Positive Solutions for an Elliptic Equation in an Annulus with a Superlinear Nonlinearity with Zeros

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Abstract

We study existence, multiplicity, and the behavior, with respect to λ, of positive radially symmetric solutions of $-\Delta u = \lambda h(x,u)$ in annular domains in $\mathbb{R}^N$, $N \geq 2$. The nonlinear term has a superlinear local growth at infinity, is nonnegative, and satisfies $h(x, a(x)) = 0$ for a suitable positive and concave function a. For this, we combine several methods such as the sub and supersolutions method, a priori estimates and degree theory.

1 Introduction

Papers on existence and multiplicity of positive radial solutions of elliptic equations in annular bounded domains, imposing Dirichlet and/or Neumann boundary conditions, have been widely considered in the literature. For example, see [7], [10], [12], [13], [14], [17], [18] and references therein. In these papers, existence of positive solutions is studied under

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several behaviors of the nonlinearity, in particular, nonlinearities which are superlinear at infinity and with different behaviors at the origin are considered.

On the other hand, more details about the geometry of the nonlinearity become important when one is interested in the multiplicity of the solutions, see for example the nice review in [15]; as was remarked in this paper, the presence of zeros in the nonlinearity usually provides multiple solutions.

For our purpose, we shall restrict our attention to the ordinary boundary value problem

$$(P_\lambda) \qquad \begin{cases} v''(t) + \lambda q(t) f(t, v(t)) = 0, & \text{for } \ t \in (0,1), \\ v(0) = v(1) = 0, & \end{cases}$$

where the function $q(t)$ is continuous and positive on the interval $[0,1]$, while for f we consider the following four assumptions:

$(H1)$ The function $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and there exists a continuous function $a : [0,1] \to (0,+\infty)$, which is concave, such that $f(t,0) = f(t,a(t)) = 0$ and $f(t,v) > 0$ if $0 < v < a(t)$.

$(H2)$ There exists a continuous function $b : [0,1] \to (0,+\infty)$ such that

$$\lim_{v\to 0^+} \frac{f(t,v)}{v} = b(t) \ \text{ uniformly in } \ t \in [0,1].$$

$(H3)$ There exist positive constants $0 < \alpha < \beta < 1$ such that

$$\lim_{v\to +\infty} \frac{f(t,v)}{v} = +\infty \ \text{ uniformly in } \ t \in [\alpha,\beta].$$

$(M2)$ (a) The function $f_v := \frac{\partial f}{\partial v}$ exists and is continuous in the set $\{(t,v) \,:\, t \in [0,1], v \in [0,a(t)]\}$,
(b) $f_v(t,v) < v^{-1} f(t,v)$ in the set $\{(t,v) \,:\, t \in (0,1), v \in (0,a(t))\}$.

Simple examples of functions satisfying our hypotheses are $f(t,u) = u|1-u|^2$ or $f(t,u) = u|2 - t^2 - u|^2$.

Observe that in [9], we studied the $p-$Laplace equation $-\Delta_p u = \lambda\, h(x,u)$ with zero Dirichlet boundary conditions, in a general domain Ω, where h is nonnegative but have a zero at a variable positive value, like we assume in $(H1)$. We obtained the existence of at least one solution for small positive λ and at least two solutions for large values of λ, assuming a superlinear and subcritical growth at infinity and a p-linear behavior at the origin for the nonlinearity h. We also studied the asymptotical behavior of the solutions as λ tends to zero and to infinity. Later, in [8], we extended these results to supercritical nonlinearities provided Ω is convex and h does not depend on $x \in \Omega$.

Our goal in this paper is to study the effect of these variable zeros on the existence and multiplicity of radial solutions on an annulus; since the radial problem in the annulus may be reduced to a boundary value problem for a suitable ordinary differential equation, it will be possible to take advantage of some techniques available in dimension one, and in particular of the fact that any positive solution is concave, in order to obtain the same kind of results but under weaker assumptions than those considered in [9] and [8].

More precisely, the hypothesis $(H3)$ says that the nonlinearity is locally superlinear at $+\infty$, that is, it is required to be superlinear only in a small interval (which means in a small annulus when considering the radial case). This assumption constitutes one of the main differences when compared to the results in [9] and [8] we cited previously: it is not necessary here to assume superlinearity in the whole of Ω.

We observe that other more technical hypotheses were needed in [9] and [8] and are not imposed here, in particular, it is not necessary to assume subcritical growth (as in [9]) nor the convexity of the domain and that the nonlinear term is independent of $x \in \Omega$ (as in [8]).

An example of a function f which satisfies our broader hypotheses, but not those in [9] and [8], could be

$$f(t,u) = \begin{cases} u(1-u)^2 & \text{for } u \le 1\,, \\ (e^{u-1}-u)\phi(t) & \text{for } u > 1\,, \end{cases}$$

where $\phi \in \mathcal{C}^0([0,1])$ is nonnegative, may be null in some set, but is positive in $[\alpha,\beta]$.

We remark that the hypothesis $(M2)$ is used to obtain the second solution, by a nice homotopy argument. It also provides the monotonicity required to apply the sub and supersolutions method.

