# Existence, nonexistence and multiplicity of positive solutions for the poly-Laplacian and nonlinearities with zeros.

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## Abstract

In this paper we consider the equation $(-\Delta)^k u = \lambda f(x,u) + \mu g(x,u)$ with Navier boundary conditions, in a bounded and smooth domain. The main interest is when the nonlinearity is nonnegative but admits a zero and $f, g$ are, respectively, identically zero above and below the zero. We prove the existence of multiple positive solutions when the parameters lie in a region of the form $\lambda > \overline{\lambda}$ and $0 < \mu < \overline{\mu}(\lambda)$, then we provide further conditions under which, respectively, the bound $\overline{\mu}(\lambda)$ is either necessary, or can be removed.

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## 1  Introduction

In this paper, we obtain results concerning the existence, non existence and multiplicity of positive solutions for the problems

$$\begin{cases} (-\Delta)^k\, u = \lambda f(x,u) + \mu g(x,u) & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ (-\Delta)^i u = 0 & \text{on} \quad \partial\Omega,\ i = 0,..,k-1, \end{cases} \qquad (P^k_{\lambda,\mu})$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\lambda, \mu \geq 0$ are two parameters, $f, g$ are nonnegative functions and $k \in \mathbb{N}$.

The operators in $(P^k_{\lambda,\mu})$ are usually called poly-Laplacian and are the prototypes of linear elliptic operators of order $2k$. The boundary conditions assumed here are called Navier conditions and have the important property that, with them, $(P^k_{\lambda,\mu})$ becomes equivalent to a system of $k$ equations with the Laplacian operator and Dirichlet boundary conditions (see for example in [14]). Among these operators, the cases $(P^1_{\lambda,\mu})$ (Laplacian) and $(P^2_{\lambda,\mu})$ (bi-Laplacian) have been largely studied in literature.

Our main interest is to consider nonlinearities with zeros, as for example in [24, 25] and in particular as in our previous works [21, 19, 20, 11, 18], where second order operators (Laplacian and $p$-Laplacian) were considered. In particular, we consider the following condition:

$(Z_1)$ there exists a function $a \in L^1(\Omega)$, $a \geq 0$ and $a \not\equiv 0$ such that

$$\begin{cases} f(x,t) = 0 & \text{if } t \geq a(x), \\ g(x,t) = 0 & \text{if } 0 \leq t \leq a(x). \end{cases}$$

Condition $(Z_1)$ means that the nonlinearity $\lambda f(x,t) + \mu g(x,t)$ has a zero at $t = a(x)$ and then $g$ describes the nonlinearity above the zero while $f$ describes it below.

One important consequence of condition $(Z_1)$ is that the half-line $\lambda_1 t$ ($\lambda_1$ being the first eigenvalue of the operator) intersects the nonlinearity at least two times for any $\lambda, \mu > 0$. The existence of a positive intersection is known to be a necessary condition for the existence of a positive solution, and in many cases one expects to be able to obtain as many positive solutions as the number of these intersections. Actually, this was the case in [21, 19, 20, 11, 18].

In this paper we first prove, by variational techniques, three existence and multiplicity results for $(P^k_{\lambda,\mu})$ when the parameters lie in a region of the form $\lambda > \overline{\lambda}$ and $0 < \mu < \overline{\mu}(\lambda)$ (see the Theorems 1.1..1.3). The three results consider different behaviors of the nonlinearity near the origin and do not require to assume condition $(Z_1)$. Part of these results is already known, for instance, Theorem 1.1 could be seen as a consequence of [5] when $k = 1$ and of [34] when $k = 2$, and is proved by similar arguments.

However, as stated above, our main interest will be to analyze the effect of $(Z_1)$ on these results. In particular, we first prove, in Theorem 1.4, that in some cases existence

and multiplicity can be extended to hold without the bound on $\mu$, under $(Z_1)$. Then, in the Theorems 1.5 and 1.6, we prove that in other cases existence is actually lost for large values of $\mu$, even under this condition.

The problem of finding positive solutions with nonlinearities that are superlinear at infinity and have different behaviors at the origin have been extensively studied. For the Laplacian, see for example [7, 1, 5]. In most of these works, the nonlinearity is strictly positive. Problems with a nonlinearity which is nonnegative but has a zero at a positive value were first considered in [24], where it was pointed out that the presence of zeros usually provides multiple solutions; in particular, two solutions were obtained there, through topological degree arguments and a-priori estimates, in the subcritical case. It was further shown that one solution lies strictly below the zero, while the other one exceeds it. Similar problems were also studied in [25]; see also [16], which deals with nonlinearities that may have multiple zeros or even be negative in some regions.

Equations with the bi-Laplacian or poly-Laplacian operator and several kinds of non-linearities were studied in many works [10, 23, 30, 33, 29, 28, 17, 22, 2, 26, 27, 13, 9, 34, 35, 36]: among them we emphasize those more related to our setting. [2] considered the problem of the existence of two positive solutions with a concave-convex nonlinearity similar to the one in [1]. A complement of this result was proved in [36]. Two positive solutions are also found in [34], in a situation similar to [5], while in [35] nonlinearities concave in the origin are again considered and the results in [4] are extended.

In the next section we state our results and then, in section 1.2, we give a deeper discussion of our conditions and results, comparing them with the existing literature.

## 1.1 The results

For the existence and multiplicity results we will assume the following hypotheses on the functions $f, g$:

$(H_0)$ The functions $f, g : \overline{\Omega} \times [0, +\infty) \longrightarrow [0, +\infty)$ are Carathéodory functions and satisfy

  (i) $f(x, 0) = g(x, 0) = 0$,

  (ii) $f(x, t),\ g(x, t) \le C_0(1 + t^\sigma)$,

  for constants $C_0 > 0$ and $\sigma \in (1, 2_{N,k}^* - 1)$, where $2_{N,k}^* = \frac{2N}{N-2k}$ if $N > 2k$, and we may set $2_{N,k}^* = \infty$ if $N \le 2k$.

$(H_1)$ The primitive of $f$: $F(x, t) = \int_0^t f(x, s)\, ds$, is bounded by a $L^1$ function, namely

$$F(x, t) \le c_F(x) \in L^1(\Omega), \qquad \|c_F\|_{L^1} = C_F.$$

$(H_2)$ One of the following three hypotheses holds:

  $(H_{2a})$ There exists an open set $\omega \subseteq \Omega$ such that

$$\lim_{t \longrightarrow 0^+} \frac{f(x, t)}{t} = +\infty \quad \text{uniformly in } x \in \omega,$$

($H_{2a*}$) There exists an open set $\omega \subseteq \Omega$ such that

$$\liminf_{t \longrightarrow 0^+} \frac{f(x,t)}{t} > 0 \quad \text{uniformly in } x \in \omega,$$

($H_{2b}$)  (i)

$$\lim_{t \longrightarrow 0^+} \frac{f(x,t)}{t} = \lim_{t \longrightarrow 0^+} \frac{g(x,t)}{t} = 0 \quad \text{uniformly in } x \in \Omega,$$

(ii)     there exist an open set $\omega \subseteq \Omega$ and $0 < t_1 < t_2$ such that $f(x,t) > 0$ in $\omega \times (t_1, t_2)$.

($H_3$) There exists an open set $\omega_2 \subseteq \Omega$ such that

$$\lim_{t \to +\infty} \frac{g(x,t)}{t} = +\infty \quad \text{uniformly in } x \in \omega_2.$$

($H_4$) There exist $\Theta > 2$ and $C_4 > 0$ such that

$$\Theta G(x,t) - g(x,t)t \leq C_4 \quad \text{for } t \geq 0.$$

Our existence and multiplicity results are the following.

**Theorem 1.1.** *Under Hypotheses ($H_0$,$H_1$,$H_{2a}$,$H_3$,$H_4$), there exists a function $M : (0, \infty) \to (0, \infty]$ such that the problem ($P_{\lambda,\mu}^k$), $k \in \mathbb{N}$, has*
   *a) at least one positive solutions for $\lambda > 0$ and $\mu = 0$,*
   *b) at least two positive solutions for $\lambda > 0$ and $0 < \mu < M(\lambda)$.*

**Theorem 1.2.** *Under Hypotheses ($H_0$,$H_1$,$H_{2a*}$,$H_3$,$H_4$), there exist $\Lambda_0 > 0$ and a function $M : [0, \infty) \to (0, \infty]$ such that the problem ($P_{\lambda,\mu}^k$), $k \in \mathbb{N}$, has*
   *a) at least one positive solutions for $\lambda > \Lambda_0$ and $\mu = 0$,*
   *b) at least two positive solutions for $\lambda > \Lambda_0$ and $0 < \mu < M(\lambda)$,*
   *c) at least one positive solution for $\lambda \geq 0$ and $0 < \mu < M(\lambda)$.*

**Theorem 1.3.** *Under Hypotheses ($H_0$,$H_1$,$H_{2b}$,$H_3$,$H_4$), there exist $\Lambda_0 > 0$ and a function $M : (\Lambda_0, \infty) \to (0, \infty]$ such that the problem ($P_{\lambda,\mu}^k$), $k \in \mathbb{N}$, has*
   *a) at least two positive solutions for $\lambda > \Lambda_0$ and $\mu = 0$,*
   *b) at least three positive solutions for $\lambda > \Lambda_0$ and $0 < \mu < M(\lambda)$,*
   *c) at least one positive solution for $\lambda \geq 0$ and $\mu > 0$.*

These results are not completely satisfying if one compares them with the typical results in the case where the nonlinearity has zeros. Actually, as we remarked above, under condition ($Z_1$) one would expect to be able to obtain at least two solutions without the bound on $\mu$ given in the above Theorems, as was the case in [21]. In other words, one would expect $M(\lambda) = \infty$ in the Theorems 1.1-1.3. For this reason, the second part of the paper is devoted to studying in which conditions, under ($Z_1$), we can obtain $M(\lambda) = \infty$, but also to show that this is not always the case.

