Three solutions for an elliptic system near resonance with the principal eigenvalue

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Abstract

We consider an elliptic system of Hamiltonian type with linear part depending on two parameters and a sublinear perturbation. We obtain the existence of at least three solutions when the linear part is near resonance with the principal eigenvalue, either from above or from below. For two of these solutions we also obtain information on the sign of its components. The system is associated to

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a strongly indefinite functional and the solutions are obtained trough saddle point theorem, after truncating the nonlinearity.

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1 Introduction

In this paper we consider an elliptic system of Hamiltonian type with linear part depending on two parameters and a sublinear perturbation. Our purpose is to improve the results obtained in [19], by proving the existence of at least three solutions when the linear part is near resonance with the principal eigenvalue.

The system we are considering is the following:

$$\begin{cases} -\Delta u = au + bv \pm (f_1(x,v) + h_1(x)) & \text{in } \Omega, \\ -\Delta v = bu + av \pm (f_2(x,u) + h_2(x)) & \text{in } \Omega, \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in particular, we will refer to problem (1.1) as (1.1+) or (1.1−), depending on the sign preceding the two nonlinearities.

In (1.1), $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $h_1, h_2 \in L^r(\Omega)$ for some $r > N$ and f_1, f_2 are sublinear nonlinearities, which in particular satisfy

$$\begin{aligned} &f_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \text{ is a continuous function and there exist constants } S > 0 \\ &\text{and } q \in (1,2), \text{ such that } |f_i(x,t)| \leq S(1 + |t|^{q-1}), \text{ for } i = 1, 2. \end{aligned} \tag{1.2}$$

We will also assume the following hypotheses on the functions $f_{1,2}$ and $h_{1,2}$:

$$\begin{cases} (i\text{-}a) & \lim_{|t|\to\infty} F_1(x,t) = +\infty, \text{ uniformly with respect to } x \in \Omega, \\ (i\text{-}b) & F_i(x,t) \geq -C_F, \text{ for } i = 1,2, \\ (ii) & \int_\Omega h_1\phi + h_2\psi = 0, \quad \text{for every } (\phi,\psi) \in Z, \end{cases} \tag{$\mathbf{F}$}$$

where $F_i(x,t) = \int_0^t f_i(x,s)ds$ and Z is a space which will be defined in the statements of the theorems. Of course, one could assume ($\mathbf{F}$-i-a) on F_2 instead of F_1 without affecting the result.

In [19] we showed the existence of at least two solutions for problem (1.1) when the linear part is near resonance: this situation is often called "almost-resonance" in literature and for system (1.1) it means that $a+b$ or $a-b$ is near to a eigenvalue of the Laplacian. Our purpose here is to consider the particular case when the almost-resonance occurs with respect to the first eigenvalue, and to show that in this case it is possible to obtain a third solution.

Throughout the paper we denote by $\sigma(-\Delta)$ the spectrum of the Laplacian in $H_0^1(\Omega)$, that is the set of the eigenvalues λ_k where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_k \leq ...$, and by ϕ_k $(k = 1,2,..)$ the corresponding eigenfunctions, which will be taken orthogonal and normalized with $\|\phi_k\|_{H_0^1} = 1$ and $\phi_1 > 0$.
Also, for $l \in \mathbb{N}$ and $\delta > 0$ we define the interval

$$I_{l,\delta} = \begin{cases} (-1/\delta, \lambda_1 - \delta) & \text{if } l = 1, \\ (\lambda_{l-1} + \delta, \lambda_l - \delta) & \text{if } l \geq 2, \end{cases}$$

where we will implicitly assume $I_{l,\delta} \neq \emptyset$.
We say that $\mathbf{u} = (u,v)$ is positive (resp. negative), when $u > 0$ and $v > 0$ (resp. $u < 0$ and $v < 0$), and we say that its components have opposite signs when $u > 0$

and $v < 0$ or $u < 0$ and $v > 0$.

Our main results are stated in the following Theorems: the first one deals with the case where $a + b$ is near λ_1, while $a - b$ is far from $\sigma(-\Delta)$.

Theorem 1.1. *Let Ω be a open and bounded set in $\mathbb{R}^N$ of regularity $\mathcal{C}^2$, $l \in \mathbb{N}$ and*

$$Z = span\,\{\,(\phi_1, \phi_1)\}\,.$$

*Suppose $h_1, h_2 \in L^r(\Omega)$ for some $r > N$, and f_1, f_2 satisfy the hypotheses (1.2) and (**F**). Then*

- *(a) given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, if $a-b \in I_{l,\delta}$ and $a+b \in (\lambda_1-\varepsilon_0, \lambda_1)$, then the problem (1.1+) has at least three distinct solutions, of which one is positive and one is negative.*
- *(b) given $\delta > 0$, there exists $\varepsilon_1 > 0$ such that, if $a-b \in I_{l,\delta}$ and $a+b \in (\lambda_1, \lambda_1+\varepsilon_1)$, then the problem (1.1–) has at least three distinct solutions, of which one is positive and one is negative.*

As a consequence of Theorem 1.1, it is possible to prove a similar result also for the case where $a - b$ is near λ_1, while $a + b$ is far from $\sigma(-\Delta)$.

Theorem 1.2. *Let Ω be a open and bounded set in $\mathbb{R}^N$ of regularity $\mathcal{C}^2$, $l \in \mathbb{N}$ and*

$$Z = span\,\{\,(\phi_1, -\phi_1)\}\,.$$

*Suppose $h_1, h_2 \in L^r(\Omega)$ for some $r > N$, and f_1, f_2 satisfy the hypotheses (1.2) and (**F**). Then*

- *(c) given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, if $a+b \in I_{l,\delta}$ and $a-b \in (\lambda_1-\varepsilon_0, \lambda_1)$, then the problem (1.1–) has at least three distinct solutions, two of which have components of opposite sign.*

(d) given $\delta > 0$, there exists $\varepsilon_1 > 0$ such that, if $a+b \in I_{l,\delta}$ and $a-b \in (\lambda_1, \lambda_1+\varepsilon_1)$, then the problem (1.1+) has at least three distinct solutions, two of which have components of opposite sign.

We observe that if we substitute hypotheses (**F**) with the following

$$\begin{cases} (i) \quad \lim_{|t|\to\infty} F_i(x,t) = +\infty, \text{ uniformly with respect to } x \in \Omega,\ i = 1,2, \\ (ii) \quad \int_\Omega h_1\phi + h_2\psi = 0, \quad \text{for every } (\phi,\psi) \in Z, \end{cases} \tag{$\mathbf{F}^*$}$$

in the Theorems 1.1 and 1.2, then the results in [19] already guarantee the existence of two solutions. What we do in this paper is to first show that hypothesis (**F**) is sufficient also for obtaining the result in [19] (see Theorem 1.3 below), and then to prove that, by exploiting the positivity of the eigenfunction ϕ_1, we can get a third solution, along with the information on the sign of two of the solutions. We remark that, with respect to [19], here we are considering slightly stronger hypotheses on the regularity of the functions in (1.1): in particular, we assume the continuity of $f_{1,2}$, the L^r regularity of $h_{1,2}$ and the $\mathcal{C}^2$ regularity of Ω. This is needed in order to obtain a better regularity for the weak solutions, which will allow us to compare them with the first eigenfunction and then to prove that they do not change sign.

The main motivation for [19] and this paper are the results obtained in [22] and [10] for the corresponding scalar problem

$$\begin{cases} -\Delta u = \lambda u \pm f(x,u) + h(x) & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{1.3}$$

It is worth noting that in the case of a single equation it was first obtained, in [22], the result of three solution near the first eigenvalue, and later, in [10], it was proved that two solutions can be obtained near any eigenvalue. In the case of the

Hamiltonian system (1.1), it turned out to be more natural to first prove a result similar to the one in [10], which we did in [19], and then to obtain a third solution near the first eigenvalue. This is due to the fact that the arguments used in [22] relied on minimization and the Mountain Pass Theorem, while near a general eigenvalue the two solutions were obtained, in [10], by finding two saddle point geometries. In the case of problem (1.1), one has to deal with a strongly indefinite functional, which implies that the solutions are always found as saddle points, even near the first eigenvalue, so that it is not possible to extend directly the techniques from [22].

The idea of this paper is to adapt the techniques from [22] to the more complex variational setting of [19], in order to be able to prove that the component in the almost resonant eigenspace of one of the two existing solutions becomes dominant. This will imply that such solution is positive. Finally, by symmetry, it will be easy to obtain a negative solution, which is then a third solution.

We observe that from the proof of the Theorems 1.1 and 1.2, it will be clear that two solutions still exist even if we assume the following weaker version of hypothesis (**F**):

$$\begin{cases} (i\text{-}a) \quad \lim_{t\to\infty} F_1(x,t) = +\infty, \text{ uniformly with respect to } x \in \Omega, \\ (i\text{-}b) \quad F_i(x,t) \geq -C_F, \text{ i=1,2}, \\ (ii) \quad \int_\Omega h_1\phi + h_2\psi = 0, \quad \text{for every } (\phi,\psi) \in Z, \end{cases} \tag{$\mathbf{F}^+$}$$

or its analogous with the limit being taken at $-\infty$: see Remark 2.15.

As remarked above, a byproduct of this paper is an improvement of the results in [19], in the sense that it is possible to assume hypothesis (**F**) instead of ($\mathbf{F}^*$) in most of the results proved there. In particular we will prove the following Theorem.

Theorem 1.3 (Improvement of [19]). *In the Theorems 1.1 through 1.4 proved in [19] one can assume hypothesis (***F***) in the form of this paper, provided that every*

function in $Z \setminus \{0\}$ has both components not identically zero.
This condition is always true except in the case where $\lambda_k = \lambda_l$ in the Theorems 1.3 and 1.4 (double resonance with respect to the same eigenvalue).

