Concave-convex behavior for a Kirchhoff type equation with degenerate nonautonomous coefficient

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Abstract. In this paper we study positive solutions for the Kirchhoff type equation

$$-M(x, \|u\|^2)\Delta u = \lambda f(u)$$

with Dirichlet boundary conditions in a bounded domain Ω, where $\|\cdot\|$ is the norm in $H_0^1(\Omega)$ and f, M are suitable functions.

The problem is nonvariational since the nonlocal coefficient M, possibly degenerate, depends on the point $x \in \Omega$. We show that these properties of M can produce interesting phenomena, even with simple homogeneous right hand sides, providing existence, nonexistence, and multiplicity results, due to the fact that the rate of growth with respect to u on the left hand side may change in Ω.

Several model examples are given, including one where M takes the form of the original Kirchhoff coefficient for the elastic string, but with nonhomogeneous material.

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1. Introduction

Our aim in this paper is to study problems of the form

$$\begin{cases} -M(x,\|u\|^2)\Delta u = \lambda f(u) & in \ \ \Omega\,, \\ u>0 & in \ \ \Omega\,, \\ u=0 & on \ \ \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded smooth domain, $\|\cdot\|$ is the norm in $H_0^1(\Omega)$, and f, M are suitable functions.

The main results of the paper are the Theorems 2.6 and 2.7, where we give conditions for the existence, nonexistence and multiplicity of solutions for Problem (1.1), in terms of the behavior of the nonlinearity and of the nonlocal term M. When the nonlinearity is a pure power a more precise result is given in Theorem 2.9.

The distinguishing property of Problem (1.1) is the presence of the Kirchhoff type coefficient M, which can also depend on the point $x \in \Omega$. Because of this term, Problem (1.1) is nonlocal and, due to the dependence on x, it is also non variational, which means that we cannot rely directly on many techniques used in previous works.
On the other hand, the dependence on x of the term M can produce new phenomena, even with simple (homogeneous or ever linear) right hand sides, due to the fact that the rate of growth with respect to u on the left hand side may change in Ω.

It is also worth mentioning that the most interesting results will be obtained when the nonlocal term M is allowed to be degenerate, that is, to reach or tend to zero. This situation has been only recently considered in literature. We remand to Section 2.2 for a further discussion of the bibliography related to our problem.

In order to put into perspective our results, we may consider the model problem

$$\begin{cases} -M(x,\|u\|^2)\Delta u = \lambda u^{p-1} & in \ \ \Omega\,, \\ u>0 & in \ \ \Omega\,, \\ u=0 & on \ \ \partial\Omega\,, \end{cases} \tag{1.2}$$

where $1<p<2^*=\frac{2N}{N-2}$ (or any $p>1$ if $N=1,2$).

Observe that, when $M\equiv 1$ (the local case) and $p\neq 2$, Problem (1.2) admits a positive solution for every $\lambda>0$. A first consequence of our results is that this is still true if M is bounded and bounded away from zero (see Theorem 2.9 case (a)).
Something similar happens when M only depends on s and is homogeneous: if the differential equation in (1.2) is

$$-\|u\|^{\sigma-2}\Delta u = \lambda u^{p-1}\,, \tag{1.3}$$

then (see Example 3.1) a positive solution exists for every $\lambda>0$ if either $\sigma>p$ or $\sigma<p$. In particular, as in the local case $\sigma=2$, the solution is expected to be small when λ is large and large when λ is small, if $p>\sigma$, while the opposite happens if $p < \sigma$. In this simple example, which is still variational, this can be seen by applying minimization and Mountain Pass Theorem, but it will also be a consequence of Theorem 2.9 case (a), see also Proposition 2.1.

However, Theorem 2.9 also extends this result to more general functions M, even depending on x, provided their behavior for small and for large norms compares with the term u^{p-1} in the same qualitative way. Moreover, Theorem 2.7 considers more general nonlinearities f, in this case comparing M with the growth of f at infinity or at zero. On the other hand, the cases (b) and (c) in the Theorems 2.7 and 2.9 show that different situations can arise, also providing nonexistence and multiplicity results, in particular when the interaction of M with the nonlinearity is different in different parts of Ω: see several model problems in Section 3.

A phenomenon similar to the one described above was studied recently in [MR18]: its authors considered a Dirichlet problem for the quasilinear equation

$$-\Delta_{p(x)} u = \lambda u^{q-1}, \tag{1.4}$$

where $p(x)$ is a discontinuous function taking the values 2 and p in two complementary subsets of Ω. They were able to prove, under suitable conditions, that if $2 < q < p < 2^*$, then at least two positive solutions exist for almost every λ in a set of the form $(0, \Lambda)$, that is, for small values of $\lambda > 0$. Such a multiplicity result is not usually expected with a homogeneous right hand side, and is instead typical of the so called concave-convex problems: for instance, for the equation

$$-\Delta u = \lambda u^{q-1} + u^{p-1} \tag{1.5}$$

when $1 < q < 2 < p < 2^*$. The interest and the novelty of the result in [MR18] is indeed that it finds such concave-convex behavior, in a situation where the nonlinearity is homogeneous, so that the multiplicity of solutions is produced instead by the operator itself, which has a different homogeneity in different regions. In fact, the condition $2 < q < p$ means that the nonlinearity u^{q-1} in (1.4) has a superlinear behavior with respect to the Laplacian, but a p-sublinear one with respect to the p-Laplacian.

One of the aims of this paper is to investigate the same kind of concave-convex behavior, but produced instead by the nonlocal and nonautonomous term M, that is, to obtain two solutions for small values of the parameter λ in Problem (1.2), if M is such that the left hand side has a faster (resp. slower) growth than the right hand side, in different regions of Ω, or at different values of the norm of the solution: this is actually the behavior obtained in Theorem 2.9 point(b). A simple model for this situation is the equation

$$-\|u\|^{\sigma(x)-2} \Delta u = \lambda u^{p-1}, \tag{1.6}$$

with $\sigma(x) > p$ in some region and $\sigma(x) < p$ in another (see Example 3.2).

The paper is structured as follows. In the next Section 2, we state our main results for Problem (1.1), while in Section 2.1 we resume the particular case of Problem (1.2). These results are proved in Section 4. In Section 2.2 we present the related bibliography, while in Section 3 we discuss several model problems.

2. Main results

In this section we state our main results for Problem (1.1).

Throughout the paper we will denote by $\|\cdot\|$ the norm in $H_0^1(\Omega)$ and by $\|\cdot\|_q$ the norm in $L^q(\Omega)$; we will usually omit the indication of the set Ω in the notation for the space. Also, solutions will always be intended as positive, even when not explicitly stated. We denote by $\lambda_1 > 0$ the first eigenvalue and eigenfunction of the Laplacian in Ω and by $\varphi_1 > 0$ the corresponding eigenfunction, with the normalization $\|\varphi_1\| = 1$.

We will first state the following conditions on the nonlinearity f and its primitive $F(t) = \int_0^t f(\tau)d\tau$:

(f_1) $f : \mathbb{R} \to [0, \infty)$ is continuous, $f(t) = 0$ for $t \le 0$ and there exists $t_0 > 0$ such that $f(t) > 0$ for $t \in (0, t_0)$;

(f_2) there exist a constant C_f and $p < 2^*$ such that $f(t) \le C_f(1 + t^{p-1})$ for $t > 0$.

For p as in hypothesis (f_2), we define the two conjugate exponents

$$\begin{cases} r = \frac{2^*}{2^*-p} = \frac{N}{p+N(1-p/2)} \in \left(\frac{2N}{N+2}, \infty\right), \\ r' = \frac{2^*}{p} \in (1, 2^*), \end{cases} \tag{2.1}$$

where if $N = 1$ one can take $r = 1$, and if $N = 2$ one can take any $r > 1$.