The first result in this direction is a positive one: we restrict to the case of Theorem 1.1 and we obtain that $M(\lambda) = \infty$ under suitable additional conditions.

**Theorem 1.4.** *Under the hypotheses of Theorem 1.1 plus ($Z_1$), it is possible to guarantee that $M(\lambda) = +\infty$ in the following cases:*

*(C1)* $N < 2k$, $a \geq a_0 > 0$ *and the parameter $\lambda > 0$ is small enough;*

*(C2)* $k = 1$, $a \geq a_0 > 0$, *the parameter $\lambda > 0$ is small enough and the following hypotheses hold:*

$\quad$ *($H_1^*$) there exists $c_f \in L^r$ with $r > N/2$, such that $f(x,t) \leq c_f(x)$,*

$\quad$ *($M_2$) there exists a constant $\varsigma > 0$ such that, for all $x \in \Omega$, the maps $s \mapsto f(x,s) + \varsigma\, s$ and $s \mapsto g(x,s) + \varsigma\, s$ are increasing;*

*(C3)* $k = 1$, *hypothesis ($M_2$) above and the following one hold:*

$\quad$ *($H_a$) the function $a \in H^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is weakly superharmonic (that is, $-\Delta\, a \geq 0$ in the weak sense).*

*Moreover, in the cases C2 and C3 the two solutions are ordered, that is $0 < u_1 < u_2$ in $\Omega$.*

It is worth remarking that under the conditions of Theorem 1.4-($C3$), two solutions of problem ($P_{\lambda,\mu}^k$) exist for every $\lambda, \mu > 0$.
However, it turns out that, even under ($Z_1$), $M(\lambda)$ cannot be always $+\infty$. We will prove this in the one dimensional version of ($P_{\lambda,\mu}^k$), but it looks reasonable that a similar behavior should be found in higher dimension too.
We take $\Omega = (-1,1) \subseteq \mathbb{R}$ and we study the problem

$$\begin{cases} (-1)^k\, u^{(2k)}(x) = \lambda f(x,u) + \mu g(x,u) & \text{in} \quad (-1,1), \\ u > 0 & \text{in} \quad (-1,1), \\ u^{(2i)}(\pm 1) = 0 & i = 0,..,k-1\,, \end{cases} \qquad (\Pi_{\lambda,\mu}^k)$$

where $f, g$ satisfy the following slightly modified hypotheses:

($K_0$) The functions $f, g : \overline{\Omega} \times [0, +\infty) \longrightarrow [0, +\infty)$ are continuous functions and satisfy

$\quad$ (i) $f(x,0) = g(x,0) = 0$,

$\quad$ (ii) condition ($Z_1$), with $a(x) \in \mathcal{C}(\overline{\Omega})$.

($K_1$) $f(x, \tau t) > \tau f(x,t) > 0$ for every $x \in \Omega$, $\tau \in (0,1)$, $t \in (0, a(x))$.

($K_2$) There exists $b_0 > 0$ such that

$$\liminf_{t \to 0^+} \frac{f(x,t)}{t} \geq b_0 \quad \text{uniformly in } x \in \Omega.$$

$(K_3)$   (i) There exists $c_0 > 0$ such that

$$\liminf_{t \to +\infty} \frac{g(x.t)}{t} \geq c_0 \quad \text{uniformly in } x \in \Omega,$$

(ii) $g(x, t) > 0$ for $t > a(x)$.

We obtain the following nonexistence results.

**Theorem 1.5.** *Under hypotheses $(K_0, .., K_3)$, let $k \geq 2$. If moreover $a(x)$ satisfies $\lim_{x \to \pm 1} a(x) = a^\pm > 0$ and $a'$ exists and is bounded near $\pm 1$, then there exists $\Lambda_1 > 0$ and $N : (\Lambda_1, \infty) \to (0, \infty)$ such that problem $(\Pi_{\lambda,\mu}^k)$ has no positive solution for $\lambda > \Lambda_1$ and $\mu > N(\lambda)$.*

**Theorem 1.6.** *Under hypotheses $(K_0, .. K_3)$, for any $k \in \mathbb{N}$, if moreover $a(x)$ is not a concave function, then the same conclusion of Theorem 1.5 holds.*

While proving these theorems we will also obtain a result concerning the convergence to $a(x)$ of the (possible) solutions of $(\Pi_{\lambda,\mu}^k)$, as $\mu, \lambda \to \infty$: see Proposition 4.2 and its immediate consequence that we state here (see also Remark 4.3):

**Corollary 1.7.** *For any $k \in \mathbb{N}$, if hypotheses $(K_0, K_1, K_2 K_3)$ hold, then along any two sequences $\mu_n, \lambda_n \to \infty$, the positive solutions $u_{\lambda_n,\mu_n}$ of problem $(\Pi_{\lambda,\mu}^k)$, if exist, converge to $a(x)$ pointwise.*

This property is well known in the Laplacian or $p$-Laplacian case even in higher dimension: see [12, 31, 32, 21].

## 1.2   Literature and some remarks

In this section we will discuss our results and hypotheses.

The simplest model for the nonlinearities considered here, which also satisfies $(Z_1)$ and fits in all our previous works cited above, is

$$\begin{cases} f(x, u) = (u^+)^p [(1 - u)^+]^q, & p, q > 0, \\ g(x, u) = [(u - 1)^+]^r, & r \in (1, 2_{N,k}^* - 1): \end{cases} \tag{1.1}$$

here $a(x) \equiv 1$ and the conditions $(H_{2a})$, $(H_{2a*})$ and $(H_{2b})$ are satisfied, respectively, for $p < 1$, $p \leq 1$ and $p > 1$. A model where the zero is not a constant is

$$\begin{cases} f(x, u) = (u^+)^p [(a(x) - u)^+]^q, & p, q > 0, \\ g(x, u) = [(u - a(x))^+]^r, & r \in (1, 2_{N,k}^* - 1), \end{cases} \tag{1.2}$$

where $a(x)$ has to be chosen in order to satisfy $(H_1)$. Another interesting model, where $a(x)$ is not bounded but $(H_1)$ is satisfied, is the following:

$$\begin{cases} f(x, u) = (|x|^b u)^p [(1 - |x|^b u)^+]^q, & p, q > 0, \\ g(x, u) = [(|x|^b u - 1)^+]^r, & r \in (1, 2_{N,k}^* - 1), \end{cases} \tag{1.3}$$

where we suppose $0 \in \Omega$ and $b \in (0, N)$. Actually, in this case $f(x, t) \leq 1$ and then $F(x, t) \leq a(x) = |x|^{-b} \in L^1(\Omega)$. Moreover, since $\Omega$ is bounded, we have $|x|^b < C$ and then $g \leq (Cu)^r$ satisfies $(H_0)$. Observe that the above models still satisfy the conditions of the multiplicity results after multiplying by a compact support nonnegative and nontrivial function of $x \in \Omega$.

When working with second order problems and a nonlinearity with zeros, in [21, 19, 20, 11, 18], we studied the equation $-\Delta_p u = \lambda\, h(x, u)$, that is, we had only one parameter $\lambda$ multiplying the nonlinearity. The general idea was to obtain a first solution via sub and supersolutions method (which is the one generated by the first bump of the nonlinearity). Then one could prove that this solution did not exceed the zero $a(x)$. This fact allowed to prove the existence of a second solution without a bound on $\lambda$. In fact, in [21] one showed that the first solution was in fact a local minimum (using [3]) and then applied the Mountain Pass Theorem, while, in the other works, the first solution being below the zero allowed to apply a topological degree argument and again obtain one further solution, or more than one if the nonlinearity has multiple zeros as in [11, 18].

