On the Fučík spectrum

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A superlinear equation

$$\begin{cases} -u'' = \lambda u + g(x,u) + h(x) & in\ (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{1.1}$$

- $g \in \mathcal{C}^0([0,1] \times \mathbb{R})$,
- $\lim_{s\to-\infty} \frac{g(x,s)}{s} = 0, \quad \lim_{s\to+\infty} \frac{g(x,s)}{s} = +\infty$ uniformly with respect to $x \in [0,1]$
- $h \in L^2(0,1)$,
- Some more Technical hypotheses to achieve PS condition

Known results:

- For $\lambda < \lambda_1$: Ambrosetti-Prodi (72): 0-1-2 solutions depending on h.
- For $\lambda \in \left(\lambda_1, \frac{\lambda_2}{4}\right)$: de Figueiredo-Ruf (91), Villegas (98): existence $\forall\ h$

My problem in 2001: what for $\lambda > \lambda_2/4$?

Partial answer: if $\lambda \in \left(\frac{\lambda_2}{4}, \frac{\lambda_2}{4} + \varepsilon\right)$, existence $\forall\ h$

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The classical Fučík spectrum

Given the problem

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & in \ \Omega \\ Bu = 0 & in \ \partial\Omega \end{cases} . \qquad \text{(PF)}$$

The Fučík spectrum (First introduced by Fučík and Dancer in 1976-77):

$$\Sigma_{eq} = \left\{ (\lambda^+, \lambda^-) \in \mathbb{R}^2 \quad \text{such that (PF) has nontrivial solutions} \right\} .$$

(Here Ω is a bounded domain in $\mathbb{R}^n$, $u^\pm(x) = \max\{0, \pm u(x)\}$ and $Bu = 0$ represents Dirichlet or Neumann boundary conditions).

Fučík spectrum: PDE case

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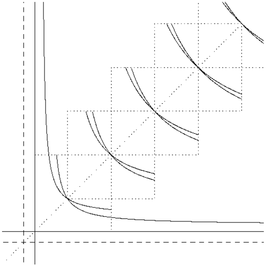

Known parts:

1. • trivial part $\lambda^{\pm} = \lambda_1$,
   • nontrivial part in $\lambda^{\pm} > \lambda_1$;
2. near the diagonal:
   in $(\lambda_{k-1}, \lambda_{k+1})^2$, two curves through (λ_k, λ_k),
   in $(\lambda_{k-1}, \lambda_k)^2$ and $(\lambda_k, \lambda_{k+1})^2$, empty;
3. first nontrivial curve (obtained variationally).

[Gallouët-Kavian (81), Ruf (81), Magalhães (90), de Figueiredo-Gossez (94), Cuesta-de Figueiredo-Gossez (99)]

Fučík spectrum: ODE Dirichlet case

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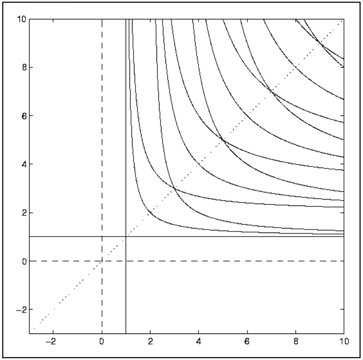
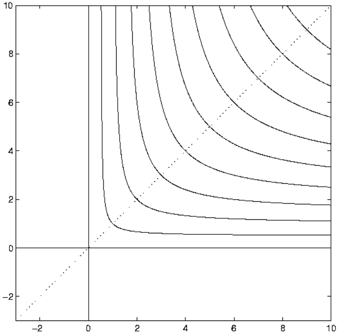


$$\Sigma_{2i} \quad : \quad \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda^+}} + \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda^-}} = 1$$

$$\Sigma^+_{2i-1} \quad : \quad \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda^+}} + \frac{(i-1)\sqrt{\lambda_1}}{\sqrt{\lambda^-}} = 1$$

$$\Sigma^-_{2i-1} \quad : \quad \frac{(i-1)\sqrt{\lambda_1}}{\sqrt{\lambda^+}} + \frac{i\sqrt{\lambda_1}}{\sqrt{\lambda^-}} = 1$$

Fučík spectrum: ODE Neumann/Periodic case



$$\Sigma_k \quad : \quad \frac{(k-1)\sqrt{\lambda_2}}{2\sqrt{\lambda^+}} + \frac{(k-1)\sqrt{\lambda_2}}{2\sqrt{\lambda^-}} = 1$$

- Periodic superlinear problem (1.1), existence $\forall h$ for $\lambda \in \left(\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4}\right)$: de Figueiredo-Ruf (93)
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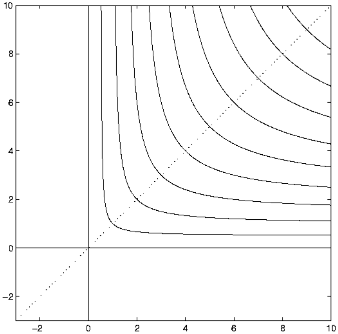

