On almost resonant elliptic problems [1]

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The main problem

We consider the following system:

$$\begin{cases} -\Delta u = au + bv \pm (f_1(x,v) + h_1(x)) & \text{in } \Omega, \\ -\Delta v = bu + av \pm (f_2(x,u) + h_2(x)) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1$\pm$}$$

where

- $\Omega \subset \mathbb{R}^N$ bounded domain,
- $a, b \in \mathbb{R}$,
- $h_1, h_2 \in L^2(\Omega)$,
- $f_1, f_2 : \Omega \times \mathbb{R} \to \mathbb{R}$ sublinear Carathéodory functions: $\exists\ S > 0,\ q \in (1,2)$, such that $|f_i(x,t)| \leq S(1 + |t|^{q-1})$, $i = 1, 2$.

Purpose: to obtain multiplicity of solutions, when the linear part is "near resonance": that is, $a + b$ or $a - b$ near some eigenvalue λ_k).

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Motivation: the scalar problem

Main motivation: de Paiva, M. [dPM08]: for the scalar equivalent

$$\begin{cases} -\Delta u = \lambda u \pm f(x,u) + h(x) & \text{in } \Omega\,, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \qquad (1.2\pm)$$

where $h \in L^2(\Omega)$, f is sublinear, and

$$\begin{cases} \lim_{|t|\to\infty} F(x,t) = +\infty \text{ uniformly } x \in \Omega\,, \\ \int_\Omega h\,\phi\,dx = 0 \quad \forall\,\phi \in H_{\lambda_k}. \end{cases}$$

it was proved that

a) there exists $\varepsilon_0 > 0$, such that, if $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$, then two solutions exist for problem (1.2+);
b) there exists $\varepsilon_1 > 0$,such that, if $\lambda \in (\lambda_k, \lambda_k + \varepsilon_1)$ then two solutions exist for problem (1.2−).

The results for the system

For our system we assume an analogous condition:

$$\begin{cases} (i) & \lim_{|t|\to\infty} F_i(x,t) = +\infty, \text{ unif. with resp. to } x \in \Omega,\ i = 1,2, \\ (ii) & \int_\Omega h_1\phi + h_2\psi = 0, \quad \text{for every } (\phi,\psi) \in Z, \end{cases} \qquad \textbf{(F)}$$

Theorem

Assume the given hypotheses, let $\lambda_k, \lambda_l \in \sigma(-\Delta)$,

$$Z = span\,\{\,(\phi,\phi):\ \phi \in H_{\lambda_k}\}\,.$$

Then

(a) *Given* $\delta > 0$, *there exists* $\varepsilon_0 > 0$ *such that, if* $a - b \in (\lambda_{l-1} + \delta, \lambda_l - \delta)$ *and* $a + b \in (\lambda_k - \varepsilon_0, \lambda_k)$, *then Problem* (1.1+) *has two distinct solutions.*

(b) *Given* $\delta > 0$, *there exists* $\varepsilon_1 > 0$ *such that, if* $a - b \in (\lambda_{l-1} + \delta, \lambda_l - \delta)$ *and* $a + b \in (\lambda_k, \lambda_k + \varepsilon_1)$, *then Problem* (1.1−) *has two distinct solutions.*

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$$Z = span\left\{ (\phi, -\phi) : \ \phi \in H_{\lambda_k} \right\}.$$

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(a) *Given* $\delta > 0$, *there exists* $\varepsilon_0 > 0$ *such that, if* $a + b \in (\lambda_{l-1} + \delta, \lambda_l - \delta)$ *and* $a - b \in (\lambda_k - \varepsilon_0, \lambda_k)$, *then Problem* (1.1−) *has two distinct solutions.*

(b) *Given* $\delta > 0$, *there exists* $\varepsilon_1 > 0$ *such that, if* $a + b \in (\lambda_{l-1} + \delta, \lambda_l - \delta)$ *and* $a - b \in (\lambda_k, \lambda_k + \varepsilon_1)$, *then Problem* (1.1+) *has two distinct solutions.*

Double resonance

Theorem

Assume the given hypotheses, let $\lambda_k, \lambda_l \in \sigma(-\Delta)$ *(may be the same) and* $Z = span\{\, (\phi, \phi) : \ \phi \in H_{\lambda_k}, \ \ (\phi, -\phi) : \ \phi \in H_{\lambda_l} \}$. *Then*

(e) *there exists* $\varepsilon_2 > 0$ *such that, if* $a - b \in (\lambda_l, \lambda_l + \varepsilon_2)$ *and* $a + b \in (\lambda_k - \varepsilon_2, \lambda_k)$, *then problem* (1.1+) *has two distinct solutions.*

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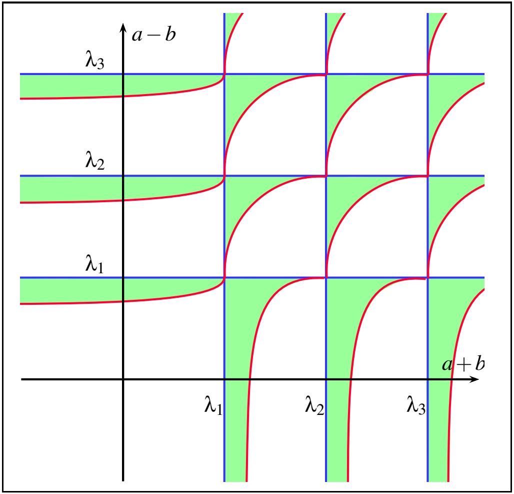
Figure: Sketch of the regions with two solutions for problem (1.1−)