1.1 Techniques and related works

Multiplicity results for almost-resonant problems as (1.3) were studied by many authors, since the work of Mawhin and Schmitt [20], where the problem (1.3) in dimension one is considered, near the first eigenvalue, using bifurcation from infinity and degree theory. We cite [3, 14], which also consider the one dimensional case, and [6, 5], which deal with the higher dimension problem; these works are all based on bifurcation theory. Results for higher eigenvalues were obtained in [14], again using bifurcation from infinity and degree theory, but only for the one dimensional case and making use of the fact that in this case all the eigenvalues are simple. The works [22, 16] were the first to analyze this kind of problem from a variational point of view: as already discussed, they obtained three solutions near the first eigenvalue, under conditions which are analogous to those assumed here. The variational approach was later exploited in [15] to obtain a similar result for the p-Laplacian operator (see also [9]).

Regarding almost resonant problems in the case of systems, we cite [24, 21, 2, 13, 19]. In particular, the case of the first eigenvalue is considered in [21], where three solutions are found for a system of gradient type, when approaching this eigenvalue from below. We remark, however, that the case of systems of gradient type is quite different from problem (1.1) (the Hamiltonian type), because the principal part of the functional has a finite dimensional negative space, so that many techniques from [22] can be applied directly.

The result in [19] was obtained by considering the functional associated to (1.1) (see the definition in (2.1)) and finding two saddle point geometries, involving two different splits of its space of definition; then one proved that these solutions were distinct since they lay at different levels.

As we observed above, in the case of a system as (1.1), the associated functional is strongly indefinite, in the sense that there exist two infinite dimensional subspaces of its space of definition such that the principal part of the functional is unbounded from above in one and from below in the other. Several techniques have been used in literature to deal with strongly indefinite problems, see [4, 8, 12, 7, 17, 18, 19]. In particular, in [19], we used a Galerkin procedure in order to obtain critical points, namely, we solved finite dimensional problems, where one could use the standard linking theorems, then we took limit on the dimension of such problems and proved that in the limit we actually found a critical point.

Here we will first truncate the nonlinearity and then, by adapting the arguments from [19], we will produce two solutions for the truncated problem, along with some improved estimates on their critical level. As a result, we will be able to prove that one of these solutions has constant sign, but in view of the truncation this sign can only be positive. In the end, by symmetry, we will also have a negative solution, giving a total of three distinct solutions, in fact, the one with no information on the sign can be distinguished by its critical level.

We remark that in [19] we also considered the following condition

$$\lim_{t\to\pm\infty} f_i(x,t) = \pm\infty, \text{ uniformly with respect to } x\in\Omega,\ i=1,2, \tag{$\mathbf{f}$}$$

as an alternative to condition (**F**). Here we only work with hypothesis (**F**), since in the case where condition (**f**) holds, one can always put the component in Z of (h_1, h_2) into the definition of f_1, f_2 in order to have condition (**F**) satisfied, that is, one defines $\widehat{f_i}(x,t) = f_i(x,t) + j_i(x)$ and $\widehat{h}_i(x) = h_i(x) - j_i(x)$, $i = 1, 2$, in such a way that they satisfy (1.2) and condition (**F**).

For further remarks regarding our problem (1.1) and the related bibliography we remand to [19].

The paper is structured as follows: in the Section 2.1 we set the notation and we state some preliminary results from [19]; in the Section 2.2 we obtain the saddle point geometries that will provide the critical points, which are obtained in the Section 2.3. Finally, the Section 2.4 will be devoted to prove the positivity of one of the solution and in the Section 2.5 we will conclude the proof of the main results. The proof of of the Theorem 1.3 is given in Remark 2.7.

2 Proof of the main results

In this section we will give the proofs of the main theorems. In particular, we give in full details the proof of the point (a) in Theorem 1.1: the point (b) and Theorem 1.2 can be proved in a similar way. Some of the arguments are analogous to those in [19] and will not be repeated, others are similar but with crucial differences and then will be exposed in details here.

2.1 Notation and preliminary Lemmas

Throughout the paper we will use the notation $H = H_0^1(\Omega)$, $E = H \times H$ and we will use the following norms: if $u \in H$ and $\mathbf{u} = (u, v) \in E$, then

$$\begin{cases} \|u\|_{L^p} = \left(\int_\Omega u^p\right)^{1/p}, & \|\mathbf{u}\|_{L^p\times L^p} = \left(\|u\|_{L^p}^p + \|v\|_{L^p}^p\right)^{1/p}, \\ \|u\|_H = \sqrt{\int_\Omega |\nabla u|^2}, & \|\mathbf{u}\|_E = \sqrt{\|u\|_H^2 + \|v\|_H^2}. \end{cases}$$

The internal products in $L^2(\Omega)\times L^2(\Omega)$ and in E, associated to the above norms, will be denoted by $\langle\cdot,\cdot\rangle_{[L^2]^2}$ and $\langle\cdot,\cdot\rangle_E$, respectively. Observe that, by Poincaré inequality, $\|u\|_{L^2} \le S\,\|u\|_H$ for some positive constant S, but we will assume throughout the paper that $S = 1$ in order to simplify the estimates. Also, we will denote by

C, C_1, C_2, ... constants whose value is not important and which may be different from line to line.

As in [19], we define the applications $\mathcal{F} : E \to \mathbb{R}$ and $\mathcal{H} : E \to \mathbb{R}$, given by

$$\mathcal{F}(u,v) = \int_\Omega F_1(x,v) + \int_\Omega F_2(x,u), \qquad \mathcal{H}(u,v) = \int_\Omega h_1 v + \int_\Omega h_2 u,$$

and the bilinear form $B_{a,b} : E \times E \to \mathbb{R}$:

$$B_{a,b}((u,v),(\phi,\psi)) = \int_\Omega \nabla u \nabla \psi + \int_\Omega \nabla v \nabla \phi - a \int_\Omega (u\psi + v\phi) - b \int_\Omega (u\phi + v\psi).$$

Then, the $\mathcal{C}^1$ functionals associated to the problems (1.1±) can be written as

$$J^{\pm}_{a,b}(\mathbf{u}) = \pm\frac{1}{2} B_{a,b}(\mathbf{u},\mathbf{u}) - (\mathcal{F}(\mathbf{u}) + \mathcal{H}(\mathbf{u})) \; . \tag{2.1}$$

Also, denoting $\mathbf{u} = (u,v)$, $\boldsymbol{\phi} = (\phi,\psi)$ and $\overline{\boldsymbol{\phi}} = (\psi,\phi)$, we have that

$$B_{a,b}(\mathbf{u},\boldsymbol{\phi}) = \langle \mathbf{u}, \overline{\boldsymbol{\phi}} \rangle_E - a\langle \mathbf{u}, \overline{\boldsymbol{\phi}} \rangle_{[L^2]^2} - b\langle \mathbf{u}, \boldsymbol{\phi} \rangle_{[L^2]^2} \; , \tag{2.2}$$

$$(J^{\pm}_{a,b})'(\mathbf{u})[\boldsymbol{\phi}] = \pm B_{a,b}(\mathbf{u},\boldsymbol{\phi}) - \left(\mathcal{F}'(\mathbf{u})[\boldsymbol{\phi}] + \mathcal{H}'(\mathbf{u})[\boldsymbol{\phi}]\right) . \tag{2.3}$$

Finally the following Lemmas contain results, proved in [19], which will be used in this paper.

Lemma 2.1. *Given f_1, f_2 satisfying (1.2) and $h_1, h_2 \in L^2(\Omega)$, there exist constant S_0 and H such that*

$$|\mathcal{H}(\mathbf{u})| \le H \left\|\mathbf{u}\right\|_{[L^2]^2} \; , \qquad\qquad |\mathcal{H}'(\mathbf{u})[\boldsymbol{\phi}]| \le H \left\|\boldsymbol{\phi}\right\|_{[L^2]^2} \; , \tag{2.4}$$

$$|\mathcal{F}(\mathbf{u})| \leq S_0 \left(1 + \|\mathbf{u}\|^q_{[L^2]^2}\right), \qquad \left|\mathcal{F}'(\mathbf{u})[\boldsymbol{\phi}]\right| \leq S_0 \left(1 + \|\mathbf{u}\|^{q-1}_{[L^2]^2}\right) \|\boldsymbol{\phi}\|_{[L^2]^2}, \tag{2.5}$$

for every $\mathbf{u} = (u, v) \in E$ *and* $\boldsymbol{\phi} = (\phi, \psi) \in E$.

Lemma 2.2. *The eigenvalues of the form* $B_{a,b}$ *are given by*

$$\mu_{\pm i} = \frac{-b \pm (\lambda_i - a)}{\lambda_i}, \; i \in \mathbb{N}, \tag{2.6}$$

with corresponding eigenfunctions (normalized in E*)*

$$\boldsymbol{\psi}_{\pm i} = \frac{(\phi_i, \pm\phi_i)}{\sqrt{2}}, \; i \in \mathbb{N}.$$

In particular,

$$\|\boldsymbol{\psi}_i\|_E = 1, \;\; \langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle_E = \delta_{i,j}, \;\; B_{a,b}(\boldsymbol{\psi}_i, \boldsymbol{\psi}_j) = \mu_i \delta_{i,j}, \;\; \langle \boldsymbol{\psi}_i, \boldsymbol{\psi}_j \rangle_{[L^2]^2} = \lambda_{|i|}^{-1} \delta_{i,j}, \tag{2.7}$$

for $i, j \in \mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$.

Moreover, if we write $\mathbf{u} = \sum_{i \in \mathbb{Z}^0} c_i \boldsymbol{\psi}_i$*, then*

$$\|\mathbf{u}\|^2_E = \sum_{i \in \mathbb{Z}^0} c_i^2, \;\; B_{a,b}(\mathbf{u}, \mathbf{u}) = \sum_{i \in \mathbb{Z}^0} \mu_i c_i^2, \;\; \|\mathbf{u}\|^2_{[L^2]^2} = \sum_{i \in \mathbb{Z}^0} \lambda_{|i|}^{-1} c_i^2. \tag{2.8}$$

2.2 Finding two saddle point geometries

In order to be able to obtain a positive solution, we will first need to truncate the nonlinearities. Actually, the purpose of this truncation is to forbid the existence of a large negative solution, which then will allow us to prove the existence of a positive one. This positive solution will be a solution also of the original problem.

