Evolution Equations in Banach Spaces. Theory of Cosine Operator Functions

S. I. Piskarev

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More than 13 years have passed since the fundamental survey [14] was prepared, which, as the author intended, should be the first part of a large work devoted to abstract differential equations and methods for their solution.

During the later 10 years, the work in the field of differential equation in abstract spaces was very active and every year, several books and a heavy number of papers devoted to this direction appear in the world.

Moreover, the author have prepared a separate edition of the bibliographical index [16], which can surve as a sufficiently complete source of information about the theory of differential equations in abstract spaces during the recent years.

The material of the present survey includes the presentation of the abstract Cauchy problem for first- and second-order equations that is not considered in the book [15].

As was already mentioned in [14], the theory of cosine operator-valued functions is philosophically very close to the operator semigroup theory and often is developed in parallel to it. Therefore, the reader easily draws analogies between the material presented here and that presented in [14]. At the same time, the theory of cosine operator-valued functions considerably differs from the operator semigroup theory. First of all, these distinctions concern with the properties inherit to the corresponding parabolic and hyperbolic partial differential equations.

We now present the main notation a certain part of which was already introduced in the book [15] and which is also used here.

The set of natural numbers is denoted by $\mathbb{N}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the set of integers by $\mathbb{Z}$, the set of reals by $\mathbb{R}$, and the set of complex numbers by $\mathbb{C}$. A tuple of numbers $1, 2, ..., m$, $m \in \mathbb{N}$, is denoted by $1, m$, the real semiaxis $(0, \infty)$ by$\mathbb{R}_+$, and $[0, \infty)$ by $\overline{\mathbb{R}_+}$.

We denote by E a Banach space over the field of complex numbers with the norm $\| \cdot \|$. For a Hilbert space with the inner product $(\cdot, \cdot)$, we use the H. .

The boundary of a set Ω is denoted by $\partial\Omega$, the interior of the set Ω by $\mathrm{int}(\Omega)$, the the closure in the strong topology by $\overline{\Omega}$, and, for example, the closure in the weak topology by w-cl-(Ω).

As usual, the space dual to E is denoted by E^*, with elements x^*, y^*, ..., and the value of a functional $x^* \in E^*$ at an element $x \in E$ is written as $\langle x, x^* \rangle$.

The domain and the range of an operator A will be written as $D(A)$ and $\mathbf{R}(A)$, respectively, and the null-space (kernel) aas $\mathbf{N}(A)$. The set of linear operators acting from $D(A) \subseteq E$ into E is denoted by $L(E)$, and the set of linear continuous operators by $B(E)$. Closed linear operators with dense domain ($\overline{D(A)} = E$) in E are distiguished into the set $\mathbf{C}(E) \subset L(E)$. In the case where operators act from one space E into another F, we write $L(E, F)$ abnd $B(E, F)$, respectively. The linear variety $D(A)$ endowed with the norm $\|x\|_A := \|x\| + \|Ax\|$ in the case of a closed linearn operator becomes a Banach space; we denote it by $\mathbf{D}(A)$.

We use the traditional notation for the resolvent set $\rho(A)$ and the spectrum $\sigma(A)$ of an operator A; as usial, the latter one is divided into the point spectrum $P\sigma(A)$, the continuous spectrum $C\sigma(A)$, and the residual spectrum $R\sigma(A)$.

Chapter I

Cauchy Problem and Resolving Families+

Before considering the theory of C_0-cosine operator functions, we describe the general picture of the statement of the well-posed Cauchy problem in a Banach space. As is easily noted, a natural generalization of the concept of solution leads to more general families: integrated semigroups and C-semigroups.

1.1 Cauchy Problem for a Complete Differential Equation

Let E be a Banach space, and let $A_0, A_1, ...A_{m-1}$ be closed linear operators on E, i.e., $A_k \in \mathbf{C}(E), k \in \overline{0, m-1}$. In the Banach space E, let us consider the following abstract Cauchy problem of order m:

$$\begin{cases} u^{(m)}(t) = \sum_{k=0}^{m-1} A_k u^{(k)}(t), & t \in \overline{\mathbb{R}}_+, \\ u^{(k)}(0) = u_k^0, & k \in \overline{0, m-1},\ m \geq 2. \end{cases} \tag{1.1}$$

Definition 1.1.1. A function $u(\cdot) \in C^m(\overline{\mathbb{R}}_+; E)$ is called a *classical solution of problem* (1.1) if $u^{(k)}(t) \in D(A_k)$, $A_k u^{(k)}(\cdot) \in C(\overline{\mathbb{R}}_+; E)$ for $t \in \overline{\mathbb{R}}_+$, $k \in \overline{0, m-1}$, and relations (1.1) hold.

As in [15], we define the propagators $\mathbf{P}_j(t)$, $j = \overline{0, m-1}$, which give a solution of the Cauchy problem (1.1) with initial conditions $u_j^{(k)}(0) = \delta_{jk} u_j^0$ (δ_{jk} is the Kronecker symbol), i.e., $u_j(t) = \mathbf{P}_j(t) u_j^0$.

Definition 1.1.2. The Cauchy problem (1.1) is said to be *uniformly well-posed* if $\mathbf{P}_k(\cdot)x \in C^k\big(\overline{\mathbb{R}}_+; E\big), \mathbf{P}_{m-1}^{(k-1)}(t)x \in D(A_k), t \in \overline{\mathbb{R}}_+$, and $A_k \mathbf{P}_{m-1}^{(k-1)} x \in C(\overline{\mathbb{R}}_+; E)$ for any $x \in E$ and $k \in \overline{0, m-1}$.

In the general case, in the Banach space E, problem (1.1) is not studied. In particular cases, some concepts are introduced, which will be considered in the next volumes in more detail.

Definition 1.1.3. We say that an operator A *generates α times integrated semigroup* with $\alpha \geq 0$ if $(\omega, \infty) \subseteq \rho(A)$ for a certain $\omega \in \mathbb{R}$ and there exists a strongly continuous function $S(\cdot) : [0, \infty) \to B(E)$ such that $\|S(t)\| \leq M e^{\omega t}$, $t \in \overline{\mathbb{R}_+}$, with a certain constant $M \geq 0$ and

$$(\lambda I - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)\, dt$$

for all $\lambda > \max\{\omega, 0\}$.

The family $S(\cdot)$ itself is called an *α times integrated semigroup.*

Theorem 1.1.1 ([114]). *Let $m \geq 2$, and let A_{m-1} generate an r times integrated semigroup. Assume that $D(A_i) \supseteq D(A_{m-1})$ for all $i \in \overline{0, m-2}$, and, moreover, $A_i D(A_{m-1}) \subseteq D(A_{m-1}^{i-m+r+2})$ for $i \geq m-r-1$. Then problem (1.1) has a unique exponentially bounded solution for $u_{m-1}^0 \in D(A_{m-1}^{r+1})$, $u_k^0 \in$*

$\bigcap_{j=0}^{m-1} D(A_{m-1}^r A_i)$, $\quad k \in \overline{0, m-2}$, *and for certain constants* $c, \omega > 0$, *we have the following estimate for* $t \in \overline{\mathbb{R}}_+$:

$$\begin{aligned}
&\|u(t)\| + \|A_{m-1}u(t)\| \\
&\leq ce^{\omega t}\Bigg\{\sum_{l=0}^{r+1} \|A_{m-1}^l u_{n-1}^0\| + \\
&+ \sum_{k=0}^{m-2}\sum_{i=0}^{m-2}\sum_{l=0}^{r} \|A_{m-1}^l A_i u_k^0\| + \\
&\sum_{k=0}^{m-2}(\|u_k^0\| + \|A_{m-1}u_k^0\|)\Bigg\}.
\end{aligned}$$

In [191], this theorem was slightly changed by extending the set of initial data and by the absence of the exponential boundedness.

Denote $P(\lambda) := \Sigma_{i=0}^m \lambda^i A_i$ with the domain $D(P) := \cap_{i=0}^{m-1} D(A_i)$.

Theorem 1.1.2 ([192]). *The propagators* $\mathbf{P}_k^{(k)}(\cdot)$, $k \in \overline{0, m-1}$, *are norm-continuous for* $t \in \mathbb{R}_+$ *iff there exists* $\tau_0 \in \mathbb{R}_+$ *such that*

$$\lim_{|\omega|\to\infty} \|(\tau_0 + i\omega)^{m-1} P^{-1}(\tau_0 + i\omega)\| = 0, \tag{1.2}$$

$$\lim_{|\omega|\to\infty} \|(\tau_0 + i\omega)^{k-1} \overline{P^{-1}(\tau_0 + i\omega)A_k}\| = 0, \quad k \in \overline{0, m-1}. \tag{1.3}$$

Corollary 1.1.1. *Let conditions* (1.2) $- -$(1.3) *hold. Then foe each* $k \in \overline{0, m-1}$, *the operator* $\mathbf{P}_k(t)$ *is norm-continuous for* $t \in \mathbb{R}_+$.

The case of time-dependence $A_k = A_k(t)$, $t \in \overline{\mathbb{R}_+}$, was considered, e.g., in [223, 224].

In [289], the conditions for the existence of a unique entire solution of problem (1.1) were presented.

consider the problem (1.1) with $A_k \in \mathbf{C}(E)$ for all $k \in \overline{0, m-1}$; let $D(A_0) \subseteq D(A_k)$ for $k \in \overline{1, m-1}$.

Theorem 1.1.3 ([222]). *Under the assumptions described above, the following conditions are equivalent for the Cauchy problem* (1.1) :

(i) the operator A_0 *generates a* C_0*-semigroup*;

(ii) for any $u^0, u^1, ..., u^{m-1} \in D(A_0)$, *the Cauchy problem* (1.1) *has a unique solution* $u(\cdot) \in C^{(m-1)}(\mathbb{R}_+, \mathbf{D}(A_0))$.

The following theorem on the uniform stability of problem (1.1) holds.

Theorem 1.1.4. *Let an operator A_0 generate a C_0-semigroup, and let $u(\cdot)$ be a solution of the Cauchy problem (1.1) with initial conditions u_l^k ($k \in \overline{1, m-1}$; $l \to \infty$), $u_l^k \in D(A)$ for $k \in \overline{1, m-1}$, $u_l^k \to 0$ in E. Then $u_l(\cdot) \to 0$ uniformly on any compact set.*

Theorem 1.1.5 ([222]). *A_0 C_0-, Let A_0 generate a C_0-semigroup, $Ak \in \mathbf{C}(E)$, $D(A) \subset D(A_k)$, and let ω be such that for $\operatorname{Re}\lambda > \omega$, there exists a generalized resolvent (pencil resolvent)*

$$R_\lambda := R(\lambda; A_0, ..., A_{m-1}) = (\lambda^m - \lambda^{m-1}A_{m-1} - ... - \lambda A_1 - A_0)^{-1}$$

(such an ω always exists!). Also, on $D(A)$, let the relation $A_k R_\lambda = R_\lambda A_k$ ($k \in \overline{1, m-1}; \operatorname{Re}\lambda > \omega$) hold. Then problem (1.1) is uniformly well-posed, and its solution has the form

$$u(t) = \sum_{k=0}^{m-1} Q_{m-1-k,m-1}(t) u^k, \tag{1.4}$$

where the operator-valued functions $Q_{m-1-k,m-1}(t)$ are strongly continuous families composing the operator semigroup

$$G(t) = \begin{bmatrix} Q_{0,0}(t) & \dots & Q_{0,m-1}(t) \\ \vdots & \ddots & \vdots \\ Q_{m,0}(t) & \dots & Q_{m-1,m-1}(t) \end{bmatrix}, \quad t \in \mathbb{R}_+,$$

with the generator

$$\Gamma = \begin{bmatrix} A_0 & I & 0 & \dots & 0 \\ A_1 & 0 & I & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & I \\ A_{m-1} & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Now let us consider the Cauchy problem for the following equation of order m having the special form:

$$\prod_{j=1}^{m} (\frac{d}{dt} - A_j) u(t) \equiv (\frac{d}{dt} - A_m) ... (\frac{d}{dt} - A_1) u(t) = 0 \tag{1.5}$$

with initial conditions

$$u^{(k)}(0) = u_k^0, \quad k \in \overline{0, m-1},\ m \geq 2,$$

and operators $A_j \in \mathbf{C}(E), j \in \overline{1, m}$.

Definition 1.1.4. The Cauchy problem (1.5) is said to be *uniformly well-posed* if the following conditions hold:,

(i) There exists a solution of the Cauchy problem (1.5) for $u^0, ..., u^{m-1}$ taken from a certain dense set D in E ;

(ii) for $u^0, ..., u^{m-1} \in D$, the solution of the Cauchy problem (1.5) has the property

$$\prod_{j=1}^{k}(\frac{d}{dt} - A_j)u(t) \in C^{m-k}(\overline{\mathbb{R}_+}, E); \tag{1.6}$$

for $k \in \overline{1, m-1}$;

(iii) the uniform stability of the solution of (1.5) is complemented by the following condition on any compact set: the convergence $\prod_{j=1}^{k}(\frac{d}{dt} - A_j)u_p(0) \to 0$ implies the convergence $\prod_{j=1}^{k}(\frac{d}{dt} - A_j)u_p(t) \to 0$ uniformly on each compact set in $\overline{\mathbb{R}_+}$ (here, $k \in \overline{1, m-1}$; $p \to \infty$).

Theorem 1.1.6 ([18]). *In the Cauchy problem* (1.5)*, let* $A_j \in \mathbf{C}(E)$ $(j \in \overline{1, m})$*, let the intersection of the resolvent sets* $\rho(A_j)$ *of the operators* A_j *be not empty, and let the set*

$$\tilde{D} \equiv \cap\{D(A_{i_1}...A_{i_m}) : \, i_k \in \overline{1, m}\} \tag{1.7}$$

be dense in E*. Then problem* (1.5) *is uniformly well-posed iff* A_j *generates a* C_0*-semigroup for each* $j \in \overline{1, m}$*.*

Theorem 1.1.7 ([18]). *Under the conditions of Theorem* 1.1.6*, let the operators* A_j *generate* C_0*-semigroups for* $j \in \overline{1, m}$*, and, moreover, let these semigroups commute*:

$$\exp(tA_i)\exp(sA_j) = \exp(sA_j)\exp(tA_i), \quad t, s \in \mathbb{R}, \quad i, j \in \overline{1, m}. \tag{1.8}$$

Then problem (1.5) *is well posed.*

Theorem 1.1.8. *For the Cauchy problem* (1.5)*, let the conditions of Theorem* 1.1.7 *hold, and let* $0 \in \rho(A_i - A_j)$ *for all* $i \neq j$*. Then the condition* $w(t) \in \mathbf{N}\Big(\prod_{i=1}^{m}(\frac{d}{dt} - A_i)\Big)$ *for* $t \in \overline{\mathbb{R}}$ *implies the relation* $w(t) = \sum_{i=1}^{m} w_i(t)$*, where* $w_i(t) \in \mathbf{N}(\frac{d}{dt} - A_i)$*,* $t \in \mathbb{R}$*.*

Condition (1.8) in Theorem 1.1.7 can be replaced by a number of conditions imposed on the domains $\mathbf{R}(A_i - A_j)$ for $i \neq j$.

Let $1, \varphi^1, \varphi^2, ..., \varphi^{p-1}$ be roots of pth degree of the unity, i.e., $\varphi^k = e^{\frac{2\pi k}{p}i}$.

Definition 1.1.5. A C_0*-function of the Mittag-Leffler type with a parameter* p is a function $\mathfrak{M} : \mathbb{C} \to B(E)$ having the following properties:

(i) $\sum_{k,l=0}^{p-1} \mathfrak{M}(\varphi^k t + \varphi^l s) = p^2 \mathfrak{M}(t)\mathfrak{M}(s)$ for any $t, s \in \mathbb{R}$;

(ii) $\mathfrak{M}(0) = I$;

(iii) the family of operators $T(t) \equiv \mathfrak{M}(\varphi^k t + \varphi^l s)$, $k, l \in \overline{0, p-1}$, with a fixed $s \in \mathbb{R}$ is strongly continuous in $t \in \mathbb{R}$.

For the Mittag-Leffler C_0-function with a parameter p, the p-generator A is defined by the relation

$$Ax = s\text{-}\lim_{t \to 0} p! \frac{\mathfrak{M}(t) - I}{t^p} x$$

for those x at which the limit exists. It is known that the generating operator of the Mittag-Leffler C_0-function with a parameter p is a linear closed densely defined operator, and the following relation holds for any $x \in D(A)$:

$$\frac{d^p}{dt^p} \mathfrak{M}(t)x = A\mathfrak{M}(t)x = \mathfrak{M}(t)Ax,$$

and, moreover, $\mathfrak{M}^{(k)}(0) = 0$ for $k \in \overline{0, p-1}$.

For the Mittag-Leffler C_0-function with a parameter p, the perturbation theorems of the Philips–Miyadera type hold (see[15]).

Theorem 1.1.9 ([30]). *Let A generate a Mittag-Leffler C_0-function with a parameter p, and let $\|\mathfrak{M}(t)\| \leq Me^{\omega t}$, $t \in \mathbb{R}$. Then for any $B \in B(E)$, the operator $A + B$ generates a Mittag-Leffler C_0-function with the parameter p.*

Proposition 1.1.1. *The Mittag-Leffler C_0-function with a parameter p is a C_0-group of operators in the case $p = 1$, and in the case $p = 2$, it is a C_0-cosine operator-valued function. A Mittag-Leffler C_0-function with a parameter p has a bounded generating operator $A \in B(E)$ for $p \geq 3$.*

In the simplest case $m = 2$, for example, the following theorems hold for problem (1.1).

Theorem 1.1.10 ([222]). *Let the Cauchy problem* (1.1) *be uniformly well-posed for $m = 2$, and let $\mathbf{P}_1'(t)E \in \mathbf{D}(A_1)$ for $t \in \overline{\mathbb{R}}_+$. Then A_0 generates a C_0-cosine operator-valued function on E.*

Theorem 1.1.11 ([222]). *Let $A_1 \in B(E)$. Then the Cauchy problem* (1.1) *is uniformly well posed for $m = 2$ iff A_0 generates a C_0-cosine operator-valued function on E.*

However, in the general case, even for $m = 2$, the Cauchy problem (1.1) turns out to be very complicated. First, in [132], H. O. Fattorini has presented an example of the Cauchy problem (1.1) that has a solution for $m = 2$, but this solution is not exponentially bounded. Second, in contrast to the Cauchy problem for $m = 1$, the case $m = 2$ admits a more flexibility in the sense of well-posedness of its statement.

As one of the variants, let us present the approach coming back to H. O. Fattorini. The constructions used in proving these theorems piratically completely repeat the techniques used in proving the assertions concerning C_0-cosine and C_0-sine operator-valued functions (see also [28]).

Consider the Cauchy problem

$$u''(t) + Bu'(t) + Au(t) = 0, \quad t \in \overline{\mathbb{R}}_+, \tag{1.9}$$

$$u(0) = u^0, \quad u'(0) = u^1,$$

with $A, B \in \mathbf{C}(E)$.

Definition 1.1.6. We say that the operators A and B generate M, N-*families of operators on* E if the following conditions hold:

(i) $M(t)$ and $BN(t)$ are strongly continuous in $t \in \overline{\mathbb{R}}_+$, and the function $N(t)x$ is strongly continuously differentiable in $t \in \overline{\mathbb{R}}_+$ for any $x \in E$;

(ii) the set $\hat{E} = \{x \in E : M(t)x$ is strongly differentiable in $t \in \overline{\mathbb{R}}_+$, and $BM(t)x$ is continuous in $t \in \overline{\mathbb{R}}_+\}$ is dense in E;

(iii) the operator $A = -M''(0)$ is B-closed, and $Bx = -N''(0)x$ for all $x \in \hat{E}$;

(iv) $M(0) = N'(0) = I$ and $N(0) = 0$;

(v) $M(t + s)x = M(t)M(s)x + N(t)M'(s)x$ for all $x \in \hat{E}$ and $t, s \in \overline{\mathbb{R}}_+$;

(vi) $N(t + s) = M(t)N(s) + N(t)N'(s)$ for all $t, s \in \overline{\mathbb{R}}_+$.

Theorem 1.1.12 ([28]). *Let A and B generate M, N-families. Then*

(i) A is closed, $D(A) \cap D(B) \subseteq \hat{E} \subseteq D(B)$, and $D(A) \cap D(B)$ is dense in E;

(ii) the families M and N are uniquely defined by the operators A and B;

(iii) $M'(0)x = 0$ for all $x \in D(M'(0))$;

(iv) $M'(t)x = -N(t)Ax$ for all $x \in D(A)$ and $t \in \overline{\mathbb{R}}_+$;

(v) $N'(t)x = M(t)x - N(t)Bx$ for all $x \in \hat{E}$ and $t \in \overline{\mathbb{R}}_+$;

(vi) $N''(t)x + N'(t)Bx + N(t)Ax = 0$ for all $x \in D(A) \cap D(B)$ and $t \in \overline{\mathbb{R}}_+$;

(vii) for all $x \in \hat{E}$ and $t \in \overline{\mathbb{R}}_+$, the element $N'(t)x \in \hat{E}$, $N(t)x \in D(A)$, and $N''(t)x - x + BN'(t)x + AN(t)x = 0$;

(viii) for all $x \in E$ and $t \in \overline{\mathbb{R}}_+$, the element $\int_0^t N(s)x\,ds \in D(A)$ and $N''(t)x - x + BN(t)x + A\int_0^t N(s)x\,ds = 0$;

(ix) for all $x \in E$ and $t \in \overline{\mathbb{R}}_+$, the element $\int_0^t N(s)x\,ds \in \hat{E}$, $M(t)x \in D(A)$ and $M''(t)x + BM'(t)x + AM(t)x = 0$;

(x) for all $x \in D(A) \bigcup \hat{E}$ and $t \in \overline{\mathbb{R}}_+$, the element $M(t)x - x \in \hat{E}$, $\int_0^t M(s)x\,ds \in D(A)$, and $M'x + B(M-I)x + A\int_0^t M(s)xds = 0$;

(xi) there exist constants $C, \omega \geq 0$ such that

$$\|M(t)\|, \|N(t)\|, \|BN(t)\|, \|N'(t)\| \leq Ce^{\omega t}, \quad t \in \overline{\mathbb{R}}_+,$$

and for all $x \in \hat{E}$, there exist constants $C, \omega \geq 0$ such that

$$\|M'(t)x\|, \|BM(t)x\| \leq C(x)e^{\omega t}, \quad t \in \overline{\mathbb{R}}_+;$$

(xii) the operator $\lambda^2 + \lambda B + A$ is closedable for all $\lambda \in \mathbb{C}$;

(xiii) there exists a constant $\omega \in \overline{\mathbb{R}}_+$ such that $\lambda \in \rho(A, B)$ for all λ with $\operatorname{Re}\lambda > \omega$ and

$$\Delta(\lambda)x := (\lambda^2 + \lambda B + A)^{-1}x = \int_0^\infty e^{-\lambda t}N(t)x\,dt \quad \textit{for} \quad x \in E;$$

$$\Delta(\lambda)(I+B)x = \int_0^\infty e^{-\lambda t}M(t)xdt \quad \textit{for} \quad x \in \hat{E};$$

(xiv) $\lambda^2\Delta(\lambda)x \to x$ as $\lambda \to \infty$ for all $x \in E$.

The following analog of Theorem 2.1.1 from [15]holds.

Theorem 1.1.13 ([288]). *Operators A and B generate M, N families iff the following conditions hold*:

(i) the operators A and B are closed, and $D(A) \bigcap D(B)$ is dense in E;

(ii) there exist constant $C, \omega \geq 0$ such that $\lambda \in \rho(A, B)$, and for $\operatorname{Re}\lambda > \omega$, the operator $\Delta(\lambda)A$ is closedable and

(1.10)

$$\|\Big(\lambda\Delta(\lambda)\Big)^{(k)}\|, \|\Big(B\Delta(\lambda)\Big)^{(k)}\|, \|\Big(\overline{\Delta(\lambda)}B\Big)^{(k)}\| \leq \frac{Ck!}{(\operatorname{Re}\lambda - \omega)^{k+1}} \qquad k \in \mathbb{N}, \quad \operatorname{Re}\lambda > \omega,$$

where $\overline{\Delta(\lambda)B}$ is a bounded extension of the operator $\Delta(\lambda)B$ with the domain $D(A) \cap D(B)$ and $(\cdot)^{(k)}$ is the derivative of order k in λ.

In the case where A and B commute, instead of the estimate with the operator B in (1.10), it can be, e.g., $\|\frac{d^k}{d\lambda^k}\Big((\lambda I - A)\Delta(\lambda)\Big)\| \leq \frac{Mk!}{(\operatorname{Re}\lambda-\omega)^{k+1}}$, $k \in \mathbb{N}_0$ (see [28]).

If $A = 0$, then A and B generate M, N families iff B generates a C_0-semigroup.

Let $D(B) \subseteq D(A)$, and let $\rho(B) \neq \varnothing$. If $(\lambda_0 - B)^{-1}A$ has a bounded extension for a certain point $\lambda_0 \in \rho(B)$, then A and B generate M, N-families iff B generates a C_0-semigroup.

Proposition 1.1.2 ([288]). *Let B be subordinated by A with exponent $0 \leq \alpha \leq 1$, i.e., $D(A) \subseteq D(B)$ and $\|Bx\| \leq C_\alpha \|x\|^{1-\alpha}\|Ax\|^\alpha$ for all $x \in D(A)$, and let A and B commute. If $-A$ generates a C_0-cosine operator-valued function and $\|(\lambda^2 + A)^{-1}\| \leq C|\lambda|^{-2}$ for $\operatorname{Re}\lambda > \omega$, then A and B generate M and N families.*

Now let us consider an analytic continuation of a solution of Eq. (1.9) to the sector $\Sigma(\theta) = \{z \in \mathbb{C} : z \neq 0, |arg\, z| < \theta\}$.

Theorem 1.1.14 ([288]). *For a given $\theta, \omega \geq 0$, the following conditions are equivalent*:

(i) the Cauchy problem (1.9) is uniformly well-posed, the families M, N can be analytically continued to the sector $\Sigma(\theta) = \{z \in \mathbb{C} : z \neq 0, |argz| < \theta\}$, for any z, we have the embedding $N(z)E \subseteq D(B)$, and $BN(\cdot)$ is analytic in $\Sigma(\theta)$. Moreover, for each $\theta' \in (0, \theta)$, $x \in E$,

$$\lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} N'(z)x = 0, \lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} BN(z)x = 0, \lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} M(z)x = x, \lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} N(z)x = 0,$$

and there exists a constant $C'_\theta > 0$ such that

$$\|N'(z)\|, \|BN(z)\|, \|M(z)\| \leq C'_\theta e^{\omega \operatorname{Re} z} \quad \text{for all} \quad z \in \Sigma(\theta);$$

(ii) the set $D(A) \cap D(B)$ is dense in E. For each $\theta' \in (0, \theta)$, there exists $M'_\theta > 0$ such that for $\lambda \in \Sigma(\theta', \omega) = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \frac{\pi}{2} + \theta'\}$, the operator $\Delta(\lambda) := (\lambda^2 + \lambda B + A)^{-1} \in B(E)$ exists, the operator $\Delta(\lambda)A$ is closedable, and

$$\|\lambda\Delta(\lambda)\| \leq \frac{M}{|\lambda - \omega|}, \quad \|B\Delta(\lambda)\| \leq \frac{M}{|\lambda - \omega|}, \quad \|\overline{\Delta(\lambda)B_0}\| \leq \frac{M}{|\lambda - \omega|},$$

where $B_0 \subseteq B$ with $D(B_0) = D(A) \cap D(B)$. Moreover, in this case, we have

$$N''(z) + BN'(z) + AN(z) = 0, \quad M''(z) + BM'(z) + AM(z) = 0,$$

where $AN(z)$ and $AM(z)$ are analytic in $\Sigma(\theta)$. For each $\theta' \in (0, \theta)$,

$$\lim_{\substack{z\in\Sigma_{\theta'}\\ z\to 0}} M'(z)x = 0 \quad \text{for any} \quad x \in D(A).$$

The existence and uniqueness of solutions of Eq. (1.9) under certain "hyperbolic" conditions is considered in [226]. Problem (1.9) in the case of nonlinear B was considered in [186].

1.2 Cauchy Problems for Differential Equations of the 1st and 2nd Orders

In this section, we present certain statements of the Cauchy problem for equations of the first and second orders. Equations of the first order were already considered in the book [15] but, however, only in connection with C_0-semigroups on the space E. Here, we consider more general interpretations preparing the reader to a more universal approach that will be presented in the survey 3.

As was already noted, different statements of the Cauchy problem are possible. We now present certain arguments that show that a solution is given not by C_0-families on the whole space E.

Definition 1.2.1. An *integrated solution* of the Cauchy problem

$$u'(t) = Au(t), \quad u(0) = x, \tag{1.11}$$

is a continuously differentiable function $v(\cdot) : \mathbb{R}_+ \to E$ such that
(i) $v(\cdot) \in C([0,\infty); \mathbf{D}(A))$ and (ii) $\frac{dv}{dt}(t) = Av(t) + x, \quad v(0) = 0.$

Definition 1.2.2. Denote by $\mathbf{Z}(A)$ the *resolving set* of the operator A, i.e., the set of all $x \in E$ for which the Cauchy problem (1.11) has an integrated solution

Proposition 1.2.1. *Let $\mathbf{Z}(A)$ be the resolving subspace endowed by the family of seminorms*

$$\|x\|_{a,b} = \sup_{t\in[a,b]} \|u(t)\|, \qquad a, b \in \mathbb{R}_+. \tag{1.12}$$

Then $\mathbf{Z}(A)$ is a Frechét space and $T(t)x = u(t)$ is a locally equicontinuous semigroup generated by the operator $A|_{\mathbf{Z}(A)}$.

Recall the definition of entire vectors, which is equivalent to [15], Definition 3.1.3.

Definition 1.2.3. Denote by $\mathfrak{U}_c(A)$the set of entire vectors of an operator A, i.e., the set of $x \in D(A^\infty)$ such that for any $t \in \overline{\mathbb{R}}_+$,

$$\sum_{k=0}^{\infty} \|A^k x\| \frac{t^k}{k!} < \infty.$$

Proposition 1.2.2 ([113]). *Any linear closed operator on E with the resolvent set $\rho(A)$ containing the semiaxis (ω, ∞) generates a C_0-semigroup on a certain maximal subspace in E.*

Proposition 1.2.3 ([113]). *For any closed linear operator A, there exists a maximal Frechét space $\mathbf{Z}(A)$ such that $\mathbf{Z}(A) \subseteq E$ and the Cauchy problem* (1.11) *is automatically well posed on $\mathbf{Z}(A)$.*

As is seen from this proposition, the requirement of existence of a C_0-semigroup is very restrictive. At the same time, namely for C_0-families of operators, the technical tools for studying approximate methods are most well elaborated.

Theorem 1.2.1 ([113]). *Let $A \in \mathbf{C}(E)$. Then $\mathfrak{U}_c(A) = \{x \in E :$ problem* (1.11) *has an entire solution and $\mathfrak{U}_c(A) \subseteq \mathbf{Z}(A)$.*

Theorem 1.2.2 ([81]). *Let A generate an analytic C_0-semigroup on E. Then $\mathfrak{U}_c(A) = \mathbf{Z}(A)$, and, moreover, the equality holds topologically and algebraically.*

as is known, a self-adjoint operator $A^* = A \leq 0$ on a Hilbert space H generates an analytic C_0-semigroup, as well as a C_0-cosine operator-valued function. Moreover, in this case, by the Stone theorem, the operator iA generates an unitary C_0-group on H. At the same time, the practical problems often require the omitting of the self-adjointnees of the operator and the Hilbert property of the initial space. Therefore, to reveal whether a concrete operator generates C_0-semigroup or no is not a simple but often a complicated independent problem. Here, we present examples showing when the verification of generation of C_0-families on the Banach space E is possible.

Theorem 1.2.3 ([41]). *For $A \in \mathbf{C}(E)$ to be a generator of an analytic C_0-semigroup, it is necessary and sufficient that there exist numbers ν, ω, and $\alpha > 1$ such that the following inequality holds for all* $\operatorname{Re}\lambda > \omega_0$:

$$\|(\lambda^\alpha I - A)^{-1}\| \leq \frac{M|\lambda|^\nu}{(\operatorname{Re}\lambda)^{\nu+\alpha-1}(\operatorname{Re}\lambda - \omega)};$$

moreover, the following representation holds for this semigroup:

$$\exp(zA) = -\frac{\alpha}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} e^{z\mu^\alpha}\mu^{\alpha-1}(\mu^\alpha I - A)^{-1}\,d\mu$$

for $z \in \{z \in \mathbb{C} : Imz < \operatorname{Re} z|ctg\frac{\alpha\pi}{2}|\}$.