We consider the following conditions on M (below the notation M^{-1} means the reciprocal $1/M$ of the real number M). Let $A, B \in [0, \infty]$ with $A < B$:

(M_1) $M : \Omega \times (A, B) \to [0, \infty)$ and, for every $s \in (A, B)$, $M(x, s^2) > 0$ for almost every $x \in \Omega$;

(M_2) there exists $\widetilde{r} > r$ such that $M^{-1}(\cdot, s^2) \in L^{\widetilde{r}}(\Omega)$ for every fixed $s \in (A, B)$;

(M_3) the map $(A, B) \to L^r(\Omega) : s \mapsto M^{-1}(\cdot, s^2)$ is continuous.

We state our first result which deals with existence of solutions for Problem (1.1).

Proposition 2.1. *Assume the conditions (f_1), (f_2), (M_1) and (M_2). Then Problem (1.1) admits a positive solution $u \in H_0^1(\Omega)$ of norm $\|u\| = s$, for*

$$\lambda = \lambda_s = \frac{s^2}{\int_\Omega M^{-1}(\cdot, s^2) f(u) u}, \qquad s \in (A, B), \tag{2.2}$$

where u is a maximizer of the problem

$$\Theta_m^s := \sup_{v \in S^s} \int_\Omega m(x) F(v), \qquad S^s = \left\{v \in H_0^1 : \|v\| = s\right\}, \tag{2.3}$$

with $m = M^{-1}(\cdot, s^2)$.

In particular, for every $s \in (A, B)$*, there exists a couple* $(u_s, \lambda_s) \in H_0^1 \times \mathbb{R}$ *with* $u_s > 0$*,* $\lambda_s > 0$ *and* $\|u_s\| = s$*, that satisfies Problem* (1.1)*.*

In the following Theorem we give nonexistence results.

Theorem 2.2. *Assume the conditions of Proposition 2.1.*

(i) Suppose

$$\begin{cases} \dfrac{f(t)}{t} \le D_f & \text{for } t > 0, \\ M(x, s^2) \ge \delta_M \ge 0 & \text{for } (x, s) \in \Omega \times (A, B), \end{cases}$$

then then there exists no positive solution of Problem (1.1) *for* $\lambda < \dfrac{\lambda_1}{D_f}\delta_M$ *.*

Suppose

$$\begin{cases} \dfrac{f(t)}{t} \ge \delta_f > 0 & \text{for } t > 0, \\ M(x, s^2) \le D_M & \text{for } (x, s) \in \Omega \times (A, B), \end{cases}$$

then there exists no positive solution of Problem (1.1) *for* $\lambda > \dfrac{\lambda_1}{\delta_f} D_M$ *.*

(ii) Suppose that for suitable $p_0, p_\infty \in [1, p]$ *and* $D > 0$ *it holds*

$$\begin{cases} f(t) \le D(t^{p_0 - 1} + t^{p_\infty - 1}), \\ \left\| M^{-1}(\cdot, s^2) \right\|_r \le \dfrac{D s^2}{s^{p_0} + s^{p_\infty}}, \end{cases} \tag{2.4}$$

then there exists $\overline{\Lambda} > 0$ *such that Problem* (1.1) *has no positive solution for* $\lambda \in (0, \overline{\Lambda})$*.*

We define now, for s_*, p_* to be specified later, the following two conditions which will play an important role in the behavior of Problem (1.1):

$$(C+) \qquad \lim_{s \to s_*} \int_{\Omega_*} s^{p_* - 2} M^{-1}(x, s^2) = \infty \text{ for some } \Omega_* \subset\subset \Omega;$$

$$(C-) \qquad \lim_{s \to s_*} \left\| s^{p_* - 2} M^{-1}(\cdot, s^2) \right\|_r = 0.$$

In the following two Theorems, by assuming some additional conditions on f and M, we will be able to control the behavior of the parameter λ_s as defined in (2.2), for s near the endpoints A, B. This study will be fundamental in order to obtain the main results of this work.

Theorem 2.3. *Assume the same hypotheses of Proposition 2.1.*
Consider couples $(\lambda_s, u_s)_{s \in (A,B)}$ *satisfying Problem* (1.1) *as obtained in Proposition 2.1, and let* $s \to s_* = A$ *(or* B*). Then* $\lambda_s \to \infty$ *while* $\|u_s\| = s \to s_*$ *in the following cases:*

α) if $0 < s_* < \infty$ *and* $(C-)$ *holds true (in this case* p_* *is not relevant);*
β) if $s_* = 0$*,*

($f\pi_0$) *there exists* $\pi_0 \le p$ *such that* $\limsup_{t\to 0} \frac{f(t)}{t^{\pi_0-1}} < \infty$
and ($C-$) *holds true with* $p_* = \pi_0$*;*

γ) if $s_* = \infty$ *and* ($C-$) *holds true with* $p_* = p$*.*

Theorem 2.4. *Assume the same hypotheses of Proposition 2.1 and the additional condition*

(f_3) *There exists* $\delta > 0$ *such that* $f(t)t > \delta F(t)$ *for every* $t > 0$*.*

Consider couples $(\lambda_s, u_s)_{s\in(A,B)}$ *satisfying Problem* (1.1) *as obtained in Proposition 2.1, and let* $s \to s_* = A$ *(or* B*). Then* $\lambda_s \to 0$ *while* $\|u_s\| = s \to s_*$ *in the following cases:*

α) if $0 < s_* < \infty$ *and* ($C+$) *holds true (in this case* p_* *is not relevant);*

β) if $s_* = 0$*,*

(fp_0) *there exists* p_0 *such that* $\liminf_{t\to 0} \frac{F(t)}{t^{p_0}} > 0$
and ($C+$) *holds true with* $p_* = p_0$*;*

γ) if $s_* = \infty$*,*

(fp_∞) *there exists* p_∞ *such that* $\liminf_{s\to \infty} \frac{F(t)}{t^{p_\infty}} > 0$
and ($C+$) *holds true with* $p_* = p_\infty$*.*

In order to obtain a more precise picture of the solvability of Problem (1.1), a key ingredient is to prove that equation (2.2) defines a continuous function λ_s. Unfortunately this continuity cannot be always guaranteed. We have the following result.

Proposition 2.5. *Assume the conditions of Proposition 2.1.*

- *If also condition* (M_3) *holds true, then* $\Theta^s_{M^{-1}(\cdot,s^2)}$ *defined in* (2.3) *is a continuous function of* s *in* (A, B)*.*
- *If moreover one of the following two conditions holds:*

(H_{pp}) $f(t) = t^{p-1}$ *for* $t > 0$*,*

(H_{dc}) *(i) Condition* (M_2) *holds with* $\widetilde{r} = \infty$*,*
(ii) Condition (f_2) *holds with* $p < 2$*,*
(iii) $f(t)/t$ *is strictly decreasing for* $t > 0$*,*

then λ_s *defined in* (2.2) *is a continuous function of* s *in* (A, B)*.*

We can finally state the main results about Problem (1.1). In the general case, where we do not have the continuity of λ_s, we can only deduce from the Theorems 2.3 and 2.4 the existence of solutions with the parameter λ arbitrarily small or arbitrarily large, but we cannot exclude that holes exist in the set of such values.

Theorem 2.6. *Assume the conditions of Proposition 2.1.*

- *If one of the three conditions in Theorem 2.3 holds at* A *(resp. at* B*), then at least one positive solution of Problem* (1.1) *exists for an infinite number of, arbitrarily large, values of* λ*.*

- *If hypothesis* (f_3) *holds true and one of the three conditions in Theorem 2.4 holds at* A *(resp. at* B*), then at least one positive solution of Problem* (1.1) *exists for an infinite number of, arbitrarily small, values of* λ*.*

The claimed solutions have norm arbitrarily near to A *(resp. to* B*).*

On the other hand, when λ_s is continuous, we are able to give a much more clear picture of the solvability of Problem (1.1).

