Existence, nonexistence and multiplicity of positive solutions for the poly-Laplacian and nonlinearities with zeros[1]

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(joint work with L. Iturriaga[b])

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Purpose

We consider the problem

$$\begin{cases} (-\Delta)^k u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ (-\Delta)^i u = 0 & \text{on } \partial\Omega, \; i = 0,..,k-1, \end{cases} \qquad (P^k_{\lambda,\mu})$$

where

- $\Omega \subset \mathbb{R}^N$ bounded smooth domain,
- $\lambda, \mu \geq 0$ are two parameters,
- f, g are nonnegative functions,
- $k \in \mathbb{N}$.

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Introduction: nonlinearities with zeros

Consider the Laplacian case:

$$\begin{cases} -\Delta\, u = \lambda h(x,u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

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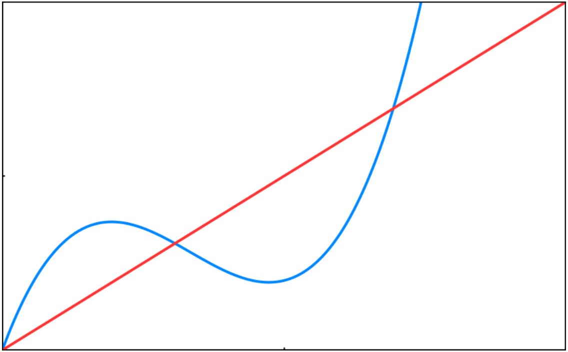
$$\text{suppose} \qquad h(x,t) = h(t) > 0 \text{ for } t > 0 \tag{1.1}$$

$$\liminf_{t\to 0^+} \frac{h(x,t)}{t} > 0, \qquad \liminf_{t\to\infty} \frac{h(x,t)}{t} > 0 \tag{1.2}$$



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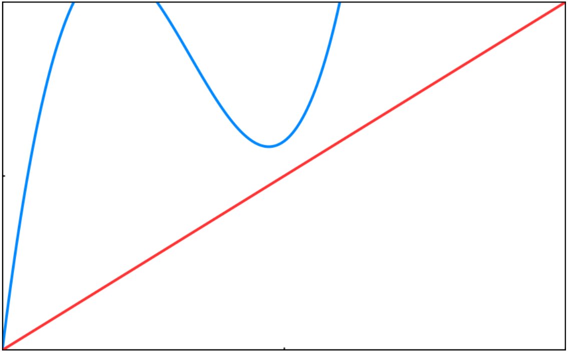

no positive solution for λ large!

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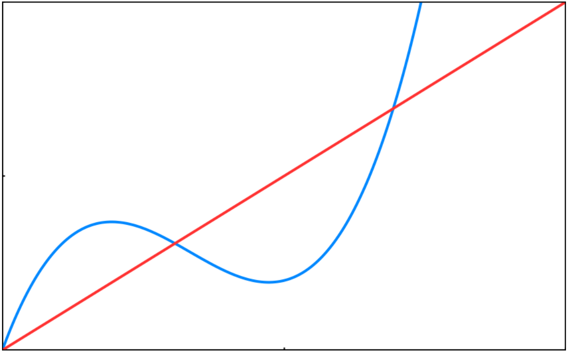
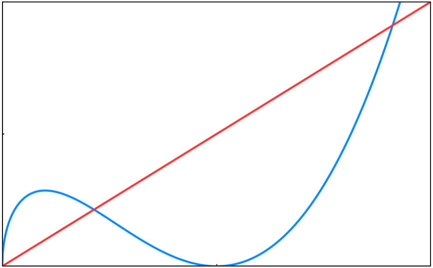


(probably) 2 solutions for λ small (for instance (Ambrosetti, Brezis, and Cerami, 1994))

A positive zero in the nonlinearity implies there always exists an intersection:



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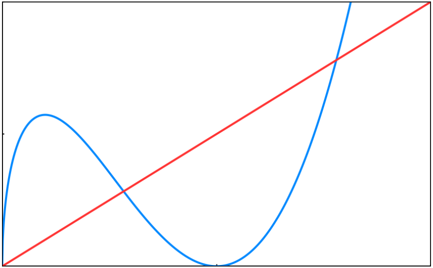

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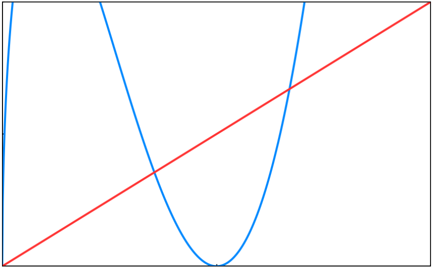

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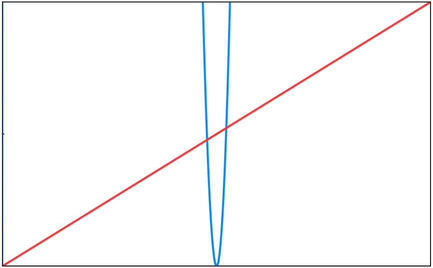

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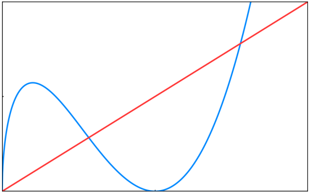
a simple case:

$$\begin{cases} -\Delta u = \lambda h(u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

model: $h(u) = u^q|a-u|^r$,
$1 < r + q < 2^*, \ q \in (0,1]$



How to find two solutions?

subsolution $\underline{u} = \varepsilon\phi_1$.

supersolution $\overline{u} = a$.

First solution $\underline{u} \leq u \leq \overline{u}$

If $u < \overline{u}$, then it is a local minimum.

Then there exists a Mountain Pass solution

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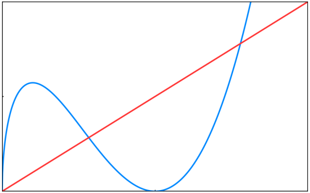

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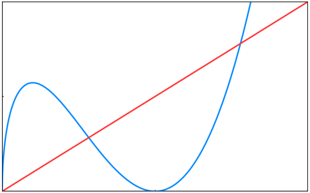

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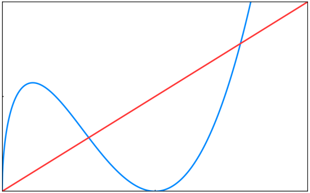

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Also, some related works:
• in (Iturriaga, Lorca, and Massa, 2010) we considered supercritical problems,
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The problem

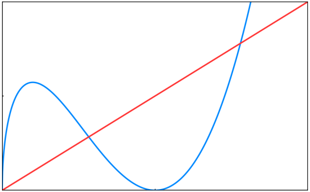
We start by considering the following problem:

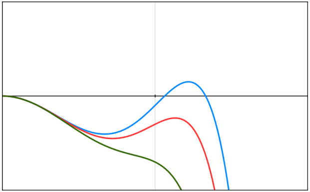
$$\begin{cases} (-\Delta)^k u = \lambda h(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ (-\Delta)^i u = 0 & \text{on } \partial\Omega, \ i = 0,..,k-1, \end{cases} \tag{2.1}$$

- Here we have no sub and supersolution method.
- We try a purely variational approach: consider

$$J_\lambda : \mathbb{H} \to \mathbb{R} : u \mapsto J_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_{\mathbb{H}}^2 - \lambda \int_\Omega H(x,u^+) \, .$$

$$\mathbb{H} = \{u \in H^k(\Omega) \text{ such that } (-\Delta)^i u = 0 \text{ on } \partial\Omega, \ i = 0,..,[(k-1)/2], \} \, ,$$





• there exists $u_0 \in \mathbb{H}$ such that, for every $\lambda > 0$ there exists $t_0(\lambda) > 0$ such that one has $J_\lambda(tu_0) < 0 \qquad \textit{for } 0 < t < t_0(\lambda)$