More literature

- Scalar problem:
  - λ_1, *ODE*, bifurcation and degree. Mawhin-Schmitt (1990), Badiale-Lupo (1989), Lupo-Ramos (1990)
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  - λ_1, *PDE*, variational techniques. Ramos-Sanchez (1997), Ma-Ramos-Sanchez (1997), Ma-Pelicer (2002) (p-Laplacian)
  - λ_k, *ODE*, bifurcation and degree. Lupo-Ramos (1990)
  - λ_k, *PDE*, variational techniques. de Paiva-M. (2008), Ke-Tang (2011)
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Background

For the problem

$$\begin{cases} -\Delta u = \lambda u + h(x) & \text{in} \quad \Omega\,, \\ \quad u = 0 & \text{in} \quad \partial\Omega, \end{cases} \tag{1.1}$$

one has

- If λ is NOT an eigenvalue: there there exists a unique solution.
- If $\lambda = \lambda_k$ is an eigenvalue: there exists no solution or infinte solutions (depending if $\int_\Omega h\,\phi\,dx \neq 0$ *or* $= 0 \quad \forall\,\phi \in H_{\lambda_k}$)

However, the solution when $\lambda \in (\lambda_k, \lambda_k + \varepsilon_1)$ and the solution when $\lambda \in (\lambda_k - \varepsilon_1, \lambda_k)$ are "different" in several ways.

Variational methods

In order to find a solutions of

$$\begin{cases} -\Delta u = f(x,u) & \text{in} \quad \Omega\,, \\ u = 0 & \text{in} \quad \partial\Omega. \end{cases} \tag{1.2}$$

We define a functional $J : H_0^1(\Omega) \to \mathbb{R}$:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(x,u)$$

(under suitable hypotheses) it turns out that J is smooth and critical points of J (i.e. $u: \; J'(u) = 0$) and solutions of the equation, are the same thing.

So: how do we find critical points of J?

Example: $J : \mathbb{R} \to \mathbb{R}$, $J \in C^1(\mathbb{R})$, $\lim_{t\to\pm\infty} J(t) = +\infty$ implies J has a global minimum, which is a crtical point.

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Variational methods

Some classical Critical point theorems:

Definition

$J \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the **PS condition:**
For each sequence $\{u_n\} \subseteq E$ such that $|J(u_n)| \leq C$ and $J'(u_n) \to 0$ there exists a (strongly) convergent subsequence.

Theorem (Mountain Pass Theorem)

1. *E Banach space; $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the PS condition;*
2. *$I(0)=0$;*
3. *$\exists \rho, \alpha > 0$ such that $I(u) \geq \alpha$ for all u such that $\|u\|_E = \rho$;*
4. *$\exists e \in E$ such that $\|e\|_E > \rho$ and $I(e) < 0$.*

Moreover, let

- $\Gamma = \{\gamma \in \mathcal{C}^0([0,1]; E)$ *such that* $\gamma(0) = 0$ *and* $\gamma(1) = e\}$;
- $c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t))$.

Then $c \geq \alpha$ and there exists a critical point at level c.

Variational methods

Theorem (Saddle Point Theorem)

1. *E Banach space; $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfies the PS condition;*
2. *$E = V \oplus W$ with $\dim(V) < \infty$;*
3. *$\exists \beta < \alpha$ and $\rho > 0$ such that*
   – *$I(u) \geq \alpha$ for all $u \in W$;*
   – *$I(u) \leq \beta$ for all $u \in B_\rho^V = \{u \in V, \|u\| = \rho\}$;*

Moreover, let

- $\Gamma = \{\gamma \in \mathcal{C}^0(B_\rho^V; E)$ *such that* $\gamma|_{\partial B_\rho^V} = id\}$
- $c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t))$.

Then $c \geq \alpha$ and there exists a critical point at level c.

Variational methods

Let $\lambda \in (\lambda_{k-1}, \lambda_k)$ and consider again

$$\begin{cases} -\Delta u = \lambda u + h(x) & \text{in } \Omega\,, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.3}$$

Let

$$V = \oplus_{i=1}^{k-1} H_{\lambda_i} \qquad W = V^{\perp}$$

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega hu$$

By properties of the eigenspaces

$$\int_\Omega |\nabla u|^2 \leq \lambda_{k-1}\int_\Omega u^2 \ \ in\ V \qquad \int_\Omega |\nabla u|^2 \geq \lambda_k \int_\Omega u^2 \ \ in\ W$$

In the end one gets the conditions of the Saddle Point Theorem.
If $\lambda \in (\lambda_{k-1}, \lambda_k)$ it is the same... but with different spaces involved!

Scalar problem

Let us go back to the scalr problem in [dPM08]:

$$\begin{cases} -\Delta u = \lambda u \pm f(x,u) + h(x) & \text{in } \Omega\,, \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1.2$\pm$}$$

where $h \in L^2(\Omega)$, f is sublinear, and

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How do we prove that there exists $\varepsilon_0 > 0$, such that, if $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$, then two solutions exist for problem (1.2+)?

The idea below this kind of problem is the following: passing the eigenvalue the saddle point geometry changes: near the eigenvalue the perturbation f makes it possible to have both saddle geometries at the same time.