Given the functions f_1, f_2 satisfying (1.2), we define, for $i = 1, 2$, two new functions

$$\widetilde{f}_i(x,s)=\begin{cases} f_i(x,s), & \text{if } s\geq -1\,,\\ (s+2)f_i(x,-1) & \text{if } s\in[-2,-1]\,,\\ 0, & \text{if } s\leq -2\,. \end{cases} \tag{2.9}$$

Observe that, by the above definition, the functions $\widetilde{f}_1, \widetilde{f}_2$ also satisfy (1.2). We also observe that if hypothesis (**F**) holds, then the following holds too

$$\begin{cases} (i\text{-}a) & \lim_{t\to\infty}\widetilde{F}_1(x,t)=+\infty, \text{ uniformly with respect to } x\in\Omega,\\ (i\text{-}b) & \widetilde{F}_i(x,t)\geq -C_F, \text{ for } i=1,2,\\ (ii) & \int_\Omega h_1\phi+h_2\psi=0, \quad \text{for every } (\phi,\psi)\in Z. \end{cases} \tag{$\widetilde{\mathbf{F}}$}$$

Now, we define

$$\widetilde{F}_i(x,t)=\int_0^t \widetilde{f}_i(x,s)ds, \quad \text{for } i=1,2 \qquad\qquad \widetilde{\mathcal{F}}(u,v)=\int_\Omega \widetilde{F}_1(x,v)+\int_\Omega \widetilde{F}_2(x,u)\,,$$

and we consider the $\mathcal{C}^1$ functional $\widetilde{J}^+_{a,b}: E\to\mathbb{R}$, given by

$$\widetilde{J}^+_{a,b}(\mathbf{u})=\frac{1}{2}B_{a,b}(\mathbf{u},\mathbf{u})-\Big(\widetilde{\mathcal{F}}(\mathbf{u})+\mathcal{H}(\mathbf{u})\Big)\,. \tag{2.10}$$

In order to find two saddle point geometries we define, as in [19], the following subspaces of E

$$\begin{cases} V=\overline{span\{\boldsymbol{\psi}_i\,:\, i\in\mathbb{Z}_0,\, \mu_i<0 \text{ and } \mu_i\neq\mu_1\}},\\ Z=\overline{span\{\boldsymbol{\psi}_i\,:\, i\in\mathbb{Z}_0,\, \mu_i=\mu_1\}},\\ W=\overline{span\{\boldsymbol{\psi}_i\,:\, i\in\mathbb{Z}_0,\, \mu_i>0 \text{ and } \mu_i\neq\mu_1\}}; \end{cases} \tag{2.11}$$

in particular, it is important to observe that if $a+b\in(\lambda_1-\varepsilon,\lambda_1)$ for some $\varepsilon>0$, then

$$0 < \mu_1 = \frac{\lambda_1 - (a+b)}{\lambda_1} < \frac{\varepsilon}{\lambda_1}\,. \tag{2.12}$$

With the definitions above, we may now state a Lemma, which is a particular case of Lemma 4.1 in [19].

Lemma 2.3. *Suppose $a-b \in I_{l,\delta}$ for some $l \in \mathbb{N}$ and $\delta > 0$, while $a+b < \lambda_1$. Then there exists a constant $K_{a+b,I_{l,\delta}} > 0$, depending on the sum $a+b$ and on $I_{l,\delta}$, such that*

$$B_{a,b}(\mathbf{u},\mathbf{u}) \leq -K_{a+b,I_{l,\delta}}\,\|\mathbf{u}\|_E^2\,, \qquad \forall\ \mathbf{u} \in V\,, \tag{2.13}$$

$$B_{a,b}(\mathbf{u},\mathbf{u}) \geq K_{a+b,I_{l,\delta}}\,\|\mathbf{u}\|_E^2\,, \qquad \forall\ \mathbf{u} \in Z \oplus W\,. \tag{2.14}$$

If, moreover, $a+b \in (\lambda_1/2, \lambda_1)$, then there exists $G_{I_{l,\delta}} > 0$, depending on $I_{l,\delta}$, such that

$$B_{a,b}(\mathbf{u},\mathbf{u}) \leq -G_{I_{l,\delta}}\,\|\mathbf{u}\|_E^2\,, \qquad \forall\ \mathbf{u} \in V\,, \tag{2.15}$$

$$B_{a,b}(\mathbf{u},\mathbf{u}) \geq G_{I_{l,\delta}}\,\|\mathbf{u}\|_E^2\,, \qquad \forall\ \mathbf{u} \in W\,. \tag{2.16}$$

Remark 2.4. *If $a+b$ is near enough to λ_1, then, for every $a-b \in I_{l,\delta}$, one has $\mu_i = \mu_1$ only for $i = 1$, that is, the Z defined in (2.11) is exactly the same as the one defined in the statement of Theorem 1.1 (compare (2.12) and (2.6)).*

We also prove the following Lemma, which will help to estimate the contribution of the nonlinearity.

Lemma 2.5. *If hypotheses (1.2) and ($\widetilde{\mathbf{F}}$) hold, then there exists a nondecreasing function*
$D : (0,+\infty) \to \mathbb{R}$ such that

$$(i) \quad \lim_{R\to+\infty} D(R) = +\infty\,, \qquad (ii) \quad \lim_{R\to+\infty} \frac{D(R)}{R^2} = 0\,, \tag{2.17}$$

and

$$\int_\Omega \widetilde{F}_1(x, Ru) \geq D(R) \qquad \textit{for every } u \in \mathcal{E}, \tag{2.18}$$

where

$$\mathcal{E} = \left\{u \in H_0^1(\Omega) : \ u > \alpha \ \textit{in} \ \Omega^u \subseteq \Omega \ \textit{and} \ |\Omega^u| > \alpha\right\} \tag{2.19}$$

and $\alpha > 0$ is such that $\phi_1 > 4\alpha$ in a set Ω_α with $|\Omega_\alpha| > 2\alpha$.

Proof. Equation (2.18) is satisfied by defining

$$D(R) := \inf\left\{\int_\Omega \widetilde{F}_1(x, \rho u) : \ \rho \geq R, \, u \in \mathcal{E}\right\}.$$

Observe that the infimum is well defined since $\int_\Omega \widetilde{F}_1(x, \rho u) \geq -|\Omega| C_F$ by condition ($\widetilde{\mathbf{F}}$-i-b), and it is a nondecreasing function of R by definition.

Now, fixed a value $H > 0$, we will show that we can find a $\widetilde{R}$ large enough so that $\int_\Omega \widetilde{F}_1(x, Ru) \geq H$ for any $R \geq \widetilde{R}$ and $u \in \mathcal{E}$: this implies (2.17-i).
In order to do this, we set $M = (H + |\Omega| C_F)\alpha^{-1}$: by ($\widetilde{\mathbf{F}}$-$i$-$a$) we have that there exists s_0 such that $\widetilde{F}_1(x, s) > M$ for $s > s_0$.
If $u \in \mathcal{E}$, then for $R > s_0/\alpha$, one has $Ru > s_0$ in Ω^u and then one gets

$$\int_{\Omega^u} \widetilde{F}_1(x, Ru) \geq M\alpha \, .$$

Since, by ($\widetilde{\mathbf{F}}$-i-b), $\int_{\Omega \setminus \Omega^u} \widetilde{F}_1(x, Ru) \geq -|\Omega| C_F$, one finally obtains

$$\int_\Omega \widetilde{F}_1(x, Ru) \geq M\alpha - |\Omega| C_F = H \, .$$

Finally, observe that $\phi_1 \in \mathcal{E}$ by the definition of α, then we can estimate, using hypothesis (1.2),

$$D(R) \leq \int_\Omega \widetilde{F}_1(x, R\phi_1) \leq C(1 + R^q) \, ,$$

which implies (2.17-*ii*). □

Using the above Lemmas, we can prove the Proposition 2.6 below, where we show that the functional $J^+_{a,b}$ satisfies a saddle point geometry between the subspaces $V \oplus Z$ and W, while the functional $\widetilde{J}^+_{a,b}$ satisfies another saddle point geometry between the subspaces V and $Z \oplus W$, when near resonance. Also, the estimates (2.24) and (2.25) below will be useful to prove that the critical levels are distinct.

These two saddle point geometries are similar to those obtained in [19], the difference is that here we will need to consider a decreasing sequence of left neighborhoods of λ_1 and produce a corresponding sequence of saddle point geometries with $a+b$ in those neighborhoods, with the property that one of the corresponding critical levels decreases to minus infinity along this sequence. Another difference with [19] is that, because of the truncation (2.9) that we have performed here, the estimate (2.25) holds only on a "halfcircle" ($k \geq 0$).