In the poly-Laplacian case, on the other hand, the method of sub and supersolutions (and also an analogous of the [3] result, see [2]) is only available when the nonlinearity is strictly increasing, so that we cannot use it in order to bound the first solution below the zero (in fact, we show in Proposition 4.6 that sometimes this solution eventually exceeds the zero). As a consequence, we had to use a different technique and it turned out to be useful to split the nonlinearity in two parts with two different parameters, obtaining the results in the Theorems 1.1-1.3, where existence is not guaranteed for large values of $\mu$.

Of course, the existence for large $\mu$ is not to be expected in general: if, for example $f \simeq u^q$, $q \in (0, 1)$ near zero and $g = u^p$, $p \in (1, 2^*_{N,k} - 1)$, then for large values of $\mu$ the nonlinearity does not intersect the half-line $\lambda_1 t$. However, when one considers nonlinearities with zeros as in $(Z_1)$, this intersection exists for every $\lambda, \mu > 0$.

For this reason, in Theorem 1.4 we considered some additional hypotheses that allow to show that $M(\lambda) = +\infty$ for some $\lambda$. In particular, the case $(C3)$ is similar to the results proved in [21]: if $k = 1$ and $a(x)$ is superharmonic then the sub and supersolutions method allows easily to prove that the first solution stays below $a$, for any value of $\lambda > 0$, and then one obtains the second solution too, for any value of $\mu > 0$. The case $(C2)$ shows that even if $a$ is not superharmonic then $M(\lambda) = +\infty$ at least for $\lambda$ small. Finally, the case $(C1)$ considers the general poly-Laplacian case $k \geq 1$ and again shows that $M(\lambda) = +\infty$ for $\lambda$ small, provided the dimension is low. In these last two cases, instead of using $a(x)$ as a supersolution in order to bound the first solution, one exploits the fact that for $\lambda$ small the first solution is small too.

If one considers the simpler model nonlinearity (1.1) (taking $q \geq 1$ in order to satisfy $(M_2)$), then Theorem 1.4 says that $M(\lambda) = +\infty$ if $k = 1$ or if $k > 1$, $N < 2k$ and $\lambda$ is small. Then a natural question was whether the bound on $\mu$ was only due to the technique we use to prove the existence and multiplicity result, or if it is possible that, when $k \geq 2$, existence is actually lost for large $\mu$. The Theorems 1.5-1.6 show that, in fact, under rather mild conditions, problem $(\Pi^k_{\lambda,\mu})$ have no positive solutions at all for $\lambda, \mu$ large.

In particular, combining Theorem 1.4 and Theorem 1.5, we see that for problem $(\Pi_{\lambda,\mu}^k)$ with the nonlinearity (1.1), $M(\lambda) = +\infty$ if $k = 1$ or if $\lambda$ is small, however, if $k > 1$, then eventually $M(\lambda)$ becomes finite for large values of $\lambda$. This fact shows a quite different behavior of the higher order problem with respect to the second order one.

It is also interesting to observe (see Proposition 4.6) that in the conditions of the Theorems 1.5-1.6, for $\lambda$ large, those solutions that exist for $\mu < M(\lambda)$ are not bounded below the zero, which may explain why they cease to exist when $\mu$ becomes too large. In fact, Corollary 1.7 forces the solution to stay close to the zero, which eventually becomes impossible if $a$ is not concave (as in Theorem 1.6), showing that the condition $(H_a)$ in Theorem 1.4 is crucial. In the case $k \geq 2$, considered in Theorem 1.5, a further obstacle to existence is given by the higher "stiffness" of the poly-Laplacian, which makes it impossible for the solution to jump from the boundary condition to near $a(x)$ without exceeding it (see also Remark 4.4). It is worth remembering that, from a physical point of view, the Laplacian models, in the one dimensional case, the deformation of a string, while the bi-Laplacian models a beam, which has a bending stiffness in addition to the tensile stiffness.

We close this section with a discussion of our hypotheses.

Condition $(H_0)$ is a standard condition in order to work with variational techniques, in particular, it implies that the functional associated to problem $(P_{\lambda,\mu}^k)$ is well defined and subcritical with respect to Sobolev embeddings.

Conditions $(H_1)$ is natural when one has the model (1.1) in mind. In fact, $(H_1)$ is satisfied under condition $(Z_1)$ if moreover $f, g$ are continuous in $\overline{\Omega} \times [0, \infty)$ and $a(x)$ is bounded. In this weaker form, it allows for more general nonlinearities, but it is still useful in order to keep the role of $\mu$ and $\lambda$ separated, that is, with $\lambda$ regulating the behavior of the functional near the origin and $\mu$ that at infinity.

The three conditions in $(H_2)$ describe the behavior of the nonlinearity near the origin: for the model autonomous nonlinearity (1.1) they correspond to the cases $p < 1$, $p \leq 1$ and $p > 1$, respectively, however, a local condition is enough for our purpose (as in [5]). Observe that $(H_{2a*})$ is a weaker version of $(H_{2a})$, while $(H_{2b})$ is almost complementary to $(H_{2a*})$. Condition $(H_{2b}$-ii) is needed in order to avoid the case of a trivial $f$. The role of $(H_{2a})$, when $(Z_1)$ holds, is to guarantee an intersection with the half-line $\lambda_1 t$ for every $\lambda > 0$, on the other hand, $(H_{2b})$ implies that the origin is a local minimum for the associated functional, and has the effect to produce a third solution above a certain value of $\lambda$ (observe that $(H_{2b})$ also contains a condition on $g$, which is needed in this case in order to avoid that $g$ dominates on $f$ near the origin; under $(Z_1)$ it would be enough to assume $a > 0$ in $\overline{\Omega}$).

Finally, hypotheses $(H_3)$ and $(H_4)$ are classical in order to obtain the PS-condition and a Mountain Pass geometry.

About the additional conditions that allow to prove that $M(\lambda) = \infty$, in Theorem 1.4, condition $(H_1^*)$ is a slightly stronger version of $(H_1)$ and is required in order to bound the first solution when $\lambda$ is small. Condition $(M_2)$ is a standard condition when one wants to use the sub and supersolution method. Finally, condition $(H_a)$ means that the function $a$ is a supersolution for the problem $(P_{\lambda,\mu}^1)$, which then allows to obtain a first solution

which is bounded by it. We observe that both model functions (1.1) and (1.3) satisfy the conditions $(H_1^*)$ and $(M_2)$ provided $q \geq 1$, moreover (1.1) also satisfies $(H_a)$.

Comparing with the results in [21], we observe that the conditions assumed here are more general. First of all, condition $(Z_1)$ is not required in the first three theorems. Moreover, more possible behaviors near the origin are considered here and the conditions at infinity are weaker, allowing for instance $g$ to be identically zero in some subset of $\Omega$. Finally, when assuming $(Z_1)$, the function $a$ describing the zero is allowed to be zero in a subset of $\Omega$, or to go to zero at the boundary, which was not possible in [21]. For these reasons, even when $k = 1$, the results here improve those in our previous works.

We finally discuss the conditions in the Theorems 1.5-1.6. Condition $(K_0)$ is similar to $(H_0)$ but requires continuity for $f, g$ and $a$, which are needed for some estimates used in the proof. Condition $(K_2)$ is similar to $(H_{2a*})$ but needs to hold uniformly in $\Omega$ and not only locally, while condition $(K_3)$ states that $g$ has at least a linear growth at infinity and that it is in fact always positive above $a(x)$.

Condition $(K_1)$ is more technical and states that, between 0 and $a(x)$, $f$ is positive and is always above the line from the origin to $(t, f(x,t))$. This condition is strictly related to asking that the ratio $f(x,t)/t$ is strictly decreasing or that the derivative $f_t(x,t) < 0$, which is a condition that was frequently assumed in similar problems, in particular it was used in order to guarantee uniqueness of the solution below $a(x)$ in [8].

The rest of the paper is organized as follows: the Theorems 1.1..1.3 are proved in section 2, Theorem 1.4 in section 3 and, finally, the nonexistence results in the Theorems 1.5 and 1.6 are proved in section 4.

In the course of the paper, $C$ will denote a generic positive constant which may vary from line to line.

# 2 Proof of the existence and multiplicity results

In this section we will prove the Theorems 1.1..1.3: we will obtain the existence and multiplicity of positive solutions of problem $(P_{\lambda,\mu}^k)$ as critical points of a suitable functional.