Let $\Omega \subset \mathbb{R}^d$ be a certain domain. Denote by $\mathfrak{C}(\Omega)$ the space of uniformly continuous bounded functions on Ω with the norm

$$\|v(\cdot)\|_{\mathfrak{C}(\Omega)} = \sup_{x\in\Omega}|v(x)|,$$

and let $\mathfrak{C}_\rho(\Omega) = \{v(\cdot) : \rho(\cdot)v(\cdot) \in \mathfrak{C}(\Omega), \rho(t) \geq 0\}$, $\|v(\cdot)\|_{\mathfrak{C}_\rho(\Omega)} = \|\rho(\cdot)v(\cdot)\|_{C(\Omega)}$.

Example 1.2.1 ([41]). Let $\Omega = [0,1]$, and let $Av = v''(\cdot)$ with $D(A) = \{v(\cdot) : v \in C(\Omega), Av \in C(\Omega), v'(0) = v'(1) = 0\}$. Then $A \in \mathbf{H}(\omega, \frac{\pi}{2})$ in $C([0,1])$.

At the same time, for the operator $A_0 v = v''(\cdot)$ with $D(A_0) = \{v(\cdot) : v \in C([0,1]), v(0) = v(1) = 0\}$, we have $A_0 \in \mathbf{H}(0,\beta)$ on $C_0([0,1]) = \{v(\cdot) : v \in C([0,1]), v(0) = v(1) = 0\}$.

Finally, the operator $A_\rho v = v''$ with $D(A_\rho) = \{v(\cdot) : \rho(\cdot)v(\cdot) \in C([0,1]),\ A_\rho v \in \mathfrak{C}_\rho([0,1])\}$ generates an analytic C_0-semigroup with the estimate

$$\|\exp(tA_\rho)\|_{\mathfrak{C}_\rho([0,1])} \le e^{-\pi^2 t}, \quad t \in \overline{\mathbb{R}_+}.$$

Example 1.2.2 ([41, 262]). The Laplace operator$\Delta v = \sum_{j=1}^{d} \frac{\partial^2 v(x)}{\partial x_j^2}, \quad x \in \mathbb{R}^d$, for $1 < p < \infty$ gives a generator of an analytic C_0-semigroup on $E = W^{2,p}(\mathbb{R}^d)$.

Here, it is appropriate to recall (see [165]) that the operator $i\Delta$ does not generate a C_0-semigroup on $L^p(\mathbb{R}^d)$ For $p \neq 2$. Moreover, the operator Δ generates a C_0-cosine operator-valued function iff $p = 2$ or $d = 1$.

Also, we note (see [122]) that the operator $-(i\Delta)^{1/2}$ does not generate a C_0-semigroup on $L^1(\mathbb{R}^1)$.

Example 1.2.3 ([122]). The Laplace operator Δ on $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, generates an α times integrated cosine operator-valued function for $\alpha > (d-1)\left|\frac{1}{2} - \frac{1}{p}\right|$.

Example 1.2.4 ([121]). Let $\tilde{A}$ be a strongly elliptic operator on $\Omega \subseteq \mathbb{R}^d$. Denote by $T_r(\cdot)$ the C_0-semigroup generated by the operators $\tilde{A}$ with the Dirichlet or Neunan conditions on the boundary in $L^r(\Omega)$. Then there exists an analytic C_0-semigroup $T_p(\cdot)$ with the angle $\pi/2$ $L^p(\Omega)$ such that $T_p(t)x = T_r(t)x$ for all $x \in L^p(\Omega) \cap L^r(\Omega)$.

Example 1.2.5 ([122]). Let $1 < p < \infty$, let the operator $\tilde{A}_p$ generate a semigroup $T_p(\cdot)$, and let $\mu(\Omega) < \infty$. Then $\tilde{A}_p$ generates an α times integrated cosine operator valued function on $L^p(\mathbb{R}^d)$ for $\alpha > \frac{d}{2}\left|\frac{1}{2} - \frac{1}{p}\right| + \frac{1}{2}$.

Example 1.2.6. Let $\Omega = \mathbb{R}_+$. The operator $(Av)(x) = v''(x) + \frac{a}{x}v'(x) + \frac{c}{x}v(x)$ generates an analytic C_0-semigroup for $a, c \in \mathbb{R}$ and $D(A) = \{v(\cdot) : v \in \mathfrak{C}(\overline{\mathbb{R}_+}), Av \in C(\overline{\mathbb{R}_+})\}$ iff $c \le 0$.

Example 1.2.7 ([41]). Let $\Omega = [-1,1]$. Then the operator $(Av)(x) = \frac{d}{dx}\left((1-x^2)\frac{dv(x)}{dx}\right)$ with $D(A) = \{v(\cdot) : v \in \mathfrak{C}(\Omega),\ Av \in \mathfrak{C}(\Omega)\}$ generates a C_0-semigroup.

Example 1.2.8 ([41]). Let $(Av)(x) = v''(x) + q(x)v(x)$, $x \in \mathbb{R}$. Denote by S_p the Banach space of Stepanov functions. i.e., the space of functions on $\mathbb{R}$ with the norm $\|v(\cdot)\|_{p,l} = \sup_{x\in\mathbb{R}}(\frac{1}{l}\int_x^{x+l}|f(s)|^p\,ds)^{\frac{1}{p}}$, $l > 0$, $p \geq 1$. It is known that fordifferent l, the norms are equivalent. For the operator $A \in \mathbf{H}(\omega,\beta)$ on $C(\mathbb{R})$, it suffices, and in the case $q(x) \geq c > -\infty$, it is necessary that $q(\cdot) \in S_1$.

Denote $H_{-1}(z) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2-2sz}\,ds$ and

$$H_\tau(t)(\lambda^2 I - A)^{-1} = -\frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} H_{-1}(\lambda\tau)e^{\lambda t}\lambda(\lambda^2 I - A)^{-1}\,d\lambda.$$

Theorem 1.2.4 ([33]). *For $A \in \mathbf{C}(M,\omega)$, it is necessary and sufficient that this operator is a generating operator of an analytic C_0-semigroup, and for each $t \in [0,T]$, the estimate*

$$\|H_\tau(t)(\lambda^2 I - A)^{-1}\| \leq M(t) \tag{1.13}$$

holds uniformly in $\tau \in (0,\varepsilon)$, $\varepsilon > 0$. In this case,

$$C(t,A) = s\text{-}\lim_{\tau\to 0}(H_\tau(t) + H_\tau(-t))(\lambda^2 I - A)^{-1}, \qquad t \in \overline{\mathbb{R}_+}.$$

Example 1.2.9 ([33]). Let the operator A be given as in Example 1.2.6. Then for the function

$$f_\tau(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ \frac{x-1}{2\tau} & \text{if } x \in [1,1+2\tau), \\ 1 & \text{if } x \in [1+2\tau,\infty,) \end{cases}$$

we have $\|H_\tau(1)(\lambda^2 I - A)^{-1}f_\tau\| > \frac{M}{\tau}$, and, therefore, by Theorem 1.2.4, such an operator A does not generate a C_0-cosine operator-valued function.

Example 1.2.10 ([41]). Consider the operator A from Example 1.2.6 but on the space $\mathfrak{C}_\rho(\overline{\mathbb{R}_+})$ with $\rho(x) = xe^{x\gamma}$, $x \in \overline{\mathbb{R}_+}$, $\gamma \in \mathbb{R}$. Then $\|H_\tau(t)(\lambda^2 I - A)^{-1}v\|_{\mathfrak{C}_\rho} \leq Me^{|\mu t|}\|v\|_{\mathfrak{C}_\rho}$, and, therefore, $A \in \mathbf{C}(M,\omega)$.

Example 1.2.11 ([41]). Let A be given as in Example 1.2.8. For $A \in \mathbf{C}(M,\omega)$ on the space $C(\mathbb{R})$ it is sufficient, and in the case $q(x) \geq c > -\infty$, it is necessary that $q(\cdot) \in S_1$.

Consider the problem

$$\frac{\partial^2 u(t,x)}{\partial t^2} = x^m\frac{\partial^2 u(t,x)}{\partial x^2} + \alpha x^{m-1}\frac{\partial u(t,x)}{\partial x} \tag{1.14}$$

with $m > 0, x > 0$ and initial conditions $\lim_{t\to 0} u(t,x) = \varphi(x)$, $\lim_{t\to 0} \frac{\partial u(t,x)}{\partial t} = \psi(x)$ for any $x \in \mathbb{R}_+$, where $\varphi, \psi \in C^{(2)}(\mathbb{R}_+) \cap E$, and E is the Banach space of functions $\varphi \in C(\mathbb{R}_+)$ such that $\lim_{x\to 0} \varphi(x) = \lim_{x\to\infty} \varphi(x) = 0$ with the norm $\|\varphi\| = \sup_{x\in\mathbb{R}_+} |\varphi(x)|$.

Definition 1.2.4. Problem (1.14) is said to be *uniformly well-posed* if for any compact set $J \subset \mathbb{R}_+$, we have $\max_{t\in J} |u(t,x)| \le M(\|\varphi\| + \|\psi\|)$.

Example 1.2.12 ([41]). For the operator $(Av)(x) = x^m v''(x) + \alpha x^{m-1} v'(x)$ with $D(A) = \{v \in E : v \in C^{(2)}(\mathbb{R}_+) \cap E,\ Av \in E\}$ on the space E just described, condition (1.13) is verified for $0 \le \frac{2\alpha-(1+\alpha)^m}{2-m} < 1$. For $\frac{2\alpha-(1+\alpha)^m}{2-m} \ge 1$, the operator A does not generate a C_0-cosine operator-valued function.

Example 1.2.13 ([41]). For the operator $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ defined on the space $C(\mathbb{R}^d)$, the operator Δ^{2k+1} on $C(\mathbb{R}^d)$ generates a C_0-cosine operator-valued function iff $d \le 4k+1$.

Here, in connection with Example 1.2.13, it is relevant to note that for any $A \in \mathbf{C}(M,\omega)$, every polynomial $P(A) = A^{2m+1} + \sum_{k=0}^{2m} c_k A^k$, $c_k \in \mathbb{R}$, generates an analytic semigroup.

Moreover, in the case of an even m, the operator $(-1)^{m+1} A^m$ does not necessary generates a C_0-cosine operator-valued function.

Proposition 1.2.4 ([175]). *Let $A \in \mathbf{C}(M,0)$. Then for any $k \in \mathbb{N}$, the operator $(-1)^k A^{2^k}$ generates an α times integrated cosine operator-valued function for a certain $\alpha > 0$. Moreover, $(-1)^k A^{2^k} \in \mathbf{H}(\omega, \pi/2)$.*

Theorem 1.2.5 ([227]). *Let $A \in \mathbf{H}(0, \frac{\pi}{2})$, and let $m \in \mathbb{N}$. Let $B_i \in B(E)$, $i \in \overline{1,m}$. Then the operator $(-1)^{m+1} A^m + B_1 A^{m-1} + ... + B_{m-1} A + B_m$ generates an analytic C_0-semigroup with the angle $\frac{\pi}{2}$.*

An analogous assertion is not true for C_0-cosine operator-valued functions!

Theorem 1.2.6 ([176]). *Let $\{A_j\}_{j=1}^m$ be resolvent commuting operators, and let $A_j \in \mathbf{C}(M,0)$, $j \in \overline{1,m}$, be given on E. Define $A_0 = \sum_{j=1}^m A_j$, $D(A_0) = \cap_{j=1}^m D(A_j)$. Then the operator A_0 is closedable and $\overline{A_0}$ generates an α times integrated cosine operator-valued function for $\alpha \ge \frac{m-1}{2}$.*

Moreover, this α times integrated semigroup satisfies the estimate $\|S(t)\| \le M_\alpha t^\alpha$, $t \in \overline{\mathbb{R}_+}$ for certain $M_\alpha > 0$ and $\alpha \ge \frac{m-1}{2}$.

Theorem 1.2.7 ([175]). *Under the conditions and notation of Theorem 1.2.6, the operator $i\overline{A_0}$ generates a β times integrated semigroup for $\beta > m/2$.*

Proposition 1.2.5 ([176]). *Under the conditions of Theorem 1.2.6, additionally, let the space $E = H$ be Hilbert. Then $\overline{A_0}$ generates a C_0-cosine operator-valued function.*

Proposition 1.2.6 ([176]). *Under the conditions of Theorem* 1.2.6*, additionally, let $E = H$ be a Hilbert space and a Banach lattice, and, moreover, let $C(t, A_j)H_+ \subseteq H_+$, $t \in \mathbb{R}$, $j \in \overline{1, m}$. Define*

$$C_k(t) = \begin{cases} \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C(s, \overline{A_0})\, ds & k \geq 1, \\ C(t, \overline{A_0}) & k = 0. \end{cases}$$

Then $C_k(\cdot)$ are positive for $k \geq \left[\frac{m}{2}\right]$.

Example 1.2.14 ([160]). Let $E = L^p(\mathbb{R}^d)$, $1 < p < \infty$. Then the Laplace operator $\Delta \quad D(\Delta) = W^{2,p}(\mathbb{R}^d)$ generates an α times integrated cosine operator-valued function iff$\alpha \geq (d-1)\left|\frac{1}{2} - \frac{1}{p}\right|$.

In [290], concrete differential operators are studied for revealing whether or not they generate a well-posed Cauchy problem for the complete second-order equation.

1.3 Resolvent Families

For functions, $k(\cdot) \in L^p_{loc}(\mathbb{R}_+)$ and $g(\cdot) \in W^{1,1}([0,T]; E)$, let us consider the Volterra equation

$$u(t) = g(t) + \int_0^t k(t-s)Au(s)\, ds, \quad t \in [0,T]. \tag{1.15}$$

Definition 1.3.1. a strongly continuous family of bounded linear operators $\{R(t) : t \in \overline{\mathbb{R}}_+\}$ on E is called a *resolvent family* for (1.15) if it commutes with the operator A and

$$R(t)x = x + \int_0^t k(t-s)AR(s)x\, ds \qquad \text{for} \quad x \in D(A),\, t \in \overline{\mathbb{R}}_+.$$

If there exists a resolvent family, then any solution of Eq. (1.15) is represented in the form

$$u(t) = R(t)g(0) + \int_0^t k(t-s)g'(s)\, ds, \quad t \in [0,T]. \tag{1.16}$$

Theorem 1.3.1 ([110, 147, 238]). *Let $R(\cdot)$ be a strongly continuous family of operators on $\overline{\mathbb{R}}_+$ such that $\|R(t)\| \leq Me^{\omega t}$ and $|k(t)| \leq Me^{\omega t}$, $t \in \overline{\mathbb{R}}_+$. Then $R(\cdot)$ ia a resolvent family iff the following conditions hold:*

(i) $\hat{k}(\lambda) \neq 0$ and $\frac{1}{\lambda \hat{k}(\lambda)} \in \rho(A)$ for all $\lambda \geq \omega$;

(ii) $\Big(I - \lambda \hat{k}(\lambda) A\Big)^{-1} x = \int_0^\infty e^{\lambda t} R(t) x \, dt$ for all $x \in E$ and $\lambda > \omega$, where $\hat{k}(\cdot)$ is the Laplace transform of the function $k(\cdot)$.

In particular, it should be noted that for $k(t) \equiv 1$, the resolvent family is a C_0-semigroup of operators, and for $k(t) = t$, it is a C_0-cosine operator-valued function. Therefore, the proof of a number of assertions on properties related to C_0-semigroups and C_0-cosine operator-valued functions can be obtained from assertions related to resolvent families.

For the kernel $k(\cdot)$ satisfying certain restrictions (positivity and bounded variation), for a resolvent family, many results that hold for C_0-semigroups and C_0-cosine operator-valued functions were reproved. So, for example, in [194]–[196], C. Lizama hes reproved the assertions on the compactness properties, uniform continuity, and periodicity In [168], Jung Chan Chang and S.-Y. Shaw have reproved the theorems on multiplicative and additive perturbations.

1.4 Incomplete Cauchy Problem

For a second-order equation, let us consider the so-called incomplete Cauchy problems

$$\begin{gathered} u''(t) = Au(t), \quad t \in \mathbb{R}_+, \\ u(0) = u^0, \quad \sup_{t \in \mathbb{R}_+} \|u(t)\| < \infty; \end{gathered} \tag{1.17}$$

$$\begin{gathered} u''(t) = Au(t), \quad t \in \mathbb{R}_+, \\ \lim_{t \to 0+} u(t) = u^0, \quad \lim_{t \to \infty} \|u(t)\| = 0; \end{gathered} \tag{1.18}$$

$$\begin{gathered} u''(t) = Au(t), \quad t \in \mathbb{R}_+, \\ u(0) = u^0, \quad \lim_{t \to \infty} \|u(t)\| = 0. \end{gathered} \tag{1.19}$$

Incomplete Cauchy problems were studied in [113], [134], [137].

Proposition 1.4.1. *An operator A has the square root $\sqrt{A}$ such that $\exp(t\sqrt{A})$ is a bounded analytic C_0-semigroup iff problem* (1.17) *has a unique solution for each $u^0 \in D(A)$ and this solution is analytically continued to a certain sector containing the semiaxis $\mathbb{R}_+$.*

Proposition 1.4.2. *Let there exist $\sqrt{A}$ generating a differentiable C_0-semigroup such that $s\text{-}\lim_{t\to\infty} \exp(t\sqrt{A}) = 0$. Then problem* (1.18) *has a unique solution for any $u^0 \in E$.*

Proposition 1.4.3. *Let $\sqrt{A}$ generate a C_0-semigroup such that $s\text{-}\lim_{t\to\infty} \exp(t\sqrt{A}) = 0$. Then problem* (1.19) *has a unique solution for each $u^0 \in D(A)$.*

Definition 1.4.1. A C_0-semigroup $\exp(\cdot A)$ is called a *C_0-semigroup stable in degree $q \in \mathbb{N}$* if $s\text{-}\lim_{t\to\infty} \exp(tA)x = 0$ for each $x \in D(A^q)$. A C_0-semigroup stable in degree 0 is said to be *uniformly stable*.

Theorem 1.4.1. *Assume that an operator $\mathfrak{B}$ generates a C_0-semigroup stable in degree 2 and a function $v(\cdot)$ has a continuous second derivative and satisfies the equation*

$$v''(t) = \mathfrak{B}^2 v(t), \quad t \in \mathbb{R}_+,$$

and, moreover, $s\text{-}\lim_{t\to\infty} v(t) = 0$. Then $v(t) = \exp(t\mathfrak{B})v(0)$, $t \in \overline{\mathbb{R}_+}$.

Theorem 1.4.2. *Let $A = \mathfrak{B}^2$, where the operator $\mathfrak{B}$ generates a C_0-semigroup. Then*

(i) if $\exp(\cdot\mathfrak{B})$ is stable in degree 2, then problem (1.19) *is well posed;*

(ii) if $\exp(\cdot\mathfrak{B})$ is stable in degree 1, then the following problem is well posed:

$$\begin{gathered} u''(t) = Au(t), \quad t \in \mathbb{R}_+, \quad u(0) = x, \\ \lim_{t\to\infty} \|u(t)\| = 0, \quad \lim_{t\to\infty} \|u'(t)\| = 0; \end{gathered} \tag{1.20}$$

(iii) if $\exp(\cdot\mathfrak{B})$ is uniformly stable, then the following problem is well posed:

$$\begin{gathered} u''(t) = Au(t), \quad t \in \mathbb{R}_+, \quad u(0) = x, \\ \lim_{t\to\infty} \|u^{(k)}(t)\| = 0, \quad k \in \mathbb{N}_0. \end{gathered} \tag{1.21}$$

Proposition 1.4.4. *Let $\rho(A) \neq \varnothing$. Then*

(i) problem (1.19) *is well posed iff the operator A has the square root $\sqrt{A}$ generating a C_0-semigroup stable in degree 2;*

(ii) problem (1.20) *is well posed iff the operator A has the square root $\sqrt{A}$ generating a C_0-semigroup stable in degree* 1*;*

(iii) problem (1.21) *is well posed iff the operator A has the square root $\sqrt{A}$ generating a stable C_0-semigroup.*

Corollary 1.4.1. *Any operator A has no more than one square root $\sqrt{A}$ generating a C_0-semigroup stable in degree* 2*.*

Theorem 1.4.3. *Let B and C be self-adjoint commuting operators on a Hilbert space H. Then there exists a closed complementable subspaces H_1 and H_2 such that if $A = B + iC$, then problems* (1.19), (1.20), *and* (1.21) *are well posed on H_1 and the problem*

$$\begin{aligned} u''(t) &= Au(t), \quad t \in \mathbb{R}_+, \\ u(0) &= u^0, \quad u'(0) = u^1. \end{aligned} \tag{1.22}$$

is well posed on H_2.

Definition 1.4.2. The regularized *fractional derivative* of order $0 < \alpha < 1$ of a function $u(\cdot)$ is the function

$$(D^{(\alpha)}u)(t) = (D^{\alpha}_{0+}u)(t) - \frac{1}{\Gamma(1-\alpha)}\frac{u(0)}{t^{\alpha}},$$

where $(D^{\alpha}_{0+}u)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}(\int_0^t \frac{u(\xi)}{(t-\xi)^{\alpha}}\,d\xi)$.

Consider the Cauchy problem

$$(D^{(\alpha)}u)(t) = Au(t), \quad 0 < t \le T,\ u(0) = u^0, \tag{1.23}$$

with a closed operator A. By a solution of problem (1.23) we mean a function $u(\cdot)$ such that

(i) $u(\cdot) \in C([0,T];E)$;

(ii) for $t \in \mathbb{R}_+$, the values $u(t) \in D(A)$;

(iii) the fractional integral $\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u(\xi)}{(t-\xi)^{\alpha}}\,d\xi$ is continuously differentiable for $t \ge 0$, and

(iv) the function $u(\cdot)$ satisfies (1.23).

Theorem 1.4.4. *[42] Let there exist the resolvent $(\lambda^{\alpha}I - A)^{-1}$ for $\lambda > \omega > 0$, and let*

$$\overline{\lim_{\lambda\to\infty}}\,\lambda^{-1/\alpha}\ln\|(\lambda I - A)^{-1}\| = 0.$$

Then a solution of problem (1.23) *is unique.*

Theorem 1.4.5. *[42] Let the resolvent* $(\lambda^\alpha I - A)^{-1}$ *exist in the half-plane* $\operatorname{Re}\lambda > \omega > 0$*, and for the same* λ*, let*

$$\|(\lambda^\alpha I - A)^{-1}\| \leq C(1 + |Im\lambda|)^{-\beta}, \quad 0 < \beta < 1.$$

Then problem (1.23) *has a unique solution. This solution is infinitely differentiable for* $t > 0$*, and for each* t*, its value continuously depends on the initial data* u^0*.*

Chapter II
Cosine and Sine Operator-Valued Functions

The existing parallelism between the theory of C_0-semigroups of operators and the theory of C_0-cosine operator-valued functions has a distinctive character. On one hand, a number of definitions and properties practically repeat each other almost literally. On the other hand, for second order equations, by the Kisynski theorem, the main object corresponding to a C_0-cosine operator-valued function is a C_0-group, which excludes the appearance of "parabolicity," despite the fact that the generating operator A of a C_0-cosine operator-valued function also generates an analytic C_0-semigroup.

2.1 Measurability of Operator Semigroups and Cosine Operator-Valued Functions

The measurability property of a cosine operator-valued function profitably differs from that of a semigroup. By the evenness, the measurability of a cosine operator-valued function implies the strong continuity at zero. We have an analogous situation for perturbation families.

Definition 2.1.1. A function $T(\cdot) : \overline{\mathbb{R}}_+ \to B(E)$ is called an *operator semigroup* if it satisfies the conditions $T(t+h) = T(t)T(h)$ for any $t, h \in \overline{\mathbb{R}}_+$ and $T(0) = I$.

Definition 2.1.2. A function $C(\cdot) : \mathbb{R} \to B(E)$ is called an *operator cosine* (or a *cosine operator-valued function)* if it satisfies the conditions $C(t+h) + C(t-h) = 2C(t)C(h)$ for any $t, h \in \mathbb{R}$ and $C(0) = I$.

Definition 2.1.3. A function $S(\cdot) : \mathbb{R} \to B(E)$ is called an *operator sine* (or a *sine operator-valued function)* if it satisfies the conditions $S(t+h) + S(t-h) = 2S(t)C(h)$ for any $t, h \in \mathbb{R}$ and $S(0) = 0$.

Theorem 2.1.1 ([184]). *Let an operator semigroup $T(\cdot)$ be strongly measurable, i.e., the function $T(\cdot)x$ is strongly measurable on $\mathbb{R}_+$ for any $x \in E$. Then it is strongly continuous on $\mathbb{R}_+$.*

We stress that in Theorem 2.1.1, the strong continuity is asserted only on $\mathbb{R}_+$ but not on $\overline{\mathbb{R}}_+$!

Proposition 2.1.1 ([184]). *Let a function $t \to T(t)x$ be strongly measurable on $\mathbb{R}_+$. Then it is locally bounded.*

Proposition 2.1.2 ([184]). *Let an operator cosine $C(\cdot)$ be strongly measurable on $\mathbb{R}_+$. Then it is strongly continuous on $\mathbb{R}$.*

Proposition 2.1.3 ([184]). *Let a function $t \to C(t)x$ be strongly measurable on $\mathbb{R}_+$. Then it is locally bounded.*

Theorem 2.1.2 ([183]). *Let a cosine operator-valued function $C(\cdot)$ be such that its restriction to a certain interval $J \subseteq \mathbb{R}$ be weakly Lebesgue measurable, and let the space E be separable and reflexive. Then $C(\cdot)$ is weakly continuous on $\mathbb{R}$.*

2.2 Multiplicative and Additive Families. Measurability and Continuity

Definition 2.2.1. let $C(\cdot, A)$ be a C_0-cosine operator valued function. A family $\{F(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is called a *multiplicative perturbation family* for $C(\cdot, A)$ if $F(0) = 0$ and

$$F(t+s) - 2F(t) + F(t-s) = 2C(t, A)F(s) \text{ for } t,\, s \in \mathbb{R}. \tag{2.1}$$

Definition 2.2.2. A family $\{G(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is called an *additive perturbation family* foe a C_0-cosine operator-valued function $C(\cdot, A)$ if $G(0) = 0$ and

$$G(t+s) - 2G(t) + G(t-s) = 2G(s)C(t, A) \text{ for } t,\, s \in \mathbb{R}. \tag{2.2}$$

If these families are strongly continuous at zero, then they are called a *multiplicative perturbation C_0-family* and an *additive perturbation C_0-family*, respectively.

Clearly, $F(\cdot)$ and $G(\cdot)$ are even functions. The terminology mentioned above is chosen by analogy with the corresponding definitions of perturbation families $U(\cdot)$ and $V(\cdot)$ for C_0-semigroups ([15], Sec. 2.2). Recall that $U(\cdot)$ satisfies the relations $U(0) = 0$ and $U(t+s) - U(t) = T(t)U(s), \quad t,\, s \in \overline{\mathbb{R}}_+$, and $V(\cdot)$ the relations $V(0) = 0$ and $V(t+s) - V(t) = V(s)T(t), \quad t,\, s \in \overline{\mathbb{R}}_+$.

The multiplicative and additive perturbation C_0-families play an important role in the perturbation theory of C_0-cosine operator-valued functions.

For example, using the multiplicative and additive perturbation C_0-families, we can consider well-posed statements of the Cauchy problem in the form

$$u''(t) = A(1 - \lambda\hat{F}(\lambda))u(t) + \lambda^3\hat{F}(\lambda)u(t),\ t \in \overline{\mathbb{R}}_+, \quad u(0) = x,\ u'(0) = y.$$

As is known, a C_0-cosine operator-valued function that is strongly (resp. uniformly) measurable on $\mathbb{R}_+$ is strongly (resp. uniformly) measurable on $\mathbb{R}$ (see Sec. 2.1). The following theorem shows that multiplicative and additive perturbation families have the same properties.

Theorem 2.2.1 ([233]). *If a multiplicative perturbation family $F(\cdot)$ is strongly (resp. uniformly) measurable on $\mathbb{R}_+$, then the function $F(\cdot)$ is strongly (resp. uniformly) continuous on $\mathbb{R}$.*

If an additive perturbation family $G(\cdot)$ is uniformly measurable on $\mathbb{R}_+$, then the function $G(\cdot)$ is uniformly continuous on $\mathbb{R}$.

Proof. first of all, the strong continuity of $F(\cdot)x$ on $\mathbb{R}_+$ implies the Lebesgue measurability of $\|F(\cdot)x\|$ on $\mathbb{R}_+$ (see[73]). Further, let us show that $\|F(\cdot)x\|$ is bounded on any compact subinteval $[a, b] \subset \mathbb{R}_+$ for any $x \in E$. Assume the contrary. Then there exist $\tilde{x} \in E$, a number $\tau > 0$, and a sequence $\tau_n \in [a, b]$ such that $\tau_n \to \tau$ and

$$\|F(\tau_n)\tilde{x}\| \geq n \quad \text{as} \quad n \to \infty.$$

By the measurability of $\|F(\cdot)\tilde{x}\|$, there exist a constant c_1 and a Lebesgue measurable set $\Lambda \subset [0, \tau]$ of measure

$$\mu(\Lambda) > \frac{3}{4}\tau$$

such that

$$\sup_{t\in\Lambda} \|F(t)\tilde{x}\| \leq c_1. \tag{2.3}$$

Now, following [98], we set

$$\mathbf{A}_k := \frac{\tau_k}{2} - \frac{\Lambda \cap [0, \tau_k]}{2}, \quad \mathbf{B}_k := \Lambda \cap [0, \tau_k/2] \tag{2.4}$$

and

$$\mathbf{A} = \frac{\tau}{2} - \frac{\Lambda}{2}, \quad \mathbf{B} = \Lambda \cap [0, \tau/2].$$

First, $\mu(\mathbf{A} \cap \mathbf{B}) > 0$. To prove this, assume that $\mu(\mathbf{A} \cap \mathbf{B}) = 0$. Then $\mu(\mathbf{A}) + \mu(\mathbf{B}) \leq \tau/2$. But $\mu(\mathbf{A}) = \mu/2(\Lambda)$ by the definition of the set $\mathbf{A}$. This means that $\mu(\Lambda) + 2\mu(\mathbf{B}) \leq \tau$. Therefore, $\frac{3}{4}\tau < \mu(\Lambda) \leq \tau - 2\mu(\mathbf{B})$, i,e.,

$$\mu(\mathbf{B}) \leq \tau/8. \tag{2.5}$$

Write

$$\Lambda = (\Lambda \cap [0, \tau/2]) \ \cup \ (\Lambda \cap [\tau/2, \tau]) = \mathbf{B} \cup \mathbf{D},$$

where $\mu(\Lambda) = \mu(\mathbf{B}) + \mu(\mathbf{D}) \ \ \mu(\mathbf{D}) \leq \tau/2$. But

$$\frac{3}{4}\tau < \mu(\Lambda) = \mu(\mathbf{B}) + \mu(\mathbf{D}) \leq \mu(\mathbf{B}) + \tau/2$$

implies $\mu(\mathbf{B}) > \tau/4$, which contradicts (2.5). We have proved that $\mu(\mathbf{A} \cap \mathbf{B}) \geq \delta > 0$.

Now define the sets

$$\mathbf{E} = \mathbf{A} \cap \mathbf{B}, \quad \mathbf{E}_n = \mathbf{A}_n \cap \mathbf{B}_n$$

and

$$H_n = \ \tau_n - \eta, \qquad \eta \in \mathbf{E}_n \ .$$

Clearly, $\mathbf{E}_n \to \mathbf{E}$ as $n \to \infty$, so that $\mu(H_n) > \delta/2$ for sufficiently large n. For the same n, if $\eta \in \mathbf{E}_n$, then η and $\tau_n - 2\eta$ belong to Λ by (2.4). Now, using (2.1) and (2.3), for $\eta \in \mathbf{E}_n$, we obtain

$$n \leq \|F(\tau_n)\tilde{x}\| \leq 2\|F(\tau_n - \eta)\tilde{x}\| + \|F(\tau_n - 2\eta)\tilde{x}\| + 2\|C(\tau_n - \eta)\|\|F(\eta)\tilde{x}\|$$

$$\leq 2\|F(\tau_n - \eta)\tilde{x}\| + c_1 + 2Me^{\omega b}c_1.$$

Therefore,

$$\|F(t)\tilde{x}\| \geq \frac{n - c_1 - 2Mc_1e^{\omega\beta}}{2}$$

for $t \in H_n$; denoting $\lim_{n\to\infty} H_n = H_\infty$, we have$\|F(t)\tilde{x}\| = \infty$ for $t \in H_\infty$ with $\mu(H_\infty) \geq \delta/2 > 0$. This contradicts the finiteness of $\|F(t)\tilde{x}\|$ for each t.