Theorem 2.7. *Assume the conditions of Proposition 2.1 plus* (M_3)*, and moreover assume that either* (H_{pp}) *or* (H_{dc}) *holds true. Then the following holds:*

a) Problem (1.1) *has at least one positive solution for every* $\lambda > 0$ *provided* (f_3) *holds, one of the three conditions in Theorem 2.4 holds at* A *and one of the three conditions in Theorem 2.3 holds at* B*, or vice versa;*
b) there exists $\Lambda > 0$ *such that Problem* (1.1) *has at least two positive solutions for every* $\lambda \in (0, \Lambda)$*, provided* (f_3) *holds true and, at both* A *and* B*, one of the three conditions in Theorem 2.4 holds;*
c) there exist $\Lambda \geq \overline{\Lambda} > 0$ *such that Problem* (1.1) *has at least two positive solutions for every* $\lambda > \Lambda$ *and no positive solution for* $\lambda \in (0, \overline{\Lambda})$*, provided, at both* A *and* B*, one of the three conditions in Theorem 2.3 holds.*

When (H_{dc}) *holds, then the cases (b) and (c) can be improved as follows:*

(b) Problem* (1.1) *has at least two positive solutions for every* $\lambda \in (0, \Lambda)$ *and no positive solution for* $\lambda > \Lambda$*;*
(c) Problem* (1.1) *has at least two positive solutions for every* $\lambda > \Lambda$ *and no positive solution for* $\lambda \in (0, \Lambda)$*.*

Remark 2.8. We collect here several remarks on the hypotheses and the results just stated.

1. The hypotheses (f_1) and (f_2) simply state that f is nonnegative but non trivial, continuous and has subcritical growth. Hypothesis (f_3), which is used in Theorem 2.4 in order to prove that $\lambda_s \to 0$ under condition $(C+)$, is rather stronger, in particular it implies that $f(t) > 0$ for $t > 0$. It is required in order to be able to estimate from above the value of λ_s in terms of the value of the critical level of problem (2.3).

   On the other hand, the conditions (fp_0)-$(f\pi_0)$-(fp_∞) state that f can be estimated by powers, at zero or at infinity, from below or from above. Such powers are then compared with the growth of M, via the conditions $(C+)$-$(C-)$.
2. Models of coefficients M satisfying in different ways the conditions $(C+)$ or $(C-)$ are given in Section 3. A remarkable example of a situation where the three behaviors described in Theorem 2.7 may arise is described in Proposition 3.5.

   The Theorems 2.6 and 2.7 show the importance of the function $s^{p_*-2}M^{-1}(x, s^2)$ in order to have a picture of the behavior of the solutions of Problem (1.1) and of the corresponding values of the parameter

λ, where p_* is taken as π_0, p, p_0, p_∞ from the hypotheses (fp_0)-(f_2)-($f\pi_0$)-(fp_∞), or is simply p when $f(t) = t^{p-1}$. In particular, if $A = 0$ or $B = \infty$, the behavior of M needs to be compared with that of f.

It is worth to remark that the two conditions ($C+$) and ($C-$) are of different nature: the condition ($C+$) is local: it would be satisfied for instance if $s^{2-p_*}M(x, s^2)$ goes to zero only in some set of positive measure (see also Remark 4.4 for a possible weakening of this condition). On the other hand, the condition ($C-$) imposes a much stronger condition, actually it implies the global property that, almost everywhere in Ω, $s^{2-p_*}M(x, s^2)$ cannot remain bounded.

3. The continuity result in Proposition 2.5, which is fundamental for the proof of the main Theorem 2.7, is obtained in tho steps: the continuity of the critical level $\Theta^s_{M^{-1}(\cdot,s^2)}$ defined in (2.3) arises naturally from its variational characterization, once that M is supposed to depend continuously on s in a suitable sense (condition (M_3)).

   The continuity of λ_s is then an immediate consequence of the continuity of the critical level when f is a pure power, actually in this case $pF(t) = f(t)t = t^p$ and then the integrals in (2.2) and (2.3) are directly related.

   When f is not a pure power, instead, the critical level (2.3) and the value of λ_s in (2.2) are not easily connected anymore. In general, λ_s could take different values for those s for which more maximizers of problem (2.3) exist, and it may not be possible to define it as a continuous function at such s. The fact that (2.2) actually defines a continuous function is then obtained only under further conditions, which allow to prove that the maximization problem (2.3) has a unique maximizer (see Lemma 4.5). Such conditions are those stated in (H_{dc}), which are known to be sufficient in order to have a unique solution of the Dirichlet problem for the equation $-\Delta u = m(x)f(u)$: see [BO86; DS87].

4. Nonexistence of positive solutions with the Kirchhoff term is usually more difficult to obtain than in the local case. For this reason we were not able to obtain, in general, the typical complete concave-convex behavior where there is a precise value of the parameter λ that divides existence from nonexistence. The results in Theorem 2.2 are obtained by the usual technique of either testing the equation with the first eigenfunction of the Laplacian or with the solution itself and then estimating with Hölder and Sobolev inequalities, in order to find a necessary condition (see also Remark 4.3 for a possible improvements of the conditions obtained).

   Further nonexistence results are obtained indirectly when it is possible to assert that solutions of certain problems are unique: see Theorem 2.7 points (b*,c*) and their proof.

   It is worth to observe that a special case of point (i) in Theorem 2.2 is when $f(u) = u$, that is, Problem (1.2) with $p = 2$, for which one gets the necessary condition $\lambda_1\delta_M \le \lambda \le \lambda_1 D_M$.

◁

2.1. The case of the pure power nonlinearity

In the case of Problem (1.2), where f is simply a pure power, the results described above take an easier form. Actually the function λ_s is always continuous under the hypotheses of Proposition 2.1 plus (M_3) (see Proposition 2.5 and also Remark 2.8 at point 3). Moreover, the conditions (f_1)-(f_2)-(f_3) hold true, and also the conditions ($f\pi_0$)-(fp_0)-(fp_∞) hold with $\pi_0 = p_0 = p_\infty = p$.

We then have the following version of Theorem (2.7).

Theorem 2.9. *Assume conditions* (M_1), (M_2) *and* (M_3) *and set* $p_* = p$ *for the conditions* ($C+$) *and* ($C-$). *Then the following holds:*

a) Problem (1.2) *has at least one positive solution for every* $\lambda > 0$ *provided*

$$\begin{cases} (C+) \text{ holds with } s_* = A, \\ (C-) \text{ holds with } s_* = B, \end{cases} \tag{2.5}$$

or vice versa;

b) there exists $\Lambda > 0$ *such that Problem* (1.2) *has at least two positive solutions for every* $\lambda \in (0, \Lambda)$ *provided*

$$(C+) \text{ holds with } s_* = A \text{ and with } s_* = B; \tag{2.6}$$

c) there exists $\Lambda \geq \overline{\Lambda} > 0$ *such that Problem* (1.2) *has at least two positive solutions for every* $\lambda > \Lambda$ *and no positive solution for* $\lambda \in (0, \overline{\Lambda})$, *provided*

$$(C-) \text{ holds with } s_* = A \text{ and with } s_* = B. \tag{2.7}$$

Moreover, if $p < 2$, *then also* (H_{dc}) *holds true and then the more precise results in (b*, c*) of Theorem 2.7 hold.*

2.2. Further remarks and related bibliography

As remarked in the Introduction, the main feature of the Problems (1.1) and (1.2) is the presence of the term $M\left(x, \|u\|^2\right)$, which is said to be nonlocal, since it depends not only on the solution at the point in Ω where the equation is evaluated, but on the Sobolev norm of the whole solution. Such problems are usually called of Kirchhoff type, as they are generalizations of the (stationary) Kirchhoff equation, originally proposed in [Kir83] as an improvement of the vibrating string equation, which takes into account the variation in the tension of the string due to the variation of its length with respect to the unstrained position. In the Kirchhoff case, the proposed function M takes the form $M(x, s^2) = M(s^2) = a + bs^2$ with $a, b > 0$.

In addition, in this paper the function M can also vary with $x \in \Omega$, which is a situation scarcely considered in literature, and has the consequence that the problem is not variational any more, that is, there exists no functional whose critical points are the solutions of (1.1) or (1.2). This fact was pointed out in [Fig+14], where in particular it was considered the case with $M(x, s^2) = a(x) + b(x)s^2$, which was motivated as representing, in dimension 1, the Kirchhoff vibrating string with nonhomogeneous physical properties.