## 2.1 The variational setting

The natural working space is defined as follows: let $B_k$ be the operator that maps $u$ to the vector of the traces on $\partial\Omega$ of the derivatives of order strictly lower than $k$ which are imposed in problem $(P_{\lambda,\mu}^k)$: then

$$\mathbb{H} = \{u \in H^k(\Omega) \text{ such that } B_k u = 0\}, \tag{2.1}$$

where $H^k(\Omega)$ is the Hilbert Sobolev space of functions with square integrable weak derivatives up to order $k$. Observe that for $k = 1$, $\mathbb{H}$ reduces to $H_0^1(\Omega)$ and for $k = 2$ to $H^2(\Omega) \cap H_0^1(\Omega)$. We will use in $\mathbb{H}$ the analogous of the Dirichlet norm in $H_0^1$, that is

$$\|u\|_{\mathbb{H}} = \left\|\nabla^k u\right\|_{L^2},$$

1  where we use the notation $\nabla^{2h}u = \Delta^h u$ and $\nabla^{2h+1}u = \nabla(\Delta^h u)$, $h = 0, 1, 2, ....$

2      It is worth noting that the eigenvalues of the operators in $(P_{\lambda,\mu}^k)$ are the $k^{th}$ power of

3  those of the Laplacian with Dirichlet boundary conditions, while the eigenfunctions are

4  the same.

5      For $N > 2k$, the exponent $2^*_{N,k}$ defined in condition $(H_0)$ is the critical Sobolev's

6  exponent of the embedding $H^k \subseteq L^p$, which means that the problems we are considering

7  here are subcritical.

8      We remand to [14] for more details about this kind of problems and their properties.

9      We will consider the functional

$$J_{\lambda,\mu} : \mathbb{H} \to \mathbb{R} : u \mapsto J_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_{\mathbb{H}}^2 - \lambda \int_\Omega F(x, u^+) - \mu \int_\Omega G(x, u^+). \qquad (2.2)$$

10  Observe that, under hypothesis $(H_0)$, the functional is well defined and of class $\mathcal{C}^1$ in $\mathbb{H}$,

11  moreover its nontrivial critical points will be positive solutions of problem $(P_{\lambda,\mu}^k)$. Actually,

12  one can write $(P_{\lambda,\mu}^k)$ as a system:

$$\begin{cases} -\Delta u = v_{k-1} & \text{in } \Omega, \\ -\Delta v_i = v_{i-1} & \text{in } \Omega, \qquad i = 2, .., k-1, \\ -\Delta v_1 = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega, \\ u = v_i = 0 & \text{on } \partial\Omega, \qquad i = 1, .., k-1. \end{cases} \qquad (S_{\lambda,\mu}^k)$$

13  By Maximum Principle $\lambda f(x, u) + \mu g(x, u) \geq 0$ implies $v_1 \geq 0$ and then sequentially

14  $v_i \geq 0$ for $i = 2, .., k-1$ and finally $u \geq 0$. In fact, if $u \not\equiv 0$ then $u, v_i > 0$ for $i = 1, .., k-1$

15  in $\Omega$.

16      When $k = 1$ (the Laplacian case) it is also easy to obtain that if the nonlinearity is

17  null above a certain value $z > 0$, then the solutions cannot exceed $z$. However, for the

18  poly-Laplacian case one cannot obtain a similar result. In fact, a strict positive maximum

19  can be reached with null forcing term because the second derivative "propagates" from

20  the nearby regions. Proposition 4.6 provides an example to this.

21  ## 2.2   The PS-condition

22  We start by proving the following result, concerning the PS-condition for the functional

23  (2.2).

24  **Lemma 2.1.** *Under hypotheses $(H_0, H_1, H_4)$, the functional $J_{\lambda,\mu}$ satisfies the PS-condition*

25  *for any $\lambda, \mu \geq 0$*

*Proof.* If $\mu = 0$ then, by hypothesis $(H_1)$, $J_{\lambda,\mu}$ is coercive, actually

$$J_{\lambda,\mu}(u) \geq \frac{1}{2}\|u\|_{\mathbb{H}}^2 - \lambda C_F \, ;$$

26  then any PS sequence is bounded.

If $\mu > 0$, then along a PS-sequence one has

$$
\begin{aligned}
C(1 + \|u_n\|_{\mathbb{H}}) \;\geq\;& \Theta J_{\lambda,\mu}(u_n) - J'_{\lambda,\mu}(u_n)[u_n] = \\
& \frac{\Theta - 2}{2} \|u_n\|_{\mathbb{H}}^2 - \lambda \int \left( \Theta F(x, u_n^+) - f(x, u_n^+) u_n \right) + \\
& \qquad\qquad -\mu \int \left( \Theta G(x, u_n^+) - g(x, u_n^+) u_n \right) \\
\geq\;& \frac{\Theta - 2}{2} \|u_n\|_{\mathbb{H}}^2 - C \,,
\end{aligned}
$$

where we used the hypotheses $(H_1)$, $(H_4)$ and also that $f \geq 0$. Then again one obtains the boundedness of the PS-sequences.

Then the existence of a convergent subsequence follows as usual since by $(H_0)$ the problem is subcritical. $\qquad\square$

## 2.3 Linear and sublinear in the origin case (hypothesis $(H_{2a})$)

The proof of the Theorems 1.1 and 1.2 will be a consequence of the geometry described in the following lemmas.

The first Lemma shows that the origin is not a local minimum.

**Lemma 2.2.** *Under Hypotheses $(H_0, H_{2a})$, there exists $u_0 \in \mathbb{H}$ such that, for every $\lambda > 0$ there exists $t_0(\lambda) > 0$ such that one has*

$$
J_{\lambda,\mu}(t u_0) < 0 \qquad for\ 0 < t < t_0(\lambda)\ and\ any\ \mu \geq 0.
$$

*If $(H_{2a})$ is substituted by the weaker $(H_{2a*})$ then there exist $u_0 \in \mathbb{H}$ and $\Lambda_0, t_0 > 0$ such that one has*

$$
J_{\lambda,\mu}(t u_0) < 0 \qquad for\ 0 < t < t_0\ and\ any\ \lambda > \Lambda_0,\ \mu \geq 0.
$$

*Proof.* Let $\omega$ be the set in hypothesis $(H_{2a})$ and $u_0 \geq 0$ be a smooth function in $\mathbb{H}$ such that $u_0 = 1$ in a set of positive measure $\varpi \subset \omega$. Then, for suitable $t_0 = t_0(\lambda) > 0$, one has $f(x,t) \geq \frac{2\|u_0\|_{\mathbb{H}}^2}{\lambda} t$ for $t \in [0, t_0]$, $x \in \varpi$. It follows that $\lambda F(x, t u_0) \geq \|u_0\|_{\mathbb{H}}^2 t^2$ for $t \in (0, t_0)$ and

$$
J_{\lambda,\mu}(t u_0) \leq \frac{t^2 \|u_0\|_{\mathbb{H}}^2}{2} - \lambda \int F(x, t u_0) < 0
$$

for $t \in (0, t_0)$.

In case of Hypothesis $(H_{2a*})$, for suitable $t_0 > 0$ and $\delta > 0$, one has $f(x,t) \geq \delta \|u_0\|_{\mathbb{H}}^2 t$ for $t \in [0, t_0]$, $x \in \varpi$. Then $\lambda F(x, t u_0) \geq \frac{\lambda \delta}{2} \|u_0\|_{\mathbb{H}}^2 t^2$ for $t \in (0, t_0)$ and

$$
J_{\lambda,\mu}(t u_0) \leq \frac{t^2 \|u_0\|_{\mathbb{H}}^2}{2} - \lambda \int F(x, t u_0) < 0
$$

for $t \in (0, t_0)$ if $\lambda > 1/\delta$. $\qquad\square$

In the next Lemma we prove the existence of a sphere in $\mathbb{H}$ where the functional is above a given value, up to a certain value of $\mu$.

**Lemma 2.3.** *Under hypotheses ($H_0$,$H_1$), given $\lambda \geq 0$ and $H \in \mathbb{R}$, there exist $\rho_H(\lambda) > 0$ and $M_H(\lambda) > 0$ such that for $0 < \mu < M_H(\lambda)$ one has*

$$J_{\lambda,\mu}(u) > H \qquad for \ \|u\|_{\mathbb{H}} = \rho_H(\lambda). \tag{2.3}$$

*Proof.* By the hypotheses ($H_0$) and ($H_1$) and Sobolev embeddings, we can estimate

$$J_{\lambda,\mu}(u) = \frac{1}{2} \|u\|_{\mathbb{H}}^2 - \lambda \int F(x, u^+) - \mu \int G(x, u^+) \geq \frac{1}{2} \|u\|_{\mathbb{H}}^2 - \lambda C_F - \mu C(1 + \|u\|_{\mathbb{H}}^{\sigma+1}).$$

Then we may take, for instance, $\rho$ such that $\rho^2/2 - \lambda C_F > H + 1$ and then $M_H$ such that $M_H C(1 + \rho^\sigma) < 1$. □

Finally, we prove the existence of a point where the functional is as negative as desired.