Before stating our results, we need to introduce some notations. Let $m : [0,1] \to \mathbb{R}$ be a continuous function and consider the eigenvalue problem

$$\begin{cases} -v''(t) = \lambda m(t) v(t) \quad \text{in} \quad (0,1), \\ v(0) = v(1) = 0\ . \end{cases} \tag{1.1}$$

It is known that there exist:

- a sequence of positive eigenvalues, which we denote by $\lambda_{i,m}$, $(i > 0)$, provided $m > 0$ in a set of positive measure,
- a sequence of negative eigenvalues, which we denote by $\lambda_{i,m}$, $(i < 0)$, provided $m < 0$ in a set of positive measure.

In particular, $\lambda_{1,m} > 0$ is called the first eigenvalue of the problem (1.1) and the associated eigenfunction will be denoted by $\phi_{1,m}$. It is known that $\phi_{1,m} > 0$ with $\phi'_{1,m}(0) > 0$ and $\phi'_{1,m}(1) < 0$, while $\phi_{i,m}$ changes sign for $|i| > 1$. In addition, we have the characterization

$$\int_0^1 |v'|^2 \ge \lambda_{1,m} \int_0^1 m\,|v|^2 \ \text{ for any } v \in H_0^1(0,1), \tag{1.2}$$

where equality holds if and only if v is a multiple of $\phi_{1,m}$ (see for instance [1], [16]).

The main results in this paper are the following three theorems.

Theorem 1.1. *Suppose $f(t,v)$ satisfies the hypotheses $(H1)$ through $(H3)$. Then there exists a positive solution of Problem (P_λ), for every $0 < \lambda < \lambda_{1,qb}$.*

Theorem 1.2. *Suppose $f(t,v)$ satisfies the hypotheses $(H1)$ through $(H3)$ and $(M2)$. Then there exist at least two ordered positive solutions of Problem (P_λ), for every $\lambda > \lambda_{1,qb}$.*

Theorem 1.3. *Suppose $f(t,v)$ satisfies the hypotheses $(H1)$ through $(H3)$ and $\{v_\lambda\}$ is a family of positive solutions of Problem (P_λ). Then*

a) one has $\|v_\lambda\|_\infty \to +\infty$ when $\lambda \to 0^+$,

b) if we assume also that $f(t,v) > 0$ for $v \neq 0$, $v \neq a(t)$, then

$$v_\lambda \to a \quad \text{pointwise in } (0,1) \text{ and } ||v_\lambda||_\infty \to ||a||_\infty\,, \qquad \text{when } \lambda \to \infty\,.$$

Application. As we said at the beginning, our results for Problem (P_λ) may be applied on the existence, multiplicity, and the asymptotic behavior with respect to λ of positive solutions of the problem

$$(A_\lambda) \qquad \begin{cases} -\Delta u = \lambda h(x,u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega = \{x \in \mathbb{R}^N : r_1 < |x| < r_2\}$ with $0 < r_1 < r_2$, $N \geq 2$, and where $\lambda > 0$ is a real parameter and h is a radial nonnegative nonlinearity which is locally superlinear at $+\infty$ and has a positive zero which may vary in the radial direction.

In fact, one can make a standard change of variables. In the case $N \geq 3$, if $t = -\frac{A}{r^{N-2}} + B$ and $v(t) = u(r)$ where

$$A = \frac{(r_1 r_2)^{N-2}}{r_2^{N-2} - r_1^{N-2}} \quad \text{and} \quad B = \frac{{r_2}^{N-2}}{r_2^{N-2} - r_1^{N-2}}\,,$$

then Problem (A_λ) transforms into the boundary value problem for the ODE

$$(P_\lambda) \qquad \begin{cases} v''(t) + \lambda q(t) f(t, v(t)) = 0\,, \quad \text{for} \quad 0 < t < 1\,, \\ v(0) = v(1) = 0\,, \end{cases}$$

where $f(t,v) = h\left((\frac{A}{B-t})^{1/(N-2)}, v\right)$ and $q(t) = (N-2)^{-2}\frac{A^{2/(N-2)}}{(B-t)^{2(N-1)/(N-2)}}$. In the case $N = 2$ one sets $r = r_2\left(\frac{r_1}{r_2}\right)^t$, and $v(t) = u(r)$, obtaining again the Problem (P_λ), this time with $q(t) = \left[r_2(\frac{r_1}{r_2})^t \log\left(\frac{r_2}{r_1}\right)\right]^2$ and $f(t,v) = h\left(r_2\left(\frac{r_1}{r_2}\right)^t, v\right)$. Note that, in both cases, the function $q(t)$ is well defined, continuous and bounded between positive constants in the interval $[0,1]$.

The paper is organized as follows. In Section 2, we establish some notation, as well as some basic facts, and we state some known results that will be used in the paper.

Section 3 contains the proof of existence of the first solution for λ small (Theorem 1.1); a Krasnosel'skii fixed point theorem in cones of expansion/compression type is used. In Section 4, we show the existence of two solutions for λ large (Theorem 1.2), using sub and supersolutions method, degree theory and a strict monotonicity property with respect to the weight m for the eigenvalue problem (1.1) (see [4]). Finally, in Section 5, we study the asymptotic behavior of the solutions (Theorem 1.3).