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Some notation

Functional:

$J^{\pm} : H_0^1(\Omega) \to \mathbb{R}$:

$$J(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right) dx \mp \int_\Omega F(x,u)\, dx - \int_\Omega h\, u\, dx$$

$V = span\{\phi_1, \dots, \phi_{k-1}\}$,

$Z = span\{\phi_k, \dots, \phi_{k+m-1}\} = H_{\lambda_k}$,

$W = (V \oplus Z)^\perp$,

S_V, S_{VZ}, S_{ZW}, the unit spheres in V, $V \oplus Z$, $Z \oplus W$

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If $\lambda \notin \sigma(-\Delta)$ there exists a solution from Saddle Point Theorem.

however, a suitable behaviour of f may give rise to a further solution.

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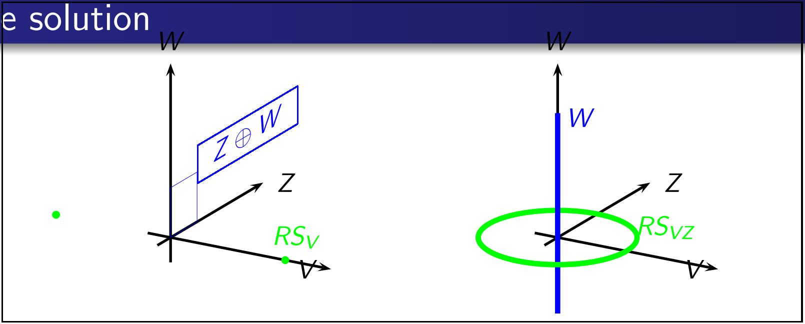
One solution



$$(\lambda < \lambda_k) \quad c_{k-1} = \inf_{\gamma \in \Gamma_{k-1}} \sup_{v \in RB_V} J(\gamma(v)) \, .$$

$$\Gamma_{k-1} = \{\gamma \in \mathcal{C}^0(RB_V; H_0^1) \ s.t. \ \gamma|_{RS_V} = Id\} \, ,$$

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The case $\lambda < \lambda_k$

Proposition

In the given hypotheses:

$$\exists\ D_W : \quad J^+(u) \geq D_W \qquad \text{for} \quad u \in W\,; \tag{1.4}$$

there exist $R^+, \varepsilon_0 > 0$ *such that, for any* $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$

$$J^+(u) < D_W \qquad \text{for} \quad u \in R^+ S_{VZ}\,, \tag{1.5}$$

$$\text{for} \quad u \in V\,,\ \|u\| \geq R^+\,; \tag{1.6}$$

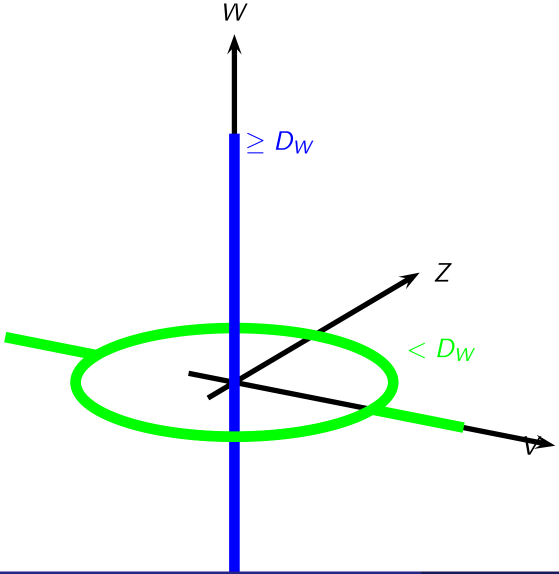
D

if now we fix $\lambda \in (\lambda_k - \varepsilon_0, \lambda_k)$ *then*

$$\exists\ D_\lambda : \quad J^+(u) \geq D_\lambda \qquad \text{for} \quad u \in Z \oplus W\,, \tag{1.7}$$

$$\exists\ \rho_\lambda^+ > R^+ : \quad J^+(u) < D_\lambda \qquad \text{for} \quad u \in \rho_\lambda^+ S_V\,. \tag{1.8}$$

D



We have

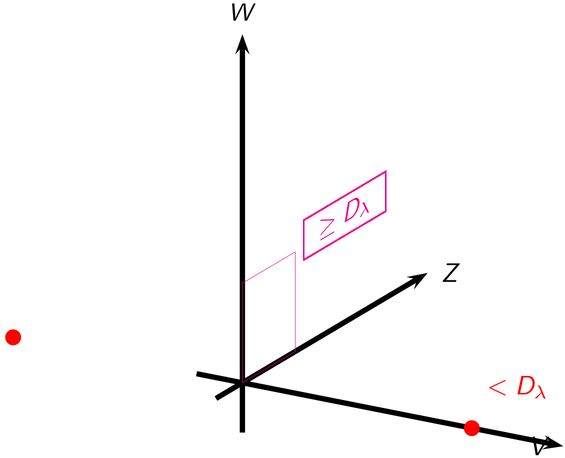
$$c_k \geq D_W \, ,$$

$$c_{k-1} \geq D_\lambda \, ,$$

but also

$$c_{k-1} < D_W \, ,$$

then the solutions are distinct.



