Multiple positive solutions for the m-Laplacian and a nonlinearity with many zeros

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Abstract

In this paper we consider the quasilinear elliptic equation $-\Delta_m u = \lambda f(u)$, in a bounded, smooth and convex domain. When the nonnegative nonlinearity f has multiple positive zeros, we prove the existence of at least two positive solutions for each of these zeros, for λ large, without any hypothesis on the behavior at infinity of f. We also prove a result concerning the behavior of the solutions as $\lambda \to \infty$.

Keywords and phrases: $m-$Laplacian; nonlinearity with zeros; multiplicity of solutions.

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1 Introduction

In this paper, we obtain a result concerning the existence of multiple positive $\mathcal{C}^1(\overline{\Omega})$-weak solutions of the Problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_m u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a convex bounded domain in $\mathbb{R}^N$ with smooth boundary, $N > m > 1$, λ is a positive parameter and f is a continuous nonnegative function which has zeros $\{z_0 = 0 < z_1 < z_2 < \ldots < z_k\}$; we will show the existence of at least $2k$ or $2k+1$ (depending on the behavior of f at the origin) positive solutions for λ large. No hypothesis will be made about the behavior at infinity of f. This result generalizes our previous work [ILM10], where we considered only one positive zero and a growth near the origin at least as u^{m-1}.

We will assume the following hypotheses on the nonlinearity f:

(F_1) $f : [0,T] \to \mathbb{R}$ is a continuous function and there exists a set $\{z_0 = 0 < z_1 < z_2 < \ldots < z_k\} \subseteq [0,T)$ such that f is locally Lipschitz continuous in $(0,T]$, $f(0) = f(z_1) = \cdots = f(z_k) = 0$ and $f(x) > 0$ for $x \in (0,T] \setminus \{z_1; \ldots; z_k\}$.

(F_2) There exist $c_j > 0$ and $\sigma_j \in (m-1, m_* - 1)$ such that

$$\lim_{t \to z_j} \frac{f(t)}{|t - z_j|^{\sigma_j}} = c_j\,, \quad j = 1, \ldots, k,$$

where $m_* = \frac{(N-1)m}{N-m}$ denotes the Serrin's exponent.

(F_3) There exists $L > 0$ such that the map $t \mapsto f(t) + Lt^{m-1}$ is increasing for $t \in [0,T]$.

About the behavior of the nonlinearity near the origin, we will assume one of the following two hypotheses:

(F_4) There exists $\sigma_0 \in (m-1, m_* - 1)$ such that

$$\lim_{t \to 0^+} \frac{f(t)}{t^{\sigma_0}} = 1;$$

or

(F_5) $\liminf_{t \to 0^+} \frac{f(t)}{t^{m-1}} \geq 1.$

Our main results are the following

Theorem 1.1. *Assume that the hypotheses (F_1) through (F_3) hold and Ω is convex smooth and bounded. If also hypothesis (F_4) holds, then there exists $\lambda^* > 0$ such that the Problem (P_λ) has at least $2k+1$ $\mathcal{C}^1$-weak positive solutions $v_{0,\lambda}$, $u_{j,\lambda}$, $v_{j,\lambda}$, $j = 1, \ldots, k$, for $\lambda > \lambda^*$.*

Moreover, these solutions satisfy $\|v_{0,\lambda}\|_\infty \to 0^+$, $\|u_{j,\lambda}\|_\infty \to z_j^-$ and $\|v_{j,\lambda}\|_\infty \to z_j^+$, $j = 1, \ldots, k$, when $\lambda \to \infty$.

Theorem 1.2. *Assume that the hypotheses (F_1) through (F_3) hold and Ω is convex smooth and bounded. If also hypothesis (F_5) holds, then there exists $\lambda^* > 0$ such that the Problem (P_λ) has at least $2k$ $\mathcal{C}^1$-weak positive solutions $u_{j,\lambda}$, $v_{j,\lambda}$, $j = 1, \dots, k$, for $\lambda > \lambda^*$.*

Moreover, these solutions satisfy $\|u_{j,\lambda}\|_\infty \to z_j^-$ and $\|v_{j,\lambda}\|_\infty \to z_j^+$, $j = 1, \dots, k$, when $\lambda \to \infty$.

Remark 1.3. *In fact, it will be clear from the proofs that we might state our result with more details: under the hypotheses of the Theorem 1.1 (resp. 1.2), there exist $\Lambda_j : \ j = 0, \dots, k$ such that for $\lambda > \Lambda_j$ Problem (P_λ) has at least $2j+1$ (resp. $2j$) $\mathcal{C}^1$-weak positive solutions.*

Moreover, it follows directly by the main theorems, that we may also consider a function f with an infinite number of zeros, all satisfying a condition as in (F_2), obtaining an arbitrary number of solutions, for sufficiently large λ. An example could be $f(x) = \sin^2(x)$.

As remarked above, this result generalizes our previous work [ILM10], where we considered only one positive zero and a growth near the origin as in hypotheses (F_5). In that work we obtained two solutions for λ large, one below the zero and another one exceeding the zero. Both solutions satisfied the property that their maximum value converged to the zero of the nonlinearity when $\lambda \to \infty$. This property was exploited in order to get a bound on the L^∞ norm which allowed to truncate the nonlinearity and then to consider very week hypotheses about its behavior at infinity.

However, this L^∞ estimate was obtained by a blowup argument and then relied on two main results: one was the Liouville-type theorem in $\mathbb{R}^N$ obtained in [IMSU10], the second, which guarantees that the blow-up procedure leads to a problem in $\mathbb{R}^N$, is contained in the following Lemma, which is a consequence of the results in [DS04].

