = \lambda^- = \mu^-$
- $\widehat{\Sigma}_{nt}$ on the diagonal is the linear spectrum: completely known

This implies (projecting to remove the variable s)

- two (may be coincident) $\mathcal{C}^1$ global surfaces $\Sigma_k^{\pm}$ for each $k \geq 2$: passing through $(\lambda_k, \lambda_k, \lambda_k)$, solutions with $k-1$ zeros, starting positive (resp. negative).
- All points in $\widehat{\Sigma}_{nt}$ belong to one of these surfaces.
- these surfaces may be represented expressing the variable λ^+ in terms of the other two, and they are monotone decreasing in the two variables and unbounded in the three directions;
- $\widehat{\Sigma}_{nt}$ restricted to the plane $\lambda^- = \mu^-$ coincide with the non trivial part of Σ_{eq}. JMP1

Symmetries of the solutions and of the surfaces

The fact that there exists only ONE surface given the initial sign and the number of zeros, may be used to study the symmetries of the surfaces and of the solutions:

(Dirichlet case) JMP2

- if k is even, the surfaces $\widehat{\Sigma}_k^{\pm}$ coincide (exists only ONE kind of solution starting positive and ONE starting negative, $\Rightarrow$ they are one the reflection of the other)
- if k is odd, the surfaces $\widehat{\Sigma}_k^{\pm}$ are distinct (since it is so for $\Sigma_{eq} \subseteq \widehat{\Sigma}_{nt}$),
  the nontrivial solutions corresponding to points in $\widehat{\Sigma}_k^{\pm}$ are always symmetric, in the sense that $(u,v)(x) = (u,v)(1-x)$ (exists only ONE solution starting positive $\Rightarrow$ it coincides with its reflection).

(Neumann case)

- the two surfaces $\widehat{\Sigma}_k^{\pm}$ coincide for any $k \geq 2$,
- if $j \geq 0$ and $k \equiv 1 \pmod{2^{j+1}}$, then (u,v) are symmetric and $(1/2)^j$ periodic.

Conclusions and summary

- Our Fučík spectrum for systems maintain useful properties of classical one (Degree properties.. relation with solvability of asymptotically linear problems..).
- It is richer (solutions unrelated to the case of a single equation).
- surfaces near the diagonal and a "first non trivial surface" may be described as for the classical problem.
- In the ODE case much more detailed results may be obtained but not an explicit description of $\widehat{\Sigma}_{nt}$ as is possible for the ODE problem with one equation.
  In fact, in the regions u and v have opposite sign much more complicated patterns arise.

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Some open problem

- Asymptotic behavior of the surfaces $\widehat{\Sigma}_k^{\pm}$: it would be interesting in view of solvability of linear-superlinear problems
- Do the nontrivial solutions have more symmetries than those proved?

References

- The PDE case:
  E. Massa and B. Ruf, *On the Fucík Spectrum for Elliptic Systems*, Topol. Methods Nonlinear Anal. 27 (2006), no. 2, 195–228.
- The ODE case:
  E. Massa and B. Ruf, *A global characterization of the Fucík spectrum for a system of ordinary differential equations*, J. Differential Equations. 234 (2007), no. 1, 311–336.