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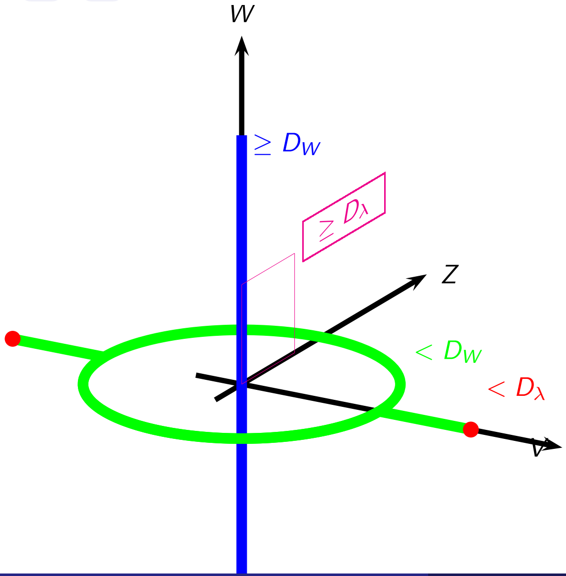

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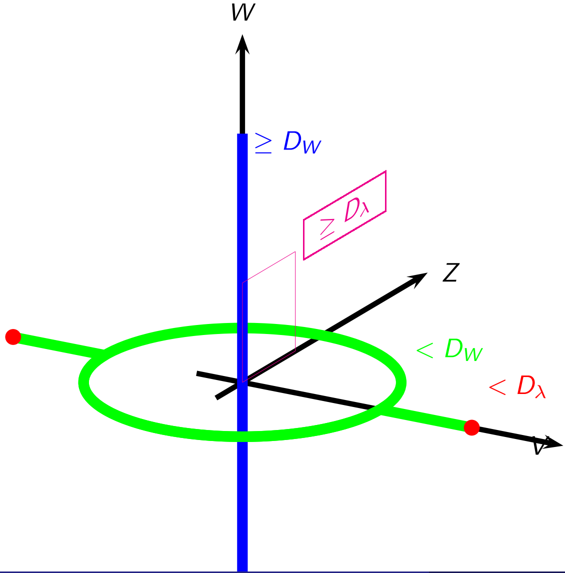

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Remarks:

- The system is of Hamiltonian type; then variational but with strongly indefinite functional: we use a Galerkin approximation.
- In the scalar case we had a different proof for the case above or below the eigenvalue (saddle point theorems or linking spheres theorem, depending if the almost resonant eigenspace is part of the finite / infinite dimensional subspace in the splitting). For the system, having to use Galerkin aproximation, the geometry is always the same: a finite dimensional saddle point.

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Idea of the Proof

We consider the functionals

$$J^{\pm}_{a,b}(u,v) = \pm\frac{1}{2}B_{a,b}((u,v),(u,v)) - \mathcal{F}(u,v) - \mathcal{H}(u,v). \tag{2.1}$$

defined in $E = H^1_0(\Omega) \times H^1_0(\Omega)$, where

$$\mathcal{F}(u,v) = \int_\Omega F_1(x,v) + \int_\Omega F_2(x,u), \qquad \mathcal{H}(u,v) = \int_\Omega h_1 v + \int_\Omega h_2 u,$$

$$B_{a,b}((u,v),(\phi,\psi)) = \int_\Omega \nabla u \nabla \psi + \int_\Omega \nabla v \nabla \phi - a\int_\Omega (u\psi + v\phi) - b\int_\Omega (u\phi + v\psi).$$

We consider the eigenvalues and eigenfunctions of B:

$$\mu_{\pm i} = \frac{-b \pm (\lambda_i - a)}{\lambda_i}, \qquad \psi_{\pm i} = \frac{(\phi_i, \pm\phi_i)}{\sqrt{2\lambda_i}}, \qquad i \in \mathbb{N}.$$

Then

$$\|\psi_i\|_E = 1, \quad \langle \psi_i, \psi_j \rangle_E = \delta_{i,j}, \quad B(\psi_i, \psi_j) = \mu_i \delta_{i,j}, \qquad i, j \in \mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}.$$

$$a \pm b = \lambda_i \Rightarrow \mu_{\pm i} = 0 \tag{2.2}$$

So we define

$V = span\{\psi_i : i \in \mathbb{Z}_0, \mu_i < 0\}$: negative subspace

$Z = span\{\psi_i : i \in \mathbb{Z}_0, \mu_i = \mu_k\}$: almost resonant subspace

$W = span\{\psi_i : i \in \mathbb{Z}_0, \mu_i > 0 \text{ e } \mu_i \neq \mu_k\}$

B_V, B_{VZ}, B_W, B_{ZW} closed unitary balls in V, $V \oplus Z$, W e $Z \oplus W$

S_V, S_{VZ}, S_W e S_{ZW} their relative boundaries

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$$\mu_{\pm i} = \frac{-b \pm (\lambda_i - a)}{\lambda_i}, \qquad \psi_{\pm i} = \frac{(\phi_i, \pm\phi_i)}{\sqrt{2\lambda_i}}, \qquad i \in \mathbb{N}.$$

Then

$$\|\psi_i\|_E = 1, \quad \langle \psi_i, \psi_j \rangle_E = \delta_{i,j}, \quad B(\psi_i, \psi_j) = \mu_i \delta_{i,j}, \qquad i, j \in \mathbb{Z}_0 = \mathbb{Z} \backslash \{0\}.$$

$$a \pm b = \lambda_i \Rightarrow \mu_{\pm i} = 0 \tag{2.2}$$

So we define

$V = span\{\psi_i : i \in \mathbb{Z}_0, \mu_i < 0\}$: negative subspace

$Z = span\{\psi_i : i \in \mathbb{Z}_0, \mu_i = \mu_k\}$: almost resonant subspace

$W = span\{\psi_i : i \in \mathbb{Z}_0, \mu_i > 0 \text{ e } \mu_i \neq \mu_k\}$

B_V, B_{VZ}, B_W, B_{ZW} closed unitary balls in V, $V \oplus Z$, W e $Z \oplus W$