Lemma 1.4. *Assume that Ω is a convex and smooth bounded domain, then there exists $\delta_\Omega > 0$ (depending only on Ω), such that if $h : [0, +\infty) \to [0, \infty)$ is a continuous function which is positive and locally Lipschitz continuous in $(0, \infty)$, then the $\mathcal{C}^1(\overline{\Omega})$-weak solutions u of*

$$\begin{cases} -\Delta_m u = h(u), & u \geq 0, \quad in \quad \Omega\,, \\ u = 0 & \qquad\quad\; on \ \ \partial\Omega\,, \end{cases}$$

satisfy the property that there exists a point $x \in \Omega$ such that $dist(x, \partial\Omega) \geq \delta_\Omega$ and $u(x) = \|u\|_\infty$.

Instead of the Liouville-type theorem from [IMSU10], we will use here the following Lemma, which is a consequence of Theorem 3.12 from [DM10].

Lemma 1.5. *Assume that $f : [0, \infty) \to [0, \infty)$ is a continuous function, $N > m$ and u is a nonconstant $\mathcal{C}^1(\overline{\Omega})-$weak solution of*

$$-\Delta_m u \geq f(u), \qquad u \geq 0, \qquad in\ \mathbb{R}^N, \tag{1.1}$$

with $\gamma = \inf_{\mathbb{R}^N}(u)$.

Then $f(\gamma) = 0$ *and*

$$\liminf_{t\to\gamma^+} \frac{f(t)}{(t-\gamma)^{m_*-1}} = 0\,. \tag{1.2}$$

Actually, by the hypotheses (F_2), (F_4) and (F_5), no zero of f satisfies condition (1.2), so that every solution of (1.1) must be constant.

With respect to [ILM10], this Lemma allows us to deal with multiple zeros in the nonlinearity, but we still need to use Lemma 1.4: in order to apply it, as in [ILM10], we will need to work with locally Lipschitz and strictly positive nonlinearities and with a convex domain Ω. In the case of the nonlinearity f (which of course is not strictly positive), we will obtain that there exist maximum points of the solutions which are uniformly bounded away from the boundary of Ω, by taking the limit of solutions of perturbed problems with positive nonlinearities.

Before going to the proofs, we give some remarks on the hypotheses. Hypothesis (F_2) is required for applying Lemma 1.5 as described above, but also in order to obtain a family of supersolutions near each zero of f (see Lemma 2.2). In fact, it would be possible to weaken this hypothesis (and (F_4) in a similar way), by assuming the following instead:

(F_2^*) There exist $\sigma_j \in (m-1, m_*-1)$ such that

$$\liminf_{t\to z_j^+} \frac{f(t)}{|t-z_j|^{m_*-1}} > 0\,, \quad j = 1, \dots, k,$$

and

$$\lim_{t\to z_j} \frac{f(t)}{|t-z_j|^{m-1}} = 0\,, \quad j = 1, \dots, k;$$

the first condition is used in order to apply Lemma 1.5 and the second in order to obtain the family of supersolutions. However, we will assume hypothesis (F_2) throughout the proofs for simplicity of the exposition.

Hypothesis (F_3) is standard in order to use the method of sub- and supersolutions. Hypothesis (F_4) states that f has a similar behavior near the origin as it has near its other zeros: as a consequence we will be able to obtain one additional solution near the origin. On the other hand, Hypothesis (F_5) makes it impossible to have a solution near the origin for λ large, but implies the existence of a subsolution, that can be used to start the sub- and supersolutions method.

In the semilinear case $m = 2$, our result is similar to the results in [Hes81] (see also [GMI15]). For the case with only one positive zero in the nonlinearity we cite also [Lio82, Liu99, IMSU10]. We remand to [IMSU10, ILM10] for further observations and literature related to this kind of problems.

The paper is organized as follows: in Section 2 we state and prove all the results needed to prove the main Theorems, which are then proved, respectively, in the Sections 3 and 4.

2 Preliminary Lemmas

From now on we will always assume that Ω is a bounded and smooth domain; the convexity will be explicitly assumed only when it is required.

Our first step will be to build a family of auxiliary problems with the nonlinearity truncated and a positive term added.

Assuming (F_1) and (F_2), we take $R_j \in (z_j, z_{j+1})$ for $j = 1, \dots, k-1$ and $R_k \in (z_k, T)$. Also, when (F_4) holds, there exist $\varepsilon_0 \in (0,1)$ and $R_0 \in (0, z_1)$ such that for any $t \in [0, R_0]$ we have

$$(1-\varepsilon_0)t^{\sigma_0} \leq f(t) \leq (1+\varepsilon_0)t^{\sigma_0}\,. \tag{2.1}$$

When we do not assume (F_4) we just set arbitrary $R_0 \in (0, z_1)$ and $\sigma_0 \in (m-1, m_*-1)$.

Then we define, for $j = 0, \dots, k$,

$$f_j(t) = \begin{cases} f(t^+), & \text{if } t \leq R_j\,, \\ \frac{f(R_j)}{R_j^{\sigma_j}} t^{\sigma_j}, & \text{if } t > R_j\,, \end{cases} \tag{2.2}$$

where $t^+ = \max\{0, t\}$, and we consider the auxiliary problems

$$(Q_{j,\lambda,\tau}) \qquad \begin{cases} -\Delta_m u = \lambda f_j(u) + \tau (u^+)^{m-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tau \geq 0$ is a real parameter.