We now want to prove that the strong measurability, together with the boundedness, implies the continuity of $F(\cdot)x$ for each $t \in \mathbb{R}_+$ and each $x \in E$. For this purpose, we choose four positive numbers α, β, ϵ, and γ such that $\beta < t - \epsilon$ and $0 < \alpha < \gamma < \beta < t$. We have from (2.1) that

$$F(t)x = 2F(t - \gamma/2)x - F(t - \gamma)x + 2C(t - \gamma/2, A)F(\gamma/2)x. \tag{2.6}$$

The left-hand side, being independent of γ, is integrable in γ, and we have

$$(\beta - \alpha)(F(t \pm \epsilon)x - F(t)x)$$

$$= \int_\alpha^\beta 2\Big(F(t\pm\epsilon-\gamma/2)-F(t-\gamma/2)x\Big)\,d\gamma - \int_\alpha^\beta \Big(F(t\pm\epsilon-\gamma)-F(t-\gamma)\Big)x\,d\gamma$$

$$+\int_\alpha^\beta 2\Big(C(t\pm\epsilon-\gamma/2,A)-C(t-\gamma/2,A)\Big)F(\gamma/2)x\,d\gamma.$$

Therefore,

$$\|(F(t\pm\epsilon)-F(t))x\| \le$$

$$\le \frac{1}{\beta-\alpha}\Bigg[\int_{t-\beta/2}^{t-\alpha/2}\|(F(\zeta\pm\epsilon)-F(\zeta))x\|d\zeta + \int_{t-\beta}^{t-\alpha}\|(F(\zeta\pm\epsilon)-F(\zeta))x\|d\zeta$$

$$+\,2\int_\alpha^\beta \|(C(t\pm\epsilon-\gamma/2,A)-C(t-\gamma/2,A))F(\gamma/2)x\|\,d\gamma\Bigg]. \tag{2.7}$$

By Theorem 3.8.3 from [73],

$$\int_{t-\beta/2}^{t-\alpha/2}\to 0 \qquad \int_{t-\beta}^{t-\alpha}\to 0 \quad \text{as} \quad \epsilon\to 0.$$

The last summand in (2.7) tends to zero by the Lebesgue theorem on the dominated convergence (see [73], Theorem 3.7.9).

We obtain that $F(\cdot)x$ is continuous for $t\in\mathbb{R}_+$. Replacing t by $t+s$ in (2.1), we obtain that for all t, $s\in\mathbb{R}_+$, the function

$$F(t)x = 2C(t+s,A)F(s)x - F(t+2s)x + 2F(t+s)x,$$

tends to $2C(s,A)F(s)x - F(2s)x + 2F(s)x = F(0)x = 0$ as $t\to 0+$. Therefore, $F(\cdot)$ is strongly continuous on $\overline{\mathbb{R}}_+$ and hence on $\mathbb{R}$, since $F(\cdot)$ is an even function. The proof for the case of uniform measurability is analogous.

To prove the assertion for $G(\cdot)$, we can use the following writing of Eq. (2.2):

$$G(\tau_n) = 2G(\tau_n-\eta) - G(\tau_n-2\eta) + 2G(\eta)C(\tau_n-\eta,A)$$

in operating in the estimate of the form (2.6). The proof is analogous.

Theorem 2.2.2. *A multiplicative perturbation C_0-family and an additive perturbation C_0-family are strongly continuous on $\overline{\mathbb{R}}_+$ for a C_0-cosine operator-valued function $C(\cdot,A)$. Moreover, the uniform continuity of at 0 implies the uniform continuity on $\in\overline{\mathbb{R}}_+$.*

Proof. Following [98], we assume the contrary: the multiplicative perturbation family $F(\cdot)$ is not strongly continuous at a certain point $t_0 \in \mathbb{R}_+$, i.e., there exists x_0 such that the nonincreasing sequence

$$K_n := \sup\{\|(F(t) - F(s))x_0\| : |t - t_0|, |s - t_0| \leq \frac{t_0}{8n}\ \}$$

converges to a certain $K > 0$ as $n \to \infty$.

We can choose sequences τ_n and σ_n such that

$$|\tau_n - t_0| \leq \frac{t_0}{8n}, \quad |\sigma_n - t_0| \leq \frac{t_0}{8n}$$

and

$$\|(F(\tau_n) - F(\sigma_n))x_0\| \geq K_n - \frac{1}{n}, \quad n \in \mathbb{N}.$$

Clearly, $|\sigma_n - \tau_n| \leq \frac{t_0}{4n}$ and $|2\tau_{4n} - \sigma_{4n} - t_0| \leq \frac{t_0}{8n}, \quad n \in \mathbb{N}$. Therefore,

$$\|\Big(F(\sigma_{4n}) - F(2\tau_{4n} - \sigma_{4n})\Big)x_0\| \leq K_n, \quad n \in \mathbb{N}.$$

Using identity (2.1) in the form

$$2\Big(F(t+h) - F(t)\Big) = (F(t+h) - F(t-h)) + 2C(t, A)F(h)$$

and setting $t_0 + h = \sigma_{4n}\ \ t_0 = \tau_{4n}$, we obtain

$$2\|\Big(F(\sigma_{4n}) - F(\tau_{4n})\Big)x_0\| \leq K_n + 2Me^{\omega t_0}\|F(\sigma_{4n} - \tau_{4n})x_0\|.$$

Therefore,

$$2(K_{4n} - \frac{1}{4n}) \leq K_n + 2Me^{\omega t_0}\|F(\sigma_{4n} - \tau_{4n})x_0\|,$$

and, thus,

$$K_{4n} + (K_{4n} - K_n) \leq \frac{1}{2n} + 2Me^{\omega t_0}\|F(h)x_0\|.$$

By the convergence $F(h)x_0 \to 0$ as $h \to 0$ (recall that $h = \sigma_{4n} - \tau_{4n}$) and $K_{4n} - K_n \to 0$ as $n \to \infty$m we have $K_n \to 0$ as $n \to \infty$, $n \in \mathbb{N}$, which is a contradiction to our assumption that $K_n \to K,\ K > 0$.

To prove the same assertion for $G(\cdot)$, we can use the identity

$$2\Big(G(t+h)-G(t)\Big) = \Big(G(t+h)-G(t-h)\Big)+2G(t)\Big(C(h,A)-I\Big)+2G(h),$$

which is obtained from (2.2) and Proposition 2.4.1 (i).

In the same way as in Proposition 2.3.2 in [15], we can prove the following assertion.

Proposition 2.2.1. *Let a multiplicative perturbation family $F(\cdot)$ and a C_0-cosine operator-valued function $C(\cdot, A)$ commute. i.e., $F(t)C(t,A) = C(t,A)F(t)$ for all $t \in \overline{\mathbb{R}}_+$. Then the multiplicative perturbation C_0-family $F(\cdot)$ is an additive perturbation family and it is commutative, i.e., $F(t)F(s) = F(s)F(t)$ for all $s, t \in \mathbb{R}$.*

2.3 Main Properties of C_0-Cosine and C_0-Sine Operator-Valued Functions

Definition 2.3.1. A *C_0-cosine operator-valued function* is defined as a one-parameter family of operators, $\{C(t),\, t \in \mathbb{R}\}$, $C(t) \in B(E),\ t \in \mathbb{R}$, having the following properties:
(i) $C(t+s) + C(t-s) = 2C(t)C(s)$ for any $t, s \in \mathbb{R}$
(d'Alembert aquation);
(ii) $C(0) = I$ is the identity operator on E ;
(iii) $s\text{-}\lim_{h\to 0} C(h)x = x$ for any $x \in E$.

With a C_0-cosine operator-valued function $C(\cdot)$, we associate the *C_0-sine operator-valued function*

$$S(t)x := \int_0^t C(s)x\, ds, \qquad x \in E,\ t \in \mathbb{R}, \tag{2.8}$$

and the linear varieties

$$E^k := \{x \in E :\ C(\cdot)x \in C^k(\mathbb{R}; E)\}, \qquad k = 1, 2. \tag{2.9}$$

Definition 2.3.2. A linear operator A with the domain $D(A)$ consisting of all x for which there exists athe limit

$$Ax := s\text{-}\lim_{h\to 0+} 2\frac{C(h)-I}{h^2}x, \tag{2.10}$$

is called a *generating (infinitesimal) operator (generator)* of a C_0-cosine function $C(\cdot)$.

The property that A is a generating operator of a C_0-cosine operator-valued function $C(\cdot)$ is written as $C(\cdot, A)$ (and $S(\cdot, A)$ for a C_0-sine operator-valued function $S(\cdot)$).

Let us present a simplest example of a C_0-cosine operator-valued function.

Example 2.3.1. Let A be the operator of multiplication by a complex number on the space $\mathbb{R}$. Then A is the generator of the C_0-cosine operator-valued function $(C(t, A)f)(s) = \cos(it\sqrt{A})f(s), \quad t \in \mathbb{R}$.

Proposition 2.3.1 ([84]). *Define the operator*

$$A_1 x := s\text{-}\lim_{h\to 0} \frac{C(2h, A) - 2C(h, A) + I}{h^2} x$$

with a natural domain (i.e., on those $x \in E$ at which this limit does exist). Then for $x \in D(A_1) \cap D(A)$, we have $Ax = A_1 x$.

For a C_0-cosine operator-valued function $C(\cdot, A)$, we can also define the first generating operator

$$\overset{\circ}{\mathbf{C}} x := s\text{-}\lim_{h\to 0+} \frac{C(h, A)x - x}{h}$$

with a natural domain.

Proposition 2.3.2 ([258]). *For a C_0-cosine operator-valued function $C(\cdot, A)$, we have $D(A) \subseteq D(\overset{\circ}{\mathbf{C}})$ and $\overset{\circ}{\mathbf{C}}\, x = 0$ for any $x \in D(A)$.*

Proposition 2.3.3 ([258]). *The operators $C(t, A), C(s, A), S(t, A)$, and $S(s, A)$ commute for any $t, s \in \mathbb{R}$.*

Proposition 2.3.4. *The C_0-sine operator-valued function $S(\cdot, A)$ is continuous in the uniform operator topology.*

Proposition 2.3.5 ([258, 266]). *For all $t, s \in \mathbb{R}$, we have the relations*

$$(i) \quad C(t, A) = C(-t, A), \quad S(-t, A) = -S(t, A), \quad S(0, A) = 0;$$

$$(ii) \quad S(t + s, A) + S(t - s, A) = 2S(t, A)C(s, A);$$

$$(iii) \quad S(t + s, A) = S(t, A)C(s, A) + S(s, A)C(t, A);$$

$$(iv) \quad C(t + s, A) - C(t - s, A) = 2AS(t, A)S(s, A);$$

$$(v) \quad C(2t, A) = 2C(t, A)^2 - I, \qquad C(t, A)^2 - AS(t, A)^2 = I;$$

$$(vi) \quad C((n + 1)t, A) = b_0 I + b_1 C(t, A) + ... + b_{n+1} C^{n+1}(t, A),$$

where $b_0 + b_1 z + ... + b_{n+1} z^{n+1}$ is a Chebyshev polynomial of the first kind of degree $n + 1$.

Proposition 2.3.6 ([258]). *For any C_0-cosine operator function $C(\cdot, A)$, there exist constants $M \geq 1$ and $\omega \geq 0$ such that for all $t \in \mathbb{R}$, we have the estimate*

$$\|C(t, A)\| \leq M ch(\omega t), \quad t \in \mathbb{R}, \tag{2.11}$$

where $ch(\omega t) := \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$ is the hyperbolic cosine.

Definition 2.3.3. The greatest lower bound of the numbers ω from (2.11) is called the *type* of a C_0-cosine operator-valued function and is denoted by $\omega_c(A)$.

Proposition 2.3.7 ([205]). *The minimum ω satisfying* (2.11) *for an appropriate constant M_ω may not exist, i.e., the greatest lower bound of $\omega_c(A)$ is not attained in general*

Proposition 2.3.8 ([130, 139, 258]). *Let an operator A generate a C_0-cosine operator-valued function $C(t, A)$, and let $\|C(t, A)\| \leq M ch(\omega t)$, $t \in \mathbb{R}$. Then $A \in \mathbf{G}(M, \omega^2)$, the C_0-semigroup $exp(\cdot A)$ is analytically continued to the right half-plane, and*

$$\exp(tA) = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-\frac{s^2}{4t}} C(s, A) ds, \quad t \in \overline{\mathbb{R}}_+. \tag{2.12}$$

Proposition 2.3.9. *The representation of the analytic semigroup in Proposition* 2.3.8 *can be written in the form*

$$\exp(tA)x = \frac{1}{2^k \sqrt{\pi} t^{(k+1)/2}} \int_0^\infty P_k\left(\frac{s}{2\sqrt{t}}\right) e^{-\frac{s^2}{4t}} C_k(s) x \, ds, \quad t \in \mathbb{R}_+.$$

Here, P_k is a polynomial of degree k and $C_k(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} C(s, A) \, ds$, where $t \in \overline{\mathbb{R}_+}$, $k \in \mathbb{N}$, and $C_0(t) = C(t, A)$.

Remark 2.3.1 ([219]). There are examples of analytic C_0-semigroups whose generating operators do mot generate C_0-cosine operator-valued functions.

Definition 2.3.4. Denote by E^k, $k \in \mathbb{N}$, the set of elements x of the initial space E for which the function $C(t, A)x : \mathbb{R} \to E$ is k times continuously differentiable in t.

Proposition 2.3.10 ([177]). *Obviously, $D(A) \subseteq E^1$ for $A \in \mathbf{C}(M, \omega)$, and, therefore, the set E^1 is dense in E.*

Proposition 2.3.11 ([268]). *For any $x \in E$ and $t, s \in \mathbb{R}$, we have*

$$(i) \quad y := \int_s^t S(\tau, A)x d\tau \in D(A) \qquad Ay = C(t, A)x - C(s, A)x; \tag{2.13}$$

$$(ii) \quad z := \int_0^t \int_0^s C(\tau, A)C(\zeta, A)x d\tau d\zeta \in D(A) \quad \text{and} \tag{2.14}$$

$$Az = \frac{1}{2}\Big(C(t+s, A) - C(t-s, A)\Big)x; \tag{2.15}$$

$$(iii) \quad S(t, A)x \in E^1. \tag{2.16}$$

Proposition 2.3.12 ([268]). *If elements x vary over the whole E, and the numbers t, s vary over $\mathbb{R}$, then the set of elements of the form $y = \int_s^t S(\tau, A)x d\tau$ is dense in E .*

Proposition 2.3.13. *For any $x \in E$, the following relations hold:*

$$s\text{-}\lim_{t\to 0} t^{-1} S(t, A)x = x \quad \text{and} \quad s\text{-}\lim_{t\to 0} 2t^{-2} \int_0^t S(\tau, A)x d\tau = x. \tag{2.17}$$

Proposition 2.3.14 ([268]). *If $x \in E^1$, then for any $t \in \mathbb{R}$,*

(2.18)

$$(i) \quad C(t, A)x \in E^1, \quad S(t, A)x \in D(A) \quad \text{and} \quad C'(t, A)x = AS(t, A)x;$$

$$(ii) \quad s\text{-}\lim_{\tau\to 0} AS(\tau, A)x = 0 \quad \text{and} \quad S''(t, A)x = AS(t, A)x. \tag{2.19}$$

Proposition 2.3.15 ([268]). *Let $x \in D(A)$. Then for all $t \in \mathbb{R}$,*

$$(i) \quad C(t, A)x \in D(A) \quad \text{and} \quad C''(t, A)x = AC(t, A)x = C(t, A)Ax; \tag{2.20}$$

$$(ii) \quad S(t, A)x \in D(A) \quad \text{and} \quad S''(t, A)x = AS(t, A)x = S(t, A)Ax. \tag{2.21}$$

Proposition 2.3.16 ([266]). *For all $t, s \in \mathbb{R}$, the following relations hold:*

$$(i) \quad C(2t, A) = C(t, A)^2 + C'(t, A)S(t, A); \tag{2.22}$$

$$(ii) \quad C'(t, A)S(s, A) = C'(s, A)S(t, A); \tag{2.23}$$

$$(iii) \quad C(t + s, A) - C(t - s, A) = 2C'(t, A)S(s, A); \tag{2.24}$$

$$(iv) \quad (C(t, A) - I) \int_0^h S(s, A)ds = (C(h, A) - I) \int_0^t S(s, A)ds; \tag{2.25}$$

$$(v) \quad (A-\lambda^2 I) \int_0^t \sinh\Big(\lambda(t-s)\Big) C(s, A)ds = \lambda\Big(C(t, A) - \cosh(\lambda t)I\Big); \tag{2.26}$$

here, $\sinh(\cdot)$ *and* $\cosh(\cdot)$ *are the hyperbolic sine and the hyperbolic cosine, respectively.*

Proposition 2.3.17 ([132]). *The domain of the generating operator of a C_0-cosine function $C(\cdot, A)$ coincides with E^2, and for each $x \in D(A)$,*

$$Ax = s\text{-}\lim_{\tau \to 0} C''(\tau, A)x. \tag{2.27}$$

Sometimes the generating operator of a C_0-cosine operator-valued function is defined by (2.27).

The set of generating operators of a C_0-cosine operator-valued function with bound (2.11) will be denoted by $\mathbf{C}(M, \omega)$.

Proposition 2.3.18 ([258]). *Let $A, G \in \mathbf{C}(M, \omega)$. Then if $D(A) \subseteq D(G)$ and $Ax = Gx$ for all $x \in D(A)$, we have $C(t, A) = C(t, G)$ for all $t \in \mathbb{R}$.*

Theorem 2.3.1 ([109, 128, 258, 263]). *For an operator $A \in \mathbf{C}(E)$ to be a generating operator of a C_0-cosine operator-valued function, it is necessary and sufficient that for a certain constants $M, \omega \geq 0$, the resolvent $(\lambda^2 I - A)^{-1}$ exists for $\operatorname{Re}\lambda > \omega$ and the following inequalities hold:*

$$\left\| \frac{d^n}{d\lambda^n} \Big(\lambda(\lambda^2 I - A)^{-1} \Big) \right\| \leq \frac{Mn!}{(\operatorname{Re}\lambda - \omega)^{n+1}}, \qquad n \in \mathbb{N}_0. \tag{2.28}$$

Remark 2.3.2 ([258]). Sometimes, estimate (2.28) is written in the form

$$\left\| \frac{d^n}{d\lambda^n}\Big(\lambda(\lambda^2 I - A)^{-1}\Big)\right\| \le \frac{Mn!}{2}\left(\frac{1}{(\operatorname{Re}\lambda-\omega)^{n+1}} + \frac{1}{(\operatorname{Re}\lambda+\omega)^{n+1}}\right) \tag{2.29}$$

for all $\operatorname{Re}\lambda > \omega, n \in \mathbb{N}_0$.

In practice, conditions (2.28)–(2.29) turn out to be difficult to verify, and therefore other conditions for generating of a C_0-cosine operator-valued function are of interest.

Theorem 2.3.2 ([40]). *An operator $A \in \mathbf{C}(E)$ is a generating operator of a C_0-cosine operator-valued function iff there exist constants $M, \delta > 0$, and ω such that*

$$\|(\lambda^2 I - A)^{-1}\| \le \frac{M}{|\lambda|(\operatorname{Re}\lambda-\omega)} \qquad \operatorname{Re}\lambda > \omega, \tag{2.30}$$

and the following estimate holds uniformly in $\tau \in (0,\delta)$:

$$\|\int\limits_{\omega-i\infty}^{\omega+i\infty} e^{\lambda^2\tau} ch(\lambda t)\lambda(\lambda^2 I - A)^{-1}x\, d\lambda\| \le \xi(t)\|x\|, \quad t \in \overline{\mathbb{R}_+}, \tag{2.31}$$

where $\xi(\cdot) \in C(\mathbb{R})$.

Remark 2.3.3. In connection with estimate (2.30), we note (see [55]) that for any fixed $\varepsilon > 0$, the condition $\|(\lambda^2 I - A)^{-1}\| \le \frac{M}{|\lambda|^{1+\varepsilon}}, \quad \operatorname{Re}\lambda > \omega$, implies the boundedness of the spectrum $\sigma(A)$.

Proposition 2.3.19 ([208]). *In the case where A is a normal operator on a Hilbert space, it generates a C_0-cosine operator-valued function if only the conditions for location of the spectrum hold, i.e., $\{z^2 : \operatorname{Re} z > \omega\} \subseteq \rho(A)$ for a certain ω.*

Proposition 2.3.20 ([266]). *For $\operatorname{Re}\lambda > \omega_c(A)$, $\lambda^2 \in \rho(A)$ and*

$$\lambda(\lambda^2 I - A)^{-1}x = \int\limits_0^\infty e^{-\lambda t} C(t,A)x dt, \quad x \in E; \tag{2.32}$$

$$(\lambda^2 I - A)^{-1}x = \int\limits_0^\infty e^{-\lambda t} S(t,A)x dt, \quad x \in E. \tag{2.33}$$

Proposition 2.3.21 ([132]). *If $x \in D(A^3)$, $y \in D(A)$ and $\omega > \omega_c(A)$, then*

$$C(t,A)x = x + \frac{t^2}{2!}Ax + \frac{t^4}{4!}A^2x + \frac{1}{2\pi i}\int\limits_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\lambda^{-3}(\lambda^2 I - A)^{-1}A^3x\,d\lambda; \tag{2.34}$$

$$C(t,A)y = \frac{1}{2\pi i}\int\limits_{\omega-i\infty}^{\omega+i\infty} e^{\lambda t}\lambda(\lambda^2 I - A)^{-1}y\,d\lambda,\ t \in \mathbb{R}_+. \tag{2.35}$$

Writing the inverse Laplace transform in another form, we can obtain other analogous representations of the operator-valued functions $C(\cdot, A)$ and $S(\cdot, A)$.

Proposition 2.3.22 ([218]). *Let $x \in D(A^k)$ for a certain $k \in \mathbb{N}$. Then for $t \in \mathbb{R}$, the following Taylor formula holds:*

$$C(t,A)x = x + \frac{t^2}{2!}Ax + ... + \frac{t^{2k-2}}{(2k-2)!}A^{k-1}x + \int\limits_0^t \frac{(t-s)^{2k-1}}{(2k-1)!}C(s,A)A^k x ds.$$

Proposition 2.3.23 ([164]). *Let $A \in \mathbf{C}(M,\omega)$, and let $r \in \mathbb{N}$. Then*

$$\Big(C(t,A) - I\Big)^r = 2^{-r}\left[2\sum_{j=1}^{r}(-1)^{r-j}C_{r-j}^{2r}C(jt,A) + (-1)^r C_r^{2r} I\right]$$

$$\Big(C(t,A) - I\Big)^r x = A^r \int_0^t \int_0^t ... \int_0^t \Pi_{j=1}^r (t-s_j)C(s_j,A)x\,ds_1 ds_2 ... ds_r$$

for any $x \in E$.

Proposition 2.3.24 ([221]). *For any $A \in \mathbf{C}(M,\omega)$, $x \in \tilde{E}_0$, and $t \in \mathbb{R}$ the representation*

$$C(t,A)x = \sum_{k=0}^{\infty} t^{2k}A^k x/(2k)!$$

holds, and for each $\tilde{x} \in \tilde{E}_0$, the function $t \to C(t,A)\tilde{x}$ can be continued in t up to a function analytic on the whole complex plane.

Proposition 2.3.25 ([130]). *The following Widder–Post formula holds:*
(2.36)

$$C(t,A)x = \lim_{k\to\infty} \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} \frac{d^k}{d\lambda^k}\left(\lambda(\lambda^2 I - A)^{-1}x\right)\Bigg|_{\lambda=\frac{k}{t}}, \quad t \neq 0,\ x \in E,$$

where the convergence is uniform in t from any compact set in $\mathbb{R}\setminus\{0\}$.

Proposition 2.3.26 ([283]). *The expression $\mathfrak{N}(\lambda,k) := \frac{d^k}{d\lambda^k}\Big(\lambda(\lambda^2 I - A)^{-1}\Big)$ from* (2.36) *can be represented in the form*

$$(i) \quad \mathfrak{N}(\lambda,k) = k!\Big(\lambda^{k+1} + C_{k+1}^2\lambda^{k-1}A + ... + C_{k+1}^k\lambda A^{k/2}\Big)(\lambda^2 I - A)^{-(k+1)},$$

k is even;

$$(ii) \quad \mathfrak{N}(\lambda,k) = -k!\Big(\lambda^{k+1} + C_{k+1}^2\lambda^{k-1}A + ... + C_{k+1}^k\lambda A^{(k+1)/2}\Big)(\lambda^2 I - A)^{-(k+1)},$$

k is odd;

$$(iii) \quad \mathfrak{N}(\lambda,k) = \sum_{j=k/2}^{k}(-1)^j\frac{(k+1)!j!\lambda(2\lambda)^{2j-k}}{(k-j)!(2j-k+1)!}(\lambda^2 I - A)^{-(j+1)},$$

k is even;

$$(iv) \quad \mathfrak{N}(\lambda,k) = \sum_{j=\frac{k-1}{2}+1}^{k}(-1)^j\frac{(k+1)!j!\lambda(2\lambda)^{2j-k}}{(k-j)!(2j-k+1)!}(\lambda^2 I - A)^{-(j+1)},$$

k is odd;

Proposition 2.3.27 ([254]). *For a C_0-cosine operator-valued function $C(\cdot,A)$, a C_0-sine operator-valued function $S(\cdot,A)$ $x \in E$ and $t \in \mathbb{R}$, we have*

$$(i) \quad C(t,A)x = \lim_{k\to\infty}\sum_{l=0}^{k}\sum_{j=0}^{l} C_{2k}^{2l}C_l^j(-1)^{l-j}\left(I - \left(\frac{t}{2k}\right)^2 A\right)^{-(2k-l+j)} x;$$

$$(ii) \quad C(t,A)x = \lim_{k\to\infty}\sum_{l=0}^{k}\sum_{j=0}^{l} C_{2k+1}^{2l}C_l^j(-1)^{l-j}\left(I - \left(\frac{t}{2k+1}\right)^2 A\right)^{-(2k+1-l+j)} x;$$

$$(iii)\quad C(t,A)x = \lim_{n\to\infty} e^{-nt} \sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{j=0}^{k} \frac{(nt)^{2m}}{(2m)!} C_{2m}^{2k} C_k^j (-1)^{k-j}$$

$$\times\left(I + \frac{nt}{2m-2k+1}(I-n^{-2}A)^{-1}\right)(I-n^2A)^{-(2m-k+j)}x;$$

$$(iv)\quad S(t,A)x = \lim_{n\to\infty} \frac{t}{n} \sum_{m=0}^{n-1}\sum_{k=0}^{m}\sum_{j=0}^{k} C_{2m+1}^{2k} C_k^j (-1)^{k-j} \left(I - \left(\frac{t}{2n}\right)^2 A\right)^{-(n+m+1-k+j)} x;$$

moreover, in all the cases, the convergence in $t \in J \subset \mathbb{R}$ is uniform. Here, J is an arbitrary closed interval.

Proposition 2.3.28 ([254]). *Under the conditions of Proposition 2.3.27, we have the following uniformly in $t \in [0,1]$:*

$$C(t,A)x = \lim_{n\to\infty} \sum_{m=0}^{n}\sum_{k=0}^{m}\sum_{j=0}^{k} C_{2n}^{2m} C_{2m}^{2k} C_k^j (-1)^{k-j} t^{2m} (1-t)^{2n-2m-1}$$

$$\times\left((1-t) + \frac{2n-2m}{2m-2k+1} t (I-(2n)^{-2}A)^{-1}\right)^{-(2m-k+j)} x\,.$$

In [254], many other relations if an analogous form were also presented. Introduce the following notation:

$$St(A) := \left\{x \in E : \sum_{k=1}^{\infty} \|A^k x\|^{-\frac{1}{2k}} < \infty\right\} \quad \text{the Stietjes vectors,}$$

$$\mathfrak{U}_p(A) := \left\{x \in E : \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!}\|A^k x\| < \infty \quad \text{for a certain} \quad t \in \mathbb{R}_+\right\}$$

are semianalytic vectors, and

$$\mathfrak{U}_{pp}(A) := \left\{x \in E : \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\|A^k x\| < \infty \quad \text{for all} \quad t \in \mathbb{R}_+\right\} - \text{are entire vectors.}$$

Proposition 2.3.29 ([103]). *Let $A \in \mathbf{C}(M,\omega)$, and let $\tilde{E}_0$ be constructed according to the C_0-semigroup* $\exp(\cdot A)$. *Then $\tilde{E}_0 \subseteq \mathfrak{U}_p(A)$.*

Proposition 2.3.30 ([103]). *Let $A \in \mathcal{C}(M.\omega)$. We have the embeddings $(A) \subseteq \mathfrak{U}_p(A) \subseteq St(A)$.*

Proposition 2.3.31 ([103]). *Let $\overline{\mathfrak{U}_{pp}(A)} = E$. Then the set of vectors x from $D(A^\infty)$ having the property $\|A^k x\|^{1/k} = o(k)$ is dense in E .*

Proposition 2.3.32 ([103]). *Let $A \in \mathbf{C}(M,\omega)$. Then $\overline{\mathfrak{U}(A) \cap \mathfrak{U}_{pp}(A)} = E$.*

Proposition 2.3.33 ([103]). *Let $\overline{\mathfrak{U}_{pp}(A)} = E.$, and let there exists an operator $G \in \mathbf{C}(E)$ such that*
(i) $G^{-1} \in B(E)$;
(ii) $G^2 = A$, *and, moreover, the operators* $\pm G$ *are dissipative.*
Then $A \in \mathbf{C}(M,\omega)$ $C(t,A) = \Big(exp(tG) + exp(-tG)\Big)/2$, *where $G \in$* $\mathbf{GR}(1,0)$.

Definition 2.3.5. A set of elements $S \subseteq E$ is said to be *total* in E if the set of all its finite linear combinations is total in E.

Proposition 2.3.34 ([103]). *Let $A_1 \in L(E)$ be closed, $St(A_1)$ be total in E, $A_2 \in \mathbf{C}(M,\omega)$, and let $A_1 \subseteq A_2$. Then $A_1 = A_2$.*

Proposition 2.3.35 ([103]). *Let A be a closed, symmetric, and semi-bounded operator on a Hilbert space H. Then the operator A is self-adjoint iff the set $St(A)$ is total in H.*

In [142], examples of nonlinear cosine operator-valued functions are presented. However, in contrast to the theory of nonlinear operator semigroup, there are no general theory of a nonlinear cosine operator-valued functions for now.

2.4 Laplace Transform and Infinitesimal Operators

In this section, we present certain basic properties of the Laplace transform for C_0-families of multiplicative perturbation $F(\cdot)$ and additive $G(\cdot)$ perturbation. Let $\hat{F}(\cdot)$ and $\hat{G}(\cdot)$ denote their Laplace transform, respectively.

Proposition 2.4.1 ([233]). *Let $F(\cdot)$ be a multiplicative perturbation C_0-family and let $G(\cdot)$ be an additive perturbation family for a C_0-cosine operator-valued function $C(\cdot, A)$. The following properties hold:*

(i) $(C(t,A)-I)F(s) = (C(s,A)-I)F(t)$ *and*
$G(s)(C(t,A)-I) = G(t)(C(s,A)-I)$ *for* $t, s \in \overline{\mathbb{R}}_+$;

(ii) *the functions $F(\cdot)$ and $G(\cdot)$ are exponentially bounded;*

(iii) $\dfrac{d^2}{dt^2}(\lambda(\lambda^2 I - A)^{-1}F(t)x) = C(t,A)\lambda^2\hat{F}(\lambda)x$ *and*
$\dfrac{d^2}{dt^2}\Big(G(t)\lambda(\lambda^2 I - A)^{-1}x\Big) = \lambda^2\hat{G}(\lambda)C(t,A)x$ *for* $x \in E$, $\lambda > \omega$, *and* $t \in \mathbb{R}_+$;

(iv) $F(t)x = (\lambda^2 I - A)\int_0^t S(s,A)\lambda\hat{F}(\lambda)x\,ds =$
$= \int_0^t S(s,A)\lambda^3\hat{F}(\lambda)x\,ds - (C(t,A)-I)\lambda\hat{F}(\lambda)x$ *for* $x \in E,\ t \in \overline{\mathbb{R}}_+$;

(v) $G(t)x = \lambda\hat{G}(\lambda)(\lambda^2 I - A)\int_0^t S(s,A)x\,ds =$
$= \lambda^3\hat{G}(\lambda)\int_0^t S(s,A)x\,ds - \lambda\hat{G}(\lambda)(C(t,A)-I)x$ *for* $x \in E,\ t \in \overline{\mathbb{R}}_+$.