Proposition 2.6. *Under the hypotheses of Theorem 1.1, if* $a-b \in I_{l,\delta}$ *for some* $\delta > 0$*, then there exist sequences* $\varepsilon_j \to 0^+$ *and* $R_j \to +\infty$*, both depending on* $I_{l,\delta}$*, such that if* $a+b \in (\lambda_1 - \varepsilon_j, \lambda_1)$*, then there exist* $E_{I_{l,\delta}}$*,* $D_{a+b,I_{l,\delta}} \in \mathbb{R}$*,* $\rho_{a+b,I_{l,\delta}} > R_j > 0$*, satisfying*

$$\widetilde{J}^+_{a,b}(\mathbf{u}) \geq D_{a+b,I_{l,\delta}}, \qquad \forall\ \mathbf{u} \in Z \oplus W, \tag{2.20}$$

$$\widetilde{J}^+_{a,b}(\mathbf{u}) < D_{a+b,I_{l,\delta}}, \qquad \forall\ \mathbf{u} \in \rho S_V,\ \ \rho \geq \rho_{a+b,I_{l,\delta}}, \tag{2.21}$$

$$J^+_{a,b}(\mathbf{u}) \geq E_{I_{l,\delta}}, \qquad \forall\ \mathbf{u} \in W, \tag{2.22}$$

$$J^+_{a,b}(\mathbf{u}) < E_{I_{l,\delta}} - 1, \qquad \forall\ \mathbf{u} \in R_j S_{VZ}, \tag{2.23}$$

$$\widetilde{J}^+_{a,b}(\mathbf{u}) < -D(R_j)/2, \qquad \forall\ \mathbf{u} \in V \ \textit{with}\ \|\mathbf{u}\|_E > R_j, \tag{2.24}$$

$$\widetilde{J}^+_{a,b}(\mathbf{u}) < -D(R_j)/2, \qquad \forall\ \mathbf{u} = \mathbf{v} + k\boldsymbol{\psi}_1,\ \ \textit{with}\ \mathbf{v} \in V, k \geq 0\ \textit{and}\ \|\mathbf{u}\|_E = R_j, \tag{2.25}$$

where the function $D(R)$ *is defined in Lemma 2.5.*

Proof. The proof of this Proposition is similar to the proofs of the Propositions 4.4 and 4.6 in [19].

In fact, using the estimates (2.4-2.5), one can easily deduce (2.20) from (2.14) and (2.21) from (2.13).
By considering $a+b \in (\lambda_1/2, \lambda_1)$ (that is, setting $\varepsilon_j < \lambda_1/2$ for every $j \in \mathbb{N}$), we can also deduce (2.22) from (2.16).

Suppose now $\mathbf{u} \in V$, with $\|\mathbf{u}\|_E > R_j$. By (2.15) and (2.4-2.5), we have

$$\widetilde{J}^+_{a,b}(\mathbf{u}) + D(R_j)/2 \leq \widetilde{J}^+_{a,b}(\mathbf{u}) + D(\|\mathbf{u}\|_E)/2 \leq$$
$$\leq -G_{I_{l,\delta}} \|\mathbf{u}\|_E^2 + S_0\left(1 + \|\mathbf{u}\|_E^q\right) + H\|\mathbf{u}\|_E + D(\|\mathbf{u}\|_E)/2. \quad (2.26)$$

By (2.17-*ii*), there exists $R_0 > 0$, which may be chosen depending only on $I_{l,\delta}$, but independent from a, b, such that if $R_j > R_0$, then $\widetilde{J}^+_{a,b}(\mathbf{u}) + D(R_j)/2 < 0$, implying (2.24).

One can also prove that there exist ε_0, R_0 (depending on $I_{l,\delta}$), such that the estimate (2.23) will be satisfied if we take $\varepsilon_j < \varepsilon_0 < \lambda_1/2$ and $R_j > R_0$: this is a consequence of estimate (4.19) in Proposition 4.6 from [19], and it can be proved by an argument similar to the one we will give below for proving (2.25): the difference is that (2.23) needs to hold for $J^+_{a,b}$, instead of $\widetilde{J}^+_{a,b}$, then we can consider the hypothesis (**F**) and so we can take $\mathbf{u} = \mathbf{v} + k\boldsymbol{\psi}_1 \in R_j S_{VZ}$ without the restriction $k \geq 0$. More details can be found in [19], see also Remark 2.7.

In order to prove the existence of the two sequences $\{\varepsilon_j\}$ and $\{R_j\}$, satisfying (2.25), let us suppose, for sake of contradiction, that every sequences $\varepsilon_j \to 0^+$ and $R_j \to \infty$, satisfying $\varepsilon_j < \varepsilon_0$ and $R_j > R_0$, admit subsequences (which we still denote by ε_j, R_j) such that for every j, there exist $a_j, b_j \in \mathbb{R}$, $\mathbf{v}_j \in V$, $c_j > 0$, such that $\mathbf{u}_j := \mathbf{v}_j + c_j\boldsymbol{\psi}_1 \in R_j S_{VZ}$, $a_j + b_j \in (\lambda_1 - \varepsilon_j, \lambda_1)$, $a_j - b_j \in I_{l,\delta}$ and $\widetilde{J}^+_{a_j,b_j}(\mathbf{u}_j) \geq -D(R_j)/2$. Without loss of generality, we may suppose that $R_j^2\varepsilon_j \to 0$.

By (2.15) and (2.12), we get

$$B_{a_j,b_j}(\mathbf{u}_j,\mathbf{u}_j) < \frac{\epsilon_j}{\lambda_1} c_j^2 - G_{I_l,\delta} \, \|\mathbf{v}_j\|_E^2 \,, \tag{2.27}$$

and then

$$-D(R_j)/2 \le \widetilde{J}^+_{a_j,b_j}(\mathbf{u}_j) < \frac{\varepsilon_j}{\lambda_1} c_j^2 - G_{I_l,\delta} \, \|\mathbf{v}_j\|_E^2 - \widetilde{\mathcal{F}}(\mathbf{u}_j) - \mathcal{H}(\mathbf{u}_j). \tag{2.28}$$

By dividing inequality (2.28) by R_j^2 and reordering we get

$$\frac{G_{I_l,\delta}}{R_j^2} \, \|\mathbf{v}_j\|_E^2 \le \frac{\varepsilon_j}{\lambda_1 R_j^2} c_j^2 + \frac{D(R_j)}{2R_j^2} - \frac{\mathcal{F}(\mathbf{u}_j) + \mathcal{H}(\mathbf{u}_j)}{R_j^2} \,. \tag{2.29}$$

We note that $\frac{|\mathcal{F}(\mathbf{u}_j)+\mathcal{H}(\mathbf{u}_j)|}{R_j^2} \le \frac{S_0\left(1+\|\mathbf{u}_j\|^q_{[L^2]^2}\right)+H\|\mathbf{u}_j\|_{[L^2]^2}}{R_j^2} \le \frac{C\left(1+R_j+R_j^q\right)}{R_j^2} \to 0$ when $j \to \infty$, since $q \in (1,2)$. Moreover $\frac{D(R_j)}{R_j^2} \to 0$ by (2.17-*ii*) and $\frac{\varepsilon_j}{\lambda_1 R_j^2} c_j^2 \le \frac{\varepsilon_j}{\lambda_1} \to 0$. Thus it follows that

$$\frac{\|\mathbf{v}_j\|_E}{R_j} \to 0 \,, \tag{2.30}$$

and since $\|\mathbf{v}_j\|_E^2 = R_j^2 - c_j^2$, we also get

$$\frac{c_j}{R_j} \boldsymbol{\psi}_1 \to \boldsymbol{\psi}_1, \text{ uniformly, when } j \to \infty. \tag{2.31}$$

At this point we need to assume that hypotheses (**F**) holds (and then also ($\widetilde{\mathbf{F}}$) holds).

We denote by $P_1\mathbf{u}$ and $P_2\mathbf{u}$ the components of a vector $\mathbf{u} \in E$, that is, $\mathbf{u} = (P_1\mathbf{u}, P_2\mathbf{u})$.

Let $\alpha > 0$ and Ω_α be as in Lemma 2.5; by applying Egorov's Theorem to both components of $\frac{\mathbf{v}_j}{R_j}$, in view of (2.30), we get a subset F_α of Ω, with $|F_\alpha| < \alpha$, such that $\frac{\mathbf{v}_j}{R_j} \to 0$, uniformly in $\Omega \setminus F_\alpha$. As a consequence, using also (2.31), there exists

j_0 large enough, such that

$$\frac{c_j}{R_j}\frac{\phi_1}{\sqrt{2}} > 2\alpha \quad \text{and} \quad \left|\frac{\mathbf{v}_j}{R_j}\right| < \alpha \quad \text{in } \Omega_\alpha \setminus F_\alpha, \text{ for every } j \geq j_0.$$

Thus, in $\Omega_\alpha \setminus F_\alpha$ we have

$$P_i\left(\frac{\mathbf{u}_j}{R_j}\right) = \frac{c_j}{R_j}\frac{\phi_1}{\sqrt{2}} + P_i\left(\frac{\mathbf{v}_j}{R_j}\right) > 2\alpha - \alpha > \alpha\,,$$

for $i = 1, 2$. Since $|\Omega_\alpha \setminus F_\alpha| > 2\alpha - \alpha = \alpha$, we have that $P_i\left(\frac{\mathbf{u}_j}{R_j}\right) \in \mathcal{E}$ (see (2.19)) and then by Lemma 2.5

$$\int_\Omega \widetilde{F}_1(x, R_j P_2(\widehat{\mathbf{u}}_j)) \quad \geq \quad D(R_j). \tag{2.32}$$

Since $\int_\Omega \widetilde{F}_2(x, R_j P_1(\widehat{\mathbf{u}}_j))$ is bounded from below by $-C_F|\Omega|$, it follows that

$$\widetilde{\mathcal{F}}(\mathbf{u}_j) = \widetilde{\mathcal{F}}(R_j \widehat{\mathbf{u}}_j) \geq D(R_j) - C_F|\Omega|\,. \tag{2.33}$$

Moreover, by hypothesis ($\widetilde{\mathbf{F}}$-ii) we have $\mathcal{H}(\mathbf{u}_j) = \mathcal{H}(\mathbf{v}_j)$, and then

$$G_{I_{l,\delta}} \|\mathbf{v}_j\|_E^2 + \mathcal{H}(\mathbf{u}_j) = G_{I_{l,\delta}} \|\mathbf{v}_j\|_E^2 + \mathcal{H}(\mathbf{v}_j) \geq G_{I_{l,\delta}} \|\mathbf{v}_j\|_E^2 - H \|\mathbf{v}_j\|_E\,; \tag{2.34}$$

as a consequence, there exists $C_1 > 0$, such that

$$G_{I_{l,\delta}} \|\mathbf{v}_j\|_E^2 + \mathcal{H}(\mathbf{u}_j) \geq -C_1.$$

Then, from (2.28) and (2.33), we get

$$\frac{\varepsilon_j c_j^2}{\lambda_1} + C_1 \geq \widetilde{\mathcal{F}}(\mathbf{u}_j) - D(R_j)/2 \geq D(R_j)/2 - C_F|\Omega|\,, \tag{2.35}$$

which gives rise to a contradiction, since the left hand side is bounded (in fact,

$\varepsilon_j c_j^2 \leq \varepsilon_j R_j^2 \to 0$), while the right hand side tends to $+\infty$ by (2.17-i).