2 Preliminaries

We consider the Banach space $X = \mathcal{C}([0,1])$ endowed with the norm $\|v\|_\infty = \max_{t\in[0,1]} |v(t)|$ and, in view of the fact that if $u(t)$ is a nonnegative solution of Problem (P_λ) then it is a concave function, we define the cone

$$C = \{v \in X \,:\, v \text{ is concave and } v(0) = v(1) = 0\}\,.$$

Let $T : C \longrightarrow X$ be defined by

$$Tv(t) = \lambda \int_0^1 G(t,s)\, q(s) f(s, v(s))\, ds\,,$$

where $G(t,s)$ denotes the Green's function for the interval $(0,1)$, given by

$$G(t,s) = \begin{cases} s(1-t)\,, & \text{for } \; 0 \le s \le t \le 1\,, \\ t(1-s)\,, & \text{for } \; 0 \le t \le s \le 1\,. \end{cases}$$

Observe that this function satisfies

$$G(t,s) > 0 \;\text{ on }\; (0,1)\times(0,1) \;\text{ and }\; G(t,s) \le G(s,s) = s(1-s), \;\text{ for all }\; t,s \in [0,1]\,.$$

It is easy to see that T is a completely continuous operator, it maps C onto C and its nontrivial fixed points in C correspond to the positive solutions of Problem (P_λ).

Now we state the following well known result without proof (compare [2], [3], [5], [6], [11]).

Theorem A. *Let X be a Banach space endowed with a norm $\|\cdot\|$, and let $C \subset X$ be a cone in X. For $R > 0$, define $C_R = \{v \in C : \|v\| < R\}$. Let r and R be numbers satisfying $0 < r < R$. Assume that $T : \overline{C_R} \to C$ is a completely continuous operator such that*

$$\|Tv\| < \|v\|, \text{ for all } v \in \partial C_r \text{ and } \|Tv\| > \|v\|, \text{ for all } v \in \partial C_R, \text{ or}$$
$$\|Tv\| > \|v\|, \text{ for all } v \in \partial C_r \text{ and } \|Tv\| < \|v\|, \text{ for all } v \in \partial C_R,$$

where $\partial C_R = \{v \in C : \|v\| = R\}$. Then T has a fixed point $v \in C$, with $r < \|v\| < R$.

We also state some elementary properties of concave functions that will be useful throughout our arguments.

Lemma 2.1. *Given a function v in the cone C and a point $p \in (0,1)$, the following estimates hold:*

$$(i) \qquad v(t) \geq \begin{cases} \frac{t}{p}\, v(p) & t < p\,, \\ \frac{1-t}{1-p}\, v(p) & t > p \end{cases} \qquad \text{and} \qquad (ii) \qquad v(t) \leq \begin{cases} \frac{t}{p}\, v(p) & t > p\,, \\ \frac{1-t}{1-p}\, v(p) & t < p\,. \end{cases}$$

Moreover, for all $0 < t_0 < t_1 < 1$, we have

$$(iii) \qquad \min_{t\in[t_0,t_1]} v(t) \geq c_{t_0,t_1}\, ||v||_\infty\,,$$

where $c_{t_0,t_1} := \min\{t_0, 1-t_1\}$.

Proof. Inequality (i) is an immediate consequence of the concavity of $v(t)$ on $[0,1]$ and of the boundary condition, actually, if v lies below the triangle whose vertices are $(0,0)$, $(p, v(p))$ and $(1,0)$ then it can not be concave. Inequality (ii) follows readily from (i) and the last claim is obtained by choosing p in (i) where the norm is attained. □

Finally, we need to introduce the following strict monotonicity property with respect to the weight m for the eigenvalue problem (1.1) (see [4]). Here we use the notation $\leq\not\equiv$ to mean inequality a.e. together with strict inequality on a set of positive measure.

Proposition 2.2 (Strict Monotonicity Property)**.** *Consider the eigenvalue problem (1.1); let m and $\widehat{m}$ be two bounded weights with $m \leq\not\equiv \widehat{m}$ and let $j \in \mathbb{Z}_0$. Then $\lambda_{j,m} > \lambda_{j,\widehat{m}}$ whenever they exist.*

3 A solution for λ small. Proof of Theorem 1.1

In this section we will show the existence of a solution for $\lambda \in (0, \lambda_{1,qb})$, by verifying, in the next two lemmas, the hypotheses of Theorem A.