We will further discuss this problem in Section 3.1, where we also obtain new results: see Proposition 3.5.

Many other physical phenomena can be modeled through nonlocal equations similar to (1.1) (see examples in [Vil97; DH09]), and interesting mathematical questions also arise. For more recent literature about such Kirchhoff type problems we cite the works [AC01; ACM05; Ma05; CF06; ZP06; CP11; Ane11; CWL12; AC15; TC16; SCT16; AA17; AA16; SJS18; FS18; IM18; IM20; ASJS].

In the vast majority of the works about Kirchhoff type problems, the nondegeneracy condition $M \geq \delta > 0$ is assumed. However, most recently, the interest has grown in investigating the case in which M is a degenerate function, that is, when M is not required to be bounded from below by a positive constant: see [CP11; SCT16; AA16; SJS18; IM20; FMS21; ASJS]. Actually, this situation is both challenging from a mathematical point of view, and can lead to behaviors that do not arise in the nondegenerate case. Indeed, the conditions in Theorem 2.9 point (b), under which we obtain the result of two solutions for Problem (1.2) with small $\lambda > 0$, could never be satisfied under the nondegeneracy condition $M \geq \delta > 0$.

Degeneracy of M may happen in different forms: in [SCT16] the original Kirchhoff model was considered with the parameter $a = 0$, that is, $M(s^2) = bs^2$ with $b > 0$. More general terms with $M(0) = 0$ were considered for instance in [CP11; AA16; IM20], while [AA17; FMS21] also considered the case where $M(s^2) \to \infty$ as $s \to \infty$. Finally, in [SJS18; ASJS] are considered cases where M has zeros at positive values of $\|u\|$. Our results apply to each one of these kinds of degeneration, with the additional feature that the degeneracy may be limited to some subset of Ω (see the model problems in Section 3).

We conclude this bibliographic introduction by citing some of the most important works related with the classical concave-convex problems like (1.5) or its generalizations, which have been intensively studied since the works [Lio82; GAPA91; BEP95; ABC94]. In particular, it seems worth to mention the papers [dGU03; dGU06; dGU09], where it was shown, in different contexts, that the concave and convex behavior of the nonlinearity in equation (1.5) could be assumed to hold only on some small set, in order to achieve the result of existence of at least two solutions for small values of the parameter that measure the magnitude of the nonlinearity. This phenomenon closely relates with point (b) in Theorem 2.7, where we see that it is enough that $s^{2-p_*}M(x, s^2)$ goes to zero in two small sets, one as $s \to A$ and another as $s \to B$, in order to obtain this kind of behavior (see also Example 3.2).

3. Model problems

In this section we present some model problems with $f(t) = t^{p-1}$, that fit in the different situations described in Theorem 2.9.

Let us first consider cases where $M(x, s^2) > 0$ in $\Omega \times (0, \infty)$.

Example 3.1. Autonomous coefficient $M = M(s^2)$.
If M does not depend on $x \in \Omega$, then we see form Theorem 2.9 that the behavior of Problem (1.2) depends on the limits at zero and at ∞ of the single variable function $s^{2-p}M(s^2)$.

- A solution exists for every $\lambda > 0$ provided
$$\lim_{s\to 0} s^{2-p}M(s^2) = 0 \qquad \textit{and} \qquad \lim_{s\to\infty} s^{2-p}M(s^2) = \infty$$
or vice-versa: for instance, if $M(s^2) = s^{\sigma-2}$ with $\sigma \neq p$, as in equation (1.3). Another example is the original Kirchhoff term $M(s^2) = a + bs^2$, $a, b > 0$ with $p \in (1, 2)$ or, for dimensions $N = 1, 2$ or 3, with $p \in (4, 2^*)$. This last case was already contained in [ZP06].
- Two solutions exist for $\lambda > 0$ small enough provided
$$\lim_{s\to 0} s^{2-p}M(s^2) = \lim_{s\to\infty} s^{2-p}M(s^2) = 0\,.$$
A model is
$$M(s^2) = \frac{1}{s^{2-\sigma} + s^{2-q}} \qquad \text{with } \sigma > p > q. \tag{3.1}$$
Observe that in the linear case $p = 2$, this M is degenerate at both 0 and ∞.
- By point (c) in Theorem 2.9, two solutions exist for $\lambda > 0$ large enough and no solution for λ small enough, provided
$$\lim_{s\to 0} s^{2-p}M(s^2) = \lim_{s\to\infty} s^{2-p}M(s^2) = \infty\ :$$
for instance taking
$$M(s^2) = s^{\sigma-2} + s^{q-2} \qquad \textit{with } \sigma < p < q\,.$$
This model includes the case $M(s^2) = 1 + s^2$ with $p \in (2, 4)$, where M takes again the form of the original Kirchhoff term.

◁

Example 3.2. Two-regions M.
A simple but significant choice of the coefficient M, depending on x, is to choose two or more independent behaviors in complementary subsets of Ω: let $\Omega = \Omega_K \coprod \Omega_H$, with both subsets of positive measure and set
$$M(x, s^2) = \begin{cases} K(s^2)\,, & \textit{for } x \in \Omega_K\,, \\ H(s^2)\,, & \textit{for } x \in \Omega_H\,. \end{cases} \tag{3.2}$$

Equation (1.6) fits in this model, if K, H are the homogeneous functions $K(s^2) = s^{\sigma_K-2}$, $H(s^2) = s^{\sigma_H-2}$. We then have the following situation.

- If $p < \min\{\sigma_K, \sigma_H\}$ or $p > \max\{\sigma_K, \sigma_H\}$ then case (a) in Theorem 2.9 applies and there exists at least one solution for every $\lambda > 0$.
- if $\sigma_K < p < \sigma_H$ then condition ($C+$) is satisfied at zero and at infinity and then case (b) of Theorem 2.9 applies, We then have the concave-convex behavior where there exist at least two solutions for $\lambda > 0$ below a certain value Λ. If moreover $p < 2$ then also no solution exists for $\lambda > \Lambda$.

Observe that the latter result of two solutions for $\lambda > 0$ small would be achieved even if the power-like behavior of M takes place only in two small sets, for instance if

$$M(x,s^2) = \begin{cases} s^{\sigma_K - 2}, & x \in \widehat{\Omega}_K\,, \\ s^{\sigma_H - 2}, & x \in \widehat{\Omega}_H\,, \\ 1\,, & x \in \Omega \setminus \left(\widehat{\Omega}_H \cup \widehat{\Omega}_K\right)\,, \end{cases} \tag{3.3}$$

with $\widehat{\Omega}_K \coprod \widehat{\Omega}_H \subset \Omega$. Case (b) applies also if we take in (3.2) $K \equiv 1$ and H as the M in (3.1).

Again we remark that in order to have two solutions for $\lambda > 0$ small, the function M needs to have some degeneracy, in particular when $p = 2$ it cannot be bounded away from zero neither as $s \to 0$ nor as $s \to \infty$. However, the last model (3.3) shows that the degeneracy in M can occur only locally, in a way similar to the local concavity and convexity conditions assumed in [dGU06]. ◁

Example 3.3. Other kinds of dependence on x.
We discuss now two models with the aim to compare the conditions ($C+$) and ($C-$). In these models M is degenerate, but the conditions ($C+$) or ($C-$) hold depending on how strong the degeneration is. In this example we will only consider $s \to 0$, that is, we are interested in solutions of small norm, but similar situations can be constructed when $s \to \infty$.