We remark that, by the strong maximum principle (see [Váz84]), the nontrivial solutions of the Problem $(Q_{j,\lambda,\tau})$ are positive and, by hypothesis (F_1) and since $\sigma_0, \dots, \sigma_k < m_* - 1$, they are in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ (see [GV89]); moreover, since $f_j \geq 0$, $(Q_{j,\lambda,\tau})$ has no positive solution if $\tau > \lambda_1$: the first eigenvalue of the m-Laplacian in Ω.

Finally, we observe that if u is a solution of Problem $(Q_{j,\lambda,\tau})$ and $u < R_j$, then it is also a solution of Problem $(Q_{i,\lambda,\tau})$ for $i > j$. In the case $\tau = 0$, it is also a solution of Problem (P_λ).

Roughly speaking, the proof of the main theorems will go on the following lines: we will first obtain two solutions of $(Q_{j,\lambda,\tau})$ for each j, with $\tau > 0$: one is obtained via the sub- and supersolutions method, and a second one by using topological degree. Being $\tau > 0$ we can apply Lemma 1.4 to these solutions. Then we take the limit for $\tau > 0$ and we obtain solutions having their maxima bonded away from the boundary. This fact will allow us to obtain the cited L^∞ estimates that justify the truncation of the nonlinearity and also make it possible to distinguish the solutions between them.

We start by presenting several lemmas which will be used later. Some of them are taken (or adapted) from [ILM10].

The first step will be to derive some a-priori estimates for the solutions of $(Q_{j,\lambda,\tau})$, $j = 0, 1, \dots, k$; we remark that this result does not require the use of Lemma 1.4 and then it holds also for $\tau = 0$.

Lemma 2.1. *Under hypotheses (F_1), (F_2) and (F_4) or (F_5), we have*

(1) given $\widetilde{\lambda} > 0$, there exists a constant $D_{\widetilde{\lambda}}$ such that, if $u \in \mathcal{C}^1(\overline{\Omega})$ is a weak solution of Problem $(Q_{j,\lambda,\tau})$ with $j \in \{0, 1, \dots, k\}$, $\lambda > \widetilde{\lambda}$ and $\tau \geq 0$ then

$$\|u\|_\infty \leq D_{\widetilde{\lambda}}\,;$$

(2) given $\lambda > 0$, there exist constants $C_\lambda > 0$ and $\alpha \in (0, 1)$ such that one has also the estimate

$$\|u\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})} \leq C_\lambda\,. \tag{2.3}$$

Sketch of the Proof. We sketch the proof here: the details can be fond in [ILM10].

For a given $j \in \{0, 1, \dots, k\}$ one supposes, for sake of contradiction, that there exists a sequence $\{(u_n, \lambda_n, \tau_n)\}_{n\in\mathbb{N}}$ with u_n being a positive $\mathcal{C}^1$-solution of (Q_{j,λ_n,τ_n}), $\lambda_n > \widetilde{\lambda}$ and $\tau_n \geq 0$, such that $S_n := \max_{\overline{\Omega}} u_n = u_n(x_n) \xrightarrow[n\to\infty]{} \infty$, where $\{x_n\} \subset \Omega$ is a sequence of points where the maximum is attained. We also have $\tau_n \leq \lambda_1$, since no positive solution of $(Q_{j,\lambda,\tau})$ exists for $\tau > \lambda_1$.

One then makes a blow-up argument, that is, one defines a sequence w_n by suitably rescaling the solutions u_n and translating them in order to bring the maxima in the origin. Finally, one proves that, up to a subsequence, $w_n \to w$ in the $\mathcal{C}^1$ norm in compact sets, where w is a $\mathcal{C}^1$-function satisfying, in the weak sense,

$$\begin{cases} -\Delta_m w = w^{\sigma_j}, \\ w > 0, \\ w(0) = \max w = 1, \end{cases}$$

in $\mathbb{R}^N$ or in a half-space. This contradicts the Liouville-type theorem in [SZ02, Corollary II] in the case of $\mathbb{R}^N$ and in [Lor07, Zou08] for the half-space.

Observe that the limiting problem may be defined in a half-space because we are not supposing $\tau > 0$ at this point, then Lemma 1.4 does not apply and thus we do not know if the sequence of maxima x_n is bounded away from the boundary.

The contradiction proves that $\|u\|_\infty \leq C$ for any solution of Problem $(Q_{j,\lambda,\tau})$ with $\lambda > \widetilde{\lambda}$ and $\tau \geq 0$, that is, the item (1) of the Lemma, and also for any solution with a given λ and $\tau \geq 0$. In this second case, using the regularity theorem in [Lie88], one obtains the uniform bound also for the $\mathcal{C}^{1,\alpha}$ norm, as claimed in the item (2).

Finally, since we deal with a finite number of indexes j, we can make the constants not depend on this index. □

Now, we look for suitable supersolutions for the Problems $(Q_{j,\lambda,\tau})$ with $j = 1, \dots, k$: actually, the constant functions z_j are always supersolutions if $\tau = 0$, but we aim for a family of supersolutions which are near to these constants and are still supersolutions when τ is positive (and small). For this purpose, let $e \in W_0^{1,m}(\Omega)$ be the solution of

$$\begin{cases} -\Delta_m e = 1 & \text{in } \Omega \\ e = 0 & \text{on } \partial\Omega \end{cases}$$

and $n := \|e\|_\infty$.