Lemma 3.1. *Suppose conditions $(H1)$ and $(H3)$ hold. Let $\|\cdot\|$ be a norm on $\mathcal{C}([0,1])$ which is equivalent to $\|\cdot\|_\infty$. Then, for all $\Lambda, K > 0$ there exists $R > 0$ such that, for all $\lambda \geq \Lambda$ and all $v \in \{v \in C : \|v\| \geq R\}$, we have*

$$||Tv|| > K||v||\,.$$

Proof. Without loss of generality we will give the proof using the norm $\|\cdot\|_\infty$. By hypothesis $(H3)$, given $M > 0$, there exists $N > 0$ such that $v > N$ implies $f(s,v) \geq Mv$, for all $s \in [\alpha, \beta]$. By estimate (iii) in Lemma 2.1, $v(s) \geq c_{\alpha,\beta}\, ||v||_\infty$ for $s \in [\alpha,\beta]$. Then, if we choose $\|v\|_\infty \geq R > \frac{N}{c_{\alpha,\beta}}$, we have (using that $f \geq 0$ by hypothesis $(H1)$)

$$\begin{aligned} ||Tv||_\infty \geq Tv\,(1/2) &= \lambda \int_0^1 G\,(1/2,s)\, q(s) f(s,v(s))\, ds \geq \lambda \int_\alpha^\beta G\,(1/2,s)\, q(s) f(s,v(s))\, ds \\ &\geq \lambda \int_\alpha^\beta G\,(1/2,s)\, q(s)\, M\, c_{\alpha,\beta} \|v\|_\infty\, ds \\ &= \left(\lambda M\, c_{\alpha,\beta} \int_\alpha^\beta G\,(1/2,s)\, q(s)\, ds\right) \|v\|_\infty. \end{aligned} \tag{3.3}$$

By setting $M > K\left(\Lambda c_{\alpha,\beta}\int_\alpha^\beta G\,(1/2,s)\,q(s)\,ds\right)^{-1}$ one obtains $\|Tv\|_\infty > K\lambda\Lambda^{-1}\|v\|_\infty \geq K\|v\|_\infty$.

□

Lemma 3.2. *Suppose conditions* $(H1)$ *and* $(H2)$ *hold. Then, for all* $\lambda \in (0, \lambda_{1,qb})$*, there exist a norm* $\|\cdot\|_*$ *equivalent to* $||\cdot||_\infty$ *and* $r > 0$ *such that, for all* $v \in \{v \in C : \|v\|_* = r\}$*, we have*

$$||Tv||_* < ||v||_* \,.$$

Proof. Since $\lambda < \lambda_{1,qb}$, for some $\varepsilon > 0$ small enough one has

$$0 < \lambda < (1+\varepsilon)\lambda < \lambda_{1,qb}.$$

Next, take $E > 0$ such that

$$E\lambda_{1,qb}\sup_{t\in[0,1]}\frac{\psi(t)}{\phi_{1,qb}(t)} < \frac{\lambda_{1,qb}}{\lambda(1+\varepsilon)} - 1\,,$$

where $\psi(t) = \int_0^1 G\,(t,s)\,q(s)b(s)\,ds$ (observe that ψ, like $\phi_{1,qb}$, is bounded with bounded derivative at 0 and 1 so the supremum is finite).

We consider the norm

$$\|v\|_* = |v|_E = \inf\{\xi :\ \xi(\phi_{1,qb}+E) \geq v\} = ||v/(\phi_{1,qb}+E)||_\infty\ ,$$

which is equivalent to $||\cdot||_\infty$.

By hypotheses $(H1)$ and $(H2)$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $0 < v < \delta$ implies $f(s,v) < (1+\varepsilon)b(s)v$ for all $s \in [0,1]$. Let $r > 0$ be such that $r(||\phi_{1,qb}||_\infty + E) < \delta$, so that $|v|_E = r$ implies $||v||_\infty < \delta$.

If $v \in C$ with $|v|_E = r$ we get

$$\begin{aligned}
Tv\,(t) &= \lambda\int_0^1 G\,(t,s)\,q(s)f(s,v(s))\,ds \leq \lambda\int_0^1 G\,(t,s)\,q(s)(1+\varepsilon)b(s)\frac{\phi_{1,qb}(s)+E}{\phi_{1,qb}(s)+E}\,v(s)\,ds\\
&\leq \lambda\int_0^1 G\,(t,s)\,q(s)(1+\varepsilon)b(s)(\phi_{1,qb}(s)+E)|v|_E\,ds\\
&= \lambda(1+\varepsilon)\left\{\int_0^1 G\,(t,s)\,q(s)b(s)\phi_{1,qb}(s)\,ds + \int_0^1 G\,(t,s)\,q(s)b(s)E\,ds\right\}|v|_E\\
&= (1+\varepsilon)\left\{\frac{\lambda}{\lambda_{1,qb}}\,\phi_{1,qb}(t) + \lambda E\psi(t)\right\}|v|_E\\
&= (1+\varepsilon)\phi_{1,qb}(t)\frac{\lambda}{\lambda_{1,qb}}\left\{1+\lambda_{1,qb}E\frac{\psi(t)}{\phi_{1,qb}(t)}\right\}|v|_E \leq \phi_{1,qb}(t)|v|_E < (\phi_{1,qb}(t)+E)|v|_E.
\end{aligned}$$

Therefore $|Tv|_E < |v|_E$. □

With the above results we may now obtain the solution of Problem (P_λ).