We conclude that there exist sequences $\{\varepsilon_j\}$ and $\{R_j\}$, with $\varepsilon_j \to 0^+$ and $R_j \to \infty$, satisfying (2.25).

The sequences and the constants in the statement of the Proposition can be obtained as follows: we first get $E_{I_{l,\delta}} \in \mathbb{R}$ satisfying (2.22), then we obtain the sequences $\varepsilon_j \to 0^+$ and $R_j \to +\infty$, satisfying (2.23), (2.24) and (2.25). Observe that, for each fixed $j \in \mathbb{N}$, these estimates hold uniformly for any a, b with $a + b \in (\lambda_1 - \varepsilon_j, \lambda_1)$ and $a - b \in I_{l,\delta}$. Then, for every $j \in \mathbb{N}$, by fixing a and b in the above intervals, we can obtain the constants $D_{a+b,I_{l,\delta}} \in \mathbb{R}$ and $\rho_{a+b,I_{l,\delta}} > 0$, such that (2.20) and (2.21) hold. Observe that we may also choose, for every j, a radius $\rho_{a+b,I_{l,\delta}} > R_j$, as desired.

□

Remark 2.7 (Proof of Theorem 1.3)**.** *In [19] we assumed the hypothesis (*$\mathbf{F}^*$*) instead of (*$\mathbf{F}$*). However, in most cases one can proceed as we did in the above proof so that hypothesis (*$\mathbf{F}$*) is sufficient.*

Actually, in Proposition 4.6 of [19], we were dealing with a sequence $\mathbf{u}_j$ *like in the previous proof, which we split as* $\mathbf{u}_j = \mathbf{v}_j + \mathbf{z}_i$ *where* $\mathbf{z}_j$ *was the component in the almost resonant eigenspace* Z*. We were able to prove the analogous of equation (2.30) here for* $\mathbf{v}_j$*, while instead of (2.31) we proved that* $\mathbf{z}_j/R_j$ *converged to a nontrivial element of* Z*.*

Provided Z *does not contain nontrivial elements having a zero component, we can assert that both components of* $\mathbf{z}_j/R_j$ *do not tend to zero, so that it is possible to assume condition (*$\mathbf{F}^*$*-i) on only one of the two functions* $F_{1,2}$*, provided the other one is bounded from below.*

Observe that in the case we are considering in this paper the space Z *always satisfies the condition above, so our application of the results of [19] can be done under hypothesis (*$\mathbf{F}$*).*

2.3 Existence of solutions for the truncated problem

From now on we will consider a sequence of coefficients a_j, b_j such that $a_j + b_j \to \lambda_1^-$ and we will prove the existence of critical points for the functionals $\widetilde{J}^+_{a_j,b_j}$ and $J^+_{a_j,b_j}$, which correspond to solutions of almost resonant problems with coefficients a_j, b_j.

We consider the system

$$\begin{cases} -\Delta u = au + bv + (\widetilde{f}_1(x,v) + h_1(x)); & \text{in } \Omega, \\ -\Delta v = bu + av + (\widetilde{f}_2(x,u) + h_2(x)); & \text{in } \Omega, \\ u(x) = v(x) = 0; & \text{on } \partial\Omega. \end{cases} \tag{2.36}$$

We will reference by $(1.1+)_j$ and $(2.36)_j$ the systems (1.1+) and (2.36), respectively, with $a = a_j$ and $b = b_j$.

Our aim is to prove the following Proposition.

Proposition 2.8. *In the hypotheses of Theorem 1.1, given $\delta > 0$, let, for every $j \in \mathbb{N}$,*

- *$\varepsilon_j > 0$ be those obtained in Proposition 2.6,*
- *a_j, b_j, be such that $a_j + b_j \in (\lambda_1 - \varepsilon_j, \lambda_1)$ and $a_j - b_j \in I_{l,\delta}$.*

Then, for j large enough, the problem $(1.1+)_j$ has three distinct solutions, of which one is positive and another one is negative.

We will first prove the existence of two solutions. In order to do this, we proceed as in [19]: see the sections 4.2 and 4.4 there.

First of all we observe that the inequalities (2.20) and (2.21) define a Saddle Point Geometry for the functionals $\widetilde{J}^+_{a_j,b_j}$, between the subspaces $Z \oplus W$ and V, while the inequalities (2.22) and (2.23) define a Saddle Point Geometry for the functionals $J^+_{a_j,b_j}$, between the subspaces W and $V \oplus Z$.

Then, by using a Galerkin approximation and then taking the limit, in order to overcome the lack of compactness due to the fact that the functionals are strongly indefinite, one can prove that there exist $\mathbf{v}^j, \widetilde{\mathbf{u}}^j \in E$, critical points of the functionals $J^+_{a_j,b_j}$ and $\widetilde{J}^+_{a_j,b_j}$, respectively.
Finally, by using also (2.24) and (2.25), one verifies that the corresponding critical levels are, respectively, $d^j \in [E_{I_{l,\delta}}, T_{\varepsilon_j}]$ and $\widetilde{c}^j \in [D_{a_j+b_j,I_{l,\delta}}, -D(R_j)/2]$, where $E_{I_{l,\delta}}$, $D_{a_j+b_j,I_{l,\delta}}$ and R_j are given in Proposition 2.6, while T_{ε_j} are numbers depending only on ε_j. Observe that in order to take the limit, one needs to use the two kinds of (PS) conditions contained in the lemmas 4.8 and 4.13 of [19], which also apply to the functional $\widetilde{J}^+_{a_j,b_j}$ since $\widetilde{f}_1, \widetilde{f}_2$ satisfy (1.2).

As a consequence, $\mathbf{v}^j$ is a weak solution of $(1.1+)_j$, and $\widetilde{\mathbf{u}}^j$ is a weak solution of $(2.36)_j$. In the next section we will prove that $\widetilde{\mathbf{u}}^j$ is positive for j large enough; this will imply, in view of (2.9), that it is also a solution of $(1.1+)_j$.

2.4 Estimating the sign of $\widetilde{\mathbf{u}}^j$

In this section we consider the weak solutions $\widetilde{\mathbf{u}}^j$ of $(2.36)_j$, with the aim of proving that, for j large enough, they are also weak solutions of the problems $(1.1+)_j$. In view of the definition of the functions $\widetilde{f}_1, \widetilde{f}_2$ in (2.9), it will be enough to prove that $\widetilde{\mathbf{u}}^j$ is positive in Ω.

In order to prove that $\widetilde{\mathbf{u}}^j$ is positive, we will obtain suitable estimates which show that the component of $\widetilde{\mathbf{u}}^j$ orthogonal to the eigenspace $Z = span\,\{\boldsymbol{\psi}_1\}$, becomes small (in the $\mathcal{C}^1$ norm) with respect to the component in the direction of $\boldsymbol{\psi}_1$.

This section is divided in two parts: in the first we prove the Lemmas containing the necessary estimates, and then we prove, in the following one, the positivity of $\widetilde{\mathbf{u}}^j$.

A fundamental tool for proving the estimates will be the following result of L^p regularity of weak solutions, which is a particular case of Theorem 8.2 in [1].

Theorem 2.9 (From Theorem 8.2 in [1]). *Let Ω be a open and bounded set in $\mathbb{R}^N$ of class $\mathcal{C}^2$, $L > 0$ and $1 < p < \infty$.*
Then there exists a constant A, depending only on N, Ω, p and L, such that, if $u \in L^q(\Omega)$, with $1 < q < \infty$, is a weak solution of

$$\begin{cases} -\Delta u + cu = f(x), & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.37}$$

with $|c| \leq L$ and $f \in L^p(\Omega)$, then $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and

$$\|u\|_{W^{2,p}} \leq A \left(\|u\|_{L^p} + \|f\|_{L^p} \right). \tag{2.38}$$

2.4.1 Auxiliary Lemmas

In the following we will denote by P be the orthogonal projection of $L^2(\Omega)$ onto *span* $\{\phi_1\}$ and by $Q = I - P$ the complementary projection. We first state a Lemma from [22].

Lemma 2.10 (Lemma 2.1 from [22]). *Let $p \geq 2$ be fixed. Then there exist $\overline{\varepsilon} > 0$ and $k_1 > 0$, both depending on p, such that if $\lambda \in (\lambda_1 - \overline{\varepsilon}, \lambda_1]$, $g \in L^p(\Omega)$ and $u \in H_0^1(\Omega)$ is a solution of problem*

$$\begin{cases} -\Delta u = \lambda u + g(x), \quad \text{in} \ \ \Omega, \\ u = 0, \quad \text{on} \ \ \partial\Omega, \end{cases} \tag{2.39}$$

where $u = \zeta\phi_1 + w$ with $\zeta \in \mathbb{R}$ and $w \in Q(L^2(\Omega))$, then the following estimate holds:

$$\|w\|_{W^{2,p}} \leq k_1 \|g\|_{L^p}. \tag{2.40}$$

Idea of the proof (see also [23]). First observe that, by the Fredholm's alternative, the linear problem

$$\begin{cases} -\Delta w = \lambda_1 w + \widetilde{g}, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.41}$$

has a unique solution $w \in H_0^1(\Omega) \cap Q(L^2(\Omega))$, for every $\widetilde{g} \in Q(L^2(\Omega))$.