Lemma 2.2. *Under hypothesis (F_2), for any $\lambda > 0$, $j \in \{1, \dots, k\}$ there exist $\tau^*_{j,\lambda}, \delta_{j,\lambda} > 0$ such that $\overline{u}_{j,\xi} = z_j + \xi + \frac{\delta_{j,\lambda}}{4n} e$ is a supersolution for $(Q_{j,\lambda,\tau})$ for any $\xi \in [-\delta_{j,\lambda}, \delta_{j,\lambda}/2]$ and $\tau \in [0, \tau^*_{j,\lambda})$. Moreover, we may choose $\delta_{j,\lambda}$ as nonincreasing functions of λ and such that $z_j - \delta_{j,\lambda} > R_{j-1}$.*

Proof. The proof is analogous to that of Lemma 3.2 in [ILM10]. In particular, the result is a consequence of $\sigma_j > m - 1$, which implies $\lim_{t \to z_j} \frac{f(t)}{|t - z_j|^{m-1}} = 0$. □

In view of Lemma 1.4, we give the following definition:

Definition 2.3. *We say that a family of nonnegative functions defined in Ω satisfies the δ_Ω-property if for every u in the family there exists a point $x \in \Omega$ such that $dist(x, \partial\Omega) \geq \delta_\Omega$ and $u(x) = \|u\|_\infty$, where δ_Ω is given in Lemma 1.4.*

Remark 2.4. *Under the hypotheses (F_1) and (F_2), if Ω is convex, then Lemma 1.4 implies that the family of the $\mathcal{C}^1(\overline{\Omega})$-weak solutions of the Problems $(Q_{j,\lambda,\tau})$ with $j \in \{0, , \dots, k\}$ and $\lambda, \tau > 0$, satisfies the δ_Ω-property. In the case $j = 0$ this is true also for $\tau = 0$.*

The following Lemma is crucial for our argument: it states that if we know that a family of solutions of the Problems $(Q_{j,\lambda,\tau})$ satisfy the δ_Ω-property, then their infinity norm must converge to the set of the zeros of f_j, when $\lambda \to \infty$. This fact will be used in order to prove that, for λ large, the solutions we obtain are distinct, and also in order to prove that they stay below the point where f_j is truncated, so that they are in fact solutions of the original Problem (P_λ). In the proof we have to combine the a-priori estimate in Lemma 2.1, the Liouville-type result in Lemma 1.5, and the δ_Ω-property which will guarantee that we converge to a limiting problem defined in $\mathbb{R}^N$ and not in a half-space, since this is necessary in order to apply Lemma 1.5.

Lemma 2.5. *Assume hypotheses (F_1), (F_2) and (F_4) or (F_5). For a given $j \in \{0, \dots, k\}$ if u_{j,λ_n,τ_n} are solutions of the corresponding Problems (Q_{j,λ_n,τ_n}) which satisfy the δ_Ω-property, and $\lambda_n \to \infty$ while $\tau_n \geq 0$ is bounded, then the sequence $\|u_{j,\lambda_n,\tau_n}\|_\infty$ has the limit set, for $n \to \infty$, contained in $\{0, z_1, \dots, z_j\}$.*

Proof. Let us denote $u_n = u_{j,\lambda_n,\tau_n}$ and let $x_n \in \Omega$ be such that $d_n := dist(x_n, \partial\Omega) \geq \delta_\Omega$ and $u_n(x_n) = \|u_n\|_\infty$.

Letting $w_n(x) = u_n(x_n + \lambda_n^{-\frac{1}{m}} x)$ we see that w_n satisfies

$$-\Delta_m w_n(x) = f_j(w_n) + \lambda_n^{-1} \tau_n (w_n^+)^{m-1} \quad \text{in} \quad B(0, d_n \lambda_n^{1/m}) \tag{2.4}$$

and $w_n(0) = u_n(x_n)$.

As in the proof of point (2) in Lemma 2.1, we obtain (since w_n is bounded in L^∞ by the point (1) in the same Lemma) also a uniform bound in the $\mathcal{C}^{1,\alpha}$ norm in compact sets, for some $\alpha \in (0,1)$; then, up to a subsequence, $w_n \to w$ in the $\mathcal{C}^1$ norm in compact sets, where now w is a $\mathcal{C}^1$ function defined in $\mathbb{R}^N$, since $d_n\lambda_n^{1/m} \to \infty$. Thus, w is a $\mathcal{C}^1(\mathbb{R}^N)$-weak solution of the Problem

$$\begin{cases} -\Delta_m w = f_j(w) \ \text{ in } \ \mathbb{R}^N, \\ w \geq 0 \,. \end{cases} \tag{2.5}$$

In view of the Hypotheses (F_2), (F_4) and (F_5), we can apply Lemma 1.5 to conclude that w is constant, that is, w must be a zero of f_j.

This proves that, up to a subsequence, $w_n(0) = u_n(x_n) = \|u_n\|_\infty \to z \in \{0, z_1, \dots, z_j\}$; since this is true for any subsequence, we proved our claim. □

We prove now a lemma which gives us two solutions of $Q_{j,\lambda,\tau}$ with $\tau > 0$, one below z_j and the other one exceeding it; the requirement is that we have a subsolution whose infinity norm is between z_{j-1} and z_j. One solution will be obtained by the sub- and supersolution method; for the second one we apply a topological degree argument, adapting a result obtained, for $m = 2$, by de Figueiredo and Lions in [dFL85], see also [ILS08, ILM10] for the general case.