**Lemma 2.4.** *Under Hypotheses ($H_0$,$H_3$), there exists $e \in \mathbb{H}$ such that, for any $\lambda \geq 0$ and $\mu > 0$,*

$$J_{\lambda,\mu}(te) \to -\infty \qquad when \ t \to +\infty.$$

*Proof.* Let $\omega_2$ be the set in hypothesis ($H_3$) and let $e \in \mathbb{H}$ be such that $e > 1$ in some open set $\varpi \subseteq \omega_2$. Then there exists $t_0$ such that, in $\varpi$, $g(x, t) \geq 2\frac{\|e\|_{\mathbb{H}}^2}{\mu |\varpi|} t$ for $t > t_0$ and then $G(x, t) \geq \frac{\|e\|_{\mathbb{H}}^2}{\mu |\varpi|}(t^2 - t_0^2)$.

Then we can estimate

$$J_{\lambda,\mu}(te) \leq \frac{t^2}{2} \|e\|_{\mathbb{H}}^2 - \mu \int_\varpi G(x, te^+) \leq \frac{t^2}{2} \|e\|_{\mathbb{H}}^2 - \|e\|_{\mathbb{H}}^2 (t^2 - t_0^2)$$

for $t > t_0$, and the claim follows. □

At this point we are able to prove:

*Proof of Theorem 1.1.* • Point a) Let $\lambda > 0$ and $\mu = 0$.
By hypothesis ($H_1$), $\int F$ is bounded and then $J_{\lambda,\mu}$ is coercive. On the other hand, Lemma 2.2 implies that there exists a minimum at a negative level, which is then a nontrivial solution.

• Point b) Let $\lambda > 0$.
By the Lemmas 2.2 and 2.3 (where we can set $H = 1$), for $0 < \mu < M_1(\lambda)$, we have a minimum at a negative level in the ball $\overline{B_{\rho_1(\lambda)}}$. Then, in view of equation (2.3) and Lemma 2.1, since for $e$ as in Lemma 2.4 $J_{\lambda,\mu}(te) \to -\infty$ as $t \to \infty$, we can apply the Mountain Pass Theorem to obtain a second nontrivial solution at level $c \geq 1$.

As a consequence we have two nontrivial (then positive) solutions.

□

1 *Proof of Theorem 1.2.* Under Hypothesis $(H_{2a*})$, one can prove points (a) and (b) as
2 above, provided now $\lambda > \Lambda_0$ obtained in Lemma 2.2.
3     For case (c) one only needs to apply the Mountain Pass Theorem, actually Lemma 2.3
4 with $H = 1$ provides a "range of mountains" around the origin and Lemma 2.4 completes
5 the mountain pass geometry. $\qquad\square$

## 2.4    Superlinear in the origin case (hypothesis $H_{2b}$)

7 For the proof of Theorem 1.3 we will use again the Lemmas 2.3-2.4, but the geometry
8 near the origin is different and is described in the following lemmas.

9 **Lemma 2.5.** *Under Hypotheses ($H_0$,$H_{2b}$) for $\lambda \geq 0$, $\mu \geq 0$, the origin is a strict local*
10 *minimum for $J_{\lambda,\mu}$.*

*Proof.* By combining the hypotheses ($H_0$) and ($H_{2b}$-i), one obtains that for any $\varepsilon > 0$
there exists $C_\varepsilon$ such that $0 \leq f(x,t) \leq \varepsilon t + C_\varepsilon t^\sigma$ for $t \geq 0$ and then, using also Sobolev
embeddings,

$$0 \leq \int_\Omega F(x,u^+) \leq C \left( \varepsilon \frac{\|u\|_{\mathbb{H}}^2}{2} + \|u\|_{\mathbb{H}}^{\sigma+1} \right),$$

11 and the same estimate holds for $G$. Then, setting $\varepsilon < \frac{1}{2(\lambda+\mu)C}$,

$$
\begin{aligned}
J_{\lambda,\mu}(u) &= \frac{\|u\|_{\mathbb{H}}^2}{2} - \lambda \int F(x,u^+) - \mu \int G(x,u^+) \\
&\geq \frac{\|u\|_{\mathbb{H}}^2}{2} - \frac{\|u\|_{\mathbb{H}}^2}{4} - (\lambda+\mu)C \|u\|_{\mathbb{H}}^{\sigma+1}.
\end{aligned}
$$

12 For fixed $\lambda, \mu$, by comparing the exponents one obtains the conclusion. $\qquad\square$

13     In the next Lemma we prove that, despite the fact that the origin is now a local
14 minimum, one can still find a point (not depending on $\mu$ and $\lambda$) where the functional is
15 negative, provided $\lambda$ is large enough.

**Lemma 2.6.** *Under Hypotheses ($H_0$,$H_{2b}$), there exists $u_0$ and $\Lambda_0 > 0$ such that*

$$J_{\lambda,\mu}(u_0) < 0 \qquad for \ \lambda > \Lambda_0, \ \mu \geq 0.$$

*Proof.* Let $\omega$ be the set from hypothesis ($H_{2b}$-ii) and let $u_0 \geq 0$ be a smooth function in
$\mathbb{H}$ such that $t_1 < u_0 < t_2$ in an open set $\varpi \subset \omega$. Then $\int_\Omega F(x,u_0) > 0$, and so

$$J_{\lambda,\mu}(u_0) \leq \frac{\|u_0\|_{\mathbb{H}}^2}{2} - \lambda \int F(x,u_0) < 0$$

16 for $\lambda > \frac{\|u_0\|_{\mathbb{H}}^2}{2 \int_\Omega F(x,u_0)}$. $\qquad\square$

17     We are now able to prove:

*Proof of Theorem 1.3.*        • Point a) As in the proof of Theorem 1.1 point (a), $J_{\lambda,\mu}$ is coercive for $\mu = 0$. If $\lambda > \Lambda_0$ form Lemma 2.6, then

$$\inf_{u \in \mathbb{H}} J_{\lambda,\mu}(u) < 0.$$

Then one has a global minimum at a negative level which is a first nontrivial (then positive) solution.

Since the origin is another local minimum by Lemma 2.5, in view of Lemma 2.1, one can obtain a second positive solution at a positive level by applying the Mountain Pass Theorem between the two minima.

• Point b)

Let $\lambda > \Lambda_0$ and $u_0$ be the function given in Lemma 2.6. Define

$$K := \max_{t \in [0,1]} J_{\lambda,0}(tu_0) :$$

observe that $K > 0$ by Lemma 2.5.

Applying Lemma 2.3 with $H = K + 1$ we obtain numbers $M_H(\lambda), \rho_H(\lambda)$ such that for $0 < \mu < M_H(\lambda)$ and $\|u\|_{\mathbb{H}} = \rho_H(\lambda)$ one has $J_{\lambda,\mu}(u) > H$.

Observe that, by construction, $\rho_H(\lambda) > \|u_0\|_{\mathbb{H}}$. Actually, by comparing the computations in the proofs of the Lemmas 2.3 and 2.6, one sees that $\|u_0\|_{\mathbb{H}}^2 \leq 2\lambda C_F < \rho_H(\lambda)^2$. Then by Lemma 2.6 there exists a minimum at a negative level in the ball $B_{\rho_H(\lambda)}$, which is a first nontrivial (then positive) solution.

Now we define two different Mountain Pass geometries:

$$c_i = \inf_{\gamma \in \Gamma_i} \sup_{t \in [0,1]} J_{\lambda,\mu}(\gamma(t)) \qquad i = 1, 2,$$

where

$$\Gamma_1 = \{\gamma : [0,1] \to \mathbb{H} : \ \gamma(0) = u_0, \ \gamma(1) = 0\},$$

$$\Gamma_2 = \{\gamma : [0,1] \to \mathbb{H} : \ \gamma(0) = u_0, \ \gamma(1) = \theta e\},$$

$e$ is as in Lemma 2.4 and $\theta$ is such that $\|\theta e\|_{\mathbb{H}} > \rho_H(\lambda)$ and $J_{\lambda,\mu}(\theta e) \leq J_{\lambda,\mu}(u_0)$.

In both cases $J_{\lambda,\mu}(\gamma(\{0;1\})) \leq 0$ by construction. Since every $\gamma \in \Gamma_2$ intersects $\partial B_{\rho_H(\lambda)}$ one has that $c_2 \geq H > 0$. On the other hand, since the origin is a strict local minimum and the path $\gamma(t) = tu_0|_{t \in [0,1]}$ is in $\Gamma_1$ one gets $0 < c_1 \leq K < H$.

In view of Lemma 2.1, we obtain two critical points which correspond to two new distinct nontrivial (then positive) solutions, since their levels are positive and distinct.

Then we got a total of three positive solutions as claimed.

- Point c): Let $\lambda \geq 0$, $\mu > 0$. Similar to the proof of Theorem 1.2 we obtain a solution via Mountain Pass Theorem: in this case the mountain pass geometry is given by Lemma 2.5 and Lemma 2.4.