• there exists $e \in \mathbb{H}$ such that, for any $\lambda \geq 0$

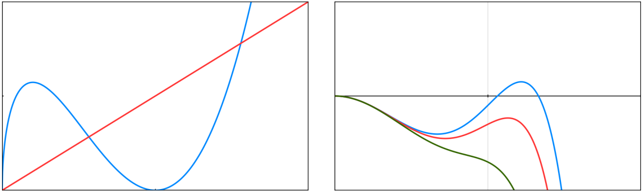
$J_\lambda(te) \to -\infty \qquad \textit{when } t \to +\infty\,.$

• We need something like:

— given $\lambda > 0$ and $H \in \mathbb{R}$, there exist $\rho_H(\lambda) > 0$ such that

$$J_\lambda(u) > H \qquad \textit{for } \|u\|_{\mathbb{H}} = \rho_H(\lambda). \tag{2.2}$$

The presence of the zero is not enough to guarantee this!



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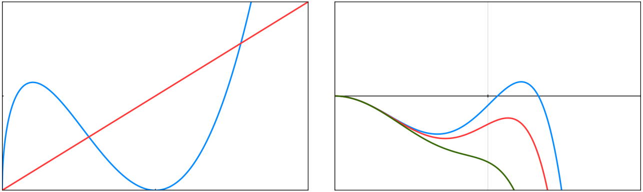

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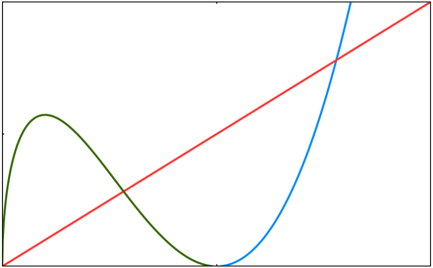
Split the nonlinearity:

$$\lambda f(x,u) + \mu g(x,u)\,,$$

where $f, g \geq 0$ and

$$\begin{cases} f(x,t) = 0 & \text{if } t \geq a(x), \\ g(x,t) = 0 & \text{if } 0 \leq t \leq a(x). \end{cases} \qquad (Z)$$

Model: $\begin{cases} f(x,u) = (u^+)^q[(a(x)-u)^+]^p, & q \in (0,1), p > 0, \\ g(x,u) = [(u-a(x))^+]^r, & r \in (1, 2^*_{N,k}-1): \end{cases}$



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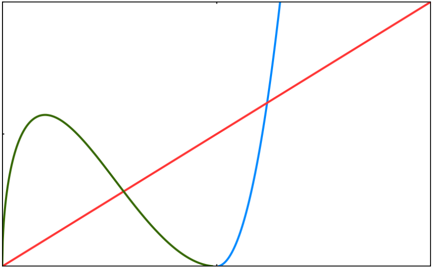

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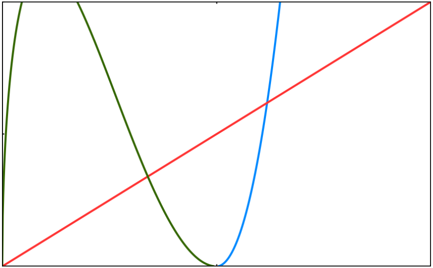

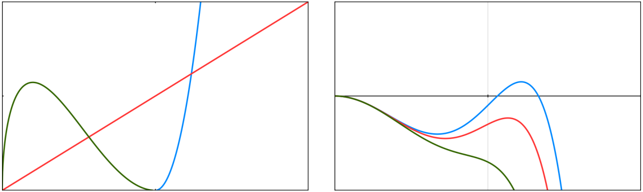

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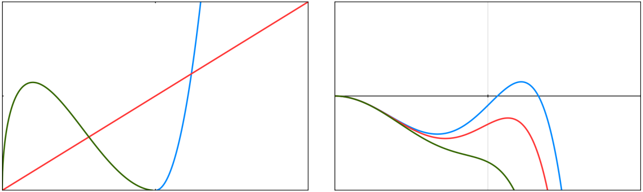