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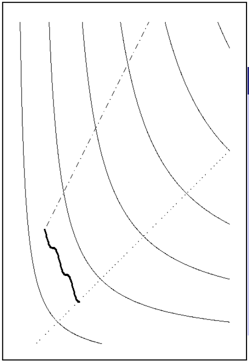
Problem (1.1) solved



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Massa (04)

Theorem

Let $(\alpha^+,\alpha^-) \notin \Sigma$ with $\alpha^+ \geq \alpha^-$ be such that $\exists a \in (\lambda_k, \lambda_{k+1})$ and a $\mathcal{C}^1$ function $\alpha : [0,1] \to \mathbb{R}^2$ such that:

a) $\alpha(0) = (a,a)$, $\alpha(1) = (\alpha^+,\alpha^-)$;

b) $\alpha([0,1]) \cap \Sigma = \emptyset$.

Then we can find and characterize one intersection of the Fučík spectrum with the halfline $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$, for each value of $r \in (0,1]$.

Theorem

Under the given hypotheses, if $\lambda \in (\frac{\lambda_k}{4}, \frac{\lambda_{k+1}}{4})$ for some $k \geq 1$, then there exists a solution of problem (1.1) for all $h \in L^2(0,1)$.

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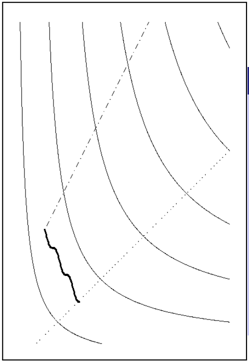

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Some variations of the Fučík spectrum

- *p*-Laplacian: Drabek (92), Cuesta-de Figueiredo-Gossez (99), Reichel-Walter (99), Micheletti-Pistoia (01), Perera (04), others..
- with weights: $-\Delta u = \lambda^+ m(x) u^+ - \lambda^- n(x) u^- \quad in\ \Omega$: Alif-Gossez (01) Arias-Campos-Cuesta-Gossez (02), others..
- higher order operators: Campos-Dancer (01) Rynne (01).
- nonlinearity in the boundary condition: $\lambda^+ u^+ - \lambda^- u^- = \partial u / \partial n \quad in\ \partial\Omega$: Martinez-Rossi (04)
- Systems: Massa-Ruf (06-07)

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A Fučík spectrum for a system

Given the problem:

$$\begin{cases} -\Delta u = \lambda^+ v^+ - \lambda^- v^- & \text{in } \Omega \\ -\Delta v = \lambda^+ u^+ - \mu^- u^- & \text{in } \Omega \\ Bu = Bv = 0 & \text{in } \partial\Omega \end{cases} \tag{PFS}$$

We define the Fučík Spectrum for this system as

$$\Sigma_{sy} = \left\{ (\lambda^+, \lambda^-, \mu^-) \in \mathbb{R}^3 \quad \text{such that (PFS) has nontrivial solutions} \right\} .$$

Some results for systems:

- Trivial part: when u, v do not change sign;
- "scalar" points:
  If $(\lambda^+, \lambda^-) \in \Sigma_{eq}$, then $(\lambda^+, \lambda^-, \lambda^-) \in \Sigma_{sy}$ and $u = v$ (in particular $(\lambda_k, \lambda_k, \lambda_k) \in \Sigma_{sy}$);
- the Fučík spectrum for the system is much richer than that for one equation: not every point is like above: u, v independent. (In contrast with the linear spectrum: $u = \pm v$)

Some results for systems

Local surfaces near λ_k

Through each point $(\lambda_k, \lambda_k, \lambda_k) \in \Sigma_{sy}$ (λ_k simple) there pass two (maybe coincident) "Fučík surfaces", in Σ_{sy}, parameterized by $\lambda^+ = \lambda_{k+}(\lambda^-, \mu^-)$ and $\lambda^+ = \lambda_{k-}(\lambda^-, \mu^-)$ with $|\lambda^- - \lambda_k|, |\mu^- - \lambda_k|$ small enough.

(topological degree and Lyapunov-Schmidt reduction:, similar to Ruf (81))

Some results for systems

A global surface near λ_2

We can find and characterize variationally a surface in Σ_{sy} of the form

$$(\lambda_1 + d(r,s), \lambda_1 + s\, d(r,s), \lambda_1 + r\, d(r,s)) : \quad r, s \in (0, +\infty)\,,$$

which pass through $(\lambda_2, \lambda_2, \lambda_2)$

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Results for ODE systems

In the ODE case: no explicit description but..

- for each $k \geq 2$, there exists two (may be coincident) $\mathcal{C}^1$ global surfaces in Σ_{sy} passing through $(\lambda_k, \lambda_k, \lambda_k)$, corresponding to solutions with $k-1$ zeros, starting positive (resp. negative).
  In particular, they coincide for Neumann and for Dirichlet if k is even, they are distinct for Dirichlet if k is odd (as scalar problem). JMP1
- All nontrivial points in Σ_{sy} belong to one of these surfaces.