S_V, S_{VZ}, S_W e S_{ZW} their relative boundaries

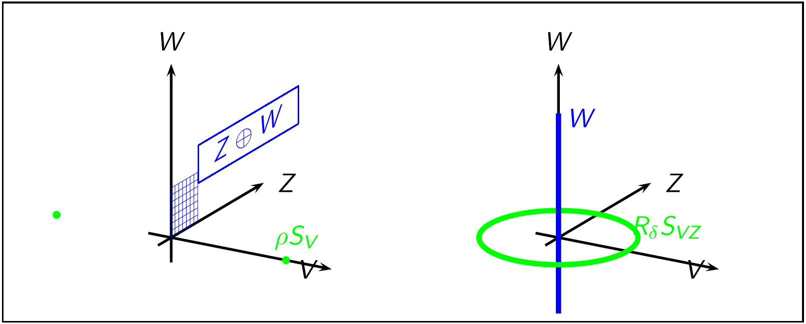
$E_n = span[\psi_{-n}, \ldots, \psi_n] \subseteq E,$

$V_n = V \cap E_n$ e $W_n = W \cap E_n$
($Z \subseteq E_n$, for every $n > k + m$)

J_n^+ the functional J^+ restricted to the subspace E_n.

Two Saddle Geometries

Let λ_k be the first eigenvalue above $a+b$ and $dist(a-b,\sigma) > \delta$.



$a+b \notin \sigma(-\Delta)$
$J^+(\mathbf{u}) \geq D_{a+b,\delta},\ \mathbf{u} \in Z \oplus W$
$J^+(\mathbf{u}) < D_{a+b,\delta},\ \mathbf{u} \in \rho S_V,\ \rho \geq \rho_{a+b,\delta}$

$a+b \nearrow \lambda_k \iff \mu_k \searrow 0$
$J^+(\mathbf{u}) \geq E_\delta,\ \mathbf{u} \in W,$
$J^+(\mathbf{u}) < E_\delta,\ \mathbf{u} \in R_\delta S_{VZ}$

Proposition

Let λ_k be the first eigenvalue above $a+b$ and $dist(a-b, \sigma(-\Delta)) > \delta$. Then

- *There exists $E_\delta \in \mathbb{R}$, such that*

$$J^+(\mathbf{u}) \geq E_\delta, \qquad \forall \mathbf{u} \in W,$$

- *There exist $\varepsilon_0 > 0$ e $R_\delta > 0$,such that, $a+b \in (\lambda_k - \varepsilon_0, \lambda_k)$,*

$$J^+(\mathbf{u}) < E_\delta - 1, \qquad \forall\ \mathbf{u} \in R_\delta S_{VZ},$$
$$J^+(\mathbf{u}) < E_\delta - 1, \qquad \forall\ \mathbf{u} \in V \text{ with } \|\mathbf{u}\|_E > R_\delta.$$

A1

- *For given a, b with $a+b \in (\lambda_k - \varepsilon_0, \lambda_k)$ and $dist(a-b, \sigma(-\Delta)) > \delta$, there exist $D_{a+b,\delta} \in \mathbb{R}$ and $\rho_{a+b,\delta} > R_\delta$ such that*

$$J^+(\mathbf{u}) \geq D_{a+b,\delta}, \qquad \forall\ \mathbf{u} \in Z \oplus W,$$
$$J^+(\mathbf{u}) < D_{a+b,\delta}, \qquad \forall\ \mathbf{u} \in \rho S_V,\ \ \rho \geq \rho_{a+b,\delta}.$$

A2

Proposition

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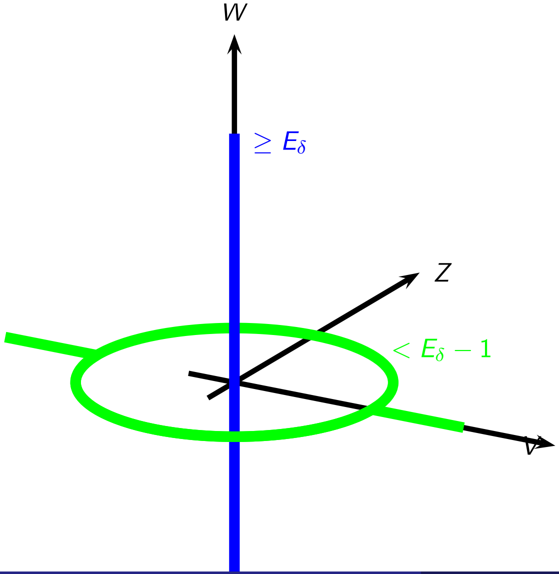
B1 B2



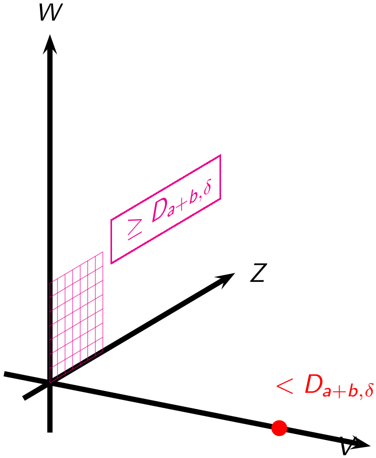
V e W have infinite dimension

Para $n > k + m$:
J_n^+ satisfies (PS).