Let us suppose $0 \in \Omega$ and, for the first model,

$$M(x,s^2) = |x|^\alpha + s^q\,, \qquad \text{with } \alpha > 0,\ q > 0. \tag{3.4}$$

In this case

$$M^{-1}(x,s^2) = \frac{1}{|x|^\alpha + s^q} \to x^{-\alpha}$$

monotonically, then by Fatou's Lemma,

$$\left\|M^{-1}(x,s^2)\right\|_\rho \to \begin{cases} \|x^{-\alpha}\|_\rho & \text{if } x^{-\alpha} \in L^\rho\,, \\ +\infty & \text{otherwise,} \end{cases}$$

in particular, by straightforward computations, one can estimate the norm $\left\|M^{-1}(x,s^2)\right\|_\rho = \left[\int_\Omega \left(\frac{1}{x^\alpha + s^q}\right)^\rho\right]^{1/\rho}$ as

$$\begin{cases} Cs^{q\left(\frac{N}{\alpha\rho} - 1\right)} & \text{if } \rho\alpha > N, \\ C|\ln s| & \text{if } \rho\alpha = N, \\ C & \text{if } \rho\alpha < N, \end{cases} \qquad \text{as } s \to 0.$$

We take for this model $p = q > 2$: then $s^{p-2}M^{-1}(x,s^2) \to 0$ except at the origin, but the situation changes depending on the strength of the singularity:

- if $\alpha \le \frac{N}{r}$ then $s^{p-2}\left\|M^{-1}(x,s^2)\right\|_r \to 0$, so by Theorem 2.3 $\lambda_s \to \infty$ as $s \to 0$;

- if $\alpha > \frac{pN}{2}$ then
$$s^{p-2}\int_{\Omega_*} M^{-1}(x,s^2) \simeq Cs^{p-2+pN/\alpha-p} = Cs^{pN/\alpha-2} \to \infty$$
provided Ω_* is a small ball centered at 0, and so now by Theorem 2.4 $\lambda_s \to 0$ as $s \to 0$;
- for the intermediate cases $\frac{N}{r} < \alpha \leq \frac{pN}{2}$ neither ($C+$) nor ($C-$) is satisfied at zero, so that the behavior of λ_s as $s \to 0$ should be assessed by a deeper analysis of (2.2).

In the second model we show that the situation changes completely if $|x|$ in (3.4) is replaced by the distance from x to a small ball B_0 centered in 0:
$$M(x,s^2) = d(x,B_0)^\alpha + s^q\,. \tag{3.5}$$
In this case $\int_{B_0} s^{p-2}M^{-1}(x,s^2) = |B_0|s^{p-2}s^{-q}$ and $\left\|s^{p-2}M^{-1}(x,s^2)\right\|_r \leq |\Omega|^{1/r}s^{p-2}s^{-q}$, so that the Theorems 2.3-2.4 now imply that $\lambda_s \to \infty$ if $q < p-2$ while $\lambda_s \to 0$ if $q > p-2$. In particular, when $p = q > 2$, as we assumed for (3.4), $\lambda_s \to 0$ for any $\alpha > 0$. ◁

Another kind of problems that can be considered are those where M has multiple zeros. Zeros in the nonlinearity are known to produce multiplicity of solutions, see for instance [Hes81; ILM17]. In a similar way, also zeros in the nonlocal term M have been shown to have a similar effect. In particular we cite [SJS18] and [ASJS], where two solutions are found for every bump of M, with a nonlinearity which also has a zero. A similar effect was also seen in [GSJ19], with a different kind of nonlocal term, depending on the Lebesgue norm of the solution.

Example 3.4. *M with zeros.*
From point (b) in Theorem 2.9 we can see that if $M = M(s^2)$ is continuous and satisfies $M(A) = M(B) = 0$ with $0 < A < B$, then two solutions exist for Problem (1.2) for $\lambda > 0$ small enough, having norm between A and B. From this, it follows that if M has multiple zeros then a couple of solutions is obtained for each bump of M.

This can be generalized to the case where M depends also on $x \in \Omega$. For instance, if in the model (3.2) we take $K \equiv 1$ and $H(s^2) = \sin(\pi s)$, then ($C+$) holds at every zero of H, so an arbitrarily large number of solutions exists for $\lambda > 0$ small enough, in fact, there exist $\Lambda_k > 0$, $k \in \mathbb{N}$ such that at least $2k$ solutions exist for $\lambda \in (0,\Lambda_k)$, with the additional information that there is a couple of solutions having norm in each interval $(j-1,j):\ j = 1,...,k$.

However, the multiple zeros do not have to be in a fixed set as in the last example: for instance, consider a sequence of disjoint sets $\{\Omega_k \subset \Omega\}_{k=0,1,...}$, each of positive measure, and let
$$M(x,s^2) = (s-k)^2 \qquad \text{for } x \in \Omega_k,\ \ k = 0,1,...\,;$$
then as above ($C+$) holds at every $s_* = k$ and an arbitrarily large number of solutions exists for $\lambda > 0$ small enough.

Finally, it can be deduced from Proposition 2.1 and Theorem 2.4, that the two solutions obtained whose norm lies between two consecutive zeros A and B, have norms which converge to these two points as the parameter λ tends to zero: a phenomenon which was already observed in [ASJS]. ◁

3.1. Application to nonlinear Kirchhoff equation with nonhomogeneous material

As remarked in the introduction, the Problem (1.2) was studied in [Fig+14] in the linear and sublinear cases $p \leq 2$, but only considering M in the form $M(x, s^2) = a(x) + b(x)s^2$ with continuous $a, b \geq 0$ and with $a \geq a_0 > 0$. The problem was motivated as a model for small transversal vibrations of an elastic string with fixed ends, which is composed by a nonhomogeneous material. Actually, the term M considered is analogous to the original one proposed by Kirchhoff [Kir83], but with the added dependence on the point x.

For $p = 2$ a unique positive solution was obtained if and only if, roughly speaking, λ lies between the eigenvalues of two asymptotic problems. In particular, the lower bound corresponds to the asymptotic problem in zero and the upper bound to that at infinity: this upper bound turns out to be infinity if also $b \geq b_0 > 0$, finite if $b \equiv 0$ in a suitable set. In the sublinear case $p \in (1, 2)$ instead, they obtained a unique positive solution for every $\lambda > 0$, in fact, a continuum of positive solutions emanates from $(\lambda, u) = (0, 0)$ and has unbounded projection on the positive λ axis.

Both existence results can also be deduced from Theorem 2.9 and Proposition 2.1. Uniqueness is a consequence of the uniqueness for the Dirichlet problem for $-\Delta u = m(x)u^{p-1}$ when $p < 2$ (see [DS87]), the simplicity of the first eigenvalue of the Laplacian and the monotonicity of M with respect to s. However, our results allow to consider also more general nonlinearities f, the superlinear subcritic pure power case $p \in (2, 2^*)$, and also to avoid the nondegeneracy condition $a \geq a_0 > 0$: we will only need to assume that M does not degenerate completely in some region, that is, we assume

(M_{ab}) there exists $c_0 > 0$ such that $a(x) + b(x) \geq c_0$ in Ω.

We summarize in the following proposition what can be said, using our results, about the problem from [Fig+14].

Proposition 3.5. *Consider Problem (1.2) with $M(x, s^2) = a(x) + b(x)s^2$, $a, b \in \mathcal{C}\left(\overline{\Omega}\right)$ (then bounded) and $a, b \geq 0$. Moreover assume condition (M_{ab}).*

- *If $p \in (1, 2)$ then at least one positive solution exists for every $\lambda > 0$, in fact, the function λ_s defined in (2.2) goes from 0 to $+\infty$ as s increases in $(0, \infty)$.*
- *If $p \in (4, 2^*)$ then at least one positive solution exists for every $\lambda > 0$, in fact, the function λ_s goes from $+\infty$ to 0 as s increases in $(0, \infty)$.*
- *If $p \in (2, 4)$ then*
  - *there exist at least two positive solutions for $\lambda > 0$ small enough if both $a \equiv 0$ and $b \equiv 0$ in (distinct) sets of positive measure;*

- *there exist at least two positive solutions for $\lambda > 0$ large enough and no solution for $\lambda > 0$ small, if both $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$ in Ω;*
- *there exists at least one positive solution for every $\lambda > 0$ if $a \equiv 0$ in a sets of positive measure and $b(x) \geq b_0 > 0$ in Ω, or if $b \equiv 0$ in a sets of positive measure and $a(x) \geq a_0 > 0$ in Ω.*

Remark 3.6. Of course, the cases $p \geq 4$ are only possible in dimension $N = 1, 2, 3$. ◁

Proof. In view of Proposition 2.1 and Theorem 2.9, all we need to do is to compute the limits of λ_s, by studying the behavior at zero and at infinity of

$$s^{p-2} M^{-1}(x, s^2) = \frac{s^{p-2}}{a(x) + b(x)s^2},$$

in order to check the conditions ($C+$), ($C-$).