The problem on a Torus

Consider Fučík spectrum of $-\Delta$ on the torus $T^2 = (0,1) \times (0,r)$, that is, for the problem

$$\begin{cases} -\Delta u = \lambda^+ u^+ - \lambda^- u^- & \text{in } \mathbb{R}^2 \\ u(x,y) = & u(x+1,y) = u(x,y+r), \end{cases}$$

By separation of variables one obtain all the linear eigenvalues:

$$\lambda_k := \lambda_{i,j} = i^2 4\pi^2 + j^2 4\pi^2 / r^2 \ , \ i,j = 0,1,2,\dots$$

and also explicit global curve in Σ given by:

$$\Sigma_k^{\text{expl}} : \quad \frac{1}{\sqrt{\lambda^+}} + \frac{1}{\sqrt{\lambda^-}} = \frac{2}{\sqrt{\lambda_k}}$$

Remark: Qualitatively like Periodic problem in one variable.

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Variational curves

Theorem

From each eigenvalue (λ_k, λ_k)*:* $k \geq 2$*, emanates a global branch*

$$\Sigma_k^{\text{var}} \subset \Sigma$$

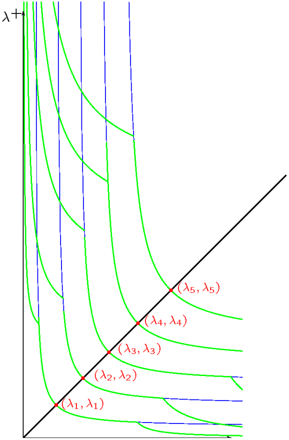
which can be characterized variationally

Proof: similar to de Figueiredo-Ruf (93):
Using the invariance of the solutions under the action of the compact group T^2 and an **index** for these actions (Marzantowicz (1989)).

A surprising result

- The variational and explicit curves coincide near to the diagonal (not surprising: here there is only one curve);
- They do not coincide globally: in fact, all the variational branches have $\lambda_1 \times \mathbb{R}$ as asymptote (the explicit ones do not). (Cuesta)-de Figueiredo-Gossez (94-99) had proven this (variationally) for the first nontrivial curve. JMP1

As a result, many crossings have to occur.



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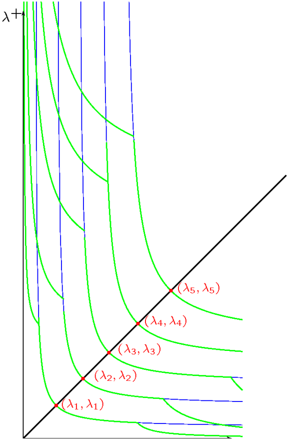

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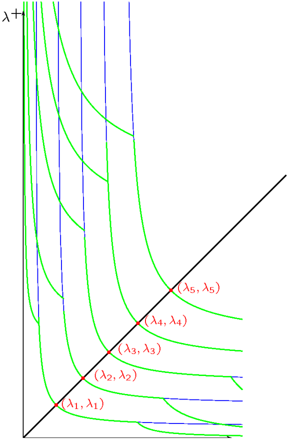

Sketch of the Proof of $\lambda^- \to \lambda_1$

Let

$$\mu = \lambda^+ - \lambda^-, \qquad S = \{u \in H^1_{(0)}(\Omega) : \int_\Omega u^2 = 1\}.$$

The variational characterization is

$$\lambda_k^- = \inf_{A \in \mathcal{A}_k} \sup_{u \in A} \int_\Omega |\nabla u|^2 - \mu \int_\Omega (u^+)^2 .$$

In Cuesta-de Figueiredo-Gossez (99)

$\mathcal{A}_1 = \{$paths in S joining $\pm\phi_1\}$
One shows that $\lambda_1^- \to \lambda_1$ when $\mu \to \infty$ by building a path where the sup is near λ_1: if $n \geq 2$ this path can be built by normalized linear combinations of ϕ_1 with an unbounded H^1 function.

In our case, $\mathcal{A}_k = \{T^2$ invariant sets $A \subseteq S$ such that $\gamma_{T^2}(A) \geq k\}$.
We build a suitable set A containing functions that are normalized linear combinations of $\phi_1 (= const)$ with k spikes concentrating near arbitrary points of T^2.

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A recent result

Recently variational curves having $\lambda_1 \times \mathbb{R}$ as asymptote have been found in much more general settings, in Molle Passaseo (C.R.2013)

- For the Dirichlet problem for $-\Delta$, with $N \geq 2$, they build a sequence of variational curves in Σ such that:
  - each curve has $\lambda_1 \times \mathbb{R}$ as asymptote,
  - each curve is associated to non-trivial solutions having k bumps,
  - the bumps concentrate near the maximum points of ϕ_1, but remain distinct.
- For the Neumann problem, they obtain a similar result, but the bumps concentrate near the boundary.

The proof once again exploit a variational characterization.

Remark: These curves are obtained only far from the diagonal.

Several other authors worked on Fučík spectrum: some of them (in no particular order) are

Schechter, Domingos, Ramos, Ben-Naoum, Fabry, Smets, Costa, Anane, Conti, Terracini, Verzini, Horák, Reichel, Castro, Chang, Omari, Tehrani...