Lemma 2.6. *Assume hypotheses (F_1), (F_2) and (F_3); fix $j \in \{1, \dots, k\}$ and $\lambda > 0$. Suppose that the Problems $(Q_{j,\lambda,\tau})$, $\tau \geq 0$ have a common subsolution $\underline{u} > 0$, satisfying $z_{j-1} < \|\underline{u}\|_\infty < R_{j-1}$.*

Then for a given $\tau_0 \in (0, \tau^*_{j,\lambda})$, Problem (Q_{j,λ,τ_0}) has 2 solutions u_{j,λ,τ_0}, v_{j,λ,τ_0} satisfying

$$z_{j-1} < \|\underline{u}\|_\infty \leq \|u_{j,\lambda,\tau_0}\|_\infty < z_j - \delta_{j,\lambda}/4 < z_j + \delta_{j,\lambda}/4 < \|v_{j,\lambda,\tau_0}\|_\infty \,.$$

Proof. For $\tau \in (0, \tau^*_{j,\lambda})$, we have the supersolution $\widetilde{u} = \overline{u}_{j,-\delta_{j,\lambda}} = z_j - \delta_{j,\lambda} + \frac{\delta_{j,\lambda}}{4n} e$ from Lemma 2.2; by construction, we have $\underline{u} < R_{j-1} < \widetilde{u} < z_j - \delta_{j,\lambda}/2$. Then the sub- and supersolutions method gives a solution u_{j,λ,τ_0} with the claimed properties.

In order to obtain the second solution we proceed as in [ILM10]. We denote by X the Banach space of the $\mathcal{C}^1$-functions on $\overline{\Omega}$ which are 0 on $\partial\Omega$, endowed with the usual $\mathcal{C}^1$-norm. Also, we will write $u << v$ to say that $u < v$ in Ω and $\frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu}$ on $\partial\Omega$, where ν denotes the unitary outward normal to $\partial\Omega$. Let L be given by (F_3) and $K_\tau : X \to X$ be defined as follows: $K_\tau v = u$, where u is the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta_m u + \lambda\, L u^{m-1} = \lambda f_j(v) + (\lambda L + \tau) v^{m-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,; \end{cases} \tag{2.6}$$

the mapping K_τ so defined is compact.

We consider the bounded open set

$$\mathcal{O} = \{u \in X : \, \|u\|_X < C_\lambda + B_\lambda + 1, \, u >> \underline{u}\}\,,$$

where $C_\lambda,\ B_\lambda > 0$ will be chosen below (see in (2.7) and (2.9), respectively)

We need that $0 \notin (I - K_\tau)(\partial\mathcal{O})$ (i.e, no solution of $(Q_{j,\lambda,\tau})$ lies on $\partial\mathcal{O}$), so that the degree $\deg(I - K_\tau, \mathcal{O}, 0)$ will be well defined and independent of τ. To obtain this we get C_λ from Lemma 2.1 part (2), so that

$$\|u\|_X \leq C_\lambda \tag{2.7}$$

for all possible solutions of $(Q_{j,\lambda,\tau})$ with $\tau \geq 0$.

Then, we claim that any solution u of $(Q_{j,\lambda,\tau})$ such that $u \geq \underline{u}$ in Ω satisfies $u >> \underline{u}$ (and then it is not on $\partial\mathcal{O}$).
Actually, we have, since $\tau > 0$ and $\underline{u} > 0$

$$\begin{cases} -\Delta_m u + \lambda L\, u^{m-1} & = \lambda f_j(u) + (\lambda L + \tau) u^{m-1} \\ -\Delta_m(\underline{u}) + \lambda L\, (\underline{u})^{m-1} & \leq \lambda f_j(\underline{u}) + \lambda L (\underline{u})^{m-1} < \lambda f_j(\underline{u}) + (\lambda L + \tau)(\underline{u})^{m-1}\,; \end{cases} \tag{2.8}$$

by hypothesis (F_3), using $\underline{u} \leq u$ and the comparison result in [dFGU09] (derived from [AR06, FPT94]), the claim is proved.

By the above computations, we obtain that

$$\deg(I - K_\tau, \mathcal{O}, 0) = 0 \qquad \text{for any } \tau > 0\,,$$

since $(Q_{j,\lambda,\tau})$ has no solutions for $\tau > \lambda_1$.

At this point we fix $\tau = \tau_0$, we consider the supersolution $\overline{u} := \overline{u}_{j,0} > z_j$ from Lemma 2.2, and we assume that no solution of (Q_{j,λ,τ_0}) touches it, otherwise such a solution would satisfy the claim we are trying to prove and then we would be finished. Using the L^∞ estimate in [Ana87] and then [Lie88] we obtain that we may choose the constant $B_\lambda > 0$ such that

$$\|K_\tau v\|_X \leq B_\lambda, \qquad \forall\, v \in X: \quad 0 \leq v \leq \overline{u}; \tag{2.9}$$

we consider the open subset of $\mathcal{O}$

$$\mathcal{O}' = \{u \in \mathcal{O} : \; u < \overline{u} \text{ in } \Omega\}$$

and we claim that $\deg(I - K_{\tau_0}, \mathcal{O}', 0) = 1$.

Observe that K_{τ_0} maps $\overline{\mathcal{O}'}$ into $\overline{\mathcal{O}'}$. Indeed, if $v \in \overline{\mathcal{O}'}$, then $\|K_{\tau_0} v\|_X \leq B_\lambda$ by (2.9), and if we consider $u = K_{\tau_0} v$ we have

$$\begin{cases} -\Delta_m \overline{u} + \lambda L\, \overline{u}^{m-1} & \geq \lambda f_1(\overline{u}) + (\lambda L + \tau_0)\overline{u}^{m-1}, \\ -\Delta_m u + \lambda L\, u^{m-1} & = \lambda f_1(v) + (\lambda L + \tau_0) v^{m-1}, \\ -\Delta_m(\underline{u}) + \lambda L\, (\underline{u})^{m-1} & \leq \lambda f_j(\underline{u}) + \lambda L (\underline{u})^{m-1} \leq \lambda f_j(\underline{u}) + (\lambda L + \tau_0)(\underline{u})^{m-1}\,, \end{cases} \tag{2.10}$$

then, since $\underline{u} \leq v \leq \overline{u}$, the comparison principle in [Tol83] implies that $\underline{u} \leq K_{\tau_0} v \leq \overline{u}$.