Proof. Property (i) is easily implied by (2.1) and (2.2). To prove that the multiplicative perturbation C_0-family $F(\cdot)$ is exponentially bounded, we choose $L \geq 1,\ \tau \in \mathbb{R}_+$ such that $\|C(s,A)\| \leq L,\ \|F(s)\| \leq L$ for $0 \leq s \leq \tau$. Using the relation

$$F(k\tau + s) = 2F(k\tau) - F(k\tau - s) + 2C(k\tau, A)F(s),$$

for $0 \leq s \leq \tau$, we have

$$\|F(\tau + s)\| \leq \|2F(\tau)\| + \|F(\tau - s)\| + \ 2\ Me^{\tau\omega}\|F(s)\|$$

$$\leq 2L + L + 2Me^{\tau\omega}L \leq Me^{\tau\omega}5L \leq Me^{2\tau\omega_1} \leq Me^{\tau\omega_1}e^{(\tau+s)\omega_1},$$

where $5L \leq e^{\tau\omega_1}$ and $\omega \leq \omega_1$,

$$\|F(2\tau + s)\| \leq 2\|F(2\tau)\| + \|F(2\tau - s)\| + 2Me^{2\tau\omega_1}\|F(s)\|$$

$$\leq 2Me^{2\tau\omega_1} + Me^{2\tau\omega_1} + 2LMe^{2\tau\omega_1} \leq Me^{3\tau\omega_1} \leq \tau\omega_1 e^{(2\tau+s)\omega_1}.$$

By induction,

$$\|F(k\tau + s)\| \leq 2\|F(k\tau)\| + \|F(k\tau - s)\| + 2\|C(k\tau, A)\|\ \|F(s)\|$$

$$\leq 2Me^{k\tau\omega_1} + Me^{k\tau\omega_1} + 2LMe^{k\tau\omega_1} \leq 5LMe^{k\tau\omega_1} \leq Me^{(k+1)\tau\omega_1} \leq Me^{\tau\omega_1}e^{(k\tau+s)\omega_1}$$

for all $s \in [0, \tau]$. Therefore, $\|F(t)\| \leq M_1 e^{\omega_1 t}$ for $M_1 = Me^{\tau\omega_1}$ and all $t \in \overline{\mathbb{R}}_+$.

To prove (iii), we set $\Theta(t,\lambda) = \lambda(\lambda^2 - A)^{-1}F(t)$ $\Upsilon(t,\lambda) = G(t)\lambda(\lambda^2 - A)^{-1}, \lambda > \omega, t \geq 0$. It follows from (2.1) and (2.2) that

$$\Theta''_t(t,\lambda) = C(t,A)\lim_{s\to 0} 2s^{-2}\lambda(\lambda^2 - A)^{-1}F(s) = C(t,A)\Theta''_t(0,\lambda)$$

if $\Theta''_t(0,\lambda)$ exists, and

$$\Upsilon''_t(t,\lambda) = \lim_{s\to 0} 2s^{-2}\lambda(\lambda^2 - A)^{-1}G(s)C(t,A) = \Upsilon''_t(0,\lambda)C(t,A)$$

if $\Upsilon''_t(0,\lambda)$ exists. Therefore, it suffices to prove that $\Theta''_t(0,\lambda) = \lambda^2\hat{F}(\lambda)$ $\Upsilon''_t(0,\lambda) = \lambda^2\hat{G}(\lambda)$.

Taking the Laplace transform in t in (2.1), we have

$$(e^{\lambda s} - 2 + e^{-\lambda s})\hat{F}(\lambda) - e^{\lambda s}\int_0^s e^{-\lambda\tau}F(\tau)d\tau + e^{-\lambda s}\int_0^s e^{\lambda\tau}F(\tau)d\tau$$

$$= 2\lambda(\lambda^2 - A)^{-1}F(s) = 2\Theta(s,\lambda).$$

Now, taking the derivative, we obtain

$$2\Theta'_s(s,\lambda) = \lambda(e^{\lambda s} - e^{-\lambda s})\hat{F}(\lambda) - \lambda e^{\lambda s}\int_0^s e^{-\lambda\tau}F(\tau)d\tau - \lambda e^{-\lambda s}\int_0^s e^{\lambda\tau}F(\tau)d\tau,$$

and differentiating once more, we have

$$2\Theta''_{ss}(s,\lambda) = \lambda^2(e^{\lambda s} + e^{-\lambda s})\hat{F}(\lambda) - \lambda^2 e^{\lambda s}\int_0^s e^{-\lambda\tau}F(\tau)d\tau$$

$$+\lambda^2 e^{-\lambda s}\int_0^s e^{\lambda\tau}F(\tau)d\tau - 2\lambda F(s).$$

Therefore, $\Theta'_s(0,\lambda) = 0$ and $\Theta''_{ss}(0,\lambda) = \lambda^2\hat{F}(\lambda)$. Analogously, we can show that $\Upsilon'_s(0,\lambda) = 0$ and $\Upsilon''_{ss}(0,\lambda) = \lambda^2\hat{G}(\lambda)$.

Integrating $\Theta''_{tt}(t,\lambda) = C(t,A)\lambda^2\hat{F}(\lambda)$ twice from zero up to t and using the relations $F(0) = 0$ and $\Theta'_t(0,\lambda) = 0$, we obtain

$$\lambda(\lambda^2 I - A)^{-1}F(t)x = \Theta(t,\lambda)x = \int_0^t S(s,A)\lambda^2\hat{F}(\lambda)x\,ds,\ x \in E, \tag{2.37}$$

and., therefore, assertion (iv) is proved. Assertion (v) is proved analogously.

Remark 2.4.1. If $C(\cdot, A)$ is uniformly continuous, then each multiplicative perturbation C_0-family $F(\cdot)$ (resp. resp. each additive perturbation C_0-family) of a C_0-cosine operator-valued function $C(\cdot, A)$ is also uniformly continuous. This follows from formula (iv) (resp. (v)) and Proposition 2.4.1.

Definition 2.4.1. Let $F(\cdot)$ be a multiplicative perturbation C_0-family for a C_0-cosine operator-valued function $C(\cdot, A)$. The *infinitesimal operator* W_s of the family $F(\cdot)$ is defined as $W_s x = s\text{-}\lim_{h\to 0} \frac{2}{h^2} F(h)x$, with a natural domain. The *infinitesimal operator* A_s of the pair $(C(\cdot, A), F(\cdot))$ is defined as $A_s x := s\text{-}\lim_{h\to 0} \dfrac{2}{h^2}(C(h, A) + F(h) - I)x$, with a natural domain. The *infinitesimal operator* W_c of an additive perturbation C_0-family $G(\cdot)$ and the *infinitesimal operator* A_c of the pair $G(\cdot), C(\cdot, A))$ are defined in the same way as

$$W_c x = s\text{-}\lim_{h\to 0} \frac{2}{h^2} G(h)x \qquad A_c x := s\text{-}\lim_{h\to 0} \frac{2}{h^2}\Big(C(h, A) + G(h) - I\Big)x,$$

respectively.

Theorem 2.4.1 ([233]). *The operators W_s and A_s defined above are closed and*

(i) $W_s = \lambda(\lambda^2 - A)\hat{F}(\lambda),\ Re\lambda > \omega$;

(ii) $A_s = A(I - \lambda\hat{F}(\lambda)) + \lambda^3\hat{F}(\lambda),\ Re\lambda > \omega$;

(iii)

$$A_s = A\big(I - \frac{2}{t^2}\int_0^t \int_0^\tau F(s)dsd\tau\big)$$
$$+\frac{2}{t^2}\Big(\lambda^2 \int_0^t \int_0^\tau C(s, A)dsd\tau - (C(t, A) - I)\Big)\lambda\hat{F}(\lambda),$$

$t \in \mathbb{R}_+,\ Re\lambda > \omega.$

Proof. Let $A_h = \frac{2}{h^2}\Big(C(h, A) + F(h) - I\Big)$. Assertion (iv) of Proposition 2.4.1 can be rewritten in the form

$$\frac{2F(h)}{h^2}x = \frac{2}{h^2}\int_0^h S(s, A)\lambda^3\hat{F}(\lambda)xds - \frac{2}{h^2}(C(h, A) - I)\lambda\hat{F}(\lambda)x,$$

$$A_h x = 2h^{-2}\int_0^h S(s, A)\lambda^3\hat{F}(\lambda)x\, ds + 2h^{-2}(C(h, A) - I)(I - \lambda\hat{F}(\lambda))x.$$

Since the first term in the right-hand side of each of these relations converges to $\lambda^3\hat{F}(\lambda)x$ as $h \to 0$ by (2.17), we have

$$D(W_s) = D(A\hat{F}(\lambda)) \quad W_s x = \lambda(\lambda^2 - A)\hat{F}(\lambda)x \quad x \in D(W_s),$$

and also

$$D(A_s) = D\Big(A(I - \lambda\hat{F}(\lambda))\Big) \;\; A_s x = \lambda^3\hat{F}(\lambda)x + A(I - \lambda\hat{F}(\lambda))x \quad x \in D(A_s).$$

Since A is closed and $\hat{F}(\lambda)$ is bounded, it is easy to see that W_s and A_s are closed. Assertions (i) and (ii) are proved.

To prove (iii), we use relation (2.1). For all $x \in E$ and $s \in \overline{\mathbb{R}}_+$, we have

$$\frac{2}{h^2}\Big(C(h,A) + F(h) - I\Big)x$$

$$= \frac{2}{h^2}\Big(C(h,A) - I\Big)x + \frac{2}{h^2}\Bigg(F(s+h) - 2C(h,A)F(s) + F(s-h)\Bigg)x$$

$$= \frac{2}{h^2}\Big(C(h,A) - I\Big)\Big(I - F(s)\Big)x + \frac{1}{h^2}\Bigg(F(s+h) - 2F(s) + F(s-h)\Bigg)x$$

$$= \frac{2}{h^2}(C(h,A) - I)(I - F(s))x + \frac{2}{h^2}C(s,A)F(h)x.$$

Integrating twice, for any $t \in \mathbb{R}_+$, we obtain

$$\frac{2}{h^2}(C(h,A) + F(h) - I)x = \frac{2}{h^2}(C(h,A) - I)\Bigg(I - \frac{2}{t^2}\int_0^t\int_0^\tau F(s)dsd\tau\Bigg)x$$

$$+\frac{2}{t^2}(\lambda^2 I - A)\int_0^t\int_0^\tau C(s,A)dsd\tau(\lambda^2 I - A)^{-1}\frac{2}{h^2}F(h)x.$$

Since the last term converges to $\frac{2}{t^2}(\lambda^2 - A)\int_0^t\int_0^\tau C(s,A)\,dsd\tau\lambda\hat{F}(\lambda)x$ as $h \to 0+$ for all $x \in E$ (see Proposition 2.4.1 (iii)), we obtain A_s in the form (iii).

Remark 2.4.2. The definition of the infinitesimal operator via the limit s-$\lim_{h\to 0+} h^{-1}F(h)$ has no sense. Indeed, in this case, using (2.37), we obtain that such an operator is zero.

Generally speaking, the domains of the operators W_s and A_s are not necessarily dense in E. But under certain conditions on $F(\cdot)$, the operator A_s not only has a dense domain but generates a C_0-cosine operator-valued function. The domains of $D(W_c)$ and $D(A_c)$ always contain the dense set $D(A)$.

Theorem 2.4.2 ([233]). *The infinitesimal operators W_c and A_c have the following properties for $Re\lambda > \omega$:*

(i) $D(A) \subseteq D(W_c)$ $W_c x = \lambda \hat{G}(\lambda)(\lambda^2 - A)x$ *for all* $x \in D(A)$;

(ii) $D(A) \subseteq D(A_c)$, *and for* $x \in D(A)$, *we have*

$A_c x = Ax + W_c x = \Big(I - \lambda \hat{G}(\lambda)\Big)Ax + \lambda^3 \hat{G}(\lambda)x$;

(iii) $D(A) \subseteq D(A_c)$, *and for all* $x \in D(A)$ *and* $t \in \mathbb{R}_+ 4$,

$$A_c x = \Big(I - \frac{2}{t^2}\int_0^t \int_0^\tau G(s)dsd\tau\Big)Ax$$
$$+\lambda \hat{G}(\lambda)\frac{2}{t^2}\Big(\lambda^2 \int_0^t \int_0^\tau C(s,A)dsd\tau - (C(t,A) - I)\Big)x.$$

Moreover, if $G(t)$ is uniformly continuous in t, then A_c is closed, $D(A_c) = D(A)$, and $A_c = \Big(I - \lambda \hat{G}(\lambda)\Big)A + \lambda^3 \hat{G}(\lambda)$ for large $\operatorname{Re}\lambda$. *If $\hat{G}(\lambda)$ is invertible for a certain λ, then the operator W_c is closed, $D(W_c) = D(A)$, and $W_c = A_c - A = \lambda \hat{G}(\lambda)(\lambda^2 I - A)$.*

Proof. Let $A_h = \dfrac{2}{h^2}\Big(C(h,A) + G(h) - I\Big)$. By (v) of Proposition 2.4.1, we have

$$\frac{2G(h)}{h^2}x = \lambda^3 \hat{G}(\lambda)\frac{2}{h^2}\int_0^h S(s,A)xds - \lambda \hat{G}(\lambda)\frac{2}{h^2}\Big(C(h,A) - I\Big)x,$$

$$A_h x = 2\lambda^3 \hat{G}(\lambda)h^{-2}\int_0^h S(s,A)xds + 2(I - \lambda \hat{G}(\lambda))h^{-2}(C(h,A) - I)x.$$

The first identity implies $D(A) \subseteq D(W_c)$ and $W_c x = \lambda \hat{G}(\lambda)(\lambda^2 I - A)x$ for $x \in D(A)$.

The second identity implies $D(A) \subseteq D(A_c)$ and $A_c x = Ax + W_c x = (I - \lambda \hat{G}(\lambda))Ax + \lambda^3 \hat{G}(\lambda)x$ for $x \in D(A)$.

The proof of (iii) is similar to that of (iii) in Theorem 2.2.2.

If $\|G(t)\| \to 0$ as $t \to 0$, then $\|\lambda \hat{G}(\lambda)\| \to 0$ as $\lambda \to \infty$ (Proposition 2.4.2 (ii)). Therefore, the operator $I - \lambda \hat{G}(\lambda)$ is invertible for large λ, and we have $D(A_c) \subseteq D(A)$. If $\{x_n\}$ is a sequence in $D(A)$ such that $x_n \to x$ and $(I - \lambda \hat{G}(\lambda))Ax_n \to y$, then $Ax_n \to \Big(I - \lambda \hat{G}(\lambda)\Big)^{-1} y$, so that $x \in D(A)$ and $Ax = \Big(I - \lambda \hat{G}(\lambda)\Big)^{-1} y$.

Therefore, $\Big(I - \lambda\hat{G}(\lambda)\Big)A$ is closed, and hence A_c is also closed. The proofs of the assertions concerning the operator W_c are going in the line as that for A_c.

It follows from (2.1) that if $\|F(t)x\| = o(t^2)\,(t \to 0+)$ for all $x \in E$, then $F''(t) = 0$ for all $t \in \overline{\mathbb{R}}_+$, so that $F'(\cdot) \equiv F'(0) = 0$, and then $F(\cdot) \equiv F(0) = 0$.

Similarly, it follows from (2.2) that the condition $\|G(t)x\| = o(t^2)\,(t \to 0+)$ for all $x \in E$ implies $G(\cdot) \equiv 0$. Therefore, the rate of convergence to 0 in the case of a nontrivial multiplicative perturbation family or an additive perturbation family cannot exceed $O(t^2)$ for $t \to 0$.

Proposition 2.4.2 ([233]). *We have the following assertions concerning the rate of convergence to zero:*

(i) *For $n = 0, 1$, if $\|F(t)x\| = o(t^n)$ as $t \to 0+$ for all $x \in E$, then $\|\lambda^n \hat{F}(\lambda)\| = o(1)$ as $\lambda \to \infty$ and $\|\lambda^{n+1}\hat{F}(\lambda)x\| = o(1)$ as $\lambda \to \infty$ for all $x \in E$;*

(ii) *For $n = 0, 1$, if $\|G(t)x\| = o(t^n)$ as $t \to 0+$ for all $x \in E$, then $\|\lambda^n \hat{G}(\lambda)\| = o(1)$ as $\lambda \to \infty$, $\|\lambda^{n+1}\hat{G}(\lambda)x\| = o(1)$ as $\lambda \to \infty$ for all $x \in E$, and*

$\|\lambda^3 \hat{G}(\lambda)x - (A_c - A)x\| = o(\lambda^{-n})$ for all $x \in D(A)$;

(iii) *For $n = 0, 1$, if $\|F(t)\| = o(t^n)$ (resp. $\|G(t)\| = o(t^n)$) as $t \to 0+$, then $\|\lambda^{n+1}\hat{F}(\lambda)\| = o(1)$ (resp. $\|\lambda^{n+1}\hat{G}(\lambda)\| = o(1)$) as $\lambda \to \infty$;*

(iv) *For $n = 1, 2$, if $\|F(t)\| = O(t^n)\,(t \to 0+)$, then $\|\lambda^{n+1}\hat{F}(\lambda)\| = O(1)$ as $\lambda \to \infty$;*

(v) *For $n = 1, 2$, if $\|G(t)\| = O(t^n)$ as $t \to 0+$, then $\|\lambda^{n+1}\hat{G}(\lambda)\| = O(1)$ as $\lambda \to \infty$, and $\|\lambda^3 \hat{G}(\lambda)x - (A_c - A)x\| = O(\lambda^{-n})$ for all $x \in D(A)$.*

(vi) if $\|F(t)\| = O(t^2)$ as $t \to 0$, then
w^*-$\lim_{\lambda\to\infty} \lambda^3 \hat{F}(\lambda)^* x^* = (A_s^* - A^*)x^*$ for any $x^* \in D(A^*)$.

Proof. We prove only (ii); the proof of assertions (i), (iii), (iv), and (v) is analogous. For a given $\epsilon > 0$, choose $\delta > 0$ such that $\|G(t)x\| \le \epsilon t^n$ for all $t \in [0, \delta]$. Then

$$\|\lambda^{n+1}\hat{G}(\lambda)x\| \le \lambda^{n+1}\Big(\int_0^\delta + \int_\delta^\infty\Big)e^{-\lambda t}\|G(t)x\|dt$$

$$\le \epsilon\lambda^{n+1}\int_0^\infty e^{-\lambda t}t^n\,dt + \lambda^{n+1}\int_\delta^\infty e^{-\lambda t}Me^{\omega t}dt\|x\| \le \epsilon/n! + M\frac{\lambda^{n+1}}{\lambda-\omega}e^{-(\lambda-\omega)\delta}\|x\|.$$

This implies $\|\lambda^{n+1}\hat{G}(\lambda)x\| = o(1)$ as $\lambda \to \infty$ for $x \in E$. By the uniform boundedness principle, we have $\|\lambda^n \hat{G}(\lambda)\| = o(1)$ as $\lambda \to \infty$. Now, we have

from (ii) of Theorem 2.4.2 that $\|\lambda^3\hat{G}(\lambda)x - (A_c - A)x\| = o(\lambda^{-n})$ for all $x \in D(A)$.

CHAPTER 3

Chapter III
Reduction of the Cauchy Problem for a Second-Order Equation to the Cauchy Problem for a System od First-Order Equations

As for ordinary differential equations, an nth order equations can be reduced to a set of first-order equations by using matrix operators. The matrix operator theory is presented in [124] in detail. In the present chapter, we consider only problems of reducing incomplete 2nd-order equations to a set of first-order equation.

3.1 Kysinski Theorem

In a Banach space E, let us consider the following uniformly well-posed Cauchy problem:

$$u''(t) = Au(t), \quad t \in \mathbb{R}; \qquad u(0) = u^0, \quad u'(0) = u^1. \tag{3.1}$$

Define the matrix operator $\mathbf{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} : E^1 \times E \to E^1 \times E$ acting on an element $(x, y) \in E^1 \times E$ by the formula $\mathbf{A}(x, y) = (y, Ax)$ that is given on the domain of $D(\mathbf{A}) = D(A) \times E^1$. In what follows, an element $(x, y) \in E^1 \times E$ in the paragraph formulas will be written as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$.

Theorem 3.1.1 ([177]). *The space E^1 with the norm*

$$\|x\|_{E^1} := \|x\| + \sup_{0 \le t \le 1} \|C'(t, A)x\| \tag{3.2}$$

is a Banach space, and the operator $\mathbf{A}$ *generates the following* C_0*-groups of operators on the Banach space* $E^1 \times E$*:*

$$\exp(t\mathbf{A})\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} C(t,A) & S(t,A) \\ AS(t,A) & C(t,A) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} C(t,A)x + S(t,A)y \\ AS(t,A)x + C(t,A)y \end{pmatrix}, \quad t \in \mathbb{R},$$

Proposition 3.1.1 ([281]). *Let a* C_0*-cosine operator-valued function* $C(\cdot, A)$ *be given. Them* E^1 *coincides with the closure of* $D(A)$ *in the norm*

$$\|x\|^* := \|x\| + \sup_{z>\omega, n\in\mathbb{N}} \frac{1}{n!}(z-\omega)^{n+1}\left\|\frac{d^n}{dz^n}A(z^2I - A)^{-1}x\right\|. \tag{3.3}$$

Proposition 3.1.2 ([177]). *The resolvent of the operator* $\mathbf{A}$ *has the form*

$$(\lambda I - \mathbf{A})^{-1} = \begin{pmatrix} \lambda(\lambda^2 I - A)^{-1} & (\lambda^2 I - A)^{-1} \\ A(\lambda^2 I - A)^{-1} & \lambda(\lambda^2 I - A)^{-1} \end{pmatrix} \qquad \lambda^2 \in \rho(A)\,. \tag{3.4}$$

Proposition 3.1.3 ([177]). *Let* $u(\cdot)$ *be a solution of problem* (3.1)*, and let* $v(t) := u'(t), t \in \mathbb{R}$*. Then the vector* $\begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix}$ *is a solution of the following uniformly well-posed Cauchy problem in the Banach space* $E^1 \times E$*:*

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \mathbf{A}\begin{pmatrix} u \\ v \end{pmatrix}(t), \quad t \in \mathbb{R}; \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}. \tag{3.5}$$

Proposition 3.1.4 ([177]). *Let a certain Banach space* $\tilde{E}^1$ *be continuously and densely embedded into the Banach space* E*, and, moreover, let* $D(\tilde{A}) \subseteq \tilde{E}^1$ *for a certain operator* $\tilde{A} \in L(E)$*. Then if in the space* $\tilde{E}^1 \times E$*, the Cauchy problem*

(3.6)
$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \begin{pmatrix} 0 & I \\ \tilde{A} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}(t) \equiv \tilde{\mathbf{A}}\begin{pmatrix} u \\ v \end{pmatrix}(t),\ t \in \mathbb{R}, \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix},$$

is uniformly well-posed and $\rho(\tilde{\mathbf{A}}) \neq \varnothing$*, we have* $\tilde{A} \in \mathbf{C}(M, \omega)$*.*

Proposition 3.1.5 ([177]). *Under the conditions of Proposition* 3.1.4*, the* C_0*-semigroup corresponding to problem* (3.6) *on the space* $\tilde{E}^1 \times E$ *is represented in the form*

$$\exp(t\tilde{\mathbf{A}}) := \begin{pmatrix} G_{11}(t) & G_{12}(t) \\ G_{21}(t) & G_{22}(t) \end{pmatrix}, \quad t \in \overline{\mathbb{R}_+}, \tag{3.7}$$

where the family $G_{22}(\cdot)$ is a C_0-cosine operator-valued function $C(\cdot, A)$ and coincides with $G_{11}(\cdot)$ on $\tilde{E}^1$.

Proposition 3.1.6 ([177]). *Under the conditions of Proposition* 3.1.4*, the following relations hold for $x \in E$ and $y \in \tilde{E}^1$:*

$$G_{12}(t)x = S(t, A)x \quad \text{and} \quad G_{21}(t)y = C'(t, A)y = AS(t, A)y, \quad t \in \mathbb{R}.$$

Proposition 3.1.7 ([177]). *The spaces $\tilde{E}^1$ and E^1 coincide with accuracy up to a norm equivalence.*

3.2 Conditions (K) and (F)

however, we note that to study problem (3.1) by reducing it to system (3.5) is very inconvenient, since the space E^1 is defined either through the C_0-cosine operator-valued function $C(\cdot, A)$ or through infinitely many powers of the resolvent. As a rule, we have only the information about the operator A. Therefore, certain additional conditions that allows us to reduce problem (3.1) to a system without use of the space E^1 are of interest.

Proposition 3.2.1 ([268]). *Let the space E be Hilbert, and let the operator A be self-adjoint and negative-definite. Then $A \in \mathbf{C}(M, \omega)$, and the corresponding space E^1 coincides with $\mathbf{D}((-A)^{1/2})$.*

Let the uniformly well-posed problem (3.1) have the form

$$u''(t) = \mathfrak{B}^2 u(t); \quad t \in \mathbb{R}, \quad u(0) = u^0, \quad u'(0) = u^1, \tag{3.8}$$

where $\mathfrak{B} \in \mathbf{C}(E)$.

Definition 3.2.1. We say that a solution $u(\cdot)$ of problem (3.8) satisfies *Condition (K)* if $u'(\cdot) \in C\Big([0, T]; \mathbf{D}(\mathfrak{B})\Big)$.

Proposition 3.2.2 ([45]). *Problem* (3.8) *has a unique solution satisfying Condition (K) iff the following Cauchy problem is uniformly well posed on the space $E \times E$:*

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}(t), \quad t \in \mathbb{R}, \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{3.9}$$

An analog of Condition (K), which allows us to simplify the etudy of problem (3.1) by using C_0-semigroups, is the following Condition (F).

Definition 3.2.2. A C_0-cosine operator-valued function $C(\cdot, A)$ satisfies *Condition (F)* if the following conditions hold:

(i) there exists $\mathfrak{B} \in \mathbf{C}(E)$ such that $\mathfrak{B}^2 = A$, and $\mathfrak{B}$ commutes with any operator from $B(E)$ commuting with A;

(ii) $S(t, A)$ maps E into $D(\mathfrak{B})$ for any $t \in \mathbb{R}$;

(iii) the function $\mathfrak{B}S(t, A)x$ is continuous in $t \in \mathbb{R}$ for any fixed $x \in E$.

Proposition 3.2.3 ([132]). *Under Condition* (F)*, for each* $t \in \mathbb{R}$*, we have* $\mathfrak{B}S(t, A) \in B(E)$ *and* $D(\mathfrak{B}) \subseteq E^1$.

Proposition 3.2.4 ([132]). *There exist a Banach space* E *and a* C_0*-cosine operator-valued function* $C(\cdot, A)$ *(even uniformly bounded) such that Condition* (F) *does not hold.*

Proposition 3.2.5 ([132]). *Via the shift* $A_b := A - b^2 I$ *for* $b > \omega_c(A)$*, we can always construct operators* A_b *and* $\mathfrak{B}_b$ *such that* $\mathfrak{B}_b^2 = A_b$ *and* $\mathfrak{B}_b$ *commutes with any operator from* $B(E)$ *commuting with* A_b*.*

Proposition 3.2.6 ([131]). *The operator* $\mathfrak{B}_b$ *in Proposition* 3.2.5 *can be constructed, e.g., as follows:*

$$\mathfrak{B}_b x := \frac{-i}{\pi} \int_0^\infty \lambda^{-1/2} (\lambda I - A_b)^{-1} (-A_b x) d\lambda \,.$$

Theorem 3.2.1 ([266]). *Let* A *and* $\mathfrak{B}$ *be operators satisfying condition* (i) *in definition* 3.2.2*, and let* $0 \in \rho(\mathfrak{B})$*. The following conditions are equivalent:*

(i) the C_0*-cosine operator-valued function* $C(\cdot, A)$ *satisfies Condition (F);*

(ii) the operator $\mathfrak{B}$ *generates a* C_0*-semigroup* $exp(\cdot\mathfrak{B})$ *on* E*;*

(iii) the operator $\begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix}$ *with the domain* $D(A) \times D(\mathfrak{B})$ *generates a* C_0*-group on* $E \times E$*;*

(iv) the operator $\mathbf{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ *with the domain* $D(A) \times D(\mathfrak{B})$ *generates a* C_0*-group* $exp(\cdot\mathbf{A})$ $\mathbf{D}(\mathfrak{B}) \times E$*, where* $\mathbf{D}(\mathfrak{B})$ *is the Banach space of elements* $D(\mathfrak{B})$ *endowed with the graph norm;*

(v) the embedding $D(\mathfrak{B}) \subseteq E^1$ *holds;*

(vi) $D(\mathfrak{B}) = E^1$.

Proposition 3.2.7 ([100]). *Let* $A \in \mathbf{C}(M, 0)$ *, and let* E *be a UMD space. Then Condition (F) holds.*

Proposition 3.2.8 ([282]). *The following condition is equivalet to conditions (i)–(vi) of Theorem* 3.2.1 *:*

$D(\mathfrak{B})$ is dense in E, and there exist constants $M > 0$ and $\omega \geq 0$ such that $\lambda^2 \in \rho(A)$ for any $\lambda > \omega$, the operator-valued function $\lambda(\lambda^2 I - A)^{-1}$, and $\mathfrak{B}(\lambda^2 I - A)^{-1}$ is strongly infinitely many times differentiable for $\lambda > \omega$, and the following estimates hold for any $m \in \mathbb{N}_0$:

$$\left\| \frac{(\lambda-\omega)^{m+1}}{m!} \left(\frac{d}{d\lambda}\right)^m \left(\lambda(\lambda^2 I - A)^{-1}\right) \right\| \leq M,$$

$$\left\| \frac{(\lambda-\omega)^{m+1}}{m!} \left(\frac{d}{d\lambda}\right)^m \left(\mathfrak{B}(\lambda^2 I - A)^{-1}\right) \right\| \leq M.$$

Proposition 3.2.9 ([266]). *Under the conditions of Theorem* 3.2.1*, for $t \in \mathbb{R}$, we have*

$$(i) \quad \exp(t\mathfrak{B}) = C(t,A) + \mathfrak{B}S(t,A),$$

$$C(t,A) = \Big(\exp(t\mathfrak{B}) + \exp(-t\mathfrak{B})\Big)/2;$$

$$(ii) \quad \exp(t\mathbf{A}) = \begin{pmatrix} \mathfrak{B}^{-1} & 0 \\ 0 & I \end{pmatrix} \exp\left(t \begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix}\right) \begin{pmatrix} \mathfrak{B} & 0 \\ 0 & I \end{pmatrix};$$

$$(iii) \quad \exp(t\mathbf{A}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} C(t,A)x + S(t,A)y \\ AS(t,A)x + C(t,A)y \end{pmatrix}, \ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbf{D}(\mathfrak{B}) \times E.$$

In applications, there often arises the following system of the special form:

$$\begin{cases} u'(t) = -A_0 u(t) + Bv(t), & u(0) = x, \\ v'(t) = Cu(t) - A_1 v(t), & v(0) = y, \end{cases}$$

on the space $\mathbf{H} = H_0 \times H_1$ with linear operators

$$A_0 : D(A_0) \subseteq H_0 \to H_0, \quad A_1 : D(A_1) \subseteq H_1 \to H_1,$$

$$B : D(B) \subseteq H_1 \to H_0, \quad C : D(C) \subseteq H_0 \to H_1.$$

The corresponding matrix operator $\mathbf{A}$ is defined as

$$\mathbf{A} = \begin{pmatrix} -A_0 & B \\ C & -A_1 \end{pmatrix}$$

H

$$D(\mathbf{A}) = \Big(D(A_0) \cap D(C)\Big) \times \Big(D(A_1) \cap D(B)\Big).$$

Theorem 3.2.2 ([193]). *Let* $\exp(t, -A_0)$ *and* $\exp(t, -A_1)$ *be contractive* C_0*-semigroups on* H_0 *and* H_1*, respectively, and let* B *and* C *be closed, and, moreover* $\overline{D(A_0) \cap D(C)} = H_0$ *and* $\overline{D(A_1) \cap D(B)} = H_1$*. Also, let* $\mathrm{Re}\{\langle A_0 x, x\rangle + \langle A_1 y, y\rangle - \langle By, x\rangle - \langle Cx, y\rangle\} \geq 0$ *for any* $x \in D(A_0) \cap D(C)$ *and* $y \in D(A_1) \cap D(B)$*.*

Then the following conditions are equivalent:

(i) $\mathbf{A}$ *generates a contractive* C_0*-semigroup on* $\mathbf{H}$*;*

(ii) for any $\lambda > 0$*, we have*

$$(\lambda I + A_0 - B(\lambda I + A_1)^{-1}C)^{-1} \in B(H_0), \quad (\lambda I + A_1 - C(\lambda I + A_0)^{-1}B)^{-1} \in B(H_1);$$

(iii) assertions (ii) hold for a certain $\lambda > 0$*;*

(iv) for any $\lambda > 0$*, the operators*

$-(A_0 - B(\lambda I + A_1)^{-1}C)$ *and* $-(A_1 - C(\lambda I + A_0)^{-1}B)$ *generare contractive* C_0*-semigroups on* H_0 *and* H_1*, respectively;*

(v) assertions (iv) hold for a certain $\lambda > 0$*.*

Also, in [193], the conditions under which the operator $\mathbf{A}$ generates an exponentially stable, differentiable, and analytic C_0-semigroup belonging to the Gevrey class with $\delta > 0$ were obtained.