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The existence of two solutions result

- $f, g : \overline{\Omega} \times [0, +\infty) \longrightarrow [0, +\infty)$ are Carathéodory functions and satisfy
  - $f(x, 0) = g(x, 0) = 0$,
  - conditions at ∞ for PS condition,
  - local (sublinearity) condition at the origin:

    $$\lim_{t \longrightarrow 0^+} \frac{f(x,t)}{t} = +\infty \quad \text{uniformly in } x \in \omega \subset\subset \Omega,$$
  - local (superlinearity) condition at infinity :

    $$\lim_{t \to +\infty} \frac{g(x,t)}{t} = +\infty \quad \text{uniformly in } x \in \omega_2 \subset\subset \Omega.$$

Then: **there exists a function $M : (0, \infty) \to (0, \infty]$ such that the problem $(P^k_{\lambda,\mu})$, $k \in \mathbb{N}$, has at least two positive solutions for $\lambda > 0$ and $0 < \mu < M(\lambda)$.**

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Question

Is this bound on μ really necessary?

Actually

- in (Iturriaga, Massa, Sánchez, and Ubilla, 2010) there is no bound: $M(\lambda) \equiv \infty$,
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Then we investigate when $M(\lambda) = \infty$ or $M(\lambda) < \infty$.

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Existence for every μ

We can prove that $M(\lambda) = \infty$ if "the first solution has a suitable neighborhood below a(x)".
This is true for example:

1. in the (Iturriaga, Massa, Sánchez, and Ubilla, 2010) situation: $k = 1$, $a(x)$ supersolution.
2. for $k = 1$, small λ, $a(x) \geq a_0 > 0$.
3. for $k > 1$, small λ, $a(x) \geq a_0 > 0$ and $N < 2k$.

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Nonexistence

On the other hand, in some cases $M(\lambda) < \infty$, in particular, no positive solution exists for large μ.

We assume $N = 1$, with $\Omega = (-1, 1)$, and

• The functions $f, g : \overline{\Omega} \times [0, +\infty) \longrightarrow [0, +\infty)$ are continuous functions and satisfy

- $f(x, 0) = g(x, 0) = 0$,
- condition (Z), with $a(x) \in \mathcal{C}(\overline{\Omega})$.
- $g(x, t) > 0$ for $t > a(x)$.
- $f(x, \tau t) > \tau f(x, t) > 0$ for every $x \in \Omega,\ \tau \in (0, 1),\ t \in (0, a(x))$.
- There exists $b_0, c_0 > 0$ such that, uniformly in $x \in \Omega$,

$$\liminf_{t \to 0^+} \frac{f(x, t)}{t} \geq b_0 \,, \qquad \liminf_{t \to +\infty} \frac{g(x.t)}{t} \geq c_0$$

We obtain

Theorem

If

- *$k \geq 2$ and $a(x)$ satisfies $\lim_{x \to \pm 1} a(x) = a^{\pm} > 0$ and a' exists and is bounded near ± 1,*

  or

- *$k \in \mathbb{N}$ and $a(x)$ is not a concave function,*

then there exists $\Lambda_1 > 0$ and $N : (\Lambda_1, \infty) \to (0, \infty)$ such that problem $(P^k_{\lambda,\mu})$ has no positive solution for $\lambda > \Lambda_1$ and $\mu > N(\lambda)$.

Idea of the proof: one first proves (using Green function and the concavity of the solutions $u_{\lambda,\mu}$)

Lemma

(1) *for every* $p \in \Omega$, $\liminf_{\lambda\to\infty} u_{\lambda,\mu}(p) \geq a(p)$, *uniformly in* $\mu \geq 0$.

(2) *for every* $p \in \Omega$, $\limsup_{\mu\to\infty} u_{\lambda,\mu}(p) \leq a(p)$, *uniformly in* $\lambda \geq 0$.