Saddle Point geometry:
V_n e $Z \oplus W_n$
$V_n \oplus Z$ e W_n

$\exists\, \mathbf{u}_n, \mathbf{v}_n$, critical points of J_n^+, at the levels
$c_n \in [D_{a+b,\delta}, E_\delta - 1]$
$d_n \in [E_\delta, T_\delta]$

B1 B2



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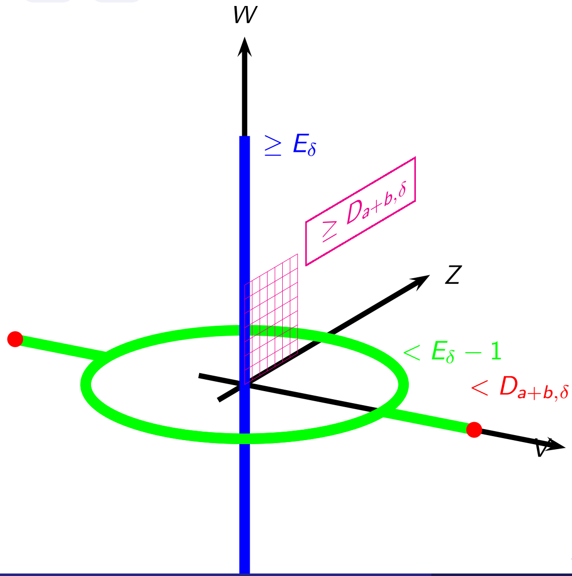

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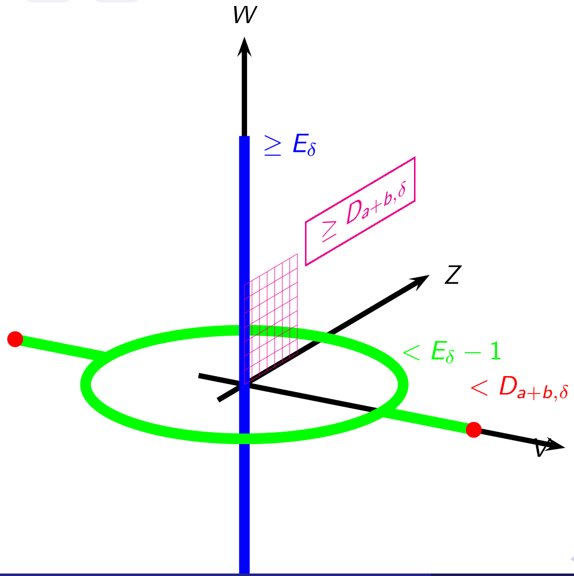

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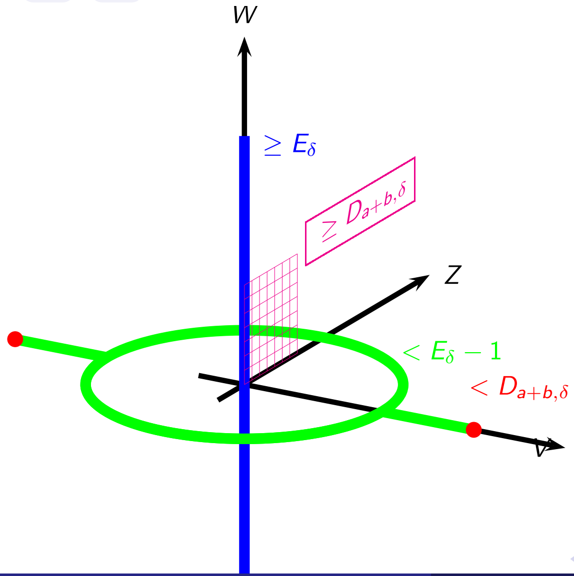

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We prove that $\exists c \in [D_{a+b,\delta}, E_\delta - 1], d \in [E_\delta, T_\delta]$, such that $c_n \to c, d_n \to d$.
Moreover there exist critical points $\mathbf{u}, \mathbf{v} \in E$, of the functional J^+, such that $\mathbf{u}_n \to \mathbf{u}$, $\mathbf{v}_n \to \mathbf{v}$, $J^+(\mathbf{u}) = c$ and $J^+(\mathbf{v}) = d$.

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Near resonance with the principal eigenvalue

Scalar case: In Ramos-Sanchez (1997) [RS97], three distinct solutions for the scalr problem (1.2±), for $\lambda \in (\lambda_1 - \epsilon, \lambda_1)$ or $\lambda \in (\lambda_1, \lambda_1 + \epsilon)$. (using minimization and Mountain Pass)

Question: Can we find a third solution for the system (1.1±), when $a + b$ or $a - b$ is near λ_1)?

In Ou-Tang (2009) three solutions are obtained for a gradient System

We need some more regularity: we assume

- Ω a $\mathcal{C}^2$ bounded domain in $\mathbb{R}^N$,
- $h_1, h_2 \in L^r(\Omega)$, where $r > N$,
- f_1, f_2 continuous functions in $\overline{\Omega} \times \mathbb{R}$, such that there exist $S > 0$, $q \in (1, 2)$, satisfying

$$|f_i(x,t)| \leq S(1 + |t|^{q-1}), \text{ for } i = 1, 2. \tag{3.1}$$

- Hypothesis (**F**).

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- Hypothesis (**F**).