By the regularity result in Theorem 2.9, it follows that $w \in W^{2,p}(\Omega) \cap H_0^1(\Omega) \cap Q(L^2(\Omega))$, provided that $\widetilde{g} \in Q(L^2(\Omega)) \cap L^p(\Omega)$. As a consequence, we may define the map

$$T : Q(L^2(\Omega)) \cap L^p(\Omega) \to W^{2,p}(\Omega) \cap H_0^1(\Omega) \cap Q(L^2(\Omega)),$$

which, to every $\widetilde{g} \in Q(L^2(\Omega)) \cap L^p(\Omega)$, associates $T(\widetilde{g}) = w \in W^{2,p}(\Omega) \cap H_0^1(\Omega) \cap Q(L^2(\Omega))$.

Using the closed graph theorem one can prove that this map is continuous and then there exists a constant $C_p > 0$, depending on p, such that

$$\|w\|_{W^{2,p}} \leq C_p \|\widetilde{g}\|_{L^p} . \tag{2.42}$$

Now, if $u = \zeta\phi_1 + w$ is a solution of (2.39), then $w \in Q(L^2(\Omega)) \cap H_0^1(\Omega)$ is a solution of

$$\begin{cases} -\Delta w = \lambda_1 w - (\lambda_1 - \lambda) w + Qg, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.43}$$

Applying Theorem 2.9 we get $w \in W^{2,p}(\Omega)$, and by (2.42),

$$\begin{aligned} \|w\|_{W^{2,p}} &\leq C_p \|(\lambda_1 - \lambda)w - Qg\|_{L^p} \\ &\leq C_p \left(|\lambda_1 - \lambda| \|w\|_{L^p} + \|Qg\|_{L^p}\right) \leq C_p \overline{\varepsilon} \|w\|_{W^{2,p}} + C \|g\|_{L^p} , \end{aligned}$$

thus

$$(1 - C_p \overline{\varepsilon}) \|w\|_{W^{2,p}} \leq C \|g\|_{L^p} , \tag{2.44}$$

and the result follows by taking $\overline{\varepsilon} > 0$, such that $C_p\overline{\varepsilon} < 1$. □

Now we prove a technical Lemma which will be needed in the following. It is an adaption of Lemma 9.17 in [11].

Lemma 2.11. *Let $p \geq 2$, $\delta > 0$ and $l \in \mathbb{N}$ be fixed. Then there exists $k_2 > 0$ depending on p and on $I_{l,\delta}$, such that if $\mu \in I_{l,\delta}$, then the following estimate holds:*

$$\|u\|_{W^{2,p}} \leq k_2 \|\Delta u + \mu u\|_{L^p}, \quad \textit{for every } u \in W^{2,p}(\Omega) \cap H_0^1(\Omega). \tag{2.45}$$

Proof. Let us suppose, for sake of contradiction, that there exist sequences

$$\{\mu_n\} \subset I_{l,\delta} \qquad \text{and} \qquad \{u_n\} \subset W^{2,p}(\Omega) \cap H_0^1(\Omega),$$

such that $\|u_n\|_{W^{2,p}} = 1$ and $\|\Delta u_n + \mu_n u_n\|_{L^p(\Omega)} \to 0$, when $n \to \infty$.
Then, since $I_{l,\delta}$ is bounded, up to a subsequence we have

$$\begin{cases} \mu_n \to \mu \in \overline{I_{l,\delta}}, \\ u_n \rightharpoonup u \in W^{2,p}(\Omega), \\ u_n \to u \in L^p(\Omega). \end{cases} \tag{2.46}$$

As a consequence, for every $\phi \in \mathcal{C}_0^\infty(\Omega)$, which implies $\phi \in L^{p/(p-1)}(\Omega)$, we get

$$0 = \lim_{n\to\infty} \int_\Omega (\Delta u_n + \mu_n u_n)\phi = \int_\Omega (\Delta u + \mu u)\phi\,.$$

Since $u \in H_0^1(\Omega)$ and then vanishes at the boundary, u is a solution of

$$\begin{cases} -\Delta u = \mu u, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.47}$$

which implies that $u = 0$, since $\mu \notin \sigma(-\Delta)$.

On the other hand, since $u_n \in W^{2,p}$, we have $f(x) := -\Delta u_n(x) - \mu_n u_n(x) \in L^p$. Then, by applying Theorem 2.9, we get the a-priori estimate

$$\|u_n\|_{W^{2,p}} \leq A(\|u_n\|_{L^p} + \|\Delta u_n + \mu_n u_n\|_{L^p}), \qquad \text{for every } n \in \mathbb{N}, \tag{2.48}$$

where the constant $A > 0$ is independent from the index n (it depends on p, Ω and $I_{l,\delta}$). Since the left hand side is equal to 1, while the right hand side goes to $A\,\|u\|_{L^p}$ when $n \to \infty$, it follows that $\|u\|_{L^p} \geq \frac{1}{A} > 0$, which contradicts $u = 0$. □

Exploiting Lemma 2.11, we can obtain a result, which is the analogous of Lemma 2.10 for the case of a linear system near resonance.

Lemma 2.12. *Let $p \geq 2$, $\delta > 0$ and $l \in \mathbb{N}$ be fixed. Then there exist $\overline{\varepsilon} > 0$ and $k_0 > 0$, depending on p and on $I_{l,\delta}$, such that, if $a - b \in I_{l,\delta}$, $a + b \in (\lambda_1 - \overline{\varepsilon}, \lambda_1)$, $\mathbf{g} = (g_1, g_2) \in L^p \times L^p$ and $\mathbf{u} = (u, v) \in E$ is a solution of the problem*

$$\begin{cases} -\Delta u = au + bv + g_1(x), & in \quad \Omega, \\ -\Delta v = bu + av + g_2(x), & in \quad \Omega, \\ u = v = 0, & on \quad \partial\Omega, \end{cases} \tag{2.49}$$

where $\mathbf{u} = \beta\boldsymbol{\psi}_1 + \boldsymbol{\omega}$ with $\beta \in \mathbb{R}$ and $\boldsymbol{\omega} \in V \oplus W$, then the following estimate holds

$$\|\boldsymbol{\omega}\|_{W^{2,p}\times W^{2,p}} \leq k_0\,\|\mathbf{g}\|_{L^p\times L^p}\,. \tag{2.50}$$

Proof. Adding and subtracting the two equations in (2.49), we get

$$\begin{cases} -\Delta s = (a+b)s + g_1(x) + g_2(x), & \text{in} \quad \Omega, \\ -\Delta d = (a-b)d + g_1(x) - g_2(x), & \text{in} \quad \Omega, \\ s = d = 0, \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.51}$$

where $s = u + v$ and $d = u - v$. For the first equation, it follows from Lemma 2.10 that there exist $\overline{\varepsilon} > 0$ and $k_1 > 0$ (depending on p), such that if $a + b \in (\lambda_1 - \overline{\varepsilon}, \lambda_1]$, then

$$\|Q(s)\|_{W^{2,p}} \leq k_1 \|g_1 + g_2\|_{L^p}. \tag{2.52}$$

For the second equation, it follows from Theorem 2.9 and Lemma 2.11 that there exists a constant $k_2 > 0$ (depending on p and $I_{l,\delta}$), such that

$$\|d\|_{W^{2,p}} \leq k_2 \|g_1 - g_2\|_{L^p}. \tag{2.53}$$

Now, writing $\boldsymbol{\omega} = (\omega_1, \omega_2)$, we have $Q(s) = \omega_1 + \omega_2$ and $d = \omega_1 - \omega_2$. Then $\boldsymbol{\omega} = \left(\frac{Q(s)+d}{2}, \frac{Q(s)-d}{2}\right)$ and so

$$\begin{aligned} \|\boldsymbol{\omega}\|_{W^{2,p}\times W^{2,p}} &\leq \left\|\frac{Q(s)+d}{2}\right\|_{W^{2,p}} + \left\|\frac{Q(s)-d}{2}\right\|_{W^{2,p}} \leq (\|Q(s)\|_{W^{2,p}} + \|d\|_{W^{2,p}}) \\ &\leq (k_1 \|g_1+g_2\|_{L^p} + k_2 \|g_1 - g_2\|_{L^p}) \leq (k_1+k_2)(\|g_1\|_{L^p} + \|g_2\|_{L^p}) \\ &\leq (k_1+k_2) 2^{\frac{p-1}{p}} \|\mathbf{g}\|_{L^p\times L^p}. \end{aligned}$$

□

Lemma 2.12 allows us to prove the following Lemma 2.13, where we obtain an estimate, in the $\mathcal{C}^1$ norm, of the component orthogonal to Z of a weak solution of (2.36), in terms of its component in Z.

Lemma 2.13. *Let f_1, f_2 satisfy (1.2) and $h_1, h_2 \in L^r(\Omega)$, for some $r > N$. Moreover, fix $\delta > 0$, $l \in \mathbb{N}$ and let $\bar{\varepsilon}$ be the one obtained in Lemma 2.12 with $p = r$. Given $\eta > 0$, there exists $\widetilde{C}_\eta > 0$, depending on $\eta, I_{l,\delta}$ and r, such that, if $a-b \in I_{l,\delta}$, $a+b \in (\lambda_1 - \bar{\varepsilon}, \lambda_1)$ and $\widetilde{\mathbf{u}} = \beta\boldsymbol{\psi}_1 + \boldsymbol{\omega}$ is a critical point of $\widetilde{J}^+_{a,b}$, where $\beta \in \mathbb{R}$ and $\boldsymbol{\omega} \in V \oplus W$, then the following estimate holds:*

$$\|\boldsymbol{\omega}\|_{\mathcal{C}^1\times\mathcal{C}^1} \leq \eta|\beta| + \widetilde{C}_\eta. \tag{2.54}$$

Proof. Let $\widetilde{\mathbf{u}} = (\widetilde{u}, \widetilde{v})$ be a critical point of the functional $\widetilde{J}^+_{a,b}$, so that it is a weak solution of problem (2.36). We denote

$$\widetilde{\mathbf{f}}(\widetilde{\mathbf{u}}) = (\widetilde{f}_1(x, \widetilde{v}(x)), \widetilde{f}_2(x, \widetilde{u}(x))) \qquad \text{and} \qquad \mathbf{h} = (h_1(x), h_2(x)).$$

Observe that, in view of the hypotheses on the functions f_1, f_2, h_1, h_2, if $\widetilde{u}, \widetilde{v} \in L^p(\Omega)$ for some $p \in [1, r]$, then $\widetilde{\mathbf{f}}(\widetilde{\mathbf{u}}), \mathbf{h} \in L^p(\Omega) \times L^p(\Omega)$, and then by Theorem 2.9 we conclude that $\widetilde{u}, \widetilde{v} \in W^{2,p}(\Omega)$.