Proof of Theorem 1.1. The existence of the positive solution is a consequence of Theorem A by virtue of the Lemmas 3.1 and 3.2 (using the equivalent norm $\|\cdot\|_* = |\cdot|_E$ from Lemma 3.2, where E depends on λ). □

4 Two solutions for λ large. Proof of Theorem 1.2

In this section we will show that Problem (P_λ) has at least two positive solutions for $\lambda > \lambda_{1,qb}$. Actually, the existence of a first solution v which satisfies $v(t) \le a(t)$ is a consequence of Theorem 1.1 in [9], under our hypotheses $(H1)$, $(H2)$ and $(M2)$. In fact, in [9] we used a hypothesis, called $(M1)$, which holds below $a(t)$ as a consequence of our hypothesis $(M2)$.

In the following we will establish, by a contradiction argument, the existence of a second solution for all $\lambda > \lambda_{1,qb}$.

From now on we assume the hypotheses of Theorem 1.2, we fix λ and we denote by $v_1 = v_1(\lambda)$ the solution of Problem (P_λ) found above.

Consider the problem

$$(Q_+)\quad \begin{cases} -(v_1+u)''(t) &= \lambda q(t) f(t, v_1+u^+)\,, \quad \text{for} \quad 0<t<1\,, \\ u(0)=u(1) &= 0. \end{cases}$$

If $u \ge 0$ is a nontrivial solution of (Q_+), then v_1+u is a second positive solution of (P_λ), which satisfies $v_1+u \ge v_1$.

For $\theta,\tau \in [0,1]$ and $\eta \ge 0$, we define the following parameterized family of operators:

$$\begin{cases} T_{\theta,\tau,\eta}u(t) = \lambda\theta \int_0^1 G\,(t,s)\,q(s)\left[\frac{f(s,v_1(s)+\tau u^+(s))-f(s,v_1(s))}{\tau}+\eta\right]\,ds, \\ T_{\theta,0,\eta}u(t) = \lambda\theta \int_0^1 G\,(t,s)\,q(s)\,[f_v(s,v_1(s))u^+(s)+\eta]\;ds. \end{cases}$$

Observe that, by hypothesis (M_2), T is continuous with respect to the parameters θ,τ,η.

With this definition, a solution of (Q_+) is a fixed point of $T_{1,1,0}$, since $-\lambda q(t) f(t,v_1) = v_1''$. On the other hand, a fixed point of $T_{1,1,\eta}$ is a solution of the more general problem

$$(Q_{1,1,\eta})\quad \begin{cases} -(v_1+u)'' &= \lambda q(t)[f(t,v_1+u^+)+\eta]\,, \\ u(0)=u(1) &= 0, \end{cases}$$

while a fixed point of $T_{\theta,0,0}$ is a solution of

$$(Q_{\theta,0,0})\quad \begin{cases} -u'' &= \lambda\theta q(t) f_v(t,v_1)u^+\,, \\ u(0)=u(1) &= 0. \end{cases}$$

Our purpose is to find a nontrivial fixed point of $T_{1,1,0}$. This will be obtained through the following lemmas.

Lemma 4.1. *Suppose conditions $(H1)$, $(H2)$ and $(M2)$ hold. Then, if u is a solution of (Q_+) or of $(Q_{\theta,0,0})$ then $u \ge 0$.*

Proof. Let u be a solution of (Q_+). By using u^- as a test function we get

$$\int_0^1 (v_1(s)+u(s))'(u^-)'(s)ds \;=\; \lambda \int_0^1 q(s) f(s,v_1(s)+u^+(s))u^-(s)\,ds\,,$$

and from this we obtain

$$\int_0^1 v_1'(s)(u^-)'(s)ds - \int_0^1 ((u^-)')^2(s)ds \;=\; \lambda \int_0^1 q(s)f(s,v_1(s))u^-(s)\,ds\,.$$

Since v_1 is a solution of (P_λ), we conclude that $\int_0^1((u^-)')^2(s)ds = 0$, then $u^- \equiv 0$. In the case that u is a solution of $(Q_{\theta,0,0})$, the same argument gives the same conclusion. □

Lemma 4.2. *Suppose conditions $(H1)$ through $(H3)$ and $(M2)$ hold. Then there exists a constant $D > 0$, which does not depend on η, such that $T_{1,1,\eta}u = u$ implies $\|u\|_\infty \leq D$ for any $\eta \geq 0$.*

Proof. Suppose, for sake of contradiction, that $\{u_n\}$ is a sequence of fixed points of $T_{1,1,\eta_n}$ with $\|u_n\|_\infty \to +\infty$ and arbitrary $\eta_n \geq 0$. Then

$$v_1(t) + u_n(t) \;=\; \lambda \int_0^1 G\,(t,s)\,q(s)[f(s, v_1(s) + u_n^+(s)) + \eta_n]\,ds\,. \tag{4.4}$$

Since $v_1 + u_n$ satisfies $(Q_{1,1,\eta})$, it is a concave and positive function. Then, using the estimate (iii) in Lemma 2.1,

$$v_1(s) + u_n^+(s) \geq v_1(s) + u_n(s) \geq c_{\alpha,\beta}\|v_1 + u_n\|_\infty \;\text{ in } [\alpha,\beta]. \tag{4.5}$$