Let $a + b \leq C$. We have the following estimates:
for $s \leq 1$

$$\begin{cases} s^{p-2}/C \leq s^{p-2}M^{-1}(x,s^2) \leq s^{p-2}/a_0 & \text{where } a(x) > a_0 > 0, \\ s^{p-4}/C \leq s^{p-2}M^{-1}(x,s^2) \leq s^{p-4}/c_0 & \text{where } a(x) = 0; \end{cases}$$

for $s \geq 1$

$$\begin{cases} s^{p-4}/C \leq s^{p-2}M^{-1}(x,s^2) \leq s^{p-4}/b_0 & \text{where } b(x) > b_0 > 0, \\ s^{p-2}/C \leq s^{p-2}M^{-1}(x,s^2) \leq s^{p-2}/c_0 & \text{where } b(x) = 0. \end{cases}$$

As a consequence, by the Theorems 2.3 and 2.4,

$$\lim_{s\to 0} \lambda_s = \begin{cases} \infty & \begin{cases} \text{if } \ a(x) \geq a_0 > 0 \text{ in } \Omega \text{ and } p > 2, \\ \text{if } p > 4\,, \end{cases} \\ 0 & \begin{cases} \text{if } p < 2, \\ \text{if } a \equiv 0 \text{ in a set of positive measure and } p < 4\,, \end{cases} \end{cases}$$

$$\lim_{s\to \infty} \lambda_s = \begin{cases} \infty & \begin{cases} \text{if } \ b(x) \geq b_0 > 0 \text{ in } \Omega \text{ and } p < 4, \\ \text{if } p < 2\,, \end{cases} \\ 0 & \begin{cases} \text{if } p > 4, \\ \text{if } b \equiv 0 \text{ in a set of positive measure and } p > 2\,. \end{cases} \end{cases}$$

□

4. Proofs of the results

In this section we prove the results stated in Section 2. Our method is related to the one used in [Fig+14] for the linear right hand side, whose argument is based on eigenvalue problems and their characterization as constrained minimizations or maximizations: in fact, we will exploit rescaling and maximization on spheres in order to obtain positive solutions with a given norm. Maximization on spheres was exploited also in [SJS18; ASJS], dealing with the Kirchhoff degenerate problem, in order to obtain suitable estimates that were then used in the construction of a minimum and a Mountain Pass geometry.

In the following, we will use the letters C, c to denote generic positive constants which may vary from line to line. In order to simplify the notation we will often substitute $m_s(x) := M^{-1}(x, s^2)$ throughout the proofs.

We first need the following technical Lemma.

Lemma 4.1. *Let*
– *G be a continuous function satisfying $|G(t)| \le C(1 + |t|^p)$ with $p < 2^*$,*
– *$u_n \rightharpoonup w$ in $H_0^1(\Omega)$,*
– *$m_n \to m$ in $L^r(\Omega)$ with $m \in L^{\widetilde{r}}(\Omega)$.*

Then, up to a subsequence, $m_n G(u_n) \to mG(w)$ in $L^1(\Omega)$.

Proof. We have that u_n is bounded in H_0^1 and L^{2^*} and, up to a subsequence, $u_n \to w$ in L^q for $q < 2^*$. Moreover, by the theory of Nemytskii operators, $G(u_n) \to G(w)$ in $L^{q/p}$.

Then we estimate

$$|m_n G(u_n) - mG(w)| \le |G(u_n) - G(w)|\ |m| + |G(u_n)|\,|m_n - m|\,.$$

If $\widetilde{r}'$ is the dual of $\widetilde{r}$, then $\widetilde{r}' < r' = 2^*/p$ and so $G(u_n) \to G(w)$ in $L^{\widetilde{r}'}$ and since $m \in L^{\widetilde{r}}$ we have that the first term of the estimate goes to zero in L^1. The same is true for the second term, since $G(u_n)$ is bounded in $L^{r'}$ while $|m_n - m| \to 0$ in L^r. □

The next step will be to prove Proposition 2.1, where the couples (λ, u) satisfying Problem (1.1) are obtained.

Proof of Proposition 2.1. For a fixed $s \in (A, B)$, set $m_s(x) := M^{-1}(x, s^2)$ and consider the maximization Problem (2.3), on the sphere S^s. By (f_2) and (M_2), the functional $\int_\Omega m_s(x)F(u)$ is of class $\mathcal{C}^1$ in H_0^1 and

$$\int_\Omega m_s(x)F(u) \le \|m_s\|_{\widetilde{r}}\, C(1 + \|u\|^p_{p\,\widetilde{r}'})\,,$$

where as before $p\,\widetilde{r}' < 2^*$. Then by standard arguments (see below) one obtains that $\Theta^s_{m_s} > 0$, is finite and attained by a positive function $w \in S^s$, which satisfies, in weak sense,

$$\begin{cases} -\Delta w = \ell\, m_s(x) f(w) & in\ \ \Omega\,, \\ w = 0 & on\ \ \partial\Omega\,, \end{cases} \tag{4.1}$$

with $\ell = s^2 / \int_\Omega m_s f(w) w$. As a consequence, $\|w\| = s$ and w also satisfies

$$-M(x, \|w\|^2)\Delta w = \ell\, f(w)\,,$$

and then it is a solution of (1.1) with norm s and $\lambda_s := s^2 / \int_\Omega M^{-1}(\cdot, s^2) f(w) w$, as claimed.

For sake of completeness, we resume here the steps used in the above argument. Let u_n be a maximizing sequence for (2.3), which we may assume to be of nonnegative functions since it may be substituted by $|u_n|$ in view of (f_1); then up to a subsequence $u_n \rightharpoonup w$ in H_0^1, where $\|w\| \leq s$ and, in view of Lemma 4.1,

$$\Theta_{m_s}^s = \lim_{n\to\infty} \int_\Omega m_s F(u_n) = \int_\Omega m_s F(w) > 0 .$$

This implies that $w \not\equiv 0$, moreover if $\|w\| < s$ then $w' := sw/\|w\| \in S^s$ and $\int_\Omega m_s F(w') > \int_\Omega m_s F(w) = \Theta_{m_s}^s$ because F is nondecreasing and strictly increasing at least near zero by (f_1), giving a contradiction. As a consequence $w \in S^s$ attains the maximum in (2.3) and then by Lagrange's multipliers rule there exists $\mu \in \mathbb{R}$ such that

$$\mu \int_\Omega \nabla w \nabla \varphi = \int_\Omega m_s f(w)\varphi \qquad \forall\, \varphi \in H_0^1 ,$$

where inserting $\varphi = w$ one gets $\mu s^2 = \int_\Omega m_s f(w) w > 0$, proving (4.1). Finally $w > 0$ by the strong maximum principle in view of (f_1). ☐

Remark 4.2. The condition (f_1) guarantees that the integral in (2.3) is positive for every nontrivial $v \geq 0$. If one admits that f can be zero up to some positive value then, for $N \geq 2$, there still always exists some $v \in S^s$ for which the integral is positive and Proposition 2.1 still holds true. If $N = 1$ then for small s the integral in (2.3) is zero on the whole sphere S^s, but Proposition 2.1 still holds true for larger values of s. ◁

We give now the proof of the nonexistence results.