□

*Remark* 2.7. Observe that in the proofs above we obtained a possible choice of the function $M(\lambda)$ which would be finite and (if chosen as in Lemma 2.3) nonincreasing. However, once proved the existence of such $M$, it can be redefined as the largest possible satisfying the claim, so it can assume infinite values and not necessarily be nonincreasing.

# 3   Existence for every $\mu > 0$

In this section we provide the proof of Theorem 1.4, which shows that, under suitable additional conditions, the function $M(\lambda)$ in Theorem 1.1 can assume an infinite value. This means that, for such $\lambda$, two solutions of problem $(P_{\lambda,\mu}^k)$ exist for any value of the parameter $\mu > 0$.

*Proof of Theorem 1.4.*      • Case ($C1$).

The condition $N < 2k$ implies that $\mathbb{H} \hookrightarrow L^\infty(\Omega)$ and then $\|u\|_\infty < C \|u\|_\mathbb{H}$. In particular, let $R > 0$ be such that $\|u\|_\mathbb{H} \leq R$ implies $\|u\|_\infty < a_0$.

For $\lambda > 0$ and $\mu = 0$, as in the proof of Theorem 1.1 point (a), we obtain a solution $u_{\lambda,0}$ which is a local minimum at a negative level.

Estimating as in the proof of Lemma 2.3, we have

$$J_{\lambda,0}(u) \geq \frac{1}{2} \|u\|_\mathbb{H}^2 - \lambda C_F \,. \tag{3.1}$$

Then we consider $\lambda < R^2/(4C_F)$, which implies

$$J_{\lambda,0}(u) > R^2/4 > 0 \qquad \text{in the complement of } B_R(0) \,. \tag{3.2}$$

Since $J_{\lambda,0}(u_{\lambda,0}) < 0$, we obtain that $\|u_{\lambda,0}\|_\mathbb{H} < R$ and then $\|u_{\lambda,0}\|_\infty < a_0$.

On the other hand, since $\|u\|_\infty < a_0$ for every $u \in \overline{B_R(0)}$, in view of hypothesis ($Z_1$), the whole $\overline{B_R(0)}$ is unaffected by $\mu$. This implies that, for every $\mu$ positive, $u_{\lambda,0}$ is a first solution of problem $(P_{\lambda,\mu}^k)$, at a negative level. Moreover, in view of (3.2), we also have that

$$J_{\lambda,\mu}(u) > R^2/4 > 0 \qquad \text{in } \partial B_R(0), \tag{3.3}$$

for every $\mu \geq 0$. As a consequence, as in point (b) of Theorem 1.1, we may apply the Mountain Pass Theorem and obtain a second solution, which is new and nontrivial since it lies at a positive level.

- Case ($C2$).
  The first step of this case is similar to the case ($C1$), but instead of exploiting the low dimension one can exploit the better properties of the Laplacian operator. As before, for $\lambda > 0$ and $\mu = 0$, we have the solution $u_{\lambda,0}$ which is a local minimum at a negative level, and then, using (3.1),

  $$\|u_{\lambda,0}\|_{\mathbb{H}} \le 2\lambda C_F .$$

  By ($H_1^*$) we have $\|f(x,u)\|_r \le \|c_f\|_r$, thus, we can use Proposition 1.3 in [15] to obtain (since $r > N/2$) the following estimate for the $L^\infty$ norm of $u_{\lambda,0}$:

  $$\|u_{\lambda,0}\|_\infty \le C(N, |\Omega|)\lambda \|c_f\|_r . \tag{3.4}$$

  This implies that for $\lambda$ small enough $u_{\lambda,0} < a_0 \le a$. As a consequence, in view of hypothesis ($Z_1$), a whole neighborhood of $u_{\lambda,0}$ in the $\mathcal{C}_1$ topology is unaffected by $\mu$, implying that, for every $\mu$ positive, $u_{\lambda,0}$ is still a solution and also a local minimum of $J_{\lambda,\mu}$ in the $\mathcal{C}^1$ topology, but then also in $\mathbb{H} = H_0^1(\Omega)$ by [3].

  At this point, just applying the Mountain Pass Theorem would provide another solution for every $\mu > 0$, but without an estimate on its level it would not be clear if it is a new solution or the trivial one. For this reason, we construct a second solution of Problem ($P_{\lambda,\mu}^1$) in the form $u_{\lambda,0} + w$, where $w$ is a nontrivial solution of the problem

  $$\begin{cases} -\Delta\left(u_{\lambda,0} + w\right) = \widetilde{h}(x, u_{\lambda,0} + w^+) & \text{in } \Omega , \\ w = 0 & \text{on } \partial\Omega , \end{cases} \tag{3.5}$$

  and we have set, for a given $\mu > 0$, $\widetilde{h}(x,t) = \lambda f(x, t^+) + \mu g(x, t^+)$.
  This technique is rather standard, it was already used in [1], see also [6]. One first notices that the equation in (3.5) is equivalent to $-\Delta w = \widetilde{h}(x, u_{\lambda,0} + w^+) - \widetilde{h}(x, u_{\lambda,0})$ and that a nontrivial solution $w$ is eventually positive (in view of ($M_2$)), then $u_{\lambda,\mu} = u_{\lambda,0} + w$ would be a second positive solution of Problem ($P_{\lambda,\mu}^1$) satisfying $u_{\lambda,\mu} > u_{\lambda,0}$. Then one shows that the functional associated to (3.5) satisfies the PS-condition (in view of the hypotheses ($H_0$-$H_4$)), that $w = 0$ is a local minimum for it, which is strict if one supposes that it is the unique solution of (3.5), and finally that the functional goes to $-\infty$ in a suitable direction (because of hypothesis ($H_3$)). As a consequence, one can apply the Mountain Pass Theorem and prove the existence of the second solution.

- Case ($C3$).
  We proceed similar to Theorems 1.1 and 1.4 in [21]. By hypotheses ($H_0$) and ($M_2$), we may use the method of sub and supersolutions. By ($H_a$), we have that $a(x)$ is always a supersolution of Problem ($P_{\lambda,\mu}^1$), in fact, it is a strict supersolution, since $a$ could only be a solution if it were identically zero, which is not compatible with condition ($H_{2a}$).

Let $\omega$ be the set from hypothesis $(H_{2a})$, $\varpi$ a smooth open subset of $\omega$ and $\lambda^*, \phi^*$ be the first eigenvalue and eigenfunction of the Laplacian in $\varpi$. Then, for a given $\lambda > 0$, there exists a sufficiently small $t_0$ such that

$$\lambda^* t < \lambda f(x,t)\,, \ \text{for} \ \ t \in (0, t_0]\,,\ x \in \varpi\,. \tag{3.6}$$

Thus, for $\varepsilon > 0$ such that $\varepsilon \|\phi^*\|_\infty < t_0$, one has

$$-\Delta(\varepsilon \phi^*) = \lambda^* \varepsilon \phi^* < \lambda f(x, \varepsilon \phi^*) \tag{3.7}$$

in $\varpi$; by continuing $\phi^*$ by zero out of $\varpi$ we obtain a weak (strict) subsolution for Problem $(P^1_{\lambda,\mu})$. Finally, (3.6) implies that $t_0 < a$ in $\varpi$ and then $\varepsilon \phi^* < a$.

At this point, the method of sub and supersolutions implies that there exists a solution $u_{\lambda,0}$ satisfying $0 < \varepsilon \phi^* \le u_{\lambda,0} \le a$ and, as in [21] Lemma 3.1, one proves that it is also a local minimum of the associated functional and that $u_{\lambda,0} < a(x)$.

Then as before a whole neighborhood of $u_{\lambda,0}$ in the $\mathcal{C}_1$ topology is unaffected by $\mu$, so we are in the same situation as in the case $(C2)$: $u_{\lambda,0}$ is a first positive solution and a local minimum for every $\mu \ge 0$, then a second positive solution is obtained as before.

□

# 4 Non existence results

In this section we will prove the nonexistence results in the Theorems 1.5 and 1.6.

Observe that, being in dimension $N = 1$, the solutions of $(\Pi^k_{\lambda,\mu})$ are easily seen to be classical for a continuous right hand side.

As remarked in the introduction, when the right hand side in $(\Pi^k_{\lambda,\mu})$ is nonnegative and not zero, by writing the problem as a system like $(S^k_{\lambda,\mu})$ and applying the Maximum Principle we deduce that $(-1)^i u^{(2i)} > 0$ in $\Omega$ for $i < k$, while $(-1)^k u^{(2k)} \ge 0$. As a consequence, all these functions are concave. In particular, $u$ attains its maximum at a point $M_u \in \Omega$ and it is nondecreasing before $M_u$ and nonincreasing afterward.