Since the integrand in (4.4) is positive, we obtain

$$\begin{aligned} v_1(t) + u_n(t) \;&\geq\; \lambda \int_\alpha^\beta G\,(t,s)\,q(s)(v_1(s) + u_n^+(s))\frac{f(s, v_1(s) + u_n^+(s))}{v_1(s) + u_n^+(s)}\,ds \\ &\geq\; \lambda \int_\alpha^\beta G\,(t,s)\,q(s)c_{\alpha,\beta}\|v_1 + u_n\|_\infty \frac{f(s, v_1(s) + u_n^+(s))}{v_1(s) + u_n^+(s)}\,ds. \end{aligned}$$

By condition $(H3)$ and (4.5), for any $M > 0$ one has $\frac{f(s,v_1(s)+u_n^+(s))}{v_1(s)+u_n^+(s)} \geq M$ in $[\alpha,\beta]$ for n suitably large, then

$$v_1(t) + u_n(t) \;\geq\; \left(M\lambda\, c_{\alpha,\beta} \int_\alpha^\beta G\,(t,s)\,q(s)\,ds\right)\|v_1 + u_n\|_\infty.$$

This leads to the contradiction

$$1 \geq \frac{v_1(1/2) + u_n(1/2)}{\|v_1 + u_n\|_\infty} \geq M\lambda\, c_{\alpha,\beta}\left\{\int_\alpha^\beta G\,(1/2,s)\,q(s)\,ds\right\}$$

for arbitrary $M > 0$. The assertion is then proved. □

Lemma 4.3. *Suppose conditions $(H1)$, $(H2)$ and $(M2)$ hold. Then, given any $\widehat{R} > 0$, there exists $\overline{\eta}(\widehat{R}) > 0$ (which does not depend on $\lambda > \lambda_{1,qb}$), such that $T_{1,1,\eta}u = u$ has no solution in $\overline{B_{\widehat{R}}}$ for $\eta \geq \overline{\eta}(\widehat{R})$. As a consequence*

$$deg(Id - T_{1,1,\eta}, B_{\widehat{R}}, 0) = 0 \quad \text{for } \eta \geq \overline{\eta}(\widehat{R})\,. \tag{4.6}$$

Proof. We have

$$\begin{aligned} v_1(t)+u(t) &= \lambda\int_0^1 G(t,s)\,q(s)[f(s,v_1(s)+u^+(s))+\eta]\,ds \\ &\geq \lambda\eta\int_0^1 G(t,s)\,q(s)\,ds\,. \end{aligned}$$

If $\|u\|_\infty \leq \widehat{R}$, then $\|v_1\|_\infty + \widehat{R} \geq v_1(1/2)+u(1/2)$ and so

$$\|v_1\|_\infty + \widehat{R} \geq \eta\lambda\int_0^1 G(1/2,s)\,q(s)\,ds\,.$$

This is impossible for large η, that is, no fixed point lies in $\overline{B_{\widehat{R}}}$ and so the degree in (4.6) must be zero.

Observe that v_1 depends on λ, but $\|v_1\|_\infty \leq \|a\|_\infty$, so $\overline{\eta}(\widehat{R})$ can be chosen uniformly with respect to $\lambda > \lambda_{1,qb}$.

□

Lemma 4.4. *Suppose conditions* $(H1)$ *through* $(H3)$ *and* $(M2)$ *hold. If we set* $\widehat{R} \geq D+1$, *where* D *is the constant from Lemma 4.2, then*

$$deg(Id - T_{1,1,0}, B_{\widehat{R}}, 0) = 0. \tag{4.7}$$

Proof. By the a priori bound in Lemma 4.2, there are no fixed points on $\partial B_{\widehat{R}}$ for any $\eta \geq 0$. Then, by using equation (4.6) and the homotopy invariance of the degree,

$$0 = deg(Id - T_{1,1,\overline{\eta}(\widehat{R})}, B_{\widehat{R}}, 0) = deg(Id - T_{1,1,0}, B_{\widehat{R}}, 0). \tag{4.8}$$

□

Lemma 4.5. *Suppose conditions* $(H1)$, $(H2)$ *and* $(M2)$ *hold. Then* $T_{\theta,0,0}u = u$ *implies* $u = 0$ *for any* $\theta \in [0,1]$.

Moreover, since $T_{0,0,0} = 0$, *we obtain*

$$deg(Id - T_{\theta,0,0}, B_\rho, 0) = deg(Id, B_\rho, 0) = 1 \quad \text{for any } \rho > 0 \text{ and } \theta \in [0,1]. \tag{4.9}$$

Proof. We may assume $\theta > 0$, for otherwise $u \equiv 0$. We define

$$m(s) := \theta q(s) f_v(s, v_1(s)) \qquad \text{and} \qquad \widehat{m}(s) := q(s)\frac{f(s,v_1(s))}{v_1(s)}\,.$$

By hypothesis $(M2)$, $m(s) \leq\not\equiv \widehat{m}(s)$ in $(0,1)$ and then, by Proposition 2.2,

$$\lambda_{j,\widehat{m}} < \lambda_{j,m}, \text{ for any } j \in \mathbb{Z}_0\,.$$

Since $v_1 > 0$ satisfies the equation

$$-z'' = \lambda\widehat{m}(s)\,z$$

and $\widehat{m}(s) \geq 0$, we deduce that $\lambda = \lambda_{1,\widehat{m}}$.