Proof of Theorem 2.2. If u is a positive solution of Problem (1.1), then for any $\varphi \in H_0^1$,

$$\int_\Omega \nabla u \nabla \varphi = \int_\Omega M^{-1}(x, \|u\|^2)\lambda f(u)\varphi . \tag{4.2}$$

If in (4.2) we take $\varphi = \varphi_1$ we get

$$\int_\Omega \nabla u \nabla \varphi_1 = \lambda_1 \int_\Omega u\varphi_1 = \int_\Omega M^{-1}(x, \|u\|^2)\lambda f(u)\varphi_1 ,$$

from which we get

$$\int_\Omega \left[M^{-1}(x, \|u\|^2)\lambda \frac{f(u)}{u} - \lambda_1 \right] u\varphi_1 = 0 .$$

From this we can see that no positive solution can exist if $M^{-1}(x, s^2)\lambda \frac{f(t)}{t} - \lambda_1$ is everywhere positive (resp. negative), so we get the necessary conditions $\lambda_1 \frac{\delta_M}{D_f} \leq \lambda \leq \lambda_1 \frac{D_M}{\delta_f}$ that give claim (i).

If in (4.2) we take $\varphi = u$ and then we use condition (2.4), Hölder and Sobolev inequalities, we get

$$\|u\|^2 = \int_\Omega M^{-1}(x, \|u\|^2)\lambda f(u) u \leq$$
$$\leq CD\lambda \left\| M^{-1}(x, \|u\|^2) \right\|_r (\|u\|^{p_0} + \|u\|^{p_\infty}) \leq CD^2\lambda \|u\|^2 ,$$

which gives the necessary condition $\lambda \geq \frac{1}{CD^2}$ for the existence of a solution. □

Remark 4.3. The claims in (i) of Theorem 2.2 can be improved if instead of λ_1, φ_1 one uses the eigencouple $\lambda_{1,m}, \varphi_{1,m}$ of the weighted problem $-\Delta u = \lambda\, m\, u$ with a suitable weight m. In this case the necessary condition becomes $\lambda_{1,m}\frac{\inf(Mm)}{D_f} \leq \lambda \leq \lambda_{1,m}\frac{\sup(Mm)}{\delta_f}$. ◁

We prove now the two theorems that study the behavior of λ_s at the endpoints A, B.

Proof of Theorem 2.3. For any $u \in H_0^1$ with $u > 0$ and $\|u\| = s \in (A, B)$, using (f_2)-(M_2) one can estimate

$$\int_\Omega m_s f(u)u \leq \int_\Omega m_s C(u+u^p) \leq \|m_s\|_r\, C\left(\|u\|_{2^*} + \|u\|_{2^*}^p\right) \leq C\,\|m_s\|_r\,(s+s^p). \tag{4.3}$$

As a consequence, from (2.2), λ_s can be estimated as

$$\lambda_s \geq \frac{c\,s^2}{(s+s^p)\,\|M^{-1}(\cdot, s^2)\|_r}\,;$$

from this inequality the claims α) and γ) follow.

For case β) observe that by joining (f_2) with ($f\pi_0$) it is possible to obtain that $f(u)u \leq C(u^{\pi_0} + u^p)$, then instead of (4.3) we get

$$\int_\Omega m_s f(u)u \leq C\,\|m_s\|_r\,(s^{\pi_0} + s^p)\,;$$

since $\pi_0 \leq p$, for s small

$$\lambda_s \geq \frac{c\,s^2}{s^{\pi_0}\,\|M^{-1}(\cdot, s^2)\|_r}$$

and then we obtain point β). □

Proof of Theorem 2.4. If u_s is a maximizer of problem (2.3) with weight m_s, we can estimate, by (f_3),

$$\int_\Omega m_s f(u_s)u_s \geq \delta \int_\Omega m_s F(u_s) \geq \delta \int_\Omega m_s F(s\varphi_1) \geq \delta\left(\inf_{\Omega_*} F(s\varphi_1)\right)\int_{\Omega_*} m_s\,.$$

Since F is positive and nondecreasing, if $s \to s_* \in (0, \infty)$, then $\inf_{\Omega_*} F(s\varphi_1)$ stays bounded away from zero.
As a consequence in case α) we have $\int_\Omega M^{-1}(\cdot, s^2) f(u_s)u_s \to \infty$ and then by (2.2), $\lambda_s \to 0$.

If $s \to 0$, since $\varphi_1 \in L^\infty$, we can estimate, by (fp_0), $F(s\varphi_1) \geq C s^{p_0} \varphi_1^{p_0}$ for s small; with this, proceeding as above, we have

$$\lambda_s^{-1} = s^{-2} \int_\Omega m_s f(u_s) u_s \geq s^{-2}\delta \int_\Omega m_s F(s\varphi_1) \geq C\delta s^{p_0-2} \int_\Omega m_s \varphi_1^{p_0} \geq \\ \geq C\delta \left(\inf_{\Omega_*} \varphi_1^{p_0}\right) \left[s^{p_0-2} \int_{\Omega_*} m_s\right] \to \infty\,,$$

proving case β).

Finally, by (fp_∞), we can find $C, t_0 > 0$ such that $F(t) \geq C t^{p_\infty}$ for every $t \geq t_0$. For s large enough $s\varphi_1 > t_0$ in Ω_*, then we may estimate as above

$$\lambda_s^{-1} \geq s^{-2}\delta \int_{\Omega_*} m_s F(s\varphi_1) \geq C\delta s^{p_\infty-2} \int_{\Omega_*} m_s \varphi_1^{p_\infty} \geq \\ \geq C\delta \left(\inf_{\Omega_*} \varphi_1^{p_\infty}\right) \left[s^{p_\infty-2} \int_{\Omega_*} m_s\right] \to \infty$$

and point γ) then follows. □

Remark 4.4. When $f(t) = t^{p-1}$, the condition $(C+)$ could also be weakened, in fact, one can see from the proof above that it would be enough to assume that, for some $\varphi \in H_0^1$, $s^{p-2} \int_\Omega M^{-1}(x, s^2)\varphi^p \to \infty$. For instance, this weaker condition would be satisfied if there exist sets Ω_s, may be converging to the boundary, then not contained in any $\Omega_* \subset\subset \Omega$, but such that one still has $\int_{\Omega_s} M^{-1}(x, s^2) s^{p-2}\varphi^p \to \infty$.

On the other hand, a simple sufficient condition for $(C+)$ is that, in some set $\widetilde{\Omega}$ of positive measure, $s^{p_*-2}M^{-1}(x, s^2) \to \infty$ uniformly (we may suppose $\widetilde{\Omega} \subset\subset \Omega$). Actually in this case

$$\int_{\widetilde{\Omega}} s^{p_*-2} M^{-1}(x, s^2) \geq s^{p_*-2} \left[\inf_{x\in\widetilde{\Omega}} M^{-1}(x, s^2)\right] |\widetilde{\Omega}| \to \infty\,.$$

◁

In order to prove the continuity results in Proposition 2.5, we first need the following Lemma, which is based on unicity results from [BO86; DS87].

Lemma 4.5. *Assume the conditions of Proposition 2.1 plus condition (H_{dc}). Then for a given $s \in (A, B)$, the maximization problem (2.3) admits exactly one maximizer, which is the unique positive solution of*

$$\begin{cases} -\Delta u = \lambda m_s f(u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{4.4}$$

with $\lambda = \lambda_s$ from (2.2). Moreover no positive solution of norm s exists for (4.4) with $\lambda \neq \lambda_s$.

Proof. Observe that if $f(t)/t$ is strictly decreasing then

$$2F(t) > f(t)t \qquad \text{for } t > 0\,. \tag{4.5}$$

Actually for $0<\tau<t$ we have $\frac{f(t)}{t}<\frac{f(\tau)}{\tau}$, then

$$F(t)=\int_0^t f(\tau)d\tau>\int_0^t \frac{f(t)}{t}\tau d\tau=\frac{f(t)}{t}\frac{t^2}{2}\,.$$

By (H_{dc}) and the results in [BO86; DS87], the problem (4.4) admits a unique positive solution for a given $\lambda>0$.