Chapter IV
Spectral Properties of C_0-Cosine Operator-Valued Functions

In the same way as for C_0-semigroup, necessary and sufficient conditions for A to generate a C_0-cosine operator-valued function are formulated in terms of conditions on the location of the spectrum and estimates for the resolvent; see [15]. For a narrow class of C_0-cosine operator-valued functions on a Hilbert space, these conditions essentially based on the location of the spectrum; see [208].

4.1 Location of the Spectrum

Proposition 4.1.1 ([217]). *Let a* C_0*-cosine operator-valued function* $C(\cdot, A)$ *be given. Then*

$$(i) \quad \cosh\left(t\sqrt{\sigma(A)}\right) \subseteq \sigma\Big(C(t, A)\Big), \quad t \in \mathbb{R};$$

$$(ii) \quad \cosh\left(t\sqrt{P\sigma(A)}\right) = P\sigma\Big(C(t,A)\Big), \quad t \in \mathbb{R};$$

$$(iii) \quad \cosh\left(t\sqrt{R\sigma(A)}\right) \subseteq R\sigma(C(t,A)), \quad t \in \mathbb{R}.$$

Proposition 4.1.2 ([3, 218]). *If $\mu \in R\sigma(C(t,A))$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ is the set of roots of the equation $\mu = e^{\lambda_n t}$, then $\lambda_{n_0} \in R\sigma(A)$ for a certain $n_0 \in \mathbb{N}$, and $\lambda_n^2 \notin P\sigma(A)$ does not hold for any $n \in \mathbb{N}$ $\mu \in P\sigma(C(t,A)^*)$.*

Proposition 4.1.3 ([3, 217]). *If $\mu \in C\sigma(C(t,A))$ and λ_n are from Proposition 4.1.2, then $\lambda_n^2 \in C\sigma(A) \cup \rho(A)$. The case where $\lambda_n^2 \in \rho(A)$ for all $n \in \mathbb{N}$ is possible.*

Proposition 4.1.4 ([136, 179]). *If $E = H$ is Hilbert and $A \in \mathbf{C}(M,0)$ or $C(\cdot,A)$ is a family of normal operators, then*

$$\sigma(C(t,A)) = \cosh\left(t\sqrt{\sigma(A)}\right), \quad t \in \mathbb{R}.$$

Proposition 4.1.5 ([101]). *Let a C_0-cosine operator-valued function $C(\cdot,A)$ satisfy Condition (F), and let $E = H$ be Hilbert. Then $\mu \in \rho(C(t,A))$ iff $\{z^2 : \cosh(zt) = \mu\} \subseteq \rho(A)$ amd $\sup\{\|zR(z^2,A)\| : \cosh(zt) = \mu\} < \infty$.*

Proposition 4.1.6 ([204]). *Let $A \in \mathbf{C}(M,0)$. Then*
(i) $\sigma(A) \subset \mathbb{R}_-$;
(ii) if $E \neq \{0\}$, then $\sigma(A) \neq \varnothing$;
(iii) the spectrum $\sigma(A)$ is bounded iff $A \in B(E)$.

A Banach space E is saif to be *hereditarily indecomposable* (in brief, an H.I. space) if whenever X_1 and X_2 are closed infinite-dimensional subspaces of E and $\delta > 0$, then there exist unit vectors $x_1 \in X_1, x_2 \in X_2$ such that $\|x_1 - x_2\| < \delta$. In other words, this property can be reformulated as follows [144]: for any two infinite-dimensional subspaces $X_1, X_2 \subset E$ such that $X_1 \cap X_2 = \{0\}$, the subspace $X_1 + X_2$ is not closed.

Proposition 4.1.7 ([242]). *Let E be an H.I. space, and let $A \in C(M,\omega)$. Then $\sigma(A)$ is either a finite set (possibly empty) in $\mathbb{C}$ or consists of a sequence $\{\mu_n\}_{n=1}^{\infty}$ that either converges to a certain some point of $\mathbb{C}$ or satisfies $\lim_{n\to\infty} \operatorname{Re}\mu_n = -\infty$.*

Proposition 4.1.8 ([242]). *Let E be an H.I. space, and let $\mathbf{C}(\cdot,A)$ be a non-quasi-analytic C_0-cosine operator family. Then $\sigma(A) \cap \mathbb{C} \neq \emptyset$.*

Proposition 4.1.9. *Let $A \in \mathbf{GR}(M, \omega)$. Then*

(i) the spectrum of A lies in the strop $-\omega < \operatorname{Re} z < \omega$ (see Fig. 2);

Fig. 2.

(ii) the operator A^2 generates a C_0-cosine operator-valued function by the formula $C(t, A^2) = \frac{1}{2}\Big(\exp(tA) + \exp(-tA)\Big),\ t \in \mathbb{R}$.

Proposition 4.1.10 ([13]). *For $A \in \mathbf{C}(M, \omega)$ and the corresponding matrix differential operator $\mathbf{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ arising in reducing the Cauchy problem (3.1) to system (3.5), the following relation holds:*

$$\{\lambda^2 : \lambda \in \sigma(\mathbf{A})\} = \sigma(A).$$

Proposition 4.1.11 ([217]). *Let $A \in \mathbf{C}(M, \omega)$. Then the spectrum $\sigma(A)$ lies on a certain parabola whose branches are directed to the left; see Fig. 3.*

Fig. 3

Proposition 4.1.12 ([3]). *There exist a C_0-cosine operator-valued function $C(\cdot, A)$ and a Banach space E such that the sets $r_1 := \{t : 0 \in \rho(C(t, A))\}$, $r_2 := \{t : 0 \in P\sigma(C(t, A))\}$, $r_3 := \{t : 0 \in C\sigma(C(t, A))\}$, are dense in $\mathbb{R}$, and, moreover, $\mathbb{R} = r_1 \cup r_2 \cup r_3$.*

Proposition 4.1.13 ([82]). *For the C_0-cosine operator-valued function $C(t, A) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} A^k$, $A \in \mathbf{B}$, given on a Banach algebra $\mathbf{B}$ with unity, we have*

(i) $0 \leq \omega_c(A) < \infty$ *;*

(ii) $\lambda R(\lambda^2, A) = \int_0^{\infty} e^{-\lambda t} C(t, A)\, dt \quad for \quad Re\lambda > \omega^c(A)$;

(iii) $R(\lambda^2, A) = \int_0^{\infty} e^{-\lambda t} S(t, A)\, dt \quad for \quad Re\lambda > \omega^c(A)$;

(iv) $C(t, A) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} \lambda (\lambda^2 I - A)^{-1}\, d\lambda, \quad t \in \mathbb{R}$;

(v) $S(t, A) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda t} (\lambda^2 I - A)^{-1}\, d\lambda, \quad t \in \mathbb{R}$,

where γ is a certain contour enclosing the spectrum of the operator $A \in \mathbf{B}$.

Proposition 4.1.14 ([82]). *Under the conditions of Proposition 4.1.13, we have*

$$\omega_c(A)^2 = \sup_{\lambda \in \sigma(A)} (|\lambda| + \operatorname{Re} \lambda)/2.$$

Theorem 4.1.1 ([102]). *Let $A \in \mathbf{C}(M, \omega)$. The following conditions are equivalent:*

(i) $1 \in \rho(C(2\pi, A))$;

(ii) $-\mathbb{N}_0^2 \subseteq \rho(A)$ *and the sequences*

$$R_N = \frac{1}{N} \Sigma_{n=0}^{N-1} \Sigma_{k=-n}^{n} (-k^2 I - A)^{-1}$$

and

$$S_N = \frac{1}{N} \Sigma_{n=0}^{N-1} \Sigma_{k=-n}^{n} A(-k^2 I - A)^{-1}$$

are bounded in $B(E)$;

(iii) $-\mathbb{N}_0^2 \subseteq \rho(A)$ *and there exist the limits*

$$Rx := s\text{-}\lim_{N \to \infty} R_N x \quad and \quad Sx := s\text{-}\lim_{N \to \infty} S_N x \tag{4.1}$$

for all $x \in E$.

Theorem 4.1.2 ([102]). *Let $A \in \mathbf{C}(M, \omega)$. In a Hilbert space $E = H$, the following condition are equivalent:*

(i) $1 \in \rho(C(2\pi; A))$ *;*

(ii) $-\mathbb{N}_0^2 \subseteq \rho(A) \quad and \quad \sup_{k \in \mathbb{Z}} \|k(-k^2 I - A)^{-1}\| < \infty$.

Chapter V
Uniformly Bounded C_0-Cosine Operator-Valued Functions

In this chapter, we collect assertins that in one way or another, are related to the boundedness of cosine operator-valued functions, although we consider cosine operator-valued functions of polynomial and sometimes exponential growth. The matter is that the asymptotic behavior of resolving families for second-order equations differs from that of operator semigroups, and the representation $C(t,A) = \frac{1}{2}\Big(\exp(t\sqrt{A}) + \exp(-t\sqrt{A})\Big)$ does not always hold.

5.1 Behavior of C_0-Cosine Operator-Valued Functions at Infinity

As was already noted, the boundedness of a C_0-cosine operator-valued function is a property that is not obtained by a shift of the generator $A_b = A + bI$.

Proposition 5.1.1 ([141]). *There exist operators $A \in \mathbf{C}(M,\omega)$, such that foe ant number $b \in \mathbb{R}$, the operator $A + b \cdot I$ does not generate a bounded C_0-cosine operator-valued function.*

With the cosine equation (i) (see p. 26) one assocoartes the hyperbolic cosine $ch(t)$ of exponential growth, as well as the bounded ordinary function $cos(t)$. In the general case, for a C_0-cosine operator-valued function, a polynomial growth in t is also possible. So, for example, we have the following.

Example 5.1.1 ([205]). Let $E = \mathbb{R}^2$. Then $C(t) = \begin{pmatrix} 1 & 0 \\ \frac{t^2}{2} & 1 \end{pmatrix}, t \in \mathbb{R}$, – is a C_0-cosine operator-valued function on E (i.e., (i) in p. 26 holds). In the Euclidean norm is given on E, then

$$\|C(t)\| = \sqrt{1 + \frac{t^2}{2} + \frac{t^4}{4}}, \quad t \in \mathbb{R}.$$

Under the conditions of the previous example, for any $\omega > 0$, there exists $M_\omega \geq 1$ such that $\|C(t)\| \leq M_\omega \cosh(\omega t)$, $t \in \mathbb{R}$, but $C(\cdot)$ is not bounded on $\mathbb{R}$.

Note by the way that Example 5.1.1 describes exactly the case where there arises the problem on the representation of $C(\cdot)$ as the half-sum of two exponential functions [105].

Example 5.1.2 ([115]). Let A be the operator on the space $C^1([a,b])$ defined by the formula

$$(Af)(s) := sf(s), \quad s \in [a,b].$$

Then A is a generator of a C_0-family of a cosine operator-valued function $(C(t,A)f)(s) = h(s,t)f(s), \quad s \in [a,b],\ t \in \mathbb{R}$, where $h(s,t) = \cos(t\sqrt{-s})$, $s \in [a,b],\ t \in \mathbb{R}$. The norm of this C_0-cosine operator-valued function is equal to

$$\|C(t,A)\| = \max\left\{ \sup_{s\in[a,b]} |h(s,t)|,\ \sup_{s\in[a,b]} \left|\frac{d}{ds}h(s,t)\right| \right\}$$

and admits the estimate

$$\|C(t_n,A)\| \geq c(1+|t_n|)$$

with certain $\{t_n\}$, $\lim_{n\to\infty} t_n = \infty$, and a constant $c > 0$.

Therefore, the spectrum of the operator A coincides with the closed interval $[a,b]$, and for any $a, b < 0$, the norm $\|C(t,A)\|$ is not bounded on $\mathbb{R}$.

Example 5.1.3 ([115]). Let A be the operator on the space $L^1(\mathbb{R})$ defined by the formula

$$(Af)(s) := \left(\frac{d}{ds}\right)^2 f(s), \quad s \in \mathbb{R}.$$

Then A is a generating operator of the C_0-cosine operator-valued function $(C(t,A)f)(s) = \frac{1}{2}\Big(f(t+s)+f(t-s)\Big), \quad s,t \in \mathbb{R}$. Moreover, the C_0-semigroup generated by A admits the estimate

$$\|\exp(zA)\| \leq \left(\frac{|z|}{Re(z)}\right)^{\frac{1}{2}} \quad \text{for any } z \text{ with } \operatorname{Re} z \geq 0.$$

Example 5.1.4 ([115]). Let A be the operator on the space $L^p(\mathbb{R}^d)$ defined by the formula

$$(Af)(s) := \Delta f(s), \quad s \in \mathbb{R}^d, \quad d \in \mathbb{N}.$$

Then A is a generator of a C_0-cosine operator-valued function $C(\cdot,A)$ only in the case $p = 2$ in general. Moreover, the C_0-semigroup generated by the operator A admits the estimate

$$\|\exp(zA)\| \leq \left(\frac{|z|}{Re(z)}\right)^{d|\frac{1}{p}-\frac{1}{2}|} \quad \text{for any } z \quad \text{with} \quad \operatorname{Re} z \geq 0.$$

Theorem 5.1.1 ([115]). *the following implication of conditions holds: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).*

(i) a function $C(\cdot, A)x$ is of exponential type less than or equal to ω for all $x \in E$;

(ii) $\{z^2 : Rez > \omega\} \subseteq \rho(A)$, and for any $\gamma > \omega$, there exists a constant $M = M(\gamma)$ such that

$$\|z(z^2 I - A)^{-1}\| \leq M \quad \textit{whenever } Rez > \gamma;$$

(iii) the function $C(\cdot, A)x$ is of exponential type not exceeding ω for all $x \in D(A)$.

Theorem 5.1.2 ([115]). *The following implication of conditions holds: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).*

(i) a C_0- cosine operator-valued function $C(\cdot, A)$ is bounded;

(ii) there exists a constant M_1 such that

$$\|\exp(zA)\| \leq M_1 \left(\frac{|z|}{Re(z)}\right)^{\frac{1}{2}}, \qquad \textit{for any } \ z \textit{ with } \operatorname{Re} z \geq 0;$$

(iii) there exists a constant M_2 such that

$$\|C(t, A)x\| \leq M_2(\|x\| + t^2\|Ax\|), \quad t \in \mathbb{R}, \quad x \in D(A).$$

An attempt to give necessary and sufficient conditions for the uniform boundedness of C_0-cosine operator-valued functions was undertaken in [104], but, as K. Bojadzhiev showed, the proof contains inaccuracies.

5.2 Uniformly Bounded C_0-Cosine Operator-Valued Functions

For all $x \in E$ and $a \in \overline{\mathbb{R}}_+$, let us define the operator

$$F_a x := \frac{2}{\pi} \int_0^\infty \left(\frac{sin(at)}{t}\right)^2 C(2t, A)x dt,$$

which, obviously, is bounded for $A \in \mathbf{C}(M, 0)$.

Proposition 5.2.1 ([239]). *Let $0 \leq a \leq b$. Then*

$$F_a F_b x = F_b F_a x = 2 \int_0^a F_u x du + (b - a) F_a x, \quad x \in E.$$

Proposition 5.2.2 ([239]). *For a certain $0 \leq a \leq b$, let the following relation hold: $2\int\limits_a^b F_t x dt = (b-a)(F_a x + F_b x)$ for any $x \in E$. Then the open interval $(-b^2, -a^2) \subseteq \rho(A)$.*

Proposition 5.2.3 ([239]). *For the operator F_a, the following relations holds for $a \in \mathbb{R}_+$:*

$$F_a^k = (k-1)k \int\limits_0^a (a-t)^{k-2} F_t dt, \quad k = 2, 3, ...,$$

$$\exp(itF_a) = I + itF_a - t^2 \int\limits_0^a e^{it(a-s)} F_s \, ds.$$

Let a C_0-cosine operator-valued function $C(\cdot, A)$ be such that the operator (5.1)

$$E_a x := s\text{-}\lim_{\alpha \to 0+} E_{a,\alpha} x := s\text{-}\lim_{\alpha \to 0+} \int\limits_{\alpha+i0}^{\alpha+ia} \left(\lambda(\lambda^2 I - A)^{-1} + \overline{\lambda} R(\overline{\lambda}^2 I - A)^{-1} \right) x d\lambda$$

(where $\lambda = \alpha + i\tau$) is linear and continuous for all $a \in \overline{\mathbb{R}}_+$.

Proposition 5.2.4 ([239]). *There are examples of uniformly bounded C_0-cosine operator-valued functions for which the family $\{E_a\}$ from (5.1) is not defined.*

Proposition 5.2.5 ([239]). *For a C_0-cosine operator-valued function $C(\cdot, A)$, let family (5.1) be defined, and let $E_a = E_b$ for certain $0 < a < b$. Then*

$$(-b^2, -a^2) \cap \Big(R\sigma(A) \cup P\sigma(A) \Big) = \varnothing.$$

Proposition 5.2.6 ([239]). *Let the function $E_{a,\alpha} x$ be bounded for $a \in [0, \overline{a}]$ and $\alpha \in [0, \overline{\alpha}]$ with any $\overline{a}, \overline{\alpha} \geq 0, \overline{\alpha} \neq 0$. Then for all $x \in E$ and $a \in [0, \overline{a}]$, there exists $E_a x$, the operator E_a is bounded, and for all $0 \leq a \leq b$, the relation $E_a E_b = E_b E_a = \pi i E_a$ holds; moreover, for almost all $a \in \overline{\mathbb{R}}_+$, we have the relation $E_a x = \frac{2}{\pi} \int\limits_0^\infty \frac{sin(at)}{t} C(t, A) x dt$ in the case where the integral converges.*

Further, for $0 \leq a \leq b$ denote $\Delta := (-b^2, -a^2)$ and $E_\Delta := E_b - E_a$.

Proposition 5.2.7 ([239]). *Under conditions of Proposition* 5.2.6*, for any two intervals* Δ_1 *and* Δ_2*, we have*

$$E_{\Delta_1}E_{\Delta_2} = E_{\Delta_2}E_{\Delta_1} = E_{\Delta_1\cap\Delta_2}.$$

Proposition 5.2.8 ([239]). *Let* $\|E_{\psi,\varphi}x\| \leq const$ *for any* $\psi \in \overline{\mathbb{R}}_+$ *and* $\varphi \in \mathbb{R}_+$ *, and. Moreover, let* $E_a = E_b$ *for certain* $0 \leq a \leq b$ *. Then* $(-b^2, -a^2) \subseteq \rho(A)$*.*

Proposition 5.2.9 ([239]). *Under the conditions of Proposition 5.2.6, let the function* $E_a x$ *be continuous at the point* $a_0 \in \overline{\mathbb{R}}_+$ *for any* $x \in E$*. Then* $-a_0^2 \notin R\sigma(A)$ *.*

Proposition 5.2.10 ([239]). *Let the space* E *be reflexive and strongly convex with the Gateaux-differentiable norm.* $A \in \mathbf{C}(M, 0)$*, and let the operator*$C(t, A)$ *have a real spectrum for any* $t \in R$*. Then* $R\sigma(A) = \varnothing$ *.*

Proposition 5.2.11 ([12]). *Let* $A \in \mathbf{C}(1, 0)$*, and let* $\mathbf{A}$ *be the operator from Theorem 3.2.1. Then for* $t \in [\ln 2, \infty)$*,*

$$\|C'(t, A)\|_{B(E^1,E)} \leq t \cdot \ln 2,\ \|S(t, A)\|_{B(E,E^1)} \leq t + 1,$$

and the resolvent $(\lambda I - \mathbf{A})^{-1}$ *satisfies the estimate*

$$\|R(\lambda, \mathbf{A})\| \leq (\operatorname{Re}\lambda - \ln 2)^{-1} \quad \textit{for} \quad \operatorname{Re}\lambda > \ln 2.$$

Proposition 5.2.12 ([4, 5]). *Let* $0 \notin \sigma(A)$*, and let* $A \in \mathbf{C}(M, 0)$*. Then*

$$\sup_{t\in\mathbb{R}} \|S(t, A)\| \leq \frac{\pi}{2} dist\left(0, \sqrt{\sigma(A)}\right) \sup_{t\in\mathbb{R}} \|C(t, A)\|.$$

Proposition 5.2.13 ([4, 5]). *For* $A \in \mathbf{C}(M, 0)$*, the set* $\{x \in E : \sup_{t\in\overline{\mathbb{R}}_+} \|S(t, A)x\| < \infty\}$ *is dense in* E *iff one of the following conditions holds:*

(i) $s\text{-}\lim_{n\to\infty} \varepsilon_n R(\varepsilon_n, A)x = 0$ *for any* $x \in E$ *and a certain sequence* $\varepsilon_n \in \mathbb{R}_+$ *such that* $\lim_{n\to\infty} \varepsilon_n = 0$ *;*

(ii) the set $\mathbf{R}(A)$ *is dense in* E *;*

(iii) $\mathbf{N}(A^*) = \{0\}$*.*

Proposition 5.2.14 ([104]). *An operator* $A \in \mathbf{C}(M, 0)$ *satisfies Condition (F) iff the following condition holds for each closed interval* $[a, b]$ *:*

$$\sup\left\{\|\exp(tG|_{D(G,\mu)})\| \ : \mu \in \mathbb{N}, t \in [a, b]\right\} < \infty.$$

Proposition 5.2.15 ([175]). *Let $A \in \mathbf{C}(M,0)$. Then the operator iA generates a an α- times integrated group for $\alpha > \frac{1}{2}$.*

Proposition 5.2.16 ([136]). *Let $A \in \mathbf{C}(M,0)$, and let $E = H$. Then there exist a self-adjoint operator Q and a constant $M > 0$ such that $(\sqrt{3}(2M+1))^{-1}I \leq Q \leq MI$ and the operator $QC(t,A)Q^{-1}$ is self-adjoint for each $t \in \mathbb{R}$. Moreover, $C(t,A) = Q^{-1}cos(tL)Q$ and $L^* = L \leq 0$, where $L := QAQ^{-1}$.*

Proposition 5.2.17 ([143]). *For $A \in \mathbf{C}(M,0)$, $x \in D(A)$, the following inequality holds:*

$$\sup_{t\in\overline{\mathbb{R}}_+} \|S(t,A)Ax\|^2 \leq 4 \sup_{t\in\overline{\mathbb{R}}_+} \|C(t,A)Ax\| \cdot \sup_{t\in\overline{\mathbb{R}}_+} \|C(t,A)x\|.$$

Proposition 5.2.18 ([204]). *Let $A \in \mathbf{C}(1,0)$, and let*

$$C_m(t) := \sum_{j=0}^{m} C_{2m}^{2j} \left(\frac{t}{2m}\right)^{2j} A^j \left(\frac{2m}{t}\right)^{4m} \left((\frac{2m}{t})^2 I - A\right)^{-2m},$$

where C_{2m}^{2j} are binomial coefficients.

Then $\lim_{m\to\infty} C_m(t)x = C(t,A)x$ for all $x \in E$ and $\|\Big(C_m(t) - C(t,A)\Big)x\| \leq t^2\|Ax\|/\sqrt{m}$ for all $x \in D(A)$, $m \geq 2$.

Proposition 5.2.19 ([217]). *The functions $C(\cdot,A)x$ and $S(\cdot,A)x$ are uniformly bounded for any $x \in E$ iff there exists a constant $M \geq 1$ such that for $\operatorname{Re} z \geq \omega$,*

$$\left\|\frac{d^k}{dz^k}(z^2I - A)^{-1}\right\|, \quad \left\|z\frac{d^k}{dz^k}R(z^2I - A)^{-1} + k\frac{d^{k-1}}{dz^{k-1}}(z^2I - A)^{-1}\right\| \leq \frac{Mk!}{|z|^{k+1}}, \quad k \in \mathbb{N}.$$

Proposition 5.2.20 ([175]). *Let an operator A generate a C_0-cosine operator-valued function such that for the corresponding C_0-sine operator-valued function $S(\cdot,A)$ the estimate $\|S(t,A)\| \leq Mt$, $t \in \overline{\mathbb{R}_+}$ holds. Then the operator iA generates an α-times integrated semigroup for $\alpha > \frac{3}{2}$.*

5.3 Asymptotics of the Functions $F(\cdot)$ and $G(\cdot)$

In this section, we study the asymptotic behavior of the C_0-families $F(t)$ and $G(t)$ as $t \to \infty$. Let us consider the case under the assumption that there

exist a real λ_0 and a nonzero bounded operator $P \in B(E)$ such that

$$\lim_{t\to\infty} 2e^{-\lambda_0 t}\, C(t,A)x = Px \qquad x \in E. \tag{5.2}$$

Clearly, in this case, there exists a constant $M_1 \geq 1$ such that

$$\|C(t,A)\| \leq M_1 e^{\lambda_0 t} \quad \text{for} \quad t \in \overline{\mathbb{R}}_+. \tag{5.3}$$

By the identity

$$2e^{-\lambda_0 2t}(C(2t,A)+I) = 2e^{-\lambda_0 t}C(t,A)2e^{-\lambda_0 t}C(t,A), \tag{5.4}$$

in case (5.2), the number λ_0 cannot be negative, since then $e^{-\lambda_0 2t} \to \infty$ as $t \to \infty$, and there is no convergence.

In the case $\lambda_0 = 0$, we have from (5.4) that $P + 2I = P^2$. On the other hand, setting $t \to \infty$ and $s \to \infty$ in the cosine equation (see (i) in p. 26), we have $2P = P^2$. Therefore, $C(t,A) \to P/2 = I$ as $t \to \infty$. Setting $t \to \infty$ om relation (i) in p. 26, we obtain $C(s,A) = I$ for all $s \in \mathbb{R}$.

In connection with these simple arguments, we note that in [78], the assumption on the convergence of $C(t,A)$ to P as $t \to \infty$, which has no meaning, was made.

We note by the way that in the case $C(t,A) \to P$ as $t \to \infty$, it follows from (iv) of Proposition 2.4.1 that $F(t)x = 2^{-1}t^2\lambda^3\hat{F}(\lambda)x$ (for any $\lambda > \omega$), which never converges as $t \to \infty$ if $x \notin \mathbf{N}(\hat{F}(\lambda))$. The same situation is true for $G(\cdot)$.

Therefore, the case $\lambda_0 = 0$ is not interesting, and in what follows, we will assume that $\lambda_0 > 0$. It is clear from (5.4) that the operator P is a projection.

It is known that a generator of a C_0-cosine operator-valued function $C(\cdot,A)$ also generates a C_0-semigroup $\exp(\cdot A)$ defined by formula (2.3.8).

As will be shown in the following theorem, the convergence of $2e^{-\lambda_0 t}\, C(t,A)$ to P as $t \to \infty$ implies the convergence of $e^{-\lambda_0^2 t}\exp(tA)$ to P in the same topology.

Theorem 5.3.1. *Let cindition* (5.2) *hold with* $\lambda_0 \in \mathbb{R}_+$. *Then* P *is a projection with the range* $\mathbf{R}(P) = \mathbf{N}(\lambda_0^2 I - A)$ *and the kernel* $\mathbf{N}(P) = \overline{\mathbf{R}(\lambda_0^2 I - A)}$. *If, moreover,* P *is of finite rank and* $\|2e^{-\lambda_0 t}C(t,A) - P\| \to 0$ *as* $t \to \infty$, *then* $\lambda_0^2 > E\omega(A)$, $B\sigma(A) = \{\lambda_0^2\}$ *and* λ_0^2 *is a simple pole of the resolvent* $(\lambda I - A)^{-1}$.

Proof. We prove that the C_0-semigroup $e^{-\lambda_0^2 t}\exp(tA)$ strongly converges to P as $t \to \infty$. Then it follows from the ergodic theorem (see [73]) that P is a projection with $\mathbf{R}(P) = \mathbf{N}(\lambda_0^2 I - A)$ and $\mathbf{N}(P) = \overline{\mathbf{R}(\lambda_0^2 I - A)}$.

We have

$$e^{-\lambda_0^2 t}\exp(tA)x$$
$$= \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{s^2}{4t}}e^{\lambda_0 s}\left(2e^{-\lambda_0 s}C(s,A)-P\right)x\,ds + \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{s^2}{4t}}e^{\lambda_0 s}\,dsPx.$$

The first term $Q_1(t)$ in the right-hand side converges to zero, and the second term $Q_2(t)$ converges to Px as $t\to\infty$. Indeed,

$$\frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{s^2}{4t}}e^{\lambda_0 s}ds = \frac{1}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{(s-2\lambda_0 t)^2}{4t}}ds = \frac{1}{\sqrt{\pi}}\int_{-\lambda_0\sqrt{t}}^\infty e^{-u^2}du$$

converges to $\frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-u^2}du = 1$ as $t\to\infty$. Therefore, $Q_2(t)$ converges to Px as $t\to\infty$.

Let $\epsilon > 0$ be sufficiently small, and let $\tau\in\mathbb{R}_+$ be so large that $\|2e^{-\lambda_0 s}C(s,A)x - Px\|\le\epsilon$ for all $s\ge\tau$. Then the quantity

$$\|Q_1(t)\|$$
$$\le \frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}}\int_0^\tau e^{-\frac{s^2}{4t}}e^{\lambda_0 s}\|(2e^{-\lambda_0 s}C(s,A)-P)x\|ds + \epsilon\frac{e^{-\lambda_0^2 t}}{2\sqrt{\pi t}}\int_0^\infty e^{-\frac{s^2}{4t}}e^{\lambda_0 s}ds\|x\|,$$

and hence is bounded by the constant 2ϵ as $t\to\infty$. That is, $Q_1(t)\to 0$ as $t\to\infty$.

If $\|2e^{-\lambda_0 t}C(t,A)-P\|\to 0$ as $t\to\infty$, then in a similar way, we prove that $\|e^{-\lambda_0^2 t}\exp(tA)-P\|\to 0$ as $t\to\infty$. When P is of finite rank, the semigroup $\exp(\cdot A)$ attains the limit with the growth exponent λ_0^2. It follows from the theorem in [285] that $\lambda_0^2 > E\omega(A)$, $B\sigma(A)=\{\lambda_0^2\}$, and λ_0^2 is a simple pole of the resolvent $(\lambda I - A)^{-1}$.

We need the following assertion.

Lemma 5.3.1 (see [15], 7.3.1). *If a strongly continuous function $f(\cdot):\overline{\mathbb{R}}_+\to E$ is such that $\lim_{t\to\infty}f(t)=\varphi$, $\varphi\in E$, then for any λ with $Re\lambda>0$, we have*

$$e^{-\lambda t}\int_0^t e^{\lambda s}f(s)\,ds \to \varphi/\lambda \quad \textit{ffor} \quad t\to\infty. \tag{5.5}$$

Proposition 5.3.1. *let a C_0-cosine operator-valued function $C(\cdot,A)$ satisfy (5.2) with $\lambda_0>0$. Then $2e^{-\lambda_0 t}S(t,A)\to P/\lambda_0$ and $2e^{-\lambda_0 t}\int_0^t S(s,A)ds\to P/\lambda_0^2$ strongly as $t\to\infty$.*

Proof. We have the following relations:

$$2e^{-\lambda_0 t}S(t,A) = e^{-\lambda_0 t}\int_0^t e^{\lambda_0 s}2e^{-\lambda_0 s}C(s,A)\,ds$$

$$2e^{-\lambda_0 t}\int_0^t S(s,A)ds = e^{-\lambda_0 t}\int_0^t e^{\lambda_0 s}e^{-\lambda_0 s}\int_0^s e^{\lambda_0 \eta}2e^{-\lambda_0 \eta}C(\eta,A)\ d\eta\ ds.$$

Now the assertion follows from Lemma 5.3.1. Now

Theorem 5.3.2 ([233]). *Let a C_0-cosine operator-valued function $C(\cdot,A)$ satisfy condition* (5.2) *with $\lambda_0 \in \mathbb{R}_+$. Then for each $\lambda > \lambda_0$ and each $x \in E$, we have*

$$s\text{-}\lim_{t\to\infty} 2e^{-\lambda_0 t}F(t)x = \lambda(\lambda^2/\lambda_0^2 - 1)P\hat{F}(\lambda)x,$$

which is equal to $\dfrac{1}{\lambda_0^2}(A_s - A)x$ if $x \in D(A)$ and $\hat{F}(\lambda)x \in \mathbf{N}(\lambda_0^2 I - A)^{-1}$, simultaneously, and is equal to zero if $\hat{F}(\lambda)x \in \overline{\mathbf{R}(\lambda_0^2 I - A)^{-1}}$.