On the other hand, if u is a nontrivial fixed point of $T_{\theta,0,0}$, then it is a solution of $(Q_{\theta,0,0})$. Then $u \geq 0$, so it is also a solution of

$$\left\{ \begin{array}{rcl} -u''(s) &=& \lambda m(s)u(s), \\ u(0) = u(1) &=& 0. \end{array} \right. \tag{4.10}$$

As a consequence, λ has to be an eigenvalue of (4.10), that is, $\lambda = \lambda_{j,m}$ for some $j \in \mathbb{Z}_0$.

However, since $\lambda > 0$, it cannot coincide with any $\lambda_{i,m}$ with $i < 0$, and since

$$\lambda = \lambda_{1,\widehat{m}} < \lambda_{1,m}\,,$$

it cannot coincide with any $\lambda_{i,m}$ with $i > 0$ neither. We conclude that $u \equiv 0$. □

Now we may prove the main result of this section, which concludes the proof of the Theorem 1.2 :

Proposition 4.6. *Suppose conditions $(H1)$ through $(H3)$ and $(M2)$ hold. Then there exists a nontrivial positive solution u of Problem (Q_+), that is, there exists a second positive solution of problem (P_λ).*

Proof. First observe that if $T_{1,\tau,0}u = u$ for some $\tau > 0$ and $u \neq 0$, then

$$\left\{ \begin{array}{rcl} -\tau u'' &=& \lambda q(t)[f(t, v_1 + \tau u^+) - f(t, v_1)]\,, \\ u(0) = u(1) &=& 0\,. \end{array} \right.$$

This implies that τu is a solution of (Q_+) and so it is positive by Lemma 4.1.

In order to conclude our argument, we suppose for sake of contradiction that $T_{1,\tau,0}u = u$ has no nontrivial solution for $\tau > 0$. This implies, by (4.9), that

$$1 = deg(Id - T_{1,0,0}, B_{\widehat{R}}, 0) = deg(Id - T_{1,1,0}, B_{\widehat{R}}, 0)\,,$$

contradicting equation (4.7). □

5 Asymptotic behavior. Proof of Theorem 1.3

In this section we will study the asymptotic behavior of the solutions with respect to the parameter λ, proving Theorem 1.3.

The first result deals with the asymptotic behavior of positive solutions of Problem (P_λ) for λ sufficiently small.

Proposition 5.1. *Suppose conditions $(H1)$ and $(H2)$ hold. If $\{v_\lambda\}$ is a family of positive solutions of Problem (P_λ), then $\|v_\lambda\|_\infty \to +\infty$ when $\lambda \to 0^+$.*

Proof. Suppose by contradiction that there exists a sequence $\lambda_n \to 0^+$ and a constant $c > 0$ such that $\|v_{\lambda_n}\|_\infty \le c$. By the continuity of f and $(H2)$, there exists a positive constant C such that $f(t,v) < Cv$ for $0 \le v \le c$. Then,

$$v_{\lambda_n}(t) \;=\; \lambda_n \int_0^1 G(t,s)q(s)f(s,v_{\lambda_n}(s))\,ds \le \lambda_n\, C\;\|v_{\lambda_n}\|_\infty \int_0^1 G(t,s)q(s)\,ds\,.$$

Therefore,

$$1 \le \lambda_n\, C \int_0^1 s(1-s)q(s)\,ds\,,$$

but this is impossible, since $\lambda_n \to 0^+$. □

In order to study the asymptotic behavior of the positive solutions when $\lambda \to \infty$, we first need to prove, in the next Lemma, a uniform estimate (from below and from above) for the L^∞ norm of the positive solutions of (P_λ), for λ large.

Lemma 5.2. *Suppose conditions $(H1)$ through $(H3)$ hold. Then there exist constants $\Gamma, D, \Lambda > 0$ such that if v_λ is a positive solution of (P_λ) with $\lambda > \Lambda$ then $\Gamma \le \|v_\lambda\|_\infty \le D$.*

Proof. By Lemma 3.1, there exists $R > 0$ such that

$$\|v_\lambda\|_\infty = \|Tv_\lambda\|_\infty > \|v_\lambda\|_\infty \qquad \text{for all } \lambda \ge 1, \text{ provided } \|v_\lambda\|_\infty \ge R.$$

As a consequence of this contradiction we obtain that $\|v_\lambda\|_\infty \le R+1 := D$ for all $\lambda \ge 1$.