Now, let $u_0 \in \mathcal{O}'$ and consider the constant mapping $C : \overline{\mathcal{O}'} \to \overline{\mathcal{O}'}$ defined by $C(u) = u_0$: one obtains that $I - \mu K_{\tau_0}(v) - (1-\mu)u_0$, $\mu \in [0,1]$, is a homotopy between $I - K_{\tau_0}$ and $I - C$ in $\overline{\mathcal{O}'}$ without zeros on $\partial\mathcal{O}'$: in fact, if $v \in \partial\mathcal{O}'$ then (since $\mathcal{O}'$ is convex)

$\mu K_{\tau_0}(v) + (1-\mu)u_0 \in \mathcal{O}'$ for $\mu \neq 1$, and then it is different from v, while for $\mu = 1$ we have $v \neq K_{\tau_0}(v)$ since we are assuming that no solution touches $\overline{u}$.

Hence $\deg(I - K_{\tau_0}, \mathcal{O}', 0) = \deg(I - C, \mathcal{O}', 0) = 1$, as we claimed.

Then, applying the excision property, it follows that $\deg(I - K_{\tau_0}, \mathcal{O} \setminus \overline{\mathcal{O}'}, 0) = -1$, so (Q_{j,λ,τ_0}) has a solution $v_{j,\lambda,\tau_0} \in \mathcal{O} \setminus \overline{\mathcal{O}'}$; in particular, if x_0 is the maximum point of $\overline{u}$ then $v_{j,\lambda,\tau_0}(x_0) > \overline{u}(x_0) = z_j + \delta_{j,\lambda}/4$, since otherwise it would be on $\partial\mathcal{O}'$: we obtained the last estimate $\|v_{j,\lambda,\tau_0}\|_\infty > z_j + \delta_{j,\lambda}/4$. □

Lemma 2.6 allows us to obtain the following Lemma, which will be used to iteratively produce two solutions of Problem (P_λ), starting with a given solution.

Lemma 2.7. *Assume hypotheses* (F_1), (F_2), (F_3), *hypothesis* (F_4) *or* (F_5), *and the convexity of* Ω. *Suppose that, for some* $j \in \{1, \dots, k\}$, *there exists* $\Lambda_{j-1} > 0$ *such that there exists a solution* v_λ *of* $(Q_{j,\lambda,0})$ *for any* $\lambda > \Lambda_{j-1}$, *satisfying* $z_{j-1} < \|v_\lambda\|_\infty < R_{j-1}$.

Then there exists $\Lambda_j \geq \Lambda_{j-1}$ *such that* $(Q_{j,\lambda,0})$ *has two more solutions* $u_{j,\lambda,0}$ *and* $v_{j,\lambda,0}$, *satisfying*

$$R_{j-1} \leq \|u_{j,\lambda,0}\|_\infty \leq z_j - \delta_{j,\lambda}/4 < z_j + \delta_{j,\lambda}/4 \leq \|v_{j,\lambda,0}\|_\infty < R_j\,. \qquad (2.11)$$

Moreover $\|u_{j,\lambda,0}\|_\infty \to z_j^-$ *and* $\|v_{j,\lambda,0}\|_\infty \to z_j^+$ *as* $\lambda \to \infty$.

Proof. First we fix $\lambda' > \Lambda_{j-1}$, and we observe that $v_{\lambda'}$ is a subsolution for Problem $(Q_{j,\lambda,\tau})$ for any $\tau \geq 0$ and $\lambda > \lambda'$.

Then we apply Lemma 2.6 with this j and this subsolution. As a result we get two solutions $u_{j,\lambda,\tau}$ and $v_{j,\lambda,\tau}$ of $(Q_{j,\lambda,\tau})$ satisfying

$$z_{j-1} < \|v_{\lambda'}\|_\infty \leq \|u_{j,\lambda,\tau}\|_\infty < z_j - \delta_{j,\lambda}/4 < z_j + \delta_{j,\lambda}/4 < \|v_{j,\lambda,\tau}\|_\infty\,,$$

for every $\lambda > \lambda'$ and $\tau \in (0, \tau^*_{j,\lambda})$.

Since Ω is convex, these solutions satisfy the δ_Ω-property by Remark 2.4; our aim is to obtain solutions for the case $\tau = 0$ which also satisfy the δ_Ω-property: since this cannot be guaranteed directly by Remark 2.4, we will obtain them by taking limit in the case $\tau > 0$.

For a fixed value of $\lambda > \lambda'$, we consider a sequence $\tau_n \to 0$ and we focus on the solutions $u_n := u_{j,\lambda,\tau_n}$ (resp. $v_n := v_{j,\lambda,\tau_n}$).