We will say

- $\mathbf{u} = (u, v)$ is positive (or negative), when $u > 0$ and $v > 0$ (or $u < 0$ and $v < 0$),

Theorem

Assume the given hypotheses and let $\lambda_h \in \sigma(-\Delta)$ and $Z = span\{(\phi_1, \phi_1)\}$. Then

(a) *given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, if $a - b \in (\lambda_{h-1} + \delta, \lambda_h - \delta)$ and $a + b \in (\lambda_1 - \varepsilon_0, \lambda_1)$, then problem (1.1+) has three distinct solutions, of which, one is positive and one is negative.*

(b) *given $\delta > 0$, there exists $\varepsilon_1 > 0$ such that, if $a - b \in (\lambda_{h-1} + \delta, \lambda_h - \delta)$ and $a + b \in (\lambda_1, \lambda_1 + \varepsilon_1)$, then problem (1.1−) has three distinct solutions, of which, one is positive and one is negative.*

Idea of the proof

First we truncate the nonlinearities: for $i = 1, 2$, we take continuous functions such that

$$\widetilde{f}_i(x,s) = \begin{cases} f_i(x,s), & \text{se } \ s \geq -1\,, \\ 0, & \text{se } \ s \leq -2\,. \end{cases} \tag{4.1}$$

As a consequence hypothesis (**F**) is satisfied, but only at $+\infty$.

Proposition

If $a - b \in (\lambda_{h-1} + \delta, \lambda_h - \delta)$ *and* $a + b < \lambda_1$, *then*

- *there exists* $E_\delta \in \mathbb{R}$, *such that* $J^+_{a,b}(\mathbf{u}) \geq E_\delta, \forall\ \mathbf{u} \in W$
- *there exist sequences* $\varepsilon_j \to 0^+$ *and* $R_j \to +\infty$, *(depending on* δ*), such that if* $a + b \in (\lambda_1 - \varepsilon_j, \lambda_1)$, *then*

$$J^+_{a,b}(\mathbf{u}) < E_\delta - 1, \qquad \forall \mathbf{u} \in R_j S_{VZ},$$

$$\widetilde{J}^+_{a,b}(\mathbf{u}) < -R_j, \qquad \forall\ \mathbf{u} \in V \text{ with } \|\mathbf{u}\|_E > R_j,$$ ▸ A4

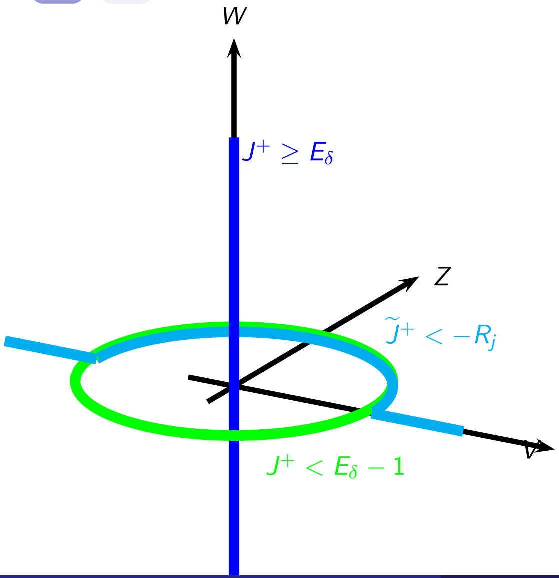
$$\widetilde{J}^+_{a,b}(\mathbf{u}) < -R_j, \qquad \forall \mathbf{u} = \mathbf{v} + k\psi_1,\ \mathbf{v} \in V, k \geq 0 \text{ and } \|\mathbf{u}\|_E = R_j.$$

- *for every* j, *fixing* $a + b \in (\lambda_1 - \varepsilon_j, \lambda_1)$ *and* $dist(a - b, \sigma(-\Delta)) > \delta$, *there exist* $D_{a+b,\delta} \in \mathbb{R}$ *and* $\rho_{a+b,\delta} > R_j$, *such that*

$$\widetilde{J}^+_{a,b}(\mathbf{u}) \geq D_{a+b,\delta}, \qquad \forall\ \mathbf{u} \in Z \oplus W,$$

$$\widetilde{J}^+_{a,b}(\mathbf{u}) < D_{a+b,\delta}, \qquad \forall\ \mathbf{u} \in \rho S_V,\ \rho \geq \rho_{a+b,\delta}.$$ ▸ A5

B4 B5



$\exists$ critical points
$\widetilde{\mathbf{u}}_n^j$ of $\widetilde{J}^+_{a_j,b_j}|_{E_n}$ and
$\mathbf{v}_n^j$ of $J^+_{a_j,b_j}|_{E_n}$,
at the levels
$\widetilde{c}_n^j \in [D_{a+b,\delta}, -R_j]$
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$n \to +\infty$:

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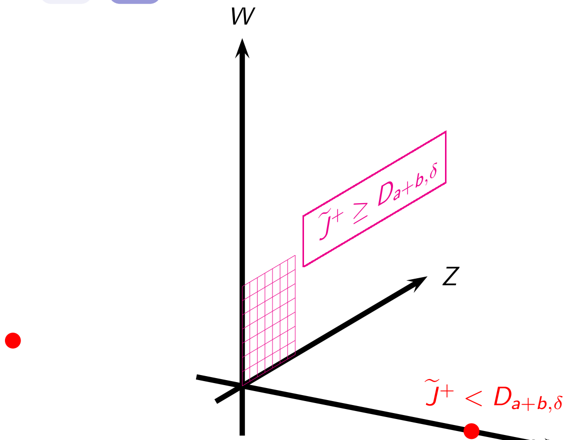