We claim that $\widetilde{\mathbf{u}} \in W^{2,r}(\Omega) \times W^{2,r}(\Omega)$.
In order to prove this, one has to apply iteratively Sobolev embeddings and Theorem 2.9, until one obtains $\widetilde{u}, \widetilde{v} \in W^{2,p}(\Omega)$ for some $p \geq N/2$. For such p one has $W^{2,p}(\Omega) \hookrightarrow L^r(\Omega)$, and then as above $\widetilde{u}, \widetilde{v} \in W^{2,r}(\Omega)$.

Since $\widetilde{\mathbf{u}}$ is a weak solution of (2.36), we already have $\widetilde{u}, \widetilde{v} \in L^2(\Omega)$ and then $\widetilde{u}, \widetilde{v} \in W^{2,2}(\Omega)$ by Theorem 2.9.
If $N \leq 4$, then we are done.
If $N > 4$, since $W^{2,2}(\Omega) \hookrightarrow L^{p_1}(\Omega)$, where $p_1 = \frac{2N}{N-4} > 2$, we get $\widetilde{u}, \widetilde{v} \in L^{p_1}(\Omega)$ and then $\widetilde{u}, \widetilde{v} \in W^{2,p_1}(\Omega)$. If $N - 2p_1 > 0$, then we use $W^{2,p_1}(\Omega) \hookrightarrow L^{p_2}(\Omega)$, where $p_2 = \frac{Np_1}{N-2p_1}$, and then $\widetilde{u}, \widetilde{v} \in W^{2,p_2}(\Omega)$. By repeating this argument we obtain that $\widetilde{u}, \widetilde{v} \in W^{2,p_i}(\Omega)$, where the sequence $\{p_i\}$ is defined recursively by $p_0 = 2$ and $p_{i+1} = \frac{Np_i}{N-2p_i}$. In order to prove our claim we only have to show that there exists

$i_0 \in \mathbb{N}$, such that $p_{i_0} \geq N/2$. Suppose for sake of contradiction that $p_i \in (2, N/2)$ for every $i \in \mathbb{N}$, which implies $p_{i+1} > p_i$. Then the sequence $\{p_i\}$ would be bounded and increasing, and then would converge to $L \in [2, N/2]$. However by taking limit in the recursion formula we get $L = \frac{NL}{N-2L}$ which implies $L = 0$, a contradiction.

We have then proved that $\widetilde{\mathbf{u}} \in W^{2,r}(\Omega) \times W^{2,r}(\Omega)$. By Lemma 2.12, there exists $k_0 > 0$ such that

$$\|\boldsymbol{\omega}\|_{W^{2,r}\times W^{2,r}} \leq k_0 \left\|\widetilde{\mathbf{f}}(\widetilde{\mathbf{u}}) + \mathbf{h}\right\|_{L^r\times L^r}. \tag{2.55}$$

Since $r > N$, we have the continuous embedding $W^{2,r}(\Omega) \hookrightarrow \mathcal{C}^1(\overline{\Omega})$ and then

$$\|\boldsymbol{\omega}\|_{\mathcal{C}^1\times\mathcal{C}^1} \leq C\|\boldsymbol{\omega}\|_{W^{2,r}\times W^{2,r}} \leq Ck_0 \left\|\widetilde{\mathbf{f}}(\widetilde{\mathbf{u}}) + \mathbf{h}\right\|_{L^r\times L^r} \leq Ck_0 \left\|\widetilde{\mathbf{f}}(\widetilde{\mathbf{u}})\right\|_{L^r\times L^r} + Ck_0 H_r, \tag{2.56}$$

where $H_r = \|\mathbf{h}\|_{L^r\times L^r}$.

Now we observe that by hypothesis (1.2), given $\gamma > 0$, there exists $D_\gamma > 0$ such that $|\widetilde{f}_i(x,t)| \leq \gamma|t| + D_\gamma$ for every $t \in \mathbb{R}$ and $i = 1, 2$. Thus

$$\left\|\widetilde{\mathbf{f}}(\widetilde{\mathbf{u}})\right\|_{L^r\times L^r} \leq C_1\left(\gamma\|\widetilde{\mathbf{u}}\|_{L^r\times L^r} + D_\gamma\right).$$

By combining this inequality with (2.56), it follows

$$\begin{aligned}
\|\boldsymbol{\omega}\|_{\mathcal{C}^1\times\mathcal{C}^1} &\leq CC_1k_0\gamma\|\widetilde{\mathbf{u}}\|_{L^r\times L^r} + Ck_0(H_r + C_1D_\gamma) \\
&\leq C_2\gamma|\beta|\,\|\boldsymbol{\psi}_1\|_{L^r\times L^r} + C_2\gamma\|\boldsymbol{\omega}\|_{L^r\times L^r} + \widetilde{D}_\gamma \\
&\leq C_2\gamma\|\boldsymbol{\psi}_1\|_{L^r\times L^r}|\beta| + C_3\gamma\|\boldsymbol{\omega}\|_{\mathcal{C}^1\times\mathcal{C}^1} + \widetilde{D}_\gamma,
\end{aligned}$$

where $\widetilde{D}_\gamma = Ck_0(H_r + C_1D_\gamma)$.
Then, given $\eta > 0$, we can find $\gamma > 0$, such that $C_2\gamma\|\boldsymbol{\psi}_1\|_{L^r\times L^r} < \frac{\eta}{2}$ and $C_3\gamma < \frac{1}{2}$; we can thus obtain the desired result with $\widetilde{C}_\eta = 2\widetilde{D}_\gamma$.

We observe that the choice of γ (and then of $\widetilde{C}_\eta$) depends on k_0 and r, that is, on $r, I_{l,\delta}$. $\square$

2.4.2 Positivity of $\widetilde{\mathbf{u}}^j$

In this section we prove that there exists j_0 such that $\widetilde{\mathbf{u}}^j$ is positive in Ω, for every $j \geq j_0$. Remember (see section 2.3) that $\widetilde{\mathbf{u}}^j$ is a critical point of the functional $\widetilde{J}^+_{a_j,b_j}$ at the critical level $\widetilde{c}^j \leq -D(R_j)/2$, and since $D(R_j) \to +\infty$, we have

$$\widetilde{J}^+_{a_j,b_j}(\widetilde{\mathbf{u}}^j) = \widetilde{c}^j \to -\infty, \qquad \text{when} \;\; j \to \infty. \tag{2.57}$$

For every $j \in \mathbb{N}$, we denote

$$\widetilde{\mathbf{u}}^j = \beta_j \boldsymbol{\psi}_1 + \boldsymbol{\omega}_j, \tag{2.58}$$

where $\beta_j \in \mathbb{R}$ and $\boldsymbol{\omega}_j \in V \oplus W$. We note that

$$|\beta_j| \to +\infty\,, \;\text{when}\; j \to \infty.$$

Actually, if (a subsequence of) $\{\beta_j\}$ were bounded, then, by Lemma 2.13, $\{\widetilde{\mathbf{u}}^j\}$ would be bounded. As a consequence, $\widetilde{J}^+_{a_j,b_j}(\widetilde{\mathbf{u}}^j)$ would be bounded too, since the coefficients a_j, b_j take values in a fixed bounded set. This contradicts (2.57).

By Lemma 2.13, it follows that, for arbitrary $\eta > 0$,

$$\limsup_{j\to\infty} \frac{\|\boldsymbol{\omega}_j\|_{\mathcal{C}^1\times\mathcal{C}^1}}{|\beta_j|} \leq \limsup_{j\to\infty} \left(\eta + \frac{\widetilde{C}_\eta}{|\beta_j|}\right) = \eta, \tag{2.59}$$

which implies

$$\frac{\boldsymbol{\omega}_j}{\beta_j} \to 0, \;\text{in}\; \mathcal{C}^1(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega}), \;\text{when}\; j \to \infty, \tag{2.60}$$

and then

$$\frac{\widetilde{\mathbf{u}}^j}{\beta_j} = \boldsymbol{\psi}_1 + \frac{\boldsymbol{\omega}_j}{\beta_j} \to \boldsymbol{\psi}_1, \text{ in } \mathcal{C}^1(\overline{\Omega}) \times \mathcal{C}^1(\overline{\Omega}), \text{ when } j \to \infty. \tag{2.61}$$

Since both components of $\boldsymbol{\psi}_1$ are positive and have positive inward derivative at the boundary, there exists j_0 such that, for every $j \geq j_0$, one has that $\widetilde{\mathbf{u}}^j$ is positive if $\beta_j > 0$ and $\widetilde{\mathbf{u}}^j$ is negative if $\beta_j < 0$. In order to prove that $\widetilde{\mathbf{u}}^j$ is positive, we need to prove the following Lemma.

Lemma 2.14. *Let β_j be given by (2.58). Then $\beta_j \to +\infty$, when $j \to \infty$.*

Proof. We already know that $|\beta_j| \to \infty$. Thus we suppose, for sake of contradiction, that (for some subsequence) $\beta_j \to -\infty$. In this case, by (2.61), $\widetilde{\mathbf{u}}^j$ is negative for j large enough.