Then one obtains a contradiction:

- since $u_{\lambda,\mu}$ is concave, in can not approximate a non-concave function
- if $k \geq 2$, $u_{\lambda,\mu}$ cannot satisfy the boundary condition and approximate a since $\lim_{x\to\pm 1} a(x) = a^{\pm} > 0$.

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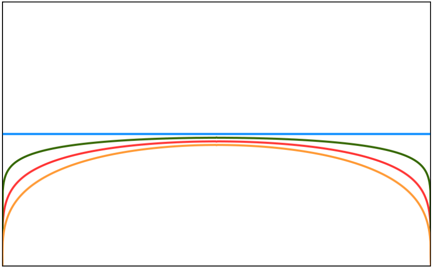
Consider the model $\begin{cases} f(x,u) = (u^+)^q[(a(x)-u)^+]^p, & q \in (0,1), p > 1, \\ g(x,u) = [(u-a(x))^+]^r, & r > 1: \end{cases}$

---

case $k = 1$, $a(x)$ constant:

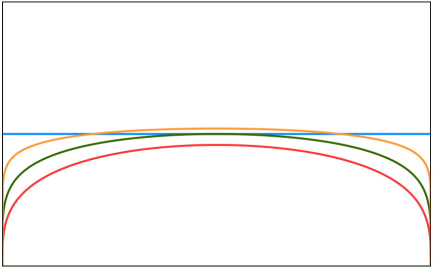


u stays below a, then it is not affected by μ.

Consider the model $\begin{cases} f(x,u) = (u^+)^q[(a(x)-u)^+]^p, & q \in (0,1), p > 1, \\ g(x,u) = [(u-a(x))^+]^r, & r > 1: \end{cases}$

---

case $k \geq 2$, $a(x)$ constant:



eventually u exceeds $a(x)$ for λ large
then for μ large there is no solution

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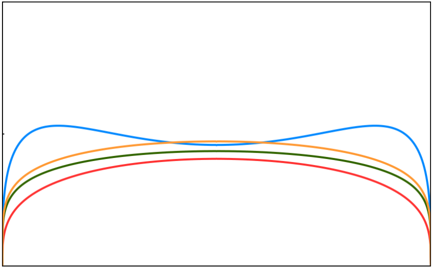
---

case any k, $a(x)$ not concave:



eventually u exceeds $a(x)$ for λ large
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Comparison in the model case

In particular, for the model autonomous problem in dimension $N = 1$

$$\begin{cases} f(x,u) = (u^+)^q[(a-u)^+]^p, & q \in (0,1), p > 1, \\ g(x,u) = [(u-a)^+]^r, & r > 1 : \end{cases}$$

- if $k = 1$ then $M(\lambda) \equiv \infty$ ([2]): two solutions for every $\lambda, \mu > 0$
- if $k \geq 2$ then
  - $M(\lambda) \equiv \infty$ for small λ
  - $M(\lambda) < \infty$ for large λ

---

for the model problem in dimension $N = 1$ with non-concave $a(x) \geq a_0 > 0$,

$$\begin{cases} f(x,u) = (u^+)^q[(a(x)-u)^+]^p, & q \in (0,1), p > 1, \\ g(x,u) = [(u-a(x))^+]^r, & r > 1 : \end{cases}$$

- for any $k \in \mathbb{N}$,
  - $M(\lambda) \equiv \infty$ for small λ
  - $M(\lambda) < \infty$ for large λ

[2] Iturriaga, Massa, Sánchez, and Ubilla (2010).

Preprint

E. Massa, L. Iturriaga, *Existence, nonexistence and multiplicity of positive solutions for the poly-Laplacian and nonlinearities with zeros*. 2017.

Still working..

- Obtain the same behavior (loss of existence) in dimension $N > 1$
- Obtain $M(\lambda) = \infty$ for small λ, in more cases: $k > 1$, $N \geq 2k$.

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