Consider now the functional (with parameter λ)

$$J_\lambda(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\lambda\int_\Omega m_sF(u)\,,$$

which is well defined by (M_2)-(f_1)-(f_2), moreover using (H_{dc})-(ii) we have $J_\lambda(u)\geq\frac{1}{2}\left\|u\right\|^2-\lambda\left\|m_s\right\|_r C(1+\left\|u\right\|^p)$, where $p<2$ and then J_λ is coercive, for every $\lambda>0$.

Let u be a maximizers of (2.3), then by Proposition 2.1 it is a positive solution of (4.4) for $\lambda=\lambda^u$. Let also, for sake of contradiction, $w\in S^s$, $w\neq u$, be a positive solution of (4.4) for some $\lambda=\lambda^w$, where $\lambda^w\neq\lambda^u$ by the unicity of solutions. Then we have

$$\|u\|=\|w\|=s\,,\qquad \int_\Omega m_sF(u)\geq\int_\Omega m_sF(w)\,, \tag{4.6}$$

and testing (4.4) with u and w

$$\lambda^u=\frac{s^2}{\int_\Omega m_sf(u)u},\qquad \lambda^w=\frac{s^2}{\int_\Omega m_sf(w)w}\,. \tag{4.7}$$

Then by (4.5)

$$J_{\lambda^u}(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\lambda^u\int_\Omega m_sF(u)=s^2\left(\frac{1}{2}-\frac{\int_\Omega m_sF(u)}{\int_\Omega m_sf(u)u}\right)<0$$

and by the same computation also $J_{\lambda^w}(w)<0$. This means that the global minima of both functionals are nontrivial, then global minima are positive solutions of (4.4) with $\lambda=\lambda^u$ (resp. $\lambda=\lambda^w$). Since only one of each can exist, the minimum of J_{λ^u} must be u and the minimum of J_{λ^w} is w. However, by (4.6), $J_{\lambda^w}(w)\geq J_{\lambda^w}(u)$, which contradicts that w is the unique global minimizer of J_{λ^w}.

We have thus proved that (4.4) has a unique solution of norm s, for a unique value of λ. Then they correspond to the maximizer of (2.3) and the corresponding λ_s from (2.2). This also means that the maximizer of (2.3) is unique. □

We can now prove the result on the continuity of the critical level (2.3) and of the function λ_s.

Proof of Proposition 2.5. The continuity of $\Theta^s_{M^{-1}(\cdot,s^2)}$ as a function of $s>0$ follows by its variational characterization and hypothesis (M_3).

Actually, if a maximizers of (2.3) with weight m_s is denoted by u_s, then for $t, s \in [a, b] \subset (A, B)$ we have

$$\Theta^s_{m_s} = \int_\Omega m_s F(u_s) \quad \text{and} \quad \Theta^t_{m_t} = \int_\Omega m_t F(u_t) \geq \int_\Omega m_t F(tu_s/s)$$

and then

$$\Theta^s_{m_s} - \Theta^t_{m_t} \leq \int_\Omega (m_s - m_t) F(u_s) + \int_\Omega m_t \left[F(u_s) - F(tu_s/s)\right] .$$

The first integral can now be estimated with $\|m_s - m_t\|_r C(1 + b^p)$ in view of condition (f_2), and then goes to zero as $|t - s| \to 0$ in $[a, b]$, by (M_3). The second integral also goes to zero since $\|m_t\|_r$ is bounded in $[a, b]$ by (M_3), $tu_s/s - u_s \to 0$ in L^{2^*} and then $F(u_s) - F(tu_s/s) \to 0$ in $L^{r'}$. By reasoning symmetrically one gets the estimate from below and proves the continuity.

Now, if (H_{pp}) holds, then $f(u)u = u^p = pF(u)$ and $\lambda_s = \dfrac{s^2}{p\Theta^s_{M^{-1}(\cdot,s^2)}}$, which is therefore also a continuous function.

If instead we assume condition (H_{dc}), then by Lemma 4.5 the maximization problem (2.3) admits exactly one maximizer.
Suppose now that $s_n \to s_0 > 0$ and let u_n, u_0 be the corresponding maximizers for (2.3).
Since $\|u_n\| = s_n$ it is bounded in H^1_0 and up to a subsequence we may assume $u_n \rightharpoonup w$ in H^1_0. By Lemma 4.1 and hypothesis (M_3) we then have (up to a further subsequence)

$$\int_\Omega m_{s_n} F(u_n) \to \int_\Omega m_{s_0} F(w) \quad \textit{and} \qquad \int_\Omega m_{s_n} f(u_n) u_n \to \int_\Omega m_{s_0} f(w) w \, . \tag{4.8}$$

Observe that

$$\Theta^{s_n}_{M^{-1}(\cdot,{s_n}^2)} = \int_\Omega m_{s_n} F(u_n) \to \Theta^{s_0}_{M^{-1}(\cdot,{s_0}^2)} = \int_\Omega m_{s_0} F(u_0)$$

because of the already proved continuity of $\Theta^s_{M^{-1}(\cdot,s^2)}$, and then $\int_\Omega m_{s_0} F(u_0) = \int_\Omega m_{s_0} F(w)$. By the weak convergence we know that $\|w\| \leq s_0 = \|u_0\|$ but in fact $\|w\| = s_0$ or otherwise

$$\int_\Omega m_{s_0} F(s_0 w / \|w\|) > \int_\Omega m_{s_0} F(w) = \Theta^{s_0}_{M^{-1}(\cdot,{s_0}^2)}$$

and then $s_0 w / \|w\| \in S^{s_0}$ would be above the maximizer level (the strict inequality follows form the fact that F is nondecreasing and also strictly increasing for small t by (f_1)).

We conclude that w is actually a maximizer for $\Theta^{s_0}_{M^{-1}(\cdot,{s_0}^2)}$ and then $u_0 = w$ by Lemma 4.5. Since the argument may be applied to any subsequence, (2.2) and (4.8) imply that $\lambda_{s_n} \to \lambda_{s_0}$. □

We can finally prove our main results.

Proof of Theorem 2.6 and Theorem 2.7. Theorem 2.6 is an immediate consequence of Proposition 2.1 and the Theorems 2.3 and 2.4, in fact, the values of λ_s from (2.2), for which we know that a positive solution of Problem (1.1) exists, accumulate at infinity or at zero, respectively, when the conclusion of Theorem 2.3 or that of Theorem 2.4 holds.

In the hypotheses of Theorem 2.7 we also know, by Proposition 2.5, that λ_s is a continuous function defined in the interval (A, B). Therefore we have the following situations.

- Case a) by the Theorems 2.3 and 2.4 the assumptions on M imply that
$$\lim_{s\to A} \lambda_s = 0, \qquad \lim_{s\to B} \lambda_s = \infty$$
or vice versa. Then the continuity of λ_s implies that $Im(\lambda_s) = (0, \infty)$.
- Case b) in this case one has
$$\lim_{s\to A} \lambda_s = \lim_{s\to B} \lambda_s = 0$$
and then $Im(\lambda_s)$ is of the form $(0, \Lambda]$, but every value below Λ corresponds to at least two solutions, which have distinct norm.
- Case c) now
$$\lim_{s\to A} \lambda_s = \lim_{s\to B} \lambda_s = \infty$$
and then $Im(\lambda_s)$ is of the form $[\Lambda, \infty)$ and, as above, two distinct solutions exist for $\lambda > \Lambda$. On the other hand, the non existence of solutions for λ small follows from Theorem 2.2 point (ii), since in case c), (2.4) is satisfied for suitable p_0, p_∞. Actually we can always take $p_\infty = p$, while we take $p_0 = \pi_0$ from $(f\pi_0)$ if $A = 0$ or $p_0 = 1$ otherwise: then the estimate on f holds by $(f\pi_0)$-(f_2) while the estimate on $\left\|M^{-1}(\cdot, s^2)\right\|_r$ follows by its continuity and since condition $(C-)$ holds at A and at B.

The result in the points b*) and c*) are a consequence of the last assertion in Lemma 4.5. Actually it implies that the values of λ for which a positive solution of (1.1) exists are exactly those in the range of the function λ_s in (2.2). □

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