By the definition of $\widetilde{f}_1, \widetilde{f}_2$ and for being continuous, there exists $M > 0$ such that, for $i = 1, 2$,

$$|\widetilde{f}_i(x,t)| \leq M, \qquad \text{and} \qquad |\widetilde{F}_i(x,t)| \leq 2M, \qquad \text{for every } x \in \Omega, t \leq 0. \tag{2.62}$$

As a consequence, we have

$$\left\| \widetilde{\mathbf{f}}(\widetilde{\mathbf{u}}^j) \right\|_{[L^2]^2} \leq 2M|\Omega|^{\frac{1}{2}}, \tag{2.63}$$

and then we can apply Lemma 2.12 to obtain

$$\|\boldsymbol{\omega}_j\|_E \leq \|\boldsymbol{\omega}_j\|_{W^{2,2}\times W^{2,2}} \leq k_0 \left\| \widetilde{\mathbf{f}}(\widetilde{\mathbf{u}}^j) + \mathbf{h} \right\|_{[L^2]^2} \leq 2Mk_0|\Omega|^{\frac{1}{2}} + k_0 \|\mathbf{h}\|_{[L^2]^2} < \infty. \tag{2.64}$$

Also, one has

$$\widetilde{\mathcal{F}}(\widetilde{\mathbf{u}}^j) \leq 4M|\Omega|$$

and, in view of ($\widetilde{\mathbf{F}}$-ii) and (2.4)

$$\mathcal{H}(\widetilde{\mathbf{u}}^j) = \beta_j \mathcal{H}(\boldsymbol{\psi}_1) + \mathcal{H}(\boldsymbol{\omega}_j) = \mathcal{H}(\boldsymbol{\omega}_j) \leq H \left\|\boldsymbol{\omega}_j\right\|_E .$$

Since

$$B_{a_j,b_j}(\widetilde{\mathbf{u}}^j, \widetilde{\mathbf{u}}^j) = B_{a_j,b_j}(\beta_j \boldsymbol{\psi}_1, \beta_j \boldsymbol{\psi}_1) + B_{a_j,b_j}(\boldsymbol{\omega}_j, \boldsymbol{\omega}_j)\,, \tag{2.65}$$

we get

$$\begin{aligned} \widetilde{J}^+_{a_j,b_j}(\widetilde{\mathbf{u}}^j) &= \frac{\beta_j^2}{2} B_{a_j,b_j}(\boldsymbol{\psi}_1, \boldsymbol{\psi}_1) + \frac{1}{2} B_{a_j,b_j}(\boldsymbol{\omega}_j, \boldsymbol{\omega}_j) - \Big(\widetilde{\mathcal{F}}(\widetilde{\mathbf{u}}^j) + \mathcal{H}(\widetilde{\mathbf{u}}^j)\Big) \\ &\geq \frac{\beta_j^2}{2} B_{a_j,b_j}(\boldsymbol{\psi}_1, \boldsymbol{\psi}_1) + \frac{1}{2} B_{a_j,b_j}(\boldsymbol{\omega}_j, \boldsymbol{\omega}_j) - H \left\|\boldsymbol{\omega}_j\right\|_E - 4M|\Omega|. \end{aligned} \tag{2.66}$$

Since $B_{a_j,b_j}(\boldsymbol{\psi}_1, \boldsymbol{\psi}_1) = \frac{\lambda_1 - (a_j+b_j)}{\lambda_1} > 0$ and the terms containing $\boldsymbol{\omega}_j$ are bounded in E by (2.64), we get a contradiction with (2.57). □

As observed, the consequence of this Lemma is that $\widetilde{\mathbf{u}}^j$ is positive, for every $j \geq j_0$. By the definition of $\widetilde{f}_1, \widetilde{f}_2$, since $\widetilde{\mathbf{u}}^j$ is a weak solution of $(2.36)_j$, it follows that it is also a weak solution of $(1.1+)_j$ and that

$$J^+_{a_j,b_j}(\widetilde{\mathbf{u}}^j) = \widetilde{J}^+_{a_j,b_j}(\widetilde{\mathbf{u}}^j) \in [D_{a_j+b_j, I_{l,\delta}}, -D(R_j)/2]. \tag{2.67}$$

2.5 The three solutions

We are now in the position to prove Proposition 2.8 and then our main results.

Proof of Proposition 2.8. The functions $\mathbf{v}^j$ and $\widetilde{\mathbf{u}}^j$ are solutions of $(1.1+)_j$, the second one being positive. Since $J^+_{a_j,b_j}(\mathbf{v}^j) \geq E_{I_{l,\delta}}$ while $J^+_{a_j,b_j}(\widetilde{\mathbf{u}}^j) \to -\infty$, they are distinct, for j large enough.

In order to obtain a third solution, we consider the system

$$\begin{cases} -\Delta u = au + bv + (-f_1(x,-v) + (-h_1(x))), & \text{in } \Omega, \\ -\Delta v = bu + av + (-f_2(x,-u) + (-h_2(x))), & \text{in } \Omega, \\ u(x) = v(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.68}$$

We observe that if (u, v) is a solution of (2.68), then $(-u, -v)$ is a solution of (1.1+). We denote $g_i(x, t) = -f_i(x, -t)$ and

$$G_i(x,t) = \int_0^t g_i(x,s)ds = \int_0^t -f_i(x,-s)ds = \int_0^{-t} f_i(x,\sigma)d\sigma = F_i(x,-t), \tag{2.69}$$

for $i = 1, 2$. It follows that g_1, g_2 and $-h_1, -h_2$ satisfy (1.2) and (**F**).
Moreover, if $\widehat{J}_{a,b}$ is the functional associated to the Problem (2.68), then it is simple to verify that $\widehat{J}_{a,b}(\mathbf{u}) = J_{a,b}(-\mathbf{u})$, for every $\mathbf{u} \in E$: actually the quadratic part of the functional is even, while $F_i(x,t) = G_i(x,-t)$, and $h_i t = (-h_i)(-t)$, for every $t \in \mathbb{R}$ and $i = 1, 2$.

Thus, we can repeat all the arguments used above (it may be necessary to redefine ε_j, R_j, a_j, and b_j in such a way that they work for the problems (1.1) and (2.68) at the same time), and in the end we obtain a sequence $\{\widehat{\mathbf{u}}^j\}_{j\in\mathbb{N}} \subset E$ and $j_1 \in \mathbb{N}$, such that for every $j \geq j_1$, $\widehat{\mathbf{u}}^j$ is a weak positive solution of $(2.68)_j$ and $\widehat{J}_{a_j,b_j}(\widehat{\mathbf{u}}^j) \to -\infty$.

We conclude that $-\widehat{\mathbf{u}}^j$ is a negative solution (then distinct from $\widetilde{\mathbf{u}}^j$) of $(1.1+)_j$, for $j \geq j_1$. Moreover, $J^+_{a_j,b_j}(-\widehat{\mathbf{u}}^j) \to -\infty$, implying that $-\widehat{\mathbf{u}}^j$ is distinct also from $\mathbf{v}^j$. □

Proof of Theorem 1.1. Item (a): suppose, for sake of contradiction, that the claim of the Theorem were false. As a consequence, for some $\delta > 0$, there would exist sequences $\{a_j\}, \{b_j\}$ such that $a_j - b_j \in I_{l,\delta}$ and $a_j + b_j \to \lambda_1^-$, for which problem $(1.1+)_j$ does not have three solutions. Without loss of generality we may suppose that $a_j + b_j \in (\lambda_1 - \varepsilon_j, \lambda_1)$. However, this would contradict Proposition 2.8.

Item (b): as in [19], this case can be proved by the same technique used for the case (a), if we work with the functional

$$J^-_{a,b}(\mathbf{u}) = -1/2 B_{a,b}(\mathbf{u},\mathbf{u}) - (\mathcal{F}(\mathbf{u}) + \mathcal{H}(\mathbf{u}))\,,$$

and we define the spaces V, Z, W as in (2.11), but making use of the eigenvalues of $-B_{a,b}$, instead of those of $B_{a,b}$.

In fact, observe that the Lemma 2.10 can be extended straightforwardly to the case $\lambda \in [\lambda_1, \lambda_1 + \overline{\varepsilon})$ (actually the sign of $\lambda - \lambda_1$ is never used in the proof) and then the following Lemmas can be extended too.

Finally, when $a + b \in (\lambda_1, \lambda_1 + \varepsilon)$, the eigenvalue corresponding to $\boldsymbol{\psi}_1$ is

$$0 < \frac{(a+b) - \lambda_1}{\lambda_1} = -B_{a,b}(\boldsymbol{\psi}_1, \boldsymbol{\psi}_1) < \frac{\varepsilon}{\lambda_1}\,.$$

The positivity of this eigenvalue is all we need to prove the Lemma 2.14 in this case (see equation (2.66)). □

Proof of Theorem 1.2. This Theorem follows directly from Theorem 1.1, by performing the change of unknown $(U, V) = (u, -v)$, as in [19]. □

Remark 2.15. *As we claimed in section 1, if hypothesis (*$\mathbf{F}^+$*) holds instead of (*$\mathbf{F}$*), then one can still truncate the nonlinearities as in (2.9), so that they satisfy (*$\widetilde{\mathbf{F}}$*). As a consequence, in the case of Theorem 1.1, one still gets one positive solution* $\widetilde{\mathbf{u}}^j$ *as a critical point of* $\widetilde{J}_{a_j,b_j}$ *at a level below* $-D(R_j)/2$*. On the other hand, the solution* $\mathbf{v}^j$ *was obtained without the need of any hypothesis like (*$\mathbf{F}$*), so this solution still exists and again it is distinct from the previous one for lying at a different level.*

Also, if the limit in ($\mathbf{F}^+$*-i-a) is at* $-\infty$*, then one can get two solutions, one being negative, in the case of Theorem 1.1. In the case of Theorem 1.2 one gets two* *solution, one having components of opposite sign.*

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