On the other hand, by $(H2)$, given a suitably small $\varepsilon > 0$ there exist $\Gamma > 0$ such that $f(t,v) > (b(t)-\varepsilon)v > 0$ provided $0 < v < \Gamma$. Let v_λ be a positive solution of (P_λ) with $\|v_\lambda\|_\infty < \Gamma$. Using estimate (iii) in Lemma 2.1,

$$\begin{aligned}
||v_\lambda||_\infty &\ge v_\lambda\,(1/2) = \lambda \int_0^1 G\,(1/2,s)\,q(s)f(s,v_\lambda(s))\,ds \\
&\ge \lambda \int_0^1 G\,(1/2,s)\,q(s)(b(s)-\varepsilon)v_\lambda(s)\,ds \ge \lambda \int_{\frac14}^{\frac34} G\,(1/2,s)\,q(s)(b(s)-\varepsilon)v_\lambda(s)\,ds \\
&\ge \lambda \int_{\frac14}^{\frac34} G\,(1/2,s)\,q(s)(b(s)-\varepsilon)c_{\frac14,\frac34}||v_\lambda||_\infty\,ds \\
&= \left(\lambda c_{\frac14,\frac34} \int_{\frac14}^{\frac34} G\,(1/2,s)\,q(s)(b(s)-\varepsilon)\,ds\right) ||v_\lambda||_\infty\,.
\end{aligned}$$

This is a contradiction if the term in brackets is greater than 1, implying that for suitably large λ it is necessary that $\|v_\lambda\|_\infty \ge \Gamma$. □

Now we may conclude the Proof of Theorem 1.3, with the following result, which deals with the asymptotic behavior of the solutions when $\lambda \to \infty$.

Proposition 5.3. *Suppose conditions $(H1)$ through $(H3)$ hold, and that $f(t,v) > 0$ for $v \neq 0$, $v \neq a(t)$.*
If $\{v_\lambda\}$ is a family of positive solutions of Problem (P_λ), then

$$v_\lambda \to a \quad \textit{pointwise in } (0,1) \textit{ and } ||v_\lambda||_\infty \to ||a||_\infty\,, \qquad \textit{when } \lambda \to \infty\,.$$

Proof. Fix $t_0 \in (0,1)$, consider a sequence $\lambda_n \to \infty$ and suppose v_n are positive solutions of Problem (P_{λ_n}). Suppose also, for sake of contradiction, that there exists $\delta > 0$ such that $v_n(t_0) > a(t_0) + \delta$. We may assume $\lambda_n > \Lambda$ (where Λ is given by Lemma 5.2).

We claim that there exists $\gamma > 0$ such that $[t_0 - \gamma, t_0 + \gamma] \subseteq (0,1)$ and

$$v_n(t) > a(t_0) + 3\delta/4 > a(t_0) + \delta/4 > a(t) \text{ for } t \in [t_0 - \gamma, t_0 + \gamma] \text{ and every } n\,.$$

Actually, this follows by the continuity of a and by inequality (i) in Lemma 2.1 with $p = t_0$.

Then $G(1/2,s)q(s)f(s,v)$ is a positive function for $(s,v) \in [t_0 - \gamma, t_0 + \gamma] \times [a(t_0) + \delta/4, D]$ (where D is given by Lemma 5.2) and since this set is compact then it is larger than some $\eta > 0$ in this set, so

$$\int_0^1 G(1/2,s)q(s)f(s,v_n(s))\,ds \geq 2\gamma\eta\,.$$

However,

$$D \geq \|v_n\|_\infty \geq v_n(1/2) = \lambda_n \int_0^1 G(1/2,s)q(s)f(s,v_n(s))\,ds \geq 2\lambda_n\gamma\eta > 0\,,$$

which is a contradiction since $\lambda_n \to \infty$.

Now suppose that $v_n(t_0) < a(t_0) - \delta$. Using again the continuity of a and now inequality (ii) in Lemma 2.1, we obtain that there exists $\gamma > 0$ such that $[t_0 - \gamma, t_0 + \gamma] \subseteq (0,1)$ and

$$0 < c_{t_0-\gamma,t_0+\gamma}\,\Gamma \leq v_n(t) < a(t_0) - 3\delta/4 < a(t_0) - \delta/4 < a(t)$$

for $t \in [t_0 - \gamma, t_0 + \gamma]$ and every n, where Γ is given by Lemma 5.2.

Then we get now that $G(1/2,s)q(s)f(s,v) \geq \widetilde{\eta} > 0$ in the compact set

$$[t_0 - \gamma, t_0 + \gamma] \times [c_{t_0-\gamma,t_0+\gamma}\,\Gamma, a(t_0) - \delta/4]$$

and we reach again a contradiction, when $\lambda_n \to \infty$, since

$$D \geq \|v_n\|_\infty \geq v_n(1/2) = \lambda_n \int_0^1 G(1/2,s)q(s)f(s,v_n(s))\,ds \geq 2\lambda_n\gamma\widetilde{\eta} > 0\,.$$

Finally, in order to prove that $||v_\lambda||_\infty \to ||a||_\infty$, suppose that for a sequence t_n in $(0,1)$, one has $v_n(t_n) > ||a||_\infty + \delta$ for some $\delta > 0$. Up to a subsequence, we may assume that t_n is bounded away from 0 or from 1. In the first case (the second is analogous), we use again inequality (i) in Lemma 2.1 in order to obtain a $\gamma > 0$ such that $[t_n - \gamma, t_n] \subseteq (0,1)$ and

$$v_n(t) \geq ||a||_\infty + 3\delta/4 > ||a||_\infty + \delta/4 > a(t) \text{ for } t \in [t_n - \gamma, t_n] \text{ and every } n\,.$$

A contradiction is obtained again. □

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