In order to prove the Theorems 1.5 and 1.6, we will make use of the Green Function for the poly-Laplacian with Navier conditions in the interval $(-1, 1)$, which we denote by $\mathcal{G}_k$. In particular, $\mathcal{G}_k$ is such that a solution of $(\Pi^k_{\lambda,\mu})$ satisfies

$$u(x) \;=\; \int_{-1}^{1} \mathcal{G}_k(y,x)[\,\lambda f(y, u(y)) + \mu g(y, u(y))\,]\, dy\,. \tag{4.1}$$

As for the Green function of the Laplacian, $\mathcal{G}_k$ is always positive in $(-1, 1)^2$, as a consequence of the Maximum Principle. In the particular case $k = 2$, $\mathcal{G}_2$ takes the form

$$\mathcal{G}_2(x, p) = \frac{|x - p|^3}{6} + \left( \frac{px}{6} - \frac{1}{2} \right) \left( x^2 + p^2 + 2 \right) + \frac{4}{3}\,.$$

We will also use the following property of concave functions:

**Lemma 4.1.** *Given a function $v$ which is concave in $(-1,1)$ and satisfies $v(\pm 1) = 0$, the following estimates hold:*

$$(i) \qquad v(t) \geq \begin{cases} \frac{1+t}{1+p}\, v(p) & t < p, \\ \frac{1-t}{1-p}\, v(p) & t > p \end{cases} \qquad and \qquad (ii) \qquad v(t) \leq \begin{cases} \frac{1+t}{1+p}\, v(p) & t > p, \\ \frac{1-t}{1-p}\, v(p) & t < p. \end{cases}$$

*Moreover, for all $-1 < t_0 < t_1 < 1$, we have*

$$(iii) \qquad \min_{t \in [t_0, t_1]} v(t) \geq c_{t_0, t_1} \, \|v\|_\infty,$$

*where $c_{t_0, t_1} := \frac{1}{2} \min\{1 + t_0, 1 - t_1\}$.*

*Proof.* Inequality (i) is an immediate consequence of the concavity of $v(t)$ on $[-1, 1]$ and of the boundary condition, actually, if $v$ lies below the triangle whose vertices are $(\pm 1, 0)$ and $(p, v(p))$, then it can not be concave. Inequality (ii) follows readily from (i) and the last claim is obtained by choosing $p$ in (i) where the norm is attained. $\qquad\square$

In the following we will denote by $u_{\lambda,\mu}$ a positive solution of $(\Pi_{\lambda,\mu}^k)$ with parameters $\lambda, \mu$, however, we will often omit the subscript $\lambda, \mu$ in the notation throughout the computations.

We prove here some preliminary results that will lead to the proof of the Theorems 1.5 and 1.6.

**Proposition 4.2.** *If hypotheses ($K_0$,$K_1$,$K_2$) hold, then*

*(1)  for every $p \in \Omega$, $\liminf_{\lambda \to \infty} u_{\lambda,\mu}(p) \geq a(p)$, uniformly in $\mu \geq 0$.*

*If hypotheses ($K_0$,$K_3$) hold, then*

*(2)  $\|u_{\lambda,\mu}\|_\infty$ is bounded as $\mu \to \infty$, uniformly with respect to to $\lambda \geq 0$.*

*(3)  for every $p \in \Omega$, $\limsup_{\mu \to \infty} u_{\lambda,\mu}(p) \leq a(p)$, uniformly in $\lambda \geq 0$.*

*Remark* 4.3. It is worth observing that in the hypotheses of Proposition 4.2 and Corollary 1.7, positive solutions may not exist, so the claims are always to be intended to hold for the solutions, provided that they exist.

*Proof of Proposition 4.2.* Proof of point (1).
Hypothesis ($K_2$) implies that there exists $\theta > 0$ such that $f(x,t) > b_0 t/2$ for $t \in (0, \theta)$ and $x \in \Omega$. The necessary condition for the existence of a positive solution requires that the nonlinearity intersects the line $\lambda_1 t$. It follows that, for $\lambda > \bar\lambda := 2\lambda_1/b_0$, no positive solution can exist with $\|u\|_\infty < \theta$. As a consequence, by (iii) in Lemma 4.1, there exists $d_p > 0$ (depending continuously on $p \in \Omega$) such that

$$u(p) \geq d_p \quad \text{for} \ \ \lambda > \bar\lambda. \tag{4.2}$$

Let now $p \in \Omega$ and suppose that for some positive $\nu < a(p)$, the positive solutions $u_{\lambda,\mu}$ with $\lambda \geq \bar\lambda, \mu \geq 0$ satisfy $u_{\lambda,\mu}(p) \leq \nu$.

We suppose $M_u \geq p$, so that

$$u(y) \leq u(p) \leq \nu \qquad \text{for } y \leq p, \tag{4.3}$$

otherwise one has to estimate on the other side of $p$ but obtains analogous estimates.

Let $2\eta = f(p, \nu) > 0$ (by $(K_1)$). By the continuity of $f$ there exists $\delta_p$ (depending on $p$ and $\nu$ only) such that $p - \delta_p > -1$ and

$$f(y, \nu) \geq \eta \qquad \text{for } y \in (p - \delta_p, p]; \tag{4.4}$$

also observe that by estimate (4.2)

$$m_p \leq u(y) \leq \nu \qquad \text{for } y \in (p - \delta_p, p], \tag{4.5}$$

where $m_p = inf\{d_y : y \in (p - \delta_p, p]\}$ is positive and depends on $p$ and $\nu$ only.

We write $u(y) = \tau(y)\nu$ where, by (4.5), $\tau(y) \in [m_p/\nu, 1]$. By $(K_1)$ we get

$$f(y, u(y)) = f(y, \tau(y)\nu) \geq \tau(y)f(y, \nu) \geq \tau(y)\eta \geq \eta m_p/\nu \qquad \text{for } y \in (p - \delta_p, p].$$

With this and equation (4.1) we may estimate

$$u(p) \geq \lambda \int_{-1}^{1} \mathcal{G}_k(y, p) f(y, u(y))\, dy \geq \lambda \int_{p-\delta_p}^{p} \mathcal{G}_k(y, p)\eta m_p/\nu\, dy\,,$$

where the last integral is a positive number depending only on $p$ and $\nu$. Since $u(p) \leq \nu$ by assumption, we have reached a contradiction for $\lambda$ large enough.

As a consequence we have proved that for every $\nu \in (0, a(p))$ there exists $\lambda_0$ depending on $p$ and $\nu$ such that $\lambda > \lambda_0$ implies $u_{\lambda,\mu} > \nu$, for every $\mu \geq 0$. This concludes the proof of (1).

Proof of point (2).
By condition $(K_3$-i), there exists $\eta > 1$ such that $g(x, t) \geq c_0 t/2$ for $t > \eta$ and any $x \in \Omega$. If $\|u\|_\infty > 4\eta$ then, using (iii) in Lemma 4.1, we obtain that $u \geq \|u\|_\infty/4 > \eta$ in $[-1/2, 1/2]$, where we can thus consider $g(y, u(y)) \geq c_0 u(y)/2 \geq c_0 \|u\|_\infty/8$.

Using equation (4.1) with a fixed $p \in \Omega$ we obtain

$$\begin{aligned} u(p) &\geq \mu \int_{-1}^{1} \mathcal{G}_k(y, p) g(y, u(y))\, dy \\ &\geq \frac{c_0\mu}{8} \int_{-1/2}^{1/2} \mathcal{G}_k(y, p) \|u\|_\infty\, dy \geq \frac{c_0\mu}{8} u(p) \int_{-1/2}^{1/2} \mathcal{G}_k(y, p)\, dy\,; \end{aligned} \tag{4.6}$$

this can be written as $u(p) \geq C\,\mu\,u(p)$ and implies $u(p) = 0$ for $\mu > 1/C$. This immediately implies $u \equiv 0$ by the concavity of $u$, so we have obtained that $\|u_{\lambda,\mu}\|_\infty \leq 4\eta$ for $\mu > 1/C$ and any $\lambda \geq 0$, proving point (2).

Proof of point (3).
Fix again $p \in \Omega$ and suppose that for some small $\delta > 0$, the positive solutions $u_{\lambda,\mu}$ with $\lambda \geq 0, \mu \geq 0$ satisfy $u_{\lambda,\mu}(p) \geq a(p) + \delta$.

Observe that the conditions in ($K_3$) imply that there exists $c_\delta > 0$ such that

$$g(x,t) \geq c_\delta \quad \text{for } t > a(p) + \delta/2, \ |x - p| \leq c_\delta, \tag{4.7}$$

actually, by ($K_3$-i), for some $M > 0$ one has $g(x,t) > 1$ for $t > M$ and any $x \in \Omega$, while by ($K_3$-ii) and the continuity of $g$, one can find $c_\delta$ such that $g$ is positive for $t \in [a(p) + \delta/2, M]$ and $|x - p| \leq c_\delta$. Then $\min \{g(x,t) : \ t \geq a(p) + \delta/2, \ |x - p| \leq c_\delta\}$ is positive and by redefining $c_\delta$ if necessary we obtain (4.7).