Proof. For $\lambda_0 \in \mathbb{R}_+$, let us write the relation

$$2e^{-\lambda_0 t}F(t)x = 2e^{-\lambda_0 t}\int_0^t S(s,A)\lambda^3\hat{F}(\lambda)xds - 2e^{-\lambda_0 t}(C(t,A) - I)\lambda\hat{F}(\lambda)x.$$

Using Propositions 5.3.1 and 2.4.1 (iv) and setting $t \to \infty$, we obtain the required result.

Analogously, using Propositions 5.3.1 and 2.4.1 (v), we obtain the following assertion.

Theorem 5.3.3 ([233]). *Let a C_0-cosine operator-valued function $C(\cdot,A)$ satisfy condition* (5.2) *with $\lambda_0 \in \mathbb{R}_+$. Then for each $\lambda > \lambda_0$ and each $x \in E$, we have*

$$\lim_{t\to\infty} 2e^{-\lambda_0 t}G(t)x = \hat{G}(\lambda)\lambda(\lambda^2/\lambda_0^2 - 1)Px,$$

which is equal to $\frac{1}{\lambda_0^2}(A_c - A)x$ if $x \in \mathbf{N}(\lambda_0^2 I - A)^{-1}$ and is equal to zero if $x \in \overline{\mathbf{R}(\lambda_0^2 I - A)^{-1}}$.

As was mentioned above, if $C(t,A)$ strongly converges as $t \to \infty$, then $C(\cdot,A) \equiv I$, and $F(\cdot)$ and $G(\cdot)$ grow. In what follows, we will consider the behavior of $F(\cdot)$ and $G(\cdot)$ under the assumption that

$$\sup_{t>0}\|t^{-2}\int_0^t\int_0^s C(u,A)\ du\ ds\| < \infty \quad \text{and} \quad t^{-2}\ C(t,A) \to 0 \tag{5.6}$$

strongly as $t \to \infty$. We need the following assertion.

Proposition 5.3.2 ([252]). *Under assumption* (5.6)*, we have*

(i) *the mapping* $P : x \to \lim_{t\to\infty} 2t^{-2}\int_0^t\int_0^s C(u,A)x\,du\,ds$ *is a projection with* $\mathbf{R}(P) = \mathbf{N}(A)$, $\mathbf{N}(P) = \overline{\mathbf{R}(A)}$ *and* $D(P) = \mathbf{N}(A) \oplus \overline{\mathbf{R}(A)}$;

(ii) *there exists* $x := -\lim_{t\to\infty} 2t^{-2}\int_0^t\int_0^s\int_0^u\int_0^v C(\tau,A)y\ d\tau\ dv\ du\ ds$ *iff* $y \in A(D(A)\cap\overline{\mathbf{R}(A)})$ *(*$= \mathbf{R}(A)$ *in the case where* $C(\cdot,A)$ *is (C,2)-ergodic, i.e.,* $D(P) = E$*). Moreover, this element* x *is a unique solution of the equation* $Ax = y$ *in* $\overline{\mathbf{R}(A)}$*, i.e.,* $x = \tilde{A}^{-1}y$*, where* $\tilde{A} = A|_{\overline{\mathbf{R}(A)}}$.

Using Proposition 2.4.1 (iv) and the proposition mentioned above, we obtain

Theorem 5.3.4 ([233]). *Under assumption* (5.6)*, the following assertions hold:*

(i) *there exists the limit* $y = \lim_{t\to\infty} 2t^{-2}F(t)x$ *iff* $\hat{F}(\lambda)x \in \mathbf{N}(A)\oplus\overline{\mathbf{R}(A)}$ *for a certain (and all)* $\lambda > \omega$*. When the limit exists,* $y = \lambda^3 P\hat{F}(\lambda)x$ *and is independent of* λ;

(ii) *for* $\hat{F}(\lambda)x \in \mathbf{N}(A)\oplus\overline{\mathbf{R}(A)}$*,* $z = \lim_{t\to\infty} 2t^{-2}\int_0^t\int_0^s F(\tau)x\ d\tau ds$ *does exist iff* $\hat{F}(\lambda)x \in A(D(A)\cap\mathbf{R}(A))$ *for a certain (and all)* $\lambda > \omega$*. In this case,* $z = -\lambda(\lambda^2 I - A)\tilde{A}^{-1}\hat{F}(\lambda)x$*, which is independent of* λ.

Proof. We have from (iv) of Proposition 2.4.1that

$$\frac{2}{t^2}F(t)x = \frac{2}{t^2}(I - C(t,A))\lambda\hat{F}(\lambda)x + \frac{2}{t^2}\int_0^t\int_0^s C(\tau,A)\lambda^3\hat{F}(\lambda)x\,d\tau ds, \tag{5.7}$$

and

$$\frac{2}{t^2}\int_0^t\int_0^s F(\tau)xd\tau ds = \frac{2}{t^2}\int_0^t\int_0^s\int_0^u\int_0^v C(\tau,A)\lambda^3\hat{F}(\lambda)x\,d\tau dvduds \tag{5.8}$$
$$-\frac{2}{t^2}\int_0^t\int_0^s C(\tau,A)\lambda\hat{F}(\lambda)x\,d\tau ds + \lambda\hat{F}(\lambda)x.$$

Then assertions (i) and (ii) follow from (5.7) and (5.8), respectively, as a consequence of Proposition 5.3.2.

By Propositions 2.4.1 (v), 5.3.2, and 2.4.2 (ii), we have the following theorem.

Theorem 5.3.5. *Under assumption* (5.6)*, we have the following assertions:*

(i) *if*$x \in \mathbf{N}(A)\oplus\overline{\mathbf{R}(A)}$*, then* $s\text{-}\lim_{t\to\infty} 2t^{-2}G(t)x = A_c Px$;

(ii) *if* $x \in A(D(A)\cap\mathbf{R}(A))$*, then*
$s\text{-}\lim_{t\to\infty} 2t^{-2}\int_0^t\int_0^s G(\tau)x\ d\tau ds = -(A_c - A)\tilde{A}^{-1}x = x - A_c\tilde{A}^{-1}x$,
where $\tilde{A} = A|_{\overline{\mathbf{R}(A)}}$.

Chapter VI
Uniformly Continuous C_0-Cosine Operator-Valued Functions

As in the case of operator semigroups, the norm-continuity is a very restrictive requirement for cosine-operator-valued functions, since it implies the boundedness of the generating operator. Conditions for generation of a cosine operator-valued function for an infinitesimal operator A are more restrictive than those for generation of operator semigroups.

6.1 Norm-Continuity

It is very natural that the boundedness of A in the case of a cosine operator-valued function follows under more weaker additional assumptions than in the case of operator semigroups. So, for example, the condition $\|tA\exp(-tA)\| \leq C$ implies the boundedness of A for $C = 1/e$ (see [15]), and in the case of cosine operator-valued functions, for boundedness of A, the boundedness $\|AS(t, A)\| \leq const$ with any constant is sufficient.

Definition 6.1.1. A C_0-cosine operator-valued function $C(\cdot, A)$ is continuous in the uniform operator topology *(norm-continuous)* if the function $C(\cdot, A) : \mathbb{R} \to B(E)$ is continuous in the operator norm.

Proposition 6.1.1 ([267]). *Let a C_0-cosine operator-valued function $C(\cdot, A)$ be continuous in the uniform operator topology. Then $A \in B(E)$ and*

$$C(t, A) = \sum_{k=0}^{\infty} t^{2k} A^k/(2k)!, \quad t \in \mathbb{R}, \tag{6.1}$$

and, moreover, the series uniformly converges in t on each finite closed interval $[0, T]$.

Sometimes, in the literature, series (6.1) is written as $\cosh(t\sqrt{A})$ analogously to the scalar case.

We note that a C_0-sine operator-valued function $S(\cdot, A)$, is always uniformly continuous in $t \in \mathbb{R}$, as follows from its definition.

Proposition 6.1.2 ([266]). *Let $A \in B(E)$. Then series* (6.1) *is a C_0-cosine operator-valued function whose generating operator is A.*

Theorem 6.1.1 ([83]). *Each of the following conditions is equivalent to the norm-continuity of $C(\cdot, A)$:*

(i) $\lim_{t\to 0} \|C(t,A) - I\| = 0$ *;*

(ii) $\lim_{t\to 0} \|t^{-1}S(t,A) - I\| = 0$ *;*

(iii) the generating operator A is bounded;

(iv) $\mathbf{R}(C(t,A)) \subseteq E^1$ *for all $t \in (\alpha, \beta)$ with a certain $\alpha < \beta$;*

(v) the inclusion $\mathbf{R}(S(t,A)) \subseteq D(A)$ and the strong continuity of the function $t \to AS(t,A)$ hold for all $t \in (\alpha, \beta)$ with a certain $\alpha < \beta$.

Proposition 6.1.3 ([5]). *The generating operator A of a C_0-cosine operator-valued function with a non-quasi-analytic weight χ is bounded iff one of the following conditions hold:*

(i) for a certain $\varepsilon > 0$, we have $\sup_{0<t<\varepsilon} \|C(t,A) - I\| < 1$*;*

(ii) the C_0-cosine operator-valued function $C(\cdot, A)$ is the restriction to $\mathbb{R}$ of an entire operator-valued function $\tilde{C}(\cdot) : \mathbb{C} \to B(E)$ of exponential type (equal to $r(A)$).

Proposition 6.1.4 ([267]). *Let a C_0-cosine operator-valued function $C(\cdot, A)$ be twice strongly differentiable at zero. Then $A \in B(E)$.*

Proposition 6.1.5 ([204]). *Let $A \in \mathbf{C}(M, 0)$. Then $A \in B(E)$ iff the spectrum $\sigma(A)$ is bounded.*

Proposition 6.1.6. *Let a C_0-cosine operator-valued function $C(\cdot, A)$ be norm-continuous. Then*

(i) $\lim_{t\to 0} \|\frac{2}{t^2}\int_0^t S(s,A)ds - I\| = 0$;

(ii) for sufficiently small h, the operator $\int_0^h S(s,A)ds$ has a bounded inverse;

(iii) for sufficiently small h, we have the relation

$$A = (C(h,A) - I)\left(\int_0^h S(s,A)ds\right)^{-1}.$$

Definition 6.1.2. A *Grothendieck space* is a Banach space in which every w^*-convergent sequence in E^* is w-convergent.

Definition 6.1.3. We say that a Banach space E has the *Danford–Pettis property* if $\langle x_n, x_n^* \rangle \to 0$ whenever x_n weakly converges to zero in E and x_n^* weakly converges to zero in E^*.

Proposition 6.1.7 ([253]). *Any C_0-cosine operator-valued function $C(\cdot, A)$ given on a Grothendieck space with the Danford–Pettis property (for example, $E = L^\infty$ is such a space) is norm-continuous, i.e., $A \in B(E)$.*

Before formulating the next assertion, we recall [144] that if E is an H.I. space and $B \in B(E)$, then there exists a unique point $\lambda_B \in \sigma(B)$ such that the operator $B - \lambda_B I$ is strictly singular. Moreover, $B - \lambda_B I$ is a Riesz operator.

Proposition 6.1.8 ([242]). *Let E be an H.I. Banach space, and let $C(\cdot, A)$ satisfy condition (**??**). Then $A \in B(E)$, and there is a positive integer m such that $(A - \lambda_A I)^m$ is a compact operator.*

Proposition 6.1.9 ([5]). *If $B \in B(E)$ and a C_0-cosine operator-valued function $C(\cdot, B)$ is uniformly bounded in $t \in \mathbb{R}$, then the following Bernshtein inequality holds:*

$$\|B\| \leq r(B) \cdot \sup_{t \in \mathbb{R}} \|C(t, B)\|,$$

where $r(B)$ is the spectral radius of the operator B.

Proposition 6.1.10 ([25, 236]). *Let $B \in B(E)$. Then the functions $C(t, -B^2)x$ and $S(t, -B^2)y$ are uniformly bounded in $t \in \mathbb{R}$ for any $x, y \in E$ iff on E, there exists an equivalent norm $\|\cdot\|_*$ such that $\|exp(itB)\|_* \leq 1$ for $t \in \mathbb{R}$ (such operators B are said to be Hernitian-equivalent on E).*

Proposition 6.1.11 ([25, 236]). *An operator $B \in B(E)$ is Hermitian-equivalent on E iff there exists a constant $C > 0$ such that*

$$\|\sin(tB)\| \leq C, \quad t \in \mathbb{R}.$$

Proposition 6.1.12 ([82]). *In a Banach algebra $\mathbf{B}$ with unit, let a C_0-cosine operator-valued function $C(\cdot, A)$, $A \in \mathbf{B}$ be given. Then for $\nu^2 > \sup_{\lambda \in \sigma(A)}(|\lambda| + \operatorname{Re}\lambda)/2$, we have*

$$C(t, A) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} e^{\lambda t} \lambda R(\lambda^2, A) d\lambda, \quad t \in \overline{\mathbb{R}}_+,$$

$$S(t, A) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} e^{\lambda t} R(\lambda^2, A) d\lambda, \quad t \in \overline{\mathbb{R}}_+.$$

For the proof of the following two propositions, see Sec. **??**.

Proposition 6.1.13 ([96]). *If for a C_0-cosine operator-valued function $\mathbf{R}(S(t, A)) \subseteq D(A)$ for $t \in \mathbb{R}$ and $\|AS(t, A)\| \leq const$ for $t \in [a, b]$, $a < b$, then $A \in B(E)$.*

Proposition 6.1.14 ([96]). *If for a C_0-cosine operator-valued function $SV\Big(C(\cdot, A), t\Big) \leq const$ for a certain $t \in \mathbb{R}_+$, then $A \in B(E)$.*

6.2 Positivity of Perturbation Families

Definition 6.2.1. In the case where E is a Banach lattice with a positive cone E_+, we say that a function $\mathbf{L}(\cdot)$ is *positive* on E if for each $t \in \overline{\mathbb{R}_+}$, the operator $\mathbf{L}(t)$ is positive (we write $\mathbf{L}(t) \succeq 0$) in the sense that $\mathbf{L}(t)E_+ \subseteq E_+$.

In the case where E is a Hilbert space with inner product $(\cdot,\cdot)$, we say that $\mathbf{L}(\cdot)$ is positive (we write $\mathbf{L}(t) \geq 0$) if for each $t \in \overline{\mathbb{R}}_+$, the operator $\mathbf{L}(t)$ is positive in the sense that $(\mathbf{L}(t)x, x) \geq 0$ for all $x \in E$.

Proposition 6.2.1. *[195] A C_0-cosine operator-valued function $C(\cdot, A)$ dominates I, i.e., $C(\cdot, A) - I$ is positive in the sense of a Banach lattice or in the sense of a Hilbert space iff the generator A is bounded and positive.*

The following propositions are a reformulation of properties of a multiplicative perturbation C_0-family and an additive perturbation C_0-family.

Let $F_B^\mu(\cdot)$ $G_B^\mu(\cdot)$ be functions defined for $B \in B(E)$ by the formulas

$$F_B^\mu(t)x := (A-\mu)\int_0^t S(s,A)Bx\,ds, \quad x \in E,\ t \in \overline{\mathbb{R}}_+, \tag{6.2}$$

$$G_B^\mu(t)x := B(A-\mu)\int_0^t S(s,A)x\,ds, \quad x \in E,\ t \in \overline{\mathbb{R}}_+.$$

Then $F_B^\mu(\cdot)$ is a multiplicative perturbation C_0-family and $G_B^\mu(\cdot)$ is an additive perturbation C_0-family.

Proposition 6.2.2 ([233]). *Let E be a Banach lattice. Each multiplicative perturbation C_0-family $F_B^\mu(\cdot)$ for a C_0-cosine operator-valued function $C(\cdot, A)$ on E defined in (6.2) with $\mu \leq 0$ and $B \succeq 0$ is positive iff the operator A is positive. The same holds for an additive perturbation C_0-family.*

Proposition 6.2.3 ([233]). *Let E be a Hilbert space. Each multiplicative perturbation C_0-family $F_B^\mu(\cdot)$ for a C_0-cosine operator-valued function $C(\cdot, A)$ on E defined in (6.2) with $\mu \leq 0$ and $B \geq 0$ that commutes with $C(\cdot, A)$ is positive iff the operator A is positive. The same holds for an additive perturbation C_0-family.*

Chapter VII
Almost-Periodic C_0-Cosine Operator-Valued Functions

Let us recall in brief the definition of almost periodicity of operator-valued functions.

Definition 9.1 A function $f(\cdot) : \overline{\mathbb{R}}_+ \to E$ is said to be *almost-periodic* if for each $\epsilon > 0$, the set $J(f, \epsilon) = \{\tau > 0 : \|f(t+\tau) - f(t)\| \leq \epsilon$ for all $t \in \overline{\mathbb{R}}_+\}$ is relatively dense in $\overline{\mathbb{R}}_+$. That is, there exists $l \in \mathbb{R}_+$ such that each subinterval from $\overline{\mathbb{R}}_+$ of length l intersects $J(f, \epsilon)$. An operator-valued function $Q(\cdot) : \overline{\mathbb{R}}_+ \to B(E)$ is said to be *almost-periodic* if for each $x \in E$, the function $Q(\cdot)x$ is almost periodic.

7.1 Almost Periodicity of the Basic Families

Definition 7.1.1. A C_0-cosine operator-valued function or a C_0-sine operator-valued function are said to be *almost=periodic (a.-p)* or *uniformly a.-p* if for any $x \in E$, the functions $C(\cdot, A)x$ or $S(\cdot, A)x$ are a.-p (uniformly a.-p).

Proposition 7.1.1 ([99]). *If E is weakly sequentially complete, then a weakly a.-p. C_0-cosine operator-valued function is almost-periodic.*

Theorem 7.1.1 ([60]). *A C_0-cosine operator-valued function $C(\cdot, A)$ is almost-periodic iff the following three conditions hold:*

(i) the C_0-cosine operator-valued function $C(\cdot, A)$ is uniformly bounded;

(ii) the spectrum $\sigma(A) \subseteq \mathbb{R}_-$;

(iii) the set of eigenvectors of the generating operator A is total on the space E.

If, moreover, $\mu \in \sigma(A)$ is an isolated point of the spectrum, then μ is a simple pole of the resolvent $(\lambda I - A)^{-1}$ and $E = \overline{\mathbf{R}(\mu I - A)} \oplus \mathbf{N}(\mu I - A)$.

Theorem 7.1.2 ([60]). *The Cauchy problem* (3.1) *has an a.-p. generalized solution for any $u^0, u^1 \in E$ iff conditions $(i) = -(iii)$ of Theorem 7.1.1 hold and $0 \in \rho(A)$.*

Theorem 7.1.3 ([60]). *A C_0-cosine operator-valued function $C(\cdot, A)$ and a C_0-sine operator-valued function $S(\cdot, A)$ are uniformly a.-p. iff the following three conditions hold:*

(i) the C_0-cosine operator-valued function $C(\cdot, A)$ and the C_0-sine operator-valued function $S(\cdot, A)$ are uniformly bounded in $t \in \mathbb{R}$;

(ii) the set $\sigma(A)$ is a harmonic subset in $\mathbb{R}_-$ and $0 \in \rho(A)$;

(iii) the set of eigenvectors of the generating operator A is total on the space E.

Proposition 7.1.2 ([60]). *If a C_0-cosine operator-valued function $C(\cdot, A)$ is uniformly a.-p., then $\sigma(A)$ consists of simple poles of the resolvent $(\lambda I - A)^{-1}$. In this case, $\sigma(A) = P\sigma(A)$.*

Proposition 7.1.3 ([159]). *The following conditions are equivalent:*

(i) a C_0-cosine operator-valued function $C(\cdot, A)$ is periodic as an operator-valued function;

(ii) the C_0-cosine operator-valued function $C(\cdot, A)$ is strongly periodic;

(iii) the C_0-cosine operator-valued function $C(\cdot, A)$ is weakly periodic.

Theorem 7.1.4 ([138, 203]). *A uniformly bounded C_0-cosine operator-valued function $C(\cdot, A)$ is periodic with period 2π iff the following three conditions hold:*

(i) the spectrum $\sigma(A) \subseteq \{l : l = -k^2, k \in \mathbb{Z}\}$;

(ii) the spectrum $\sigma(A)$ consists of simple poles of the resolvent;

(iii) the set of eigenvectors of the generating operator A is total in the space E.

Under conditions $(i) - -(iii)$, the Riesz projections are given by the formulas

$$P(-k^2)x = \begin{cases} \frac{1}{\pi} \int\limits_0^{2\pi} \cos(ks) C(s, A) x ds & \text{for} \quad k \neq 0, \\ \frac{1}{2\pi} \int\limits_0^{2\pi} C(s, A) x ds & \text{for} \quad k = 0 \end{cases}$$

and, moreover, for $x \in D(A)$, we have the relation

$$C(t, A)x = \sum_{k=0}^{\infty} \cos(kt) P(-k^2)x, \tag{7.1}$$

where the series converges uniformly in $t \in \mathbb{R}$.

Proposition 7.1.4 ([138]). *In the case where $E = H$ and $C(\cdot, A) - 2$ is π-periodic, relation (7.1) holds for all $x \in E$ and the convergence of the series is uniform in $t \in \mathbb{R}$.*

Theorem 7.1.5 ([61]). *The function $C(t, A)u^0 + S(t, A)u^1$ is 2π-periodic for any $u^0, u^1 \in E$ iff conditions $(i) - -(iii)$ of Theorem 7.1.4 hold and $0 \in \rho(A)$.*

Proposition 7.1.5 ([138]). *A C_0-cosine operator-valued function $C(\cdot, A)$ is periodic with period T iff the function $F(z) := (1 - e^{-Tz})zR(z^2, A)$ can be analytically continued up to an entire function $\tilde{F}(z)$ such that the following estimate holds for $|z| > r$:*

$$\|\tilde{F}(z)\| \leq M e^{(q|z|^{2-\varepsilon})}, \qquad q, M, r, \varepsilon > 0. \tag{7.2}$$

Proposition 7.1.6 ([217]). *The uniformly well-posed Cauchy problem (3.1) has periodic solutions with period T iff $A \in \mathbf{C}(M, 0)$ and the function $F(z)/z$ can be analytically continued up to an entire function $Q(z)$ such that estimate (7.2) for $Q(z)$ holds for $|z| > r$.*

Proposition 7.1.7 ([136]). *Let a C_0-cosine operator-valued function $C(\cdot, A)$ be given on a Hilbert space H, and let $C(\cdot, A)$ be weakly a.-p. Then we have the relation $C(t, A) = Q^{-1}C(t, V)Q$, where V is a self-adjoint operator given by $V := \sum_{\lambda \geq 0} \lambda P(\lambda)$ and $P(\lambda)$ is a set of mutually orthogonal projections.*

Proposition 7.1.8 ([203]). *The periodicity of a C_0-cosine operator-valued function $C(\cdot, A)x$ for each $x \in D(A)$ implies the periodicity of $C(\cdot, A)$.*

Proposition 7.1.9 ([5]). *Let $\sqrt{\sigma(-A)} \cap \mathbb{R}_+$ be a no more than countable set. Then all solutions of problem (3.1) are almost-periodic iff the following conditions hold:*

(i) the C_0-cosine operator-valued function $C(\cdot, A)$ is uniformly bounded in $t \in \mathbb{R}$;

(ii) $0 \in \rho(A)$;

(iii) for each limit point $\lambda_0 \in \sqrt{\sigma(A)}$, there is a sequence $\varepsilon_n \in \mathbb{R}$ converging to zero as $n \to \infty$ such that $s\text{-}\lim_{n\to\infty} \varepsilon_n(\varepsilon_n + i\lambda_0)\Big((\varepsilon_n + i\lambda_0)\cdot I - A\Big)^{-1} x = 0$ for each $x \in E$.

Proposition 7.1.10 ([5]). *A C_0-cosine operator-valued function $C(\cdot, A)$ is a.-p. in the uniform operator topology iff it is uniformly bounded on $\mathbb{R}$ and $\sqrt{\sigma(A)}$ is a harmonic subset of $i\mathbb{R}$.*

Proposition 7.1.11 ([5]). *Let $A \in \mathbf{C}(M, \omega)$, and let $\sqrt{\sigma(-A)}$ have no limit points in $\mathbb{R}_+$. Then:*

(i) the linear span of eigenvectors and root vectors of A is dense E if there exists a function $\chi(t)$ such that

$$\|C(t, A)\| \leq \chi(t) \qquad \chi(t) \leq C(1 + |t|)^{\gamma} \quad \text{for} \quad t \in \mathbb{R}, \gamma \geq 0; \tag{7.3}$$

(ii) under the condition $\lim_{t\to\infty} \chi(t)/t = 0$, where $\chi(\cdot)$ is the function from (7.3), the C_0-cosine operator-valued function $C(\cdot, A)$ is periodic with period 1 iff $\sigma(A) \subseteq \{-(2\pi k)^2 : \quad k \in \mathbb{N}\}$.

In [158], asymptotic almost-periodic in the sense of Stepanov operator semigroups and cosine operator-valued functions were considered.

7.2 Almost-Periodicity of the Families $F(\cdot)$ and $G(\cdot)$

Proposition 7.2.1. *If a continuous function $f(\cdot) : \overline{\mathbb{R}}_+ \to E$ converges to a certain element $\varphi \in E$ as $t \to \infty$, then*

$$2t^{-2} \int_0^t s f(s)\, ds \ \to \varphi \ \text{as } t \to \infty. \tag{7.4}$$

Proof. It is clear that as in the case of Lemma 5.3.1 (see also [15], Lemma 7.3.1.), it suffices to consider the case $\varphi = 0$. We set $t = \tau + \zeta$ and write

$$\frac{2}{t^2}\int_0^t sf(s)\ ds = \frac{2}{(\tau+\zeta)^2}\int_0^\tau sf(s)\ ds + \frac{2}{(\tau+\zeta)^2}\int_\tau^{\tau+\zeta} sf(s)\ ds. \tag{7.5}$$

Since

$$\left\| \frac{2}{(\tau+\zeta)^2}\int_\tau^{\tau+\zeta} sf(s)ds \right\| \leq \sup_{t\geq\tau} \|f(t)\|$$

for all ζ and τ and $f(t) \to 0$ as $t \to \infty$, we can choose τ so large that the second term in (7.5) becomes less than a certain $\varepsilon > 0$. Then we can choose ζ so large that the first term in (7.5) also becomes less than ϵ. This proves (7.4) with $\varphi = 0$.

The next theorem yields necessary and sufficient conditions for each multiplicative perturbation C_0-family (or each additive perturbation C_0-family) to be almost-periodic.

Theorem 7.2.1 ([233]). *Each multiplicative perturbation C_0-family $F(\cdot)$ for $C(\cdot, A)$ is almost periodic iff $C(\cdot, A)$ is almost periodic and $0 \in \rho(A)$. The same assertion also holds for an additive perturbation C_0-family.*

Proof. Let $C(\cdot, A)$ be slmost-periodic. Then the condition $0 \in \rho(A)$ implies the almost-periodicity of the function $\int_0^t S(s, A)\, ds = \int_0^t S(s, A)AA^{-1}ds = (C(t, A) - I)A^{-1},\ t \in \mathbb{R}$. Therefore, Proposition 2.4.1 (iv) implies the almost periodicity of $F(\cdot)$.

Conversely, if each multiplicative perturbation C_0-family is almost-periodic, then two particular multiplicative perturbation C_0-families $C(t, A) - I$ and $\int_0^t S(s, A)ds$ are almost-periodic functions. If $x \in \mathbf{N}(A)$, then $x = C(s, A)x - \int_0^s S(u, A)Axdu = C(s, A)x$ for all $s \in \overline{\mathbb{R}}_+$ and $x = 2t^{-2}\int_0^t S(s, A)xds \to 0$ as $t \to \infty$, since an almost-periodic function is bounded. Therefore, A is injective. Then since an almost-periodic function is ergodic (see, e.g., [75], p. 21), the limit $\frac{1}{s}\int_0^s\int_0^u S(v, A)x\ dv\ du$ does exist as $s \to \infty$ for each $x \in E$. By Proposition 7.2.1, the limit

$$\frac{2}{t^2}\int_0^t s\frac{1}{s}\int_0^s\int_0^u\int_0^v C(\tau, A)xd\tau dvduds$$

exists as $t \to \infty$ for any $x \in E$. Since $C(\cdot, A)$ is uniformly bounded, we have from Proposition 5.3.2 (i.e., [234], Theorem 3.7) that $\mathbf{R}(A) = E$. Therefore, $0 \in \rho(A)$.

Remark 7.2.1 ([60]). The assumptions that $C(\cdot, A)$ is almost periodic and $0 \in \rho(A)$ are equivalent to the condition that each generalized solution of the Cauchy problem (3.1) is almost-periodic.

We can deduce the following theorem from Theorem 7.2.1.

Theorem 7.2.2. *Each multiplicative perturbation C_0-family $F(\cdot)\ C(\cdot, A)$ is periodic iff $C(\cdot, A)$ is periodic and $0 \in \rho(A)$. In this case, $F(\cdot)$ and $C(\cdot, A)$ have the same period. The same assertion holds for an additive perturbation C_0-family.*

Chapter VIII
Compactness in the Theory of C_0-Cosine Operator-Valued Functions

Compactness properties are widely used in various aspects of the theory of resolving families. We denote by $B_0(E)$ (or $B_0(E, F)$ in the case of distinct spaces) the set of compact operators acting on E.

8.1 Compact Basic Families

Definition 8.1.1. A C_0-cosine operator-valued function $C(\cdot, A)$ is said to be *compact* (we write $C(\cdot, A) \in B_0(E)$) if the operator $C(t, A) \in B_0(E)$ for any $t \in \mathbb{R}_+$. A C_0-sine operator-valued function $S(\cdot, A)$ is said to be *compact* if the operator $S(t, A) \in B_0(E)$ for any $t \in \mathbb{R}$.

Proposition 8.1.1 ([267]). *If an operator $C(t, A) \in B_0(E)$ for each $t \in (\alpha, \beta)$ for certain $\alpha < \beta$, then the C_0-cosine operator-valued function $C(\cdot, A) \in B_0(E)$ and the C_0-sine operator-valued function $S(\cdot, A) \in B_0(E)$.*

Proposition 8.1.2 ([267]). *If an operator $S(t, A) \in B_0(E)$ for each $t \in (\alpha, \beta)$ and for certain $\alpha < \beta$, then the C_0-sine operator-valued function $S(\cdot, A) \in B_0(E)$, $t \in \mathbb{R}$.*

Proposition 8.1.3 ([62]). *If $\dim E = \infty$, then for no $t_0 > 0$, the operators $C(t_0, A)$ and $C(2t_0, A)$ cannot be compact, simultaneously.*

Proposition 8.1.4 ([267]). *Under the condition of Proposition 8.1.1, we necessarily have $dimE < \infty$.*

Theorem 8.1.1 ([267]). *The following conditions are equivalent:*
(i) a C_0-sine operator-valued function $S(\cdot, A) \in B_0(E)$;
(ii) the resolvent $(\lambda^2 I - A)^{-1} \in B_0(E)$ for any λ with $\operatorname{Re}\lambda > \omega_c(A)$.

Theorem 8.1.2 ([62]). *The following conditions are equivalent:*
(i) a generating operator $A \in B_0(E)$;
(ii) the operator $\lambda^2(\lambda^2 I - A)^{-1} - I \in B_0(E)$ for each $\lambda > \omega_c(A)$;
(iii) the operator $S(t, A) - tI \in B_0(E)$ for any $t \in \mathbb{R}$;
(iv) the operator $C(t, A) - I \in B_0(E)$ for any $t \in \mathbb{R}$.

Proposition 8.1.5 ([157]). *Let $C(t, A) - I \in B_0(E)$ for any $t \in \mathbb{R}$. Then $\lambda(\lambda I - A)^{-1} - \mu(\mu I - A)^{-1} \in B_0(E)$ for all $\lambda, \mu \in \rho(A)$ such that $\operatorname{Re}\lambda, \operatorname{Re}\mu > \omega_c(A)$.*

Proposition 8.1.6 ([157]). *Let $C(t, A) - I \in B_0(E)$ for each $t \in (\alpha, \beta)$ and for certain $\alpha < \beta$. Then $C(t, A) - I \in B_0(E)$ for any $t \in \mathbb{R}$.*

Proposition 8.1.7 ([157]). *If $S(t, A) - tI \in B_0(E)$ for each $t \in (\alpha, \beta)$ and for certain $\alpha < \beta$, then $S(t, A) - tI \in B_0(E)$ for any $t \in \mathbb{R}$.*

Proposition 8.1.8 ([59]). *Let a C_0-sine operator-valued function $S(\cdot, A)$ and the operator-valued function $C(t, A) - I$ be compact for each $t \in \mathbb{R}$. Then the space E is necessarily finite-dimensional.*

Proof. By our assumptions, it follows from Theorems 8.1.1 and 8.1.2that the resolvent $(\lambda I - A)^{-1}$ and the operator $\lambda(\lambda I - A)^{-1} - I$ are compact for certain $\lambda \neq 0$. This implies that I is a compact operator, i.e., E is finite-dimensional.