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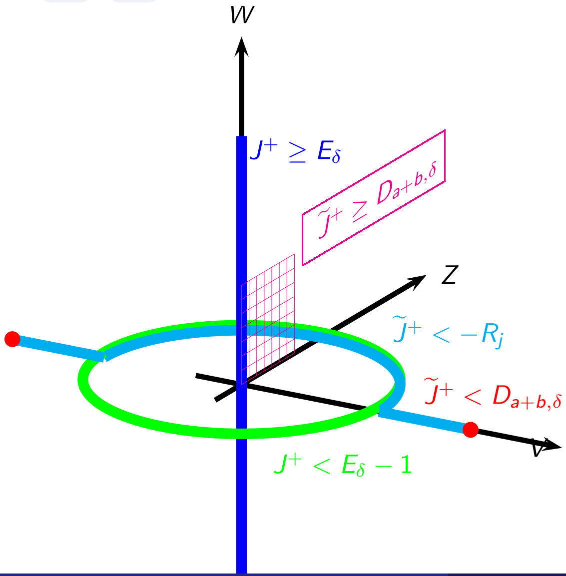

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A sequence of solutions

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$\mathbf{v}^j$ is a solution of

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Next step: prove that, for j large, $\widetilde{\mathbf{u}}^j$ is positive.

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We denote, for every $j \in \mathbb{N}$,

$$\widetilde{\mathbf{u}}^j = \beta_j \psi_1 + \omega_j, \tag{4.4}$$

where $\beta_j \in \mathbb{R}$ and $\omega_j \in V \oplus W$.

- Given $\eta > 0$, there exists $\widetilde{C}_\eta > 0$ (not depending on j), such that

$$\|\omega_{\mathbf{j}}\|_{\mathcal{C}^1 \times \mathcal{C}^1} \leq \eta|\beta_j| + \widetilde{C}_\eta. \tag{4.5}$$

- $|\beta_j| \to +\infty$ (since $\widetilde{J}^+_{a_j,b_j}(\widetilde{\mathbf{u}}^j) = \widetilde{c}^j \to -\infty$)
-

$$\frac{\widetilde{\mathbf{u}}^j}{\beta_j} = \psi_1 + \frac{\omega_j}{\beta_j} \to \psi_1, \text{ in } \mathcal{C}^1(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega})$$

Then, $\exists j_0$, such that $\forall j \geq j_0$, $\widetilde{\mathbf{u}}^j$ is positive, if $\beta_j > 0$, or negative, if $\beta_j < 0$.

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- $$\frac{\widetilde{\mathbf{u}}^j}{\beta_j} = \boldsymbol{\psi}_1 + \frac{\boldsymbol{\omega}_j}{\beta_j} \to \boldsymbol{\psi}_1, \text{ in } \mathcal{C}^1(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega})$$

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- $\beta_j \to +\infty$, when $j \to \infty$.

- Then, for every $j > j_0$, we have that $\widetilde{\mathbf{u}}^j$ is positive and then it is a solution of (1.1+).
- At this point we have two solutions: $\widetilde{\mathbf{u}}^j$ e $\mathbf{v}^j$: they are distinct since they lie at different levels.

For the third solutions, we consider the system

$$\begin{cases} -\Delta u = au + bv + (-f_1(x,-v) + (-h_1(x))); & \text{in } \Omega, \\ -\Delta v = bu + av + (-f_2(x,-u) + (-h_2(x))); & \text{in } \Omega, \\ u(x) = v(x) = 0; & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

- if (u,v) is a solution of (4.6), then $(-u,-v)$ is a solution of (1.1+)
- if $\widehat{J}_{a,b}$ is the functional associated to (4.6), then $\widehat{J}_{a,b}(\mathbf{u}) = J^+_{a,b}(-\mathbf{u})$, $\forall \mathbf{u} \in E$
- $g_i(x,t) = -f_i(x,-t)$ satisfy the same hypotheses as f_i.

- Then, for every $j > j_0$, we have that $\widetilde{\mathbf{u}}^j$ is positive and then it is a solution of (1.1+).
- At this point we have two solutions: $\widetilde{\mathbf{u}}^j$ e $\mathbf{v}^j$: they are distinct since they lie at different levels.

For the third solutions, we consider the system

$$\begin{cases} -\Delta u = au + bv + (-f_1(x,-v) + (-h_1(x))); & \text{in } \Omega, \\ -\Delta v = bu + av + (-f_2(x,-u) + (-h_2(x))); & \text{in } \Omega, \\ u(x) = v(x) = 0; & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

- if (u, v) is a solution of (4.6), then $(-u, -v)$ is a solution of (1.1+)
- if $\widehat{J}_{a,b}$ is the functional associated to (4.6), then $\widehat{J}_{a,b}(\mathbf{u}) = J^+_{a,b}(-\mathbf{u})$, $\forall \mathbf{u} \in E$
- $g_i(x,t) = -f_i(x,-t)$ satisfy the same hypotheses as f_i.

Proceeding as before, we get

- there exist $\widehat{\mathbf{u}}^j$ positive solutions of (4.6) for j large, moreover $\widehat{J}_{a_j,b_j}(\widehat{\mathbf{u}}^j) \to -\infty$.
- then $-\widehat{\mathbf{u}}^j$ are negative solutions of $(1.1+)_j$, and $J^+_{a_j,b_j}(-\widehat{\mathbf{u}}^j) \to -\infty$.

Conclusion:
$-\widehat{\mathbf{u}}^j \neq \widetilde{\mathbf{u}}^j$: one positive, one negative.
$-\widehat{\mathbf{u}}^j \neq \mathbf{v}^j$: $J^+_{a_j,b_j}(-\widehat{\mathbf{u}}^j) \to -\infty$ e $J^+_{a_j,b_j}(\mathbf{v}^j) > E_\delta$.

Then (1.1+) has three distinct solutions (of which, one positive and one negative), near enough to the eigenvalue.

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