By Lemma 2.1 point (2), we have a uniform bound for $\|u_n\|_{\mathcal{C}^{1,\alpha}(\overline{\Omega})}$ for some $\alpha \in (0,1)$. Then, up to a subsequence, $u_n \to u_{j,\lambda,0}$ and $v_n \to v_{j,\lambda,0}$ in $\mathcal{C}^1(\overline{\Omega})$, where $u_{j,\lambda,0}, v_{j,\lambda,0}$ are nonnegative weak solution of $(Q_{j,\lambda,0})$.
Since $z_{j-1} < \|v_{\lambda'}\|_\infty \leq \|u_n\|_\infty < z_j - \delta_{j,\lambda}/4$ we obtain $z_{j-1} < \|v_{\lambda'}\|_\infty \leq \|u_{j,\lambda,0}\|_\infty \leq z_j - \delta_{j,\lambda}/4$ (resp, $\|v_n\|_\infty > z_j + \delta_{j,\lambda}/4$ implies $\|v_{j,\lambda,0}\|_\infty \geq z_j + \delta_{j,\lambda}/4$). Thus $u_{j,\lambda,0}$ is nontrivial and distinct from $v_{j,\lambda,0}$.

Finally, we know by remark 2.4 that there exist $x_n \in \Omega$ such that $d_n := dist(x_n, \partial\Omega) \geq \delta_\Omega$ and $u_n(x_n) = \|u_n\|_\infty$ (resp. $v_n(x_n) = \|v_n\|_\infty$). Then, up to a further subsequence, $x_n \to x \in \Omega$ with $dist(x, \partial\Omega) \geq \delta_\Omega$ and taking limit $u_{j,\lambda,0}(x) = \|u_{j,\lambda,0}\|_\infty$ (resp. $v_{j,\lambda,0}(x) =$

$\|v_{j,\lambda,0}\|_\infty$). We have thus obtained solutions $u_{j,\lambda,0}$ and $v_{j,\lambda,0}$ of $(Q_{j,\lambda,0})$ satisfying the δ_Ω-property and such that

$$z_{j-1} < \|v_{\lambda'}\|_\infty \le \|u_{j,\lambda,0}\|_\infty \le z_j - \delta_{j,\lambda}/4 < z_j + \delta_{j,\lambda}/4 \le \|v_{j,\lambda,0}\|_\infty \, . \tag{2.12}$$

Then, we can apply Lemma 2.5 and, comparing with (2.12), we deduce that $\|u_{j,\lambda,0}\|_\infty \to z_j^-$ and $\|v_{j,\lambda,0}\|_\infty \to z_j^+$ as $\lambda \to \infty$.

Thus there exists Λ_j such that $R_{j-1} < \|u_{j,\lambda,0}\|_\infty , \|v_{j,\lambda,0}\|_\infty < R_j$ for $\lambda > \Lambda_j$, which completes the estimates in (2.11).

□

3 The proof of Theorem 1.1

In this section we prove Theorem 1.1. In order to apply Lemma 2.7 we need to have a first solution: this is obtained variationally via the Mountain Pass Theorem, actually, because of hypothesis (F_4), the origin is a minimum for the functional associated to our truncated problem.

Lemma 3.1. *Under the hypothesis* (F_1), (F_2) *and* (F_4), *if* Ω *is convex, there exists* $\Lambda_0 > 0$ *such that the Problem* $(Q_{0,\lambda,0})$ *has at least one positive solution* $v_{0,\lambda,0}$ *for* $\lambda > \Lambda_0$, *satisfying* $\|v_{0,\lambda,0}\|_\infty < R_0$.
Moreover, $\|v_{0,\lambda,0}\|_\infty \to 0^+$ *as* $\lambda \to \infty$.

Proof. The functional associated to Problem $(Q_{0,\lambda,0})$ is given by

$$I_\lambda(u) = \frac{\|u\|^m}{m} - \lambda \int_\Omega F_0(u).$$

where $F_0(u) = \int_0^u f_0(t)\,dt$. It is easy to verify that it satisfies the Mountain Pass geometry.

Actually, by the definition of R_0, (2.1) and (2.2), we get

$$(1-\varepsilon_0)\frac{t^{\sigma_0+1}}{\sigma_0+1} \le F_0(t) \le (1+\varepsilon_0)\frac{t^{\sigma_0+1}}{\sigma_0+1} \, ,$$

so that

$$\frac{\|u\|^m}{m} - \lambda\frac{1+\varepsilon_0}{\sigma_0+1}\int_\Omega u^{\sigma_0+1} \le I_\lambda(u) \le \frac{\|u\|^m}{m} - \lambda\frac{1-\varepsilon_0}{\sigma_0+1}\int_\Omega u^{\sigma_0+1};$$

since $\sigma_0 + 1 > m$, the norm dominates near the origin, which then is a minimum point, while the functional becomes negative in every direction far from the origin.

Finally, since the nonlinearity has a growth at infinity as t^{σ_0}, which is subcritical, it verifies the well-known Ambrosetti–Rabinowitz condition; then we may apply the Mountain Pass Theorem to obtain a positive solution $v_{0,\lambda,0}$ of Problem $(Q_{0,\lambda,0})$.

By remark 2.4 we have that $v_{0,\lambda,0}$ satisfies the δ_Ω-property, then by Lemma 2.5 we obtain that $\|v_{0,\lambda,0}\|_\infty \to 0^+$ as $\lambda \to \infty$.

As a consequence, there exists $\Lambda_0 > 0$ such that $\|v_{0,\lambda,0}\|_\infty < R_0$ for every $\lambda > \Lambda_0$. □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. The proof is by induction, where Lemma 3.1 gives the starting point and Lemma 2.7 is the induction step.

In fact, by Lemma 3.1 there exists one solution of Problem $(Q_{0,\lambda,0})$ satisfying $0 < \|v_\lambda\|_\infty < R_0$, for any $\lambda > \Lambda_0$; then it is also a solution of the Problems $(Q_{1,\lambda,0})$ and (P_λ).