Let us define the sets

$$NBI_0 = \{t > 0 : \text{ the operator } C(t, A) \text{ has no bounded inverse}\},$$

$$NBI_1 = \{t > 0 : \text{ the operator } S(t, A) \text{ has no bounded inverse}\},$$

Proposition 8.1.9 ([157]). *Let $C(t, A) - I \in B_0(E)$ for all $t \in \mathbb{R}$. Then the sets NBI_0 and NBI_1 either are empty, simultaneously, or are infinite of continuum cardinality, and there exist constants $\alpha_0, \alpha_1 > 0$ such that $NBI_j \subseteq (\alpha_j, \infty), j = 0, 1$.*

8.2 Compactness of the Difference of Cosines

Many linear distributed parameter control systems can be reduced to the form

$$v'(t) = Av(t) + Bu(t),\ v(0) = v_0,\ t \in \mathbb{R}_+, \tag{8.1}$$

where A generates a C_0-semigroup on the state Hilbert or Banach space E and B is the control operator acting from control space to the state space.

When we design a feedback control $u(t) = Fv(t)$ for some feedback operator from the state space to the control space, the closed-loop system becomes

$$v'(t) = (A + BF)v(t),\ v(0) = v_0,\ t \in \mathbb{R}_+. \tag{8.2}$$

In the context of the stabilization theory, we want to choose a feedback operator F in order to force the closed-loop system to possess stability properties that are not enjoyed by the original system. One important class in physical applications is that of operators F such that BF is compact on the state space. When BF is compact, it was first proved in [284] that the difference of the semigroups $\exp((A + BF)t$ and $\exp(tA)$ is compact for any positive t. Hence

$$\omega_{ess}(A) = \omega_{ess}(A + BF), \tag{8.3}$$

where ω_{ess} stands for the essential growth rate of the associated semigroup. Property (8.3) holds for any two C_0-semigroups whenever their difference is compact for some $t > 0$ (see Theorem 3.52 in [200]). This is the basis of the compactness method that was used in studying the stabilization of elastic systems (see [241]) and the spectral property of the transport equation (see [277]). The compactness method was first formulated in [270] for Hilbert spaces, and later on, itwas generalized to Banach spaces in [148]; it says that a compact perturbation cannot make the system exponentially stable if it is asymptotically but not exponentially stable.

This leads to the general study of necessary and sufficient conditions for compactness of the difference of two C_0-semigroups. A recent result in [188] says that $\exp(tA) - \exp(tB)$ is compact for some $t > 0$ iff $R(\lambda, A) - R(\lambda, B)$ is compact under the norm-continuity assumption.

On the other hand, it is more convenient to be write most of control hyperbolic systems in the form of a the second-order system instead of a first-order evolution equation in an abstract space (see [18]), [245]:

$$v''(t) = Av(t) + Bu(t),\ v(0) = v_0,\ v'(0) = v_1,\ t \in \mathbb{R}_+, \tag{8.4}$$

System (8.4) can be transferred into the first order equation (3.5); however, there are some problems, since $\mathbf{A}$ does not generate a C_0-semigroup on $E \times E$. In this connection, the problem of compactness of the difference of two C_0-cosine operator functions is of interest. Let $C(t, A)$ and $C(t, B)$ be two cosine functions on a Banach space E generated by A and B, respectively, and satisfying $\|C(t, A)\|,\ \|C(t, B)\| \leq Me^{w|t|}$, $t \in \mathbb{R}$, for some constants $M,\ w \geq 0$. Denote $\Delta_{A,B}(t) = C(t, A) - C(t, B)$ for all $t \in \mathbb{R}$.

Theorem 8.2.1 ([189]). *Let $\Delta_{A,B}(t)$ be norm-continuous for $t > 0$. Then for all $\lambda > w^2$, the operator $R(\lambda, A) - R(\lambda, B)$ is compact iff $\Delta_{A,B}(t)$ is compact for $t \geq 0$.*

We can characterize the norm-continuity in Hilbert space analogously to Theorem 2.5 in [188] and the proof is just a simple modification.

Proposition 8.2.1. *Let A and B generate cosine functions $C(t,A)$ and $C(t,B)$, respectively, on a Hilbert space H, and let $\|C(t,A)\|, \|C(t,B)\| \leq Me^{\omega t}$ for some constants $M \geq 1, \omega \in \mathbb{R}$. Then $C(t,A) - C(t,B)$ is norm-continuous for $t > 0$ iff for every $\sigma > \omega$,*

$$\lim_{|r|\to\infty} \|(\sigma + ir)\Big(R((\sigma+ir)^2, A) - R((\sigma+ir)^2, B)\Big)\| = 0$$

and

$$\lim_{n\to\infty} \int_n^\infty \|(\sigma \pm ir)\Big(R((\sigma \pm ir)^2, A) - R((\sigma \pm ir)^2, B)\Big)x\|^2 dr = 0,$$

$$\lim_{n\to\infty} \int_n^\infty \|(\sigma \pm ir)\Big(R((\sigma \pm ir)^2, A^*) - R((\sigma \pm ir)^2, B^*)\Big)y\|^2 dr = 0$$

uniformly for $x \in H, y \in H^*$ with $\|x\|, \|y\| \leq 1$.

The case for sine functions is easier to treat.

Theorem 8.2.2 ([189]). *Let $S(t,A)$ and $S(t,B)$ be the corresponding sine functions of $C(t,A)$ and $C(t,B)$, respectively. Then $S(t,A) - S(t,B)$ is compact for $t > 0$ iff $R(\lambda, A) - R(\lambda, B)$ is compact for $\lambda > w^2$.*

Now we show a similar result for the cosine as in [188], Proposition 2.7.

Proposition 8.2.2 ([189]). *Suppose that $\Delta_{A,B}(t)$ is compact for $t > 0$ and norm-continuous at $t = 0$. Then*

$$\lim_{h\to 0} \|\Delta_{A,B}(t+h) - 2\Delta_{A,B}(t) + \Delta_{A,B}(t-h)\| = 0 \text{ for any } t \geq 0. \tag{8.5}$$

Proof. We have

$$\begin{aligned}
& \Delta_{A,B}(t+h) + \Delta_{A,B}(t-h) - 2\Delta_{A,B}(t) \\
= \; & \Big(C(t+h,A) + C(t-h,A)\Big) - \Big(C(t+h,B) + C(t-h,B)\Big) - 2\Delta_{A,B}(t) \\
= \; & 2C(t,A)C(h,A) - 2C(t,B)C(h,B) - 2\Delta_{A,B}(t) \\
= \; & 2\Big(C(t,A)C(h,A) - C(h,A)C(t,B)\Big) + 2\Big(C(h,A)C(t,B) - C(t,B)C(h,B)\Big) - 2\Delta_{A,B}(t) \\
= \; & 2C(h,A)\Delta_{A,B}(t) + 2\Delta_{A,B}(h)C(t,B) - 2\Delta_{A,B}(t) \\
= \; & 2[C(h,A) - I]\Delta_{A,B}(t) + 2\Delta_{A,B}(h)C(t,B) \\
\to \; & 0 \qquad \text{as } h \to 0.
\end{aligned}$$

Remark 8.2.1. In the proof of Theorem 8.2.1, we used actually (8.5), but not the norm-continuity of $\Delta_{A,B}(\cdot)$.

Theorem 8.2.3 ([189]). *Let $\Delta_{A,B}(t)$ be norm-continuous in t at 0. Then $\Delta_{A,B}(t)$ is compact for $t > 0$ iff $R(\lambda, A) - R(\lambda, B)$ is compact for $\lambda > w^2$ and (8.5) holds.*

Proposition 8.2.3 ([189]). *Suppose that the assumptions of Theorem 9.2.1 (resp. Theorem 9.3.1) hold. Then $\Delta_{A,A(I+B)}(t)$ (resp. $\Delta_{A,(I+B)A}(t)$) is compact for $t > 0$ iff $\Delta_{A,A(I+B)}(t)$ (resp. $\Delta_{A,(I+B)A}(t)$) satisfies (8.5) and $R(\lambda, A) - R(\lambda, A(I + B))$ (resp. $R(\lambda, A) - R(\lambda, (I + B)A)$) is compact for λ large enough.*

We complete this section by comparing the results on the difference of semigroups and cosine operator-valued functions. We first consider bounded perturbations. It is well known that if A generates a C_0-semigroup $\exp(tA)$, then $A + B$, $B \in B(E)$, also generates a C_0-semigroup $\exp t(A + B)$. It was shown in [296] that $\exp t(A + B) - \exp(tA)$ is norm-continuous for $t > 0$ if it is compact for $t > 0$. As for the cosine case, the compactness hypotheses can be removed.

Theorem 8.2.4 ([189]). *Let A be a generator of a cosine function $C(t, A)$, and let $B \in B(E)$. Then $\Delta_{A+B,A}(t)$ is norm-continuous in $t \in \mathbb{R}$.*

Combining Theorems 8.2.1 and 8.2.4, we have the following.

Theorem 8.2.5 ([189]). *Let $B \in B(E)$, and let A generate a cosine function. Then $\Delta_{A+B,A}(t)$ is compact for $t > 0$ iff $R(\lambda, A + B) - R(\lambda, A)$ is compact for λ large enough.*

Proposition 8.2.4 ([189]). *Suppose that C_0-semigroups $\exp tA$ and $\exp(tB)$ commute and $D(B) \subseteq D(A)$. Assume that $\exp(tB)$ is a C_0-group. If $\Theta(t) := \exp(tA) - \exp(tB)$ is compact for all $t > 0$, then $A = B + K$, where the operator K is compact.*

Proof. One can write $\exp(-tB)\Theta(t) = \exp(-tB)\exp(tA) - I$, $t \in \mathbb{R}_+$. By assumption, the operator $\exp(-tB)\exp(tA) - I$ is compact for any $t > 0$, and, moreover, $\exp(-tB)\exp tA$ is a C_0-semigroup with the generator $A - B$. It follows from [107] that the operator $A - B$ is compact.

For cosines with bounded generators, we also have the following.

Proposition 8.2.5 ([189]). *Let $B \in B(E)$. Then $\Delta_{A,B}(t)$ is compact iff $A - B$ is compact.*

Proof. The compactness of $\Delta_{A,B}(t)$ for any $t > 0$ implies (see [278]) that $R(\mu, A) - R(\mu, B)$ is compact for some μ. In such a case, the operator $I-(\mu-B)R(\mu, A)$ is compact. This means that $(\mu-B)R(\mu, A)$ is a Fredholm operator of index 0, i.e., it has a closed range $\mathbf{R}((\mu - B)R(\mu, A)) = E$. Since $\mu - B$ is one-to-one on E, we obtain $\mathbf{R}(R(\mu, A)) = E$. By the Banach theorem, $\mu - A$ is bounded. Since the operator A is bounded, we have $\left\| \frac{2}{t^2} \int_0^t S(s, A)\, ds - I \right\| \to 0$ as $t \to 0$. Hence the operator $\int_0^t S(s, A)\, ds$ is invertible. If $\Delta_{A,B}(t)$ is compact, then $B \int_0^t (S(s, B) - S(s, A))\, ds$ is also compact. Now from

$$\Delta_{A,B}(t) = (A - B) \int_0^t S(s, A)\, ds - B \int_0^t (S(s, B) - S(s, A))\, ds$$

it follows that the difference $A - B$ is a compact operator.

Conversely, if $A - B$ is a compact operator, then A is bounded and

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(B - A)R(\lambda, B)$$

is compact; therefore, the compactness of $\Delta_{A,B}(t)$ follows from Theorem 8.2.1.

In the following example both A and B generate a C_0-semigroup and a cosine function. The operator $\exp(tA) - \exp(tB)$ is compact for all $t > 0$, but $C(t, A) - C(t, B)$ is not compact.

Example 8.2.1 ([189]). Let $E = l^1$ and $\{e_n\}$ be the standard basis for it, i.e. $e_n = (0, ..., 0, 1, 0, ..., 0, ...)$, where 1 is in the nth coordinate. Let

$$Ax := \sum_{n=1}^{\infty} -n(x, e_n)e_n, \quad Bx := \sum_{n=1}^{\infty} -(n + n^2)(x, e_n)e_n,$$

where $x = (x^1, x^2, ..., x^n, ...)$, $(x, e_n) = x^n$, $\|x\|_{l^1} = \sum_{i=1}^{\infty} |x^i|$. Then the C_0-semigroups generated by them are

$$\exp(tA)x = \sum_{n=1}^{\infty} e^{-nt}(x, e_n)e_n, \quad \exp(tB)x = \sum_{n=1}^{\infty} e^{-(n+n^2)t}(x, e_n)e_n.$$

And the cosine functions are given by the formula

$$C(t, A)x = \sum_{n=1}^{\infty} \cos(nt)\ (x, e_n)e_n, \quad C(t, B)x = \sum_{n=1}^{\infty} \cos(n + n^2)t\ (x, e_n)e_n.$$

Since $e^{-nt} - e^{-(n+n^2)t} \to 0$ as $n \to \infty$, the operator $\exp(tA) - \exp(tB)$ can be approximated in norm by a sequence of operators $S_N(t)x = \sum_{n=1}^{N} \Big(e^{-nt} - e^{-(n+n^2)t}\Big)(x, e_n)e_n$ with ranges of finite dimension. Therefore, the operator $\exp(tA) - \exp(tB)$ is compact for $t \geq 0$. However, $C(t, A) - C(t, B)$ is not compact. Indeed, take $t = \pi/2$ and choose $\{y_k\} := \{(\delta_n^{2k+1})\} \in l^1$, where δ_i^j is the Kronecker delta. Now, for $n = 2k + 1$, we obtain $nt = k\pi + \pi/2$ and $(n+n^2)t = 2k^2\pi + 3k\pi + \pi$. Therefore, we have $\cos(nt) - \cos((n+n^2)t) = \pm 1$; if k can be divided exactly by 2, then we choose $+$; otherwise, we choose $-$. Thus,

$$(C(\pi/2, A) - C(\pi/2, B))y_k = \{[\cos(nt) - \cos(n + n^2)t]\delta_n^{2k+1}\} = \{\pm\delta_n^{2k+1}\},$$

which means $\|[C(\pi/2, A) - C(\pi/2, B)](y_k - y_m)\|_{l^1} = 2$ for $k \neq m$. Therefore, we cannot choose a convergent subsequence from the sequence $\{[C(\pi/2, A) - C(\pi/2, B)]y_k\}$. Also, we can see that $C(t, A) - C(t, B)$ is not norm-continuous in t. Indeed, for each $t > 0$, define $s_k = t + \dfrac{1}{k + k^2}$. Then $s_k \to t$ as $k \to \infty$ and

$$\begin{aligned}
& \Big\|[C(t, A) - C(t, B)] - [C(s_k, A) - C(s_k, B)]\Big\|_{B(l^1)} \\
= \; & \Big\|\{[\cos(nt) - \cos(ns_k)] - [\cos(n + n^2)t - \cos(n + n^2)s_k]\}_{n=1}^{\infty}\Big\|_{l^\infty} \\
\geq \; & 2\Big|\sin\left(\frac{k}{2}(s_k + t)\right)\sin\left(\frac{k}{2}(s_k - t)\right) - \sin\left(\frac{k + k^2}{2}(s_k + t)\right)\sin\left(\frac{k + k^2}{2}(s_k - t)\right)\Big| \\
= \; & 2\Big|\sin\left(\frac{k}{2}(s_k + t)\right)\sin\left(\frac{1}{2(1 + k)}\right) - \sin((k + k^2)t + 1/2))\sin\frac{1}{2}\Big|.
\end{aligned}$$

It is clear that $\sin\frac{1}{2(1+k)} \to 0$ as $k \to \infty$. But $\sin((k + k^2)t + 1/2))$ does not converge to 0 as $k \to \infty$ for every $t > 0$! To prove this, we suppose the contrary: $\sin((k+k^2)t+1/2)) \to 0$ as $k \to \infty$. Then $\sin((k+1+(k+1)^2)t+1/2) \to 0$ $k \to \infty$. Now, since

$$\begin{aligned}
& \sin((k + 1 + (k + 1)^2)t + 1/2) = \sin(((k + k^2)t + 1/2) + 2(k + 1)t) \\
= \; & \sin((k + k^2)t + 1/2))\cos(2(k + 1)t) + \cos((k + k^2)t + 1/2))\sin(2(k + 1)t),
\end{aligned}$$

we obtain $\sin(2(k + 1)t) \to 0$ as $k \to \infty$, since $\cos((k + k^2)t + 1/2)$ cannot converge to 0 according to the relation $\sin^2 x + \cos^2 x = 1$. Hence $\sin(2(k+1+1)t) \to 0$ as $k \to \infty$. Thus, from $\sin(2(k + 1 + 1)t) = \sin(2(k + 1)t)\cos(2t) + \cos(2(k + 1)t)\sin(2t)$ we obtain $\sin(2t) \to 0$ as $k \to \infty$. Therefore, have

$t = n\pi/2$ for some $n \in \mathbb{N}$. But for such t, we find that $\sin((k+k^2)t+1/2)) = \pm\sin(1/2)$, which contradicts our assumption of the convergence to 0. This means that $C(t,A) - C(t,B)$ is not norm-continuous.

The converse does not hold: we have the following.

Proposition 8.2.6 ([189]). *Suppose that A and B generate cosine functions $C(t,A)$ and $C(t,B)$. If $C(t,A)-C(t,B)$ is compact for $t>0$, then $\exp(tA)-\exp(tB)$ is compact. Moreover, if $C(t,A)-C(t,B)$ is norm-continuous for $t>0$, then $C(t,A)-C(t,B)$ is compact for $t>0$ iff $\exp(tA)-\exp(tB)$ is compact.*

Proof. Since $C(t,A) - C(t,B)$ is compact, it follows from [278] that $S(t,A) - S(t,B) = \int_0^t (C(s,A) - C(s,B))ds$ is also compact, which implies $R(\lambda,A) - R(\lambda,B)$ is compact by Theorem 8.2.2. Moreover, since both $\exp(tA)$ and $\exp(tB)$ are analytic, $\exp(tA)-\exp(tB)$ is norm-continuous, and the compactness of $\exp(tA) - \exp(tB)$ follows from Theorem 2.3 of [188]. If, in addition, $C(t,A) - C(t,B)$ is norm-continuous, then the compactness of $\exp(tA) - \exp(tB)$ implies that $R(\lambda,A) - R(\lambda,B)$ is compact. Now the compactness of $C(t,A) - C(t,B)$ follows from Theorem 8.2.1.

The following extends Proposition 2.7 in [188].

Proposition 8.2.7 ([278]). *Suppose that $D(A) \subseteq D(B)$, where A generates an analytic semigroup $\exp tA$ and B is a generator of a C_0-semigroup $\exp(tB)$. If $\Theta(t) := \exp(tA) - \exp(tB)$ is compact for $t>0$, then $\Theta(t)$ is norm-continuous for $t \geq 0$.*

8.3 Compactness of the Families $F(\cdot)$ and $G(\cdot)$

Definition 8.3.1. A multiplicative perturbation C_0-family $F(\cdot)$ (resp. an additive perturbation C_0-family $G(\cdot)$) is said to be *compact* if the operators $F(t)$ (resp. $G(t)$) are compact for each $t \in \overline{\mathbb{R}}_+$.

Proposition 8.3.1. *If a C_0-sine operator-valued function $S(\cdot,A)$ is compact and if a multiplicative perturbation family $F(\cdot)$ is norm-continuous at zero, then $F(\cdot)$ is compact. The same is also true for an additive perturbation C_0-family.*

Proof. Integrating relation (2.1) in t from 0 up to τ, we obtain

$$\int_\tau^{\tau+h} F(\eta)x\,d\eta - \int_{\tau-h}^{\tau} F(\eta)x\,d\eta = 2S(\tau,A)F(h)x. \tag{8.6}$$

The compactness of $S(\cdot, A)$ implies the compactness of the left-hand side of (8.6) for any $\tau, h \in \overline{\mathbb{R}}_+$. Since $F(\cdot)$ is uniformly continuous, in (8.6), we can take the derivative in τ without loss of the compactness property, since the obtained left-hand side (8.6) remains a compact operator. Using the condition $F(0) = 0$ and the uniform continuity of $F(\cdot)$ and tending τ to zero, we obtain that $F(h)$ is compact for each $h \in \overline{\mathbb{R}}_+$.

The requirement of the uniform continuity in Proposition 8.3.1 is also necessary, as the following proposition shows.

Proposition 8.3.2 ([233]). *If a multiplicative perturbation family $F(\cdot)$ is compact, then the family $F(\cdot)$ is uniformly continuous on $\overline{\mathbb{R}}_+$.*

Proof. Since $F(\cdot)$ is compact, the Laplace transform $\hat{F}(\cdot)$ is also compact (see [278]). By formula (iv) of Proposition 2.4.1, the assertion is proved, since the strong convergence becomes uniform after postmultiplying by a compact operator.

Proposition 8.3.3 ([233]). *Let a C_0-cosine operator-valued function $C(\cdot, A)$ on a Banach space E be such that each of the multiplicative perturbation families $F(\cdot)$ (or the additive perturbation C_0-families $G(\cdot)$) for $C(\cdot, A)$ is compact. Then E is finite-dimensional.*

Proof. By assumption, two particular multiplicative perturbation families

$$F_1(t) = C(t, A) - I \quad \text{and} \quad F_2(t) = \int_0^t S(s, A)\, ds$$

are compact. Then since we know (see Theorem 8.1.2) that the operator $C(t, A) - I$ is compact for all $t \in \mathbb{R}_+$, iff the generator A is compact, the family $C(\cdot, A)$ is norm-continuous on $\overline{\mathbb{R}}_+$. Therefore, the operator $C(0, A) = I$, being the limit in norm of compact operators $2t^{-2}F_2(t)$ as $t \to 0$ is compact. This implies that E is finite-dimensional.

Chapter IX
Perturbations of C_0-Cosine Operator-Valued Functions

The perturbation theory of C_0-cosine operator-valued functions differs from that of semigroup in an interesting way. On one hand, the generating operator of a C_0-cosine operator-valued function lies in a more narrow class

of operators than $\mathbf{G}(M,\omega)$; for example, it always generates an analytic C_0-semigroup, and, therefore, its fractional powers $(-A)^\alpha$, $0 \le \alpha \le 1$ are defined in a sufficiently simple way. On the other hand, it is not clear since now whether the M. Watanabe perturbation is the strongest perturbation or not, and what happens with SV of a multiplicative perturbation family if $SV(F(\cdot),t) = O(t^\alpha)$, $t \to 0+$, for a certain $0 < \alpha < 1$.

9.1 General Multiplicative Theorems

We quote the following theorem (see [190], Theorem 3.10) for $C(\cdot,A)$, which is convenient for practical applications and which will be repeatedly used below.

Theorem 9.1.1. *A C_0-cosine operator-valued function $C(\cdot,A)$ satisfying the estimate $\|C(t,A)\| \le Me^{\omega t}$ for all $t \ge 0$ is a cosine operator-valued function with a generating operator A iff the condition $\lambda > \omega$ implies $\lambda^2 \in \rho(A)$ and we have*

$$\lambda(\lambda^2 - A)^{-1} = \int_0^\infty e^{-\lambda t} C(t,A)\,dt.$$

Proposition 9.1.1 ([234]). *Let A be a densely defined closed linear operator on a Banach space E, and let $\Re \in B(X)$. The following assertions hold:*

(i) If the operator $\Re A$ generates a cosine operator-valued function $\tilde{C}(\cdot,A)$, then the operator $A\Re$ also generates a cosine operator-valued function.

(ii) If the operator $A\Re$ generates a cosine operator-valued function $\hat{C}(\cdot,A)$, and for a certain real λ, the operator $\lambda - \Re A$ is invertible, then the operator $\Re A$ also generates a cosine operator-valued function.

(iii) If the operator $A\Re$ generates a cosine operator-valued function $\check{C}(\cdot,A)$ and $D((A\Re)^*) = D(A^*)$, then the operator $\Re A$ also generates a cosine operator-valued function.

Definition 9.1.1. We say that an operator $\Re \in B(E)$ belongs to the class $M1(A)$ of multiplicative perturbations of the generator A of a cosine operator function $C(\cdot,A)$ if the operator $B = \Re - I$ satisfies the following Condition $(M1)$:

For all continuous functions $f \in C([0,t];E)$

$(M1_a)$ $\int_0^t S(t-s,A)B\,f(s)ds \in D(A)$,

$(M1_b)$ $\|A\int_0^t S(t-s,A)Bf(s)ds\| \le M\gamma_B(t)\|f\|_{[0,t]}$,

where $\gamma_B : [0,\infty) \to [0,\infty)$ is some continuous nondecreasing function with $\gamma_B(0) = 0$ and $\|f\|_{[0,t]} = \sup\limits_{0\le s\le t} \|f(s)\|$.

Remark 9.1.1. If $B + I \in M1(A)$, then $\|C(t+h,A)B - C(t,A)B\| \to 0$ as $h \to 0$ for any t. To see this, we first set $f(t) = x$ for $t \geq 0$ in $(M1_b)$. It follows that $\|(C(h,A) - I)B\| \to 0$ as $h \to 0$. This, together with the fact that $(C(\cdot,A) - I)B$ is a C_0-step response, proves the assertion.

Theorem 9.1.2 ([234]). *Let A be the infinitesimal generator of a cosine operator-valued function $C(\cdot,A)$ on E. If an operator $\Re$ belongs to $M1(A)$, then both $A\Re$ and $\Re A$ are generators of cosine operator-valued functions. Moreover, the cosine operator-valued function $\Im(\cdot)$ generated by $A\Re$ satisfies $\|\Im(t) - C(t,A)\| = O(\gamma_B(t))\,(t \to 0^+)$.*

Remark 9.1.2. (i) If $(M1_a)$ and $(M1_b)$ hold for all functions in a dense subset of $C([0,t];E)$, then because of the closedness of A, we easily see that they actually hold for all f in $C([0,t];E)$. Hence since $(M1_a)$ holds for all f in $C^1([0,t];E)$, which is dense in $C([0,t];E)$, Condition $(M1)$ can be replaced by the equivalent condition:

$$\|A\int_0^t S(t-s,A)Bf(s)ds\| \leq M\gamma_B(t)\|f\|_{[0,t]} \quad \text{for all } f \in C^1([0,t];E). \tag{9.1}$$

Thus, we only need to verify Condition $(M1')$ in practical applications.
(ii) If $(M1)$ holds with some $\gamma_B(t) = o(t^2)$, then $\Im(\cdot) \equiv C(\cdot,A)$ (see [234] Corollary 3.6), so that $A(I+B) = A$ and AB=0. Conversely, the latter condition implies $C(\cdot,A)B = B$, and hence $A\int_0^t S(t-s,A)Bf(s)ds \equiv 0$ for all $f \in C([0,t],X)$. Thus, $(M1)$ holds with some $\gamma_B(t) = o(t^2)$ iff $AB = 0$, and in this case, $(M1)$ actually holds with $\gamma_B(\cdot) \equiv 0$.

Let $(Z, |\cdot|)$ be a Banach space satisfying Condition (Z) with respect to $C(\cdot,A)$:
(Z_a) Z is continuously embedded in E,
(Z_b) for all continuous functions $\phi \in C([0,t],Z)$

$$\int_0^t S(t-s,A)\phi(s)ds \in D(A),$$

(Z_c) $\|A\int_0^t S(t-s,A)\phi(s)ds\| \leq \gamma(t)\sup_{0\leq s\leq t}|\phi(s)|_Z,$

where $\gamma(\cdot) : [0,\infty) \to [0,\infty)$ is a continuous nondecreasing function with $\gamma(0) = 0$.

It is easy to verify Condition (Z) for the spaces $\mathbf{D}(A)$ and the Favard class $(Fav_{C(\cdot)}, |\cdot|_{Fav_{C(\cdot)}})$. Indeed, if $Z = \mathbf{D}(A)$, then (Z) holds with $\gamma(t) = O(t^2)$ as $t \to 0^+$.

Corollary 9.1.1. *If Z is a Banach space satisfying Condition (Z), then $I + B(E,Z) \subseteq M1(A)$, so that for every $B \in B(E,Z)$, both $A(I+B)$ and $(I+B)A$ are generators of cosine operator-valued functions.*

Definition 9.1.2. We say that an operator $\Re \in B(E)$ belongs to the class $M2(A)$ of multiplicative perturbations of the generator A of a cosine operator-valued function $C(\cdot, A)$ if the operator $B = \Re - I$ satisfies
(9.2)

$$\delta_B(t) := \sup\{\int_0^t \|BS(s,A)Ax\|ds : x \in D(A), \|x\| \leq 1\} \to 0 \text{ as } t \to 0^+.$$

Remark 9.1.3. As was shown by Fattorini [128], in the case of $E = L_p$, we have $\|AS(t,A)x\| = O(t^{2\alpha-1})$ as $t \to 0$ for $1/2 \leq \alpha \leq 1$ and $x \in D((A-cI)^\alpha)$. Therefore, $\Re \in M2(A)$, e.g., if $B = (A - cI)^{-\beta}$ for $\beta \geq 1/2$.

Theorem 9.1.3 ([234]). *Let A be the infinitesimal generator of a cosine operator function $C(\cdot, A)$ on E. If an operator $\Re$ belongs to $M2(A)$, then both $\Re A$ and $A\Re$ are generators of cosine operator-valued functions. Moreover, the cosine function $C_1(\cdot)$ generated by $\Re A$ satisfies $\|C_1(t) - C(t,A)\| = O(\delta_B(t))\,(t \to 0^+)$.*

9.2 Perturbations by the Family $F(\cdot)$

For any fixed λ and an operator $B \in B(E)$, let $F_{B,\lambda}(\cdot)$ and $G_{B,\lambda}(\cdot)$ be the functions defined by
(9.3)

$$F_{B,\lambda}(t)x := (\lambda^2 - A)\int_0^t S(s,A)Bx\,ds = \lambda^2 \int_0^t S(s,A)Bxds - (C(t,A) - I)Bx,\ x \in E,\ t \geq 0,$$

(9.4)

$$G_{B,\lambda}(t)x := B(\lambda^2 - A)\int_0^t S(s,A)x\,ds = \lambda^2 B \int_0^t S(s,A)xds - B(C(t,A) - I)x,\ x \in E,\ t \geq 0.$$

The next theorem gives a characterization of $M1(A)$ in terms of the semi-variation of $F_{B,\lambda}(\cdot)$.

Theorem 9.2.1 ([234]). *An operator $\Re \in B(E)$ belongs to $M1(A)$, i.e., $B = \Re - I$ satisfies Condition $(M1)$, iff $SV(F_{B,\lambda}(\cdot), t) = o(1)\,(t \to 0^+)$ for some (and all) $\lambda > \omega$. Moreover, in Condition $(M1_b)$, one can choose $\gamma_B(t) = SV(F_{B,\lambda}(\cdot), t)$ in the case $SV(F_{B,\lambda}(\cdot), t) = O(t^2)$, and $\gamma_B(t) = O(t^2)$ in the case $SV(F_{B,\lambda}(\cdot), t) = o(t^2)$.*

Now, from Theorem 9.2.1, we can deduce the following theorem on an additive perturbation.

Theorem 9.2.2. *Let A be the generator of a cosine operator-valued function $C(\cdot, A)$ on E. If $P \in B(\mathbf{D}(A), E)$ is such that*

$$\int_0^t S(t-s)Pg(s)ds \in D(A), \tag{9.5}$$

$$\|A \int_0^t S(t-s)Pg(s)ds\| \leq \gamma_P(t) \sup_{0 \leq s \leq t} \|g(s)\|_{\mathbf{D}(A)} \tag{9.6}$$

for all $g \in C([0,t], \mathbf{D}(A))$ and for some function $\gamma_P(\cdot)$ with $\gamma_P(t) = o(1)$ $(t \to 0^+)$, then the operators $A + P$ and $A + (A - \lambda)P(A - \lambda)^{-1}$ $(\lambda > \omega)$ are generators of cosine operator-valued functions.