By the concavity of $u$ and Lemma 4.1, we have that $u$ is never below the triangle with base on $\Omega$ and vertex in $(p, a(p) + \delta)$, as a consequence there exists an interval $I$ of length $\ell_\delta := \frac{\delta}{1+\delta}$, such that

$$u \geq a(p) + \delta/2 \quad in \ I. \tag{4.8}$$

Moreover, we can select a subinterval $I^* \subseteq I$ which has length $\frac{1}{2}\ell_\delta$ and distance from the boundary of $\Omega$ at least $\frac{1}{2}\ell_\delta$. Finally, let $J = I^* \cap [p - c_\delta, p - c_\delta]$: observe that $|J|$ depends on $p$ and $\delta$ only. With estimate (4.8), from equation (4.6) and using also (4.7), we get

$$\begin{aligned}
u(p) &\geq \mu \int_{-1}^{1} \mathcal{G}_k(y,p) g(y, u(y)) \, dy \geq \mu \int_J \mathcal{G}_k(y,p) c_\delta \, dy \\
&\geq \mu c_\delta \, C_{\mathcal{G}_k,\delta,p} |J|,
\end{aligned}$$

where $C_{\mathcal{G}_k,\delta,p} = \min \{\mathcal{G}_k(y,p) : \ y \in (-1 + \frac{1}{2}\ell_\delta, 1 - \frac{1}{2}\ell_\delta)\} > 0$.

As a consequence, $u(p) \to \infty$ for $\mu \to \infty$, which contradicts the boundedness in point (2). We conclude that for every $p \in \Omega$ and every $\delta > 0$ there exists $\mu_0$ such that $\mu > \mu_0$ implies $u_{\lambda,\mu}(p) < a(p) + \delta$, for every $\lambda \geq 0$. We have thus proves point (3).  $\square$

The next Lemma is fundamental in order to produce the contradiction that will prove Theorem 1.5. Roughly speaking, it states that when $k \geq 2$ it is not possible for the solutions of ($\Pi_{\lambda,\mu}^k$) to converge to $a(x)$ under the hypotheses of the Theorem, which impose a high value for $-u''$ near the boundary and then would force $-u''$ to have two separate maxima, which is impossible for a concave function.

*Remark* 4.4. Observe that when $k = 1$, on the other hand, $-u''$ does not have to be concave and may therefore have two separate maxima. Actually, the typical asymptotic behavior of the solutions in this case (see [12, 31, 32, 21]) is exactly to converge to the zero up to near the boundary and then to go very fast to the boundary condition.

**Lemma 4.5.** *Given $C, a^\pm > 0$, for sufficiently small $\gamma > 0$, no solution of ($\Pi_{\lambda,\mu}^k$) with $k \geq 2$ (in particular, no function $u$ which is concave in $(-1, 1)$ with concave $-u''$ and satisfies $u = u'' = 0$ at $\pm 1$) may satisfy*

$$\begin{cases} u(\pm(1-\gamma)) \geq a^\pm - C\gamma, \\ u(\pm(1-2\gamma)) \leq a^\pm + C\gamma. \end{cases} \tag{4.9}$$

*Proof.* Suppose $u = u_\gamma$ are functions as in the statement and satisfying (4.9). Let $\gamma \to 0$.

Since $u_\gamma'$ is nonincreasing and $u_\gamma(-1) = 0$, $u_\gamma(-1+\gamma) \geq a^- - C\gamma$, one has

$$u_\gamma'(-1) \geq \frac{a^- - C\gamma}{\gamma} \to +\infty.$$

Moreover, since $u_\gamma(-1+2\gamma) \leq a^- + C\gamma$, one obtains that

$$u_\gamma'(-1+2\gamma) \leq \frac{2C\gamma}{\gamma} = 2C.$$

This means that

$$-u_\gamma''(\xi_1) \geq \frac{\frac{a^- - C\gamma}{\gamma} - 2C}{2\gamma} \to +\infty$$

at some $\xi_1(\gamma) \in (-1, -1+2\gamma)$. By the same argument, also $-u_\gamma''(\xi_2) \to +\infty$ at a some $\xi_2(\gamma) \in (1-2\gamma, 1)$ and $u_\gamma'(1-2\gamma) \geq -2C$.

However, since $-u_\gamma''$ is concave too, its minimal value between $\xi_1$ and $\xi_2$ also goes to infinity. This gives rise to a contradiction since

$$4C \geq u_\gamma'(-1+2\gamma) - u_\gamma'(1-2\gamma) \geq \int_{-1+2\gamma}^{1-2\gamma} -u_\gamma'' \to +\infty.$$

$\square$

At this point we can conclude:

*Proof of Theorem 1.5.* Let $C$ be such that $|a'| < C/4$ in some neighborhood of $\pm 1$. Let $\gamma > 0$ (depending on $C, a^\pm$) be as small as required in Lemma 4.5 and such that the estimate above on $a'$ holds true in $\pm[1-2\gamma, 1]$. Then

$$\begin{cases} a(\pm(1-\gamma)) \geq a^\pm - C\gamma/4, \\ a(\pm(1-2\gamma)) \leq a^\pm + C\gamma/2. \end{cases}$$

By the points (1) and (3) in Proposition 4.2, we have that

$$\begin{cases} u_{\lambda,\mu}(\pm(1-\gamma)) > a^\pm - C\gamma, \\ u_{\lambda,\mu}(\pm(1-2\gamma)) < a^\pm + C\gamma, \end{cases} \tag{4.10}$$

for $\lambda, \mu$ large enough.

Then Lemma 4.5 implies that for such $\lambda, \mu$ a positive solution of $(\Pi_{\lambda,\mu}^k)$ with $k \geq 2$ cannot exist, which proves the claim of the Theorem. $\square$

The proof of Theorem 1.6 will follow a similar argument, however, the contradiction will not come from Lemma 4.5, which is only true for $k \geq 2$, but from that fact that the solutions are concave functions that must converge to $a$, which is not concave.

*Proof of Theorem 1.6.* If $a$ is not a concave function, then there exist $p_1 < p_2$, $t \in (0,1)$ and $\delta > 0$ such that

$$a(tp_1 + (1-t)p_2) + \delta < ta(p_1) + (1-t)a(p_2) - \delta \,.$$

By points (1) and (3) in Proposition 4.2 we have that

$$\begin{cases} u_{\lambda,\mu}(p_i) > a(p_i) - \delta \,, & i = 1, 2 \,, \\ u_{\lambda,\mu}(tp_1 + (1-t)p_2) < a(tp_1 + (1-t)p_2) + \delta \,, \end{cases} \tag{4.11}$$

for $\lambda, \mu$ large enough. Then

$$u_{\lambda,\mu}(tp_1 + (1-t)p_2) < a(tp_1 + (1-t)p_2) + \delta < ta(p_1) + (1-t)a(p_2) - \delta < tu_{\lambda,\mu}(p_1) + (1-t)u_{\lambda,\mu}(p_2),$$

which is a contradiction since $u_{\lambda,\mu}$ is concave. Then for such $\lambda, \mu$ a positive solution of $(\Pi_{\lambda,\mu}^k)$ cannot exist, which proves the claim of the Theorem. $\qquad\square$

As a final observation, we show below that sometimes the solutions must exceed the zero of the nonlinearity, even with $\mu = 0$, in contrast with the the case $k = 1$ when $a$ is superharmonic.

In fact, under the same conditions in which we were able to prove nonexistence for large $\mu$, we have that solutions still exist at least for $\mu < M(\lambda)$ by Theorem 1.1. However, if $\lambda$ is large, these solutions cannot stay below the zero, as shown in the next Proposition.

**Proposition 4.6.** *In the hypotheses of Theorem 1.5 or of Theorem 1.6, there exists $\Lambda_1 > 0$ such that for $\lambda > \Lambda_1$ and for every $\mu \geq 0$*

$$\text{the inequality} \quad u_{\lambda,\mu} \leq a \quad \text{must be false.}$$

*Proof.* We proceed exactly as in the proof of Theorem 1.5, until obtaining the first equation in (4.10), which in fact holds true for $\lambda$ large enough and any $\mu \geq 0$ (see point (1) in Proposition 4.2). Then we suppose, for sake of contradiction, that $u_{\lambda,\mu} \leq a$, which implies that the second equation in (4.10) holds true too. Then Lemma 4.5 implies that for such $\lambda$, and any $\mu \geq 0$, a positive solution of $(\Pi_{\lambda,\mu}^k)$ with $k \geq 2$ not exceeding $a$ cannot exist.

In the conditions of Theorem 1.6 the proof is similar: the first equation in (4.11) holds true for $\lambda$ large enough and any $\mu \geq 0$ by point (1) in Proposition 4.2, while the second equation is true if we suppose $u_{\lambda,\mu} \leq a$. Then again we reach a contradiction. $\qquad\square$

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