Thus, by applying Lemma 2.7 k times, starting with $j = 1$, we obtain two more solutions at each step; in view of the estimate (2.11), these two solutions are always distinct from the previous ones, are also solutions of Problem (P_λ), and the larger one will serve as v_λ when applying again Lemma 2.7. So we obtain a total of $2k+1$ positive solutions of Problem (P_λ), for $\lambda > \Lambda_k$. The convergence result also comes from Lemma 2.7

□

4 The proof of Theorem 1.2

In order to prove Theorem 1.2 we need again at least one solution that allows to apply Lemma 2.7. In this case, we fist obtain a subsolution, whose existence is guaranteed by Hypothesis (F_5), and then, using Lemma 2.6, we obtain two solutions.

Lemma 4.1. *Under Hypotheses (F_1), (F_2) and (F_5), given $\overline{\lambda} > \lambda_1$ there exists $\varepsilon > 0$ such that $\varepsilon\phi_1 < R_0$ and $\varepsilon\phi_1$ is a subsolution for Problem $(Q_{1,\lambda,\tau})$ for any $\tau \geq 0$ and $\lambda > \overline{\lambda}$.*

Proof. In standard way, using (F_5), we may find a $\varepsilon > 0$ (as small as desired and depending only on $\overline{\lambda}$) such that $\lambda f_1(t) > \lambda_1 t^{m-1}$ for any $t \in (0, \max\{\varepsilon\phi_1\})$ and any $\lambda > \overline{\lambda}$; then $\varepsilon\phi_1$ is a subsolution for Problem $(Q_{1,\lambda,\tau})$ for any $\tau \geq 0$ and $\lambda > \overline{\lambda}$. □

Lemma 4.2. *Under Hypotheses (F_1), (F_2), (F_3) and (F_5), if Ω is convex, there exists Λ_1 such that the Problem $(Q_{1,\lambda,0})$ has at least two positive solutions $u_{1,\lambda,0}$ and $v_{1,\lambda,0}$ for $\lambda > \Lambda_1$, satisfying*

$$0 < \|\varepsilon\phi_1\|_\infty \leq \|u_{1,\lambda,0}\|_\infty \leq z_1 - \delta_{1,\lambda}/4 < z_1 + \delta_{1,\lambda}/4 \leq \|v_{1,\lambda,0}\|_\infty < R_1 \,.$$

Moreover, $\|u_{1,\lambda,0}\|_\infty \to z_1^-$ and $\|v_{1,\lambda,0}\|_\infty \to z_1^+$ as $\lambda \to \infty$.

Proof. By Lemma 4.1 we are in the position to apply Lemma 2.6 with $j = 1$ and the subsolution $\underline{u} = \varepsilon\phi_1$.

As a result we get, for every $\lambda > \overline{\lambda}$ and $\tau \in (0, \tau^*_{1,\lambda})$, solutions $u_{1,\lambda,\tau}$ and $v_{1,\lambda,\tau}$ of $(Q_{1,\lambda,\tau})$ satisfying $0 < \|\varepsilon\phi_1\|_\infty \leq \|u_{1,\lambda,\tau}\|_\infty < z_1 - \delta_{1,\lambda}/4 < z_1 + \delta_{1,\lambda}/4 < \|v_{1,\lambda,\tau}\|_\infty$

Since Ω is convex, these solutions satisfy the δ_Ω-property by Remark 2.4.

Then, by reasoning as in the proof of Lemma 2.7 we may take the limit for $\tau \to 0$ and we obtain solutions $u_{1,\lambda,0}$ and $v_{1,\lambda,0}$ of $(Q_{1,\lambda,0})$ satisfying

$$0 < \|\varepsilon\phi_1\|_\infty \leq \|u_{1,\lambda,0}\|_\infty \leq z_1 - \delta_{1,\lambda}/4 < z_1 + \delta_{1,\lambda}/4 \leq \|v_{1,\lambda,0}\|_\infty \,, \tag{4.1}$$

which also satisfy the δ_Ω-property.

Then we may apply Lemma 2.5. Since $\|u_{1,\lambda,0}\|_\infty$ cannot tend to zero by (4.1), we deduce that $\|u_{1,\lambda,0}\|_\infty, \|v_{1,\lambda,0}\|_\infty \to z_1$, in fact, $\|u_{1,\lambda,0}\|_\infty \to z_1^-$ and $\|v_{1,\lambda,0}\|_\infty \to z_1^+$ by (4.1).

As a consequence, there exists Λ_1 such that $u_{1,\lambda,0}, v_{1,\lambda,0} < R_1$ for $\lambda > \Lambda_1$. □

Now we are in the position to prove Theorem 1.2.

Proof of Theorem 1.2. The proof is again by induction, where now the starting point is given by Lemma 4.2.

In fact, by Lemma 4.2 there exist two distinct solutions of Problem $(Q_{1,\lambda,0})$: $u_{1,\lambda,0}$, $v_{1,\lambda,0}$, for $\lambda > \Lambda_1$, where $z_1 < \|v_{1,\lambda,0}\|_\infty < R_1$; then they are also solutions of the Problems $(Q_{2,\lambda,0})$ and (P_λ).

Thus, as in the proof of Theorem 1.1, we apply the Lemma 2.7 $k-1$ times, starting with $j = 2$, and we obtain a total of $2k$ positive solutions of Problem (P_λ), for $\lambda > \Lambda_k$, and the convergence result. □

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