Proof. Without loss of generality, we may assume that A is invertible, so that $A + P = (I + PA^{-1})A$. In view of Theorem 9.2.1, we only have to verify Condition $(M1)$ for the operator $B = PA^{-1}$. Indeed, if $f \in C([0,t]; E)$, then $A^{-1}f \in C([0,t], \mathbf{D}(A))$, so that, setting $g = A^{-1}f$ in (9.5) and (9.6), we have

$$\int_0^t S(t-s, A)Bf(s)ds = \int_0^t S(t-s, A)P(A^{-1}f)(s)ds \in D(A)$$

and

$$\|A \int_0^t S(t-s, A)Bf(s)ds\| \leq \gamma_P(t) \sup_{0 \leq s \leq t} \|A^{-1}f(s)\|_{\mathbf{D}(A)} \leq \gamma_P(t)(\|A^{-1}\|+1)\|f\|_{[0,t]}.$$

Corollary 9.2.1. *Let A be the generator of a cosine operator-valued function $C(\cdot)$ on E. If P is a continuous operator acting from $D(A)$ to Z (Z being a Banach space satisfying condition (Z)), then $A+P$ and $A+(A-\lambda)P(A-\lambda)^{-1}$ $(\lambda > \omega)$ are generators of cosine operator-valued functions.*

Proof. Suppose $P \in B(\mathbf{D}(A), Z)$, and let $g \in C([0,t], D(A))$. Then we have $Pg \in C([0,t]; Z)$, and by Condition (Z), $\int_0^t S(t-s, A)Pg(s)ds \in D(A)$ and

$$\|A \int_0^t S(t-s)Pg(s)ds\|$$

$$\leq \gamma_P(t) \sup_{0 \leq s \leq t} |Pg(s)|_Z \leq \gamma_P(t)\|P\|_{B(\mathbf{D}(A),Z)} \sup_{0 \leq s \leq t} \|g(s)\|_{\mathbf{D}(A)}.$$

The conclusion now follows from Theorem 9.2.2.

9.3 Perturbations by the Family $G(\cdot)$: Additive Perturbations

The following theorem gives a characterization of $M2(A)$ in terms of the strong variation of $G_{B,\lambda}(\cdot)$ for $x \in D(A)$; more precisely, by

$$\beta(G_{B,\lambda}(\cdot), t) := \sup\{Var(G_{B,\lambda}(\cdot)x, t);\ x \in D(A), \|x\| \leq 1\}. \tag{9.7}$$

Theorem 9.3.1 ([234]). *An operator $\Re \in B(E)$ belongs to $M2(A)$, i.e., $B = \Re - I$ satisfies Condition $(M2)$, iff $\beta(G_{B,\lambda}(\cdot), t) = o(1)$ $(t \to 0^+)$ for some (and all) $\lambda > \omega$. Moreover, $\beta(G_{B,\lambda}(\cdot), t)$ and the function $\delta_B(t)$ in condition $(M2)$ has the same order of convergence at zero whenever the order does not exceed $O(t^2)$.*

Proof. From (9.4), we see that

$$\mathrm{Var}(G_{B,\lambda}(\cdot)x, t) = \mathrm{Var}(BC(\cdot, A)x, t) + \lambda^2\|B\| \int_0^t \|S(s, A)x\| ds$$

if the variation exists. Since for $x \in D(A)$,

$$\mathrm{Var}(BC(\cdot, A)x, t) = \int_0^t \|\frac{d}{ds}BC(s, A)x\| ds = \int_0^t \|BS(s, A)Ax\| ds,$$

we have $\|\beta(G_{B,\lambda}(\cdot), t) - \delta_B(t)\| \leq \lambda^2\|B\| \int_0^t \|S(s, A)\| ds \leq \lambda^2\|B\|Me^{\omega t}t^2$. Hence $\delta_B(t)$ tends to 0 as $t \to 0^+$ iff $\beta(G_{B,\lambda}(\cdot), t)$ does. They have the same order of convergence at zero whenever one of them has the order less than or equal to $O(t^2)$.

In general, $M1(A)$ and $M2(A)$ are proper subsets of $I + B(E)$. Each of the conditions $M1(A) = I + B(E)$ and $M2(A) = I + B(E)$ is equivalent to the condition that A is bounded. Indeed, if for every $B \in B(E)$, the operator $(I + B)A$ generates a cosine operator-valued function, then for $B = -2I$, we have that $-A$ also generates a cosine operator-valued function. Hence both A and $-A$ generate analytic semigroups and, consequently, A is bounded.

From Theorem 9.3.1, we deduce the following additive perturbation theorem.

Theorem 9.3.2 ([234]). *Let A be the generator of a cosine operator function $C(\cdot)$ on E. If P is an operator satisfying the following conditions:*

$$D(A) \subset D(P) \text{ and } P(\lambda^2 - A)^{-1} \in B(E) \text{ for some } \lambda > \omega; \tag{9.8}$$

(9.9)

$$\theta_P(t) := \sup\Big\{\int_0^t \|PS(s)x\| ds; x \in D(A), \|x\| \leq 1\Big\} < 1 \text{ for some } t > 0,$$

then the operators $A+P$ and $A+(A-\lambda)P(A-\lambda)^{-1}$ are generators of cosine operator-valued functions. Moreover, the cosine function $C_1(\cdot)$ generated by $A+P$ satisfies $\|C_1(t)-C(t,A)\| = O(\theta_P(t))\,(t\to 0^+)$.

Proof. We may assume that A is invertible, so that $A+P=(I+PA^{-1})A$. We set $B=PA^{-1}$. Then $\int_0^t \|BS(s)Ax\|ds \le \int_0^t \|PS(t)x\|ds$ for all $x\in D(A)$. Hence (9.9) implies $\delta_B(t)\le\theta_P(t)<1$ for some $t>0$, and the conclusion follows from Theorem 9.3.1.

From Theorem 9.2.2, we can deduce the following perturbation theorem of Watanabe ([281] Theorem 2).

Corollary 9.3.1 ([234]). *Let A be the generator of a cosine operator-valued function $C(\cdot,A)$ on E. If $P\in B(E^1,E)$, then $A+P$ and $A+(A-\lambda)P(A-\lambda)^{-1}\,(\lambda>\omega)$ are generators of cosine operator-valued functions. Moreover, the cosine function $C_1(\cdot)$ generated by $A+P$ satisfies $\|C_1(t)-C(t,A)\| = O(t)\,(t\to 0^+)$.*

Proof. It is proved in [249] that $P\in B(E^1,E)$ implies (9.8) . To show (9.9) , let $x\in D(A)$. Then for $t\in[0,1]$, we have

$$\|PS(t,A)x\| \le \|P\|_{B(E^1,E)}\|S(t,A)x\|_E \le \|P\|_{B(E^1,E)}\big[\|S(t,A)x\|+\sup_{0\le\eta\le 1}\|AS(\eta,A)S(t,A)x\|\big]\le K\|x\|.$$

Therefore, $\theta_P(t)=O(t)\,(t\to 0^+)$, and hence the conclusion follows from Theorem 9.3.2.

From Theorem 9.3.2, we can also deduce the following corollary: when $a=\infty$, it is Theorem A in [256] (see also [249], Corollary 2.1]), and when $a<\infty$, it is Corollary 2.2 in [249], which contains Theorem 3.2 in [256].

Corollary 9.3.2 ([234]). *Let A be the generator of a cosine operator-valued function $C(\cdot)$ on E. Let P be an operator satisfying conditions* (9.8) *, and let*

$$L(\lambda) := \sup\Big\{\int_0^a e^{-\lambda s}\|PS(s)x\|ds;\ x\in D(A), \|x\|\le 1\Big\} < \infty \tag{9.10}$$

for some $a\in(0,\infty]$ and $\lambda>\omega$. Let $L(\infty):=\lim_{\lambda\to\infty}L(\lambda)$. Then for each ε with $|\varepsilon|<L(\infty)^{-1}$, $A+\varepsilon P$ and $A+\varepsilon(A-\lambda)P(A-\lambda)^{-1}\,(\lambda>\omega)$ are generators of cosine operator-valued functions.

Proof. Choose numbers $0<\mu<\mu_1<\mu_2<1$ such that $|\varepsilon|=\mu L(\infty)^{-1}$. Fix λ so large that $L(\lambda)/L(\infty)<\frac{\mu_1}{\mu}$, and then fix $t\in(0,a)$ so small that $e^{\lambda t}<\frac{\mu_2}{\mu_1}$. Then for all $x\in D(A)$ with $\|x\|\le 1$, we have

$$\int_0^t \|\varepsilon PS(s,A)x\|ds \le |\varepsilon|e^{\lambda t}\int_0^t e^{-\lambda s}\|PS(s,A)x\|ds \le |\varepsilon|e^{\lambda t}L(\lambda) = e^{\lambda t}\mu L(\lambda)/L(\infty) \le \frac{\mu_2}{\mu_1}\mu\frac{\mu_1}{\mu} = \mu_2 < 1,$$

i.e., $\theta_{\varepsilon P}(t) < 1$. Now the conclusion follows from Theorem 9.3.2.

From this corollary, Shimizu and Miyadera were able to deduce a generalization (Corollary 2.2 in [256]) of the perturbation theorem of Fattorini [135] and Travis and Webb [269]. The latter theorem states that if a closed operator P satisfies $D(A) \subset D(P)$ and $PS(\cdot, A)x \in C([0,1]; E)$ for every $x \in E$, then $A + P$ is the generator of a cosine operator-valued function. This is also an immediate consequence of Theorem 9.3.2, as it is clear that $\theta_P(t) = O(t)\,(t \to 0^+)$.

Next, we consider mixed-type perturbations induced by a C_0-cosine step response and a C_0-cosine cumulative output. Thus, the following two theorems follow immediately from Theorems 9.2.1, 9.3.1, and Corollary 9.3.1.

Theorem 9.3.3 ([234]). *If a C_0-cosine step response $F(\cdot)$ for $C(\cdot, A)$ is locally of bounded semivariation and if $SV(F(\cdot), t) = o(1)\,(t \to 0^+)$, then the operator $A_1 := A(I - \lambda\hat{F}(\lambda)) + \lambda^3\hat{F}(\lambda)$, $\lambda > \omega$, is the infinitesimal generator of some cosine operator-valued function $C_1(\cdot)$.*

Theorem 9.3.4 ([234]). *If a C_0-cumulative output $G(\cdot)$ for $C(\cdot, A)$ is locally of bounded strong variation and if $\beta(F(\cdot), t) = o(1)\,(t \to 0^+)$, then the operator $A_2 := (I - \lambda\hat{G}(\lambda))A + \lambda^3\hat{G}(\lambda)$, $\lambda > \omega$, is the infinitesimal generator of some cosine operator-valued function $C_2(\cdot)$.*

9.4 Comparison of Cosine Operator-Valued Functions

In this section, we give some characterizations of the property that $\|C_1(t, A_1) - C(t, A)\| = O(t^2)\,(t \to 0^+)$, and then apply Theorem 9.1.3 to show that $\|C_1(t, A_1) - C(t, A)\| = O(t^2)\,(t \to 0^+)$ and $D(A) \subseteq D(A_1)$ iff A_1 is a bounded additive perturbation of A.

Theorem 9.4.1 ([234]). *Let $C(\cdot, A)$ be a cosine operator-valued function with a generator A, and let A_1 be a linear operator. The following statements are equivalent:*

(i) A_1 generates a cosine operator-valued function $C_1(\cdot, A_1)$ that satisfies $\|C_1(t, A_1) - C(t, A)\| = O(t^2)\,(t \to 0^+)$;

(ii) there exists $B \in B(E, Fav_{C(\cdot,A)})$ such that $A_1 = A(I - B) + \lambda^2 B$ for some $\lambda > \omega$;

(iii) there exists a $B \in B(E)$ such that the function $F(\cdot) \equiv F_{B,\lambda}(\cdot)$ defined in (9.3) is square Lipschitz continuous and $A_1 = A(I - \lambda\hat{F}(\lambda)) + \lambda^3\hat{F}(\lambda)$;

(iv) A_1 generates a cosine operator-valued function $C_1(\cdot, A_1)$, $D(A_1^*) = D(A^*)$, and $A_1^* - A^*$ is a bounded operator acting from $D(A^*)$ to E^*;

(v) A_1 generates a cosine operator-valued function $C_1(\cdot, A_1)$ and $\|(\lambda^2 - A_1)^{-1} - (\lambda^2 - A)^{-1}\| = O(\lambda^{-4})\,(\lambda \to \infty)$.

Proof. The proof of (i) $\Rightarrow$ (ii) is similar to the proof for (C_0)-semigroups. Let B be the operator defined by $Bx := x - (\lambda^2 - A)^{-1}(\lambda^2 - A_1)x$ for $x \in D(A_1)$. Since for all $x \in D(A_1)$,

$$\|Bx\| = \lim_{\eta \to 0} \ \|x - \lambda^2(\lambda^2 - A)^{-1}x + \frac{2}{\eta^2}(\lambda^2 - A)^{-1}(C_1(\eta, A_1)x - x)\|$$

$$= \lim_{\eta \to 0} \|x - \lambda^2(\lambda^2 - A)^{-1}x + \frac{2}{\eta^2}(\lambda^2 - A)^{-1}(C(\eta, A)x - x)$$

$$-\frac{2}{\eta^2}(\lambda^2 - A)^{-1}(C(\eta, A)x - C_1(\eta, A_1)x)\|$$

$$\leq \lim_{\eta \to 0} \|(\lambda^2 - A)^{-1}\| \ \frac{2}{\eta^2} \ \|C(\eta, A) - C_1(\eta, A_1)\| \ \|x\| \leq K\|(\lambda^2 - A)^{-1}\| \ \|x\|,$$

B is bounded and can be extended to a bounded operator (still denoted by B) on the whole space E. To show that B maps E to $Fav_{C(\cdot,A)}$ continuously, we set $y = Bx$ for $x \in D(A_1)$. Then

$$\limsup_{\eta \to 0} \frac{2}{\eta^2}\|C(\eta, A)y - y\|$$

$$\leq \limsup_{\eta \to 0} \frac{2}{\eta^2}\big(\|(\lambda^2\eta^2 - C(\eta, A) + I)(x + y) - (\lambda^2\eta^2 - C_1(\eta, A_1) + I)x\|$$

$$+\|C(\eta, A)x - C_1(\eta, A_1)x\|\big) + 2\lambda^2\|y\|$$

$$\leq K\|x\| + 2\lambda^2\|y\| \leq K(1 + 2\lambda^2\|(\lambda^2 - A)^{-1}\|)\|x\|,$$

so that $|Bx|_{Fav_{C(\cdot,A)}} \leq K(1 + (1 + 2\lambda^2)\|(\lambda^2 - A)^{-1}\|)\|x\|$ for all $x \in D(A_1)$ (and hence for all $x \in E$). From $(\lambda^2 - A)(x - Bx) = (\lambda^2 - A_1)x$, we have $A_1x = A(I - B)x + \lambda^2 Bx$ for all $x \in D(A_1)$, i.e., $A_1 \subseteq A(I - B) + \lambda^2 B$. Since $Fav_{C(\cdot,A)}$ satisfies Condition (Z), Corollary 9.1.1 implies that $A(I - B) + \lambda^2 B$ is a generator of cosine operator function and hence coincides with A_1.

Taking the Laplace transform of $F(\cdot)$, we obtain $\hat{F}(\mu) = \lambda^2\mu^{-1}(\mu^2 - A)^{-1}B - \mu(\mu^2 - A)^{-1}B + \mu^{-1}B$, so that $B = \lambda\hat{F}(\lambda)$. Further, using (9.3), for all $x \in E$, we have

$$|\ \|F(t)x\| - \|(C(t, A) - I)Bx\|\ | \leq \|\lambda^2 \int_0^t S(s, A)Bxds\| = O(t^2)\,(t \to 0^+),$$

which implies that $B \in B(E, Fav_{C(\cdot,A)})$ iff $\|F(t)\| = O(t^2)\,(t \to 0^+)$. Hence (ii) and (iii) are equivalent.

(iii) $\Rightarrow$ (i). In view of Theorem 9.3.3, we need only to show that if $F(t) = O(t^2)\,(t \to 0^+)$, then $Var(F(\cdot), t) = O(t^2)\,(t \to 0^+)$. But, because of (9.3), this is equivalent to showing that $\|(C(t,A) - I)B\| = O(t^2)\,(t \to 0^+)$ implies $Var(C(\cdot, A)B, t) = O(t^2)\,(t \to 0^+)$. Therefore, we suppose that $\|(C(s,A) - I)B\| \leq Ks^2$ for $0 \leq s \leq \tau$. For any subdivision $\{t_0, t_1, \cdots, t_n\}$ of $[0,t] \subseteq [0,1]$ with $h_i = t_i - t_{i-1} \leq \tau$, let n_i be the largest integer such that $n_i h_i \leq t_i$. One has

$$C(t_i, A) - C(t_{i-1}, A) = C(t_{i-1}, A) - C(t_{i-1,A} - C(t_i - t_{i-1})) + 2C(t_{i-1}, A)(C(t_i - t_{i-1}, A) - I),$$

and, therefore,

$$\|C(t_i, A) - C(t_{i-1}, A)\| \leq \|C(t_i - n_i h_i, A) - C((n_i+1)h_i - t_i, A)\| + 2KMe^{\omega t} n_i h_i^2$$

$$\leq 2KMe^{\omega t}(n_i + 2)h_i^2 \leq 4KMe^{\omega t} t h_i.$$

Therefore,

$$\sum^{n}$$

Finally, to prove (v) $\Rightarrow$ (i), we write (see [232], Theorem 3.9)

$$\|C_1(t, A_1)x - C(t, A)x\| = \lim_{\lambda \to \infty} \lambda^4 \|(\lambda^2 - A_1)^{-1}(C_1(t, A_1) - C(t, A))(\lambda^2 - A)^{-1}x\|$$

$$\leq \lim_{\lambda \to \infty} \| \int_0^t S_1(t-s, A_1)\lambda^4((\lambda^2 - A_1)^{-1} - (\lambda^2 - A)^{-1})C(s, A)x ds\| \leq Kt^2\|x\|.$$

Hence mixed type perturbations of the form $A_1 = A(I - B) + \lambda^2 B$ with $B \in B(E, Fav_{C(\cdot,A)})$ characterize those cosine functions $C_1(\cdot, A_1)$ which satisfy $\|C_1(t, A_1) - C(t, A)\| = O(t^2)\,(t \to 0^+)$. In this case, although $D(A_1^*) = D(A^*)$, the domain of A_1 may not contain the domain of A. What kind of cosine operator-valued functions $C_1(\cdot, A_1)$ have the property that $D(A) \subseteq D(A_1)$ and $\|C_1(t, A_1) - C(t, A)\| = O(t^2)\,(t \to 0^+)$? It is clear that additive perturbations of A by bounded operators generate cosine functions with this property. The next theorem shows that they actually characterize this property. Another characterization is the mixed type perturbation of the form $A_1 = (I - B)A + \lambda^2 B$ with $R(B^*) \subseteq Fav_{C^*(\cdot,A)}$.

Theorem 9.4.2. *Let $C(\cdot, A)$ be a cosine operator-valued function with the generator A. For any operator A_1, the following statements are equivalent:*

(i) $D(A) \subseteq D(A_1)$, and A_1 generates a cosine operator-valued function $C_1(\cdot)$ such that

$\|C_1(t, A_1) - C(t, A)\| = O(t^2)\,(t \to 0^+)$;

(ii) there exists an operator $B \in B(E)$ such that $R(B^*) \subseteq Fav_{C^*(\cdot,A)}$ and $A_1 = (I - B)A + \lambda^2 B$ for some $\lambda > \omega$;

(iii) there exists $B \in B(E)$ such that the function $G(\cdot) \equiv G_{B,\lambda}(\cdot)$ defined in (9.4) is square Lipschitz continuous at 0 and $A_1 = (I - \lambda\hat{G}(\lambda))A + \lambda^3\hat{G}(\lambda)$;

(iv) $A_1 = A + Q$ for some $Q \in B(E)$.

Proof. (iv) $\Rightarrow$ (1) is obvious. We first prove (i) $\Rightarrow$ (ii)+(iv). Since $D(A) \subseteq D(A_1)$, we can define the bounded operator $B := I - (\lambda^2 - A_1)(\lambda^2 - A)^{-1} = (A_1 - A)(\lambda^2 - A)^{-1}$ for $\lambda > \omega$. Then we have $A_1 x = (I - B)Ax + \lambda^2 Bx$ for $x \in D(A)$. Using (9.4), we can write

$$G(t)x = (A_1 - A)\int_0^t S(s, A)x\,ds,\ \ x \in E,\ t \geq 0.$$

Since $\|(A_1 - A)x\| \leq \overline{\lim_{t\to 0^+}} \frac{2}{t^2}\|(C_1(t, A_1) - C(t, A))x\| \leq K\|x\|$ for all $x \in D(A)$, the operator $A_1 - A$ has a bounded extension $Q \in B(E)$. Thus (iv) holds. This also implies

$$\|G(t)x\| \leq Var(G(\cdot)x, t) \leq \|Q\|\int_0^t \|S(s, A_1)x\|ds \leq \|Q\|Me^{\omega t}t^2\|x\|,\ x \in E,$$

so that $\|G(t)\| \leq \beta(G_{B,\lambda}(t) = O(t^2)$. It follows from Theorems 9.2.1 and 9.1.3 that $(I - B)A + \lambda^2 B$ is a generator of cosine function, and hence it coincides with A_1. Further, using (9.4) for all $x^* \in E^*$, we have

$$|\,\|G^*(t)x^*\| - \|(C^*(t, A) - I^*)B^*x^*\|\,| \leq \|\lambda^2\int_0^t S^*(s, A)B^*x^*ds\| = O(t^2)\,(t \to 0^+).$$

Hence $\|G(t)\| = O(t^2)\,(t \to 0^+)$ iff $R(B^*) \subseteq Fav_{C^*(\cdot,A)}$. In particular, this completes the proof of (ii).

To prove (ii) $\Leftrightarrow$ (iii), it remains to show that $B = \lambda\hat{G}(\lambda)$. This can be done by taking Laplace transform of $G_{B,\lambda}(\cdot)$ in (9.4).

(iii) $\Rightarrow$ (i). It suffices to show that $R(B^*) \subseteq Fav_{C^*(\cdot,A)}$ implies the boundedness of BA. Indeed, for all $x \in D(A)$ with $\|x\| \leq 1$ and for all $x^* \in E^*$, we have

$$|\langle BAx, x^*\rangle| = |\lim_{t\to 0} 2t^{-2}\langle (C(t, A) - I)x, B^*x^*\rangle|$$

$$\leq \overline{\lim_{t\to 0}} 2t^{-2}\|(C^*(t,A)-I^*)B^*x^*\|.$$

The uniform boundedness principle implies that $\{BAx : x \in D(A),\ \|x\| \leq 1\}$ is bounded. Hence the operator BA is bounded on $D(A)$.

9.5 Preservation of Properties under Additive Perturbations

Proposition 9.5.1 ([218, 281]). *Let $A \in \mathbf{C}(M,\omega)$, and let $B \in B(E)$. Then the operator $A+B$ generates a C_0-cosine operator-valued function and $\|C(t,A+B)-C(t,A)\| \to 0$ as $\|B\| \to 0$ uniformly on any compact set in $\mathbb{R}$.*

Proposition 9.5.2 ([217]). *Under the conditions of Proposition 9.5.1, if $\tilde{\omega} > \omega + \frac{M}{\omega}\|B\|$, then there exists a number $\tilde{M} = \tilde{M}(\omega)$ such that*

$$\|C(t,A+B)\| \leq \tilde{M}e^{\tilde{\omega}|t|}, \quad t \in \mathbb{R}. \tag{9.11}$$

Proposition 9.5.3 ([129]). *Let $A \in \mathbf{C}(M,\omega)$. Then for $x \in E$,*

$$C(t,\zeta^2 I + A)x = C(t,A)x + \zeta t \int_0^t \frac{I_1(\zeta\sqrt{t^2-s^2})}{\sqrt{t^2-s^2}} C(s,A)x\,ds, \tag{9.12}$$

where I_1 is the Bessel function and

$$\|C(t,A+\zeta^2 I)\| \leq M\,ch(\sqrt{\zeta^2+\omega^2}\,t) \quad \textit{for} \quad \zeta \in \mathbb{C}.$$

In [129], [259], [260], other more precise estimates of the expression $C(t, A \pm \zeta^2 I)$ are presented for Banach and Hilbert spaces.

Proposition 9.5.4 ([249, 282]). *Let $A \in \mathbf{C}(M,\omega)$, and let $D(A) \subseteq D(G)$. If there exist $\omega' \geq 0$ and $M' \geq 1$ such that $\rho(A)$ contains the set $\{z : z > \omega'\}$ and the function $G(z^2 I - A)^{-1}$ is infinitely many times differentiable, and, moreover,*

$$\frac{1}{n!}\|(z-\omega')^{n+1}\Big(\frac{d}{dz}\Big)^n\Big(G(z^2 I - A)^{-1}\Big)x\| \leq M'\|x\|$$

for $x \in E$ and any $n \in \mathbb{N}$ $z > \omega'$, then $A+G$ generates a C_0-cosine operator-valued function.

Proposition 9.5.5 ([141]). *If $A, G \in \mathbf{C}(M,\omega)$, then the operator $A+G$ (or its closure) may not generates a C_0-cosine operator-valued function in general, even in the case where $C(\cdot,A)$ and $C(\cdot,G)$ commute. However, we always have $\overline{A+B} \in \mathbf{H}(\pi/2,\omega)$.*

Proposition 9.5.6 ([19]). *Let $A, G \in \mathbf{C}(M,\omega)$, and let $D_1 := D(A) \cap D(G)$ be dense in E. Then on D_1, the following "generalized" cosine function (in the sense of fulfillment of conditions (i)–(ii) of Definition 2.3.1)is defined:*

$$\tilde{C}(t, A+G)x = C(t,A)x + \frac{t^2}{2}\int_0^1 j_1(t\sqrt{1-s^2}, A)C(ts, G)x\, ds,$$

where $j_1(t,A) := \frac{4}{\pi}\int_0^1 \sqrt{1-s^2}C(ts,A)x\, ds$, $x \in D_1$. If $A+G$ generates a C_0-cosine operator-valued function $C(t, A+G)$, then $C(t, A+G) = \tilde{C}(t, A+G)$, $t \in \mathbb{R}$.

Proposition 9.5.7 ([269]). *Let $A \in \mathbf{C}(M,\omega)$, and let the operator $G \in \mathbf{C}(E)$ be such that*

(i) for the C_0-sine operator valued function, $\mathbf{R}(S(t,A)) \subseteq D(G)$ for all $t \in \mathbb{R}$;

(ii) the function $GS(t,A)$ is strongly continuous in $t \in \mathbb{R}$.

Then the operator $A+G$ generates a C_0-cosine operator-valued function. Moreover,

$$C(t, A+G)x = \sum_{k=0}^{\infty} \hat{C}_k(t) \qquad S(t, A+G)x = \sum_{k=0}^{\infty} \hat{S}_k(t), \tag{9.13}$$

where $\hat{C}_0(t) := C(t,A)$, $\hat{C}_k(t) := \int_0^t C(t-s,A)G\hat{S}_{k-1}(s)\, ds$ and $\hat{S}_0(t) := S(t,A)$, $\hat{S}_k(t) := \int_0^t S(t-s,A)G\hat{S}_{k-1}(s)\, ds$ and series (9.13) converge absolutely in $B(E)$.

Proposition 9.5.8 ([269]). *Under the conditions of Proposition 9.5.7, we have the relation*

$$(\lambda I - A - G)^{-1} = (\lambda I - A)^{-1}\sum_{k=0}^{\infty}(G(\lambda I - A)^{-1})^k.$$

Proposition 9.5.9 ([267]). *Under the conditions of Proposition 9.5.7. if the C_0-sine operator-valued function $S(\cdot, A)$ id compact, the the C_0-sine operator-valued function $S(\cdot, A+G)$ is also compact.*

Proposition 9.5.10 ([263]). *Let $A_1, A_2 \in \mathbf{C}(M,\omega)$, and let $D(A_1) \subseteq D(A_2)$. Then*

$$C(t, A_1)x - C(t, A_2)x = \int_0^t S(t-s, A_2)(A_1 - A_2)S(s, A_1)x\, ds$$

for all $x \in D(A_1)$.

Theorem 9.5.1 ([263, 267]). *Let $A \in \mathbf{C}(M,\omega)$, and let $G \in \mathbf{C}(E)$ be such that*

(i) $D(A) \subseteq D(G)$;

(ii) there exists a continuous function $K(t)$ having the property

$$\|GS(t,A)x\| \leq K(t)\|x\| \quad \textit{for all} \quad x \in D(A).$$

Then the operator $A+G$ generates a C_0-cosine operator-valued function.

Proposition 9.5.11 ([267]). *Let an operator $A \in \mathbf{C}(M,\omega)$ satisfy Condition (F) with an operator $G \in \mathbf{C}(E)$, and, moreover, for a certain operator $Q \in \mathbf{C}(E)$, the condition $D(G) \subseteq D(Q)$ holds. Then $A+Q$ generates a C_0-cosine operator-valued function.*

Proposition 9.5.12 ([267]). *In Proposition 9.5.11, let the condition of inclusion of domains be replaced by the condition $D(A) \subseteq D(Q)$. If the operator Q is G-bounded, then the operator $A+Q$ generates a C_0-cosine operator-valued function.*

Proposition 9.5.13 ([256, 264]). *Let $A \in \mathbf{C}(M,\omega)$, $G \in \mathfrak{M}(C(t,A))$, $G(\lambda I - A)^{-1} \in B(E)$, and let $|\varepsilon| < K_\infty^{-1}$. Then the operator $A+\varepsilon G$ generates a C_0-cosine operator-valued function, $\lim_{\varepsilon\to 0}\|C(t,A+\varepsilon G) - C(t,A)\| = 0$ uniformly on any compact set from $\mathbb{R}$, and*

$$C(t,A+\varepsilon G) = \sum_{k=0}^{\infty} \varepsilon^k \overline{C}_k(t),$$

where $\overline{C}_0(t) := C(t,A)$; $C_k(t) := \int_0^t \overline{C}_{k-1}(t-s)GS(s,A)x\,ds$, $x \in D(A)$, $\overline{C}_k(t)$ is a continuous extension of $C_k(t)$ to the whole E and $\mathfrak{M}$ isthe class of perturbations in the sense of Miyadera.

Proposition 9.5.14 ([206]). *Let C_0-cosine operator-valued functions $C(t,A_1)$ and $C(t,A_2)$ be such that $\|C(t,A_1)\| \leq Me^{\omega t}$ and $\|C(t,A_2)\| \leq Ne^{\nu t}$ for $t \in \overline{\mathbb{R}}$ $\|(A_1 - A_2)x\| \leq a\|x\| + b\|A_1 x\|$ $x \in D(A_1)$. Then for $z > \omega$, we have*

$$\|\Big(C(t,A_1) - C(t,A_2)\Big)(z^2 I - A_1)^{-1}\| \leq \begin{cases} \frac{MN}{2\nu} Qtsh(\omega t), \\ \frac{MN}{\omega^2-\nu^2} Q\Big(ch(\omega t) - ch(\nu t)\Big), \end{cases}$$

where $Q := (1+M)b + \frac{(a+b\omega^2)M}{z^2-\omega^2}$.

Proposition 9.5.15 ([206]). *Under the conditions of Proposition 9.5.14, if $a, b \to 0$, then*

$$s\text{-}\lim_{a,b\to 0} C(t,A_2)x = C(t,A_1)x \qquad x \in D(A_1).$$

Proposition 9.5.16. *Let $K(t, A)$ be a w^*-continuous cosine operator-valued fnction, and let B be a w^*-w^* continuous operator on E^*. Then $A + B$ is the w^*-generating operator of a w^*-continuous cosine-function $K(t, A + B)$ and $\lim_{\|B\|\to 0} \|K(t, A + B) - K(t, A)\| = 0$ uniformly on any compact set $t \in [0, T]$.*

9.6 An Integral Operator on $L^p([0, T]; E)$

For C_0-groups of operators and C_0-cosine operator-valued functions, we can formulate specific perturbation theorems , which assume certain "hyperbolicity" conditions. To do this, we give certain definitions.

Let $J \subseteq \mathbb{R}$ be a certain interval. Denote by $\Sigma(J; E)$ the vector space of all linear combinations of mappings of the form $\chi_j x$, where $x \in E$ and χ_j is the characteristic function of the interval $\mathbf{T} \subseteq J$, i.e., $\Sigma(J; E)$ is the space of step functions.

Let $\{L(t)\}_{t=-\infty}^{\infty}$ be a strongly continuous family of bounded operators on E, and let $A \in \mathbf{C}(E)$. Assume that the following hypotheses hold:

H1. For each $x \in E$ and $t \in \mathbb{R}$, the integral $\int_0^t L(s)x\, ds \in D(A)$ and the mapping $t \to A \int_0^t L(s)x\, ds$ is continuous as a mapping from $\mathbb{R}$ into E;

H2. There exist a subset $D^{\odot} \subseteq D(A^*)$ and a constant $M \geq 1$ such that

(a) for any $\phi^{\odot} \in D^{\odot}$, the mapping $t \to L^*(t)A^*\phi^{\odot}$ is continuous as a mapping from $\mathbb{R}$ into E^*;

(b) for each $x \in E$, there exists $\phi^{\odot} \in D^{\odot}$ such that $\|\phi^{\odot}\| \leq M \quad \|x\| = \langle x, \phi^{\odot}\rangle$.

By H1, for any $f(\cdot) \in \Sigma(\mathbb{R}; E)$, the mapping $t \to A \int_0^t L(t - s)f(s)\, ds$ is continuous as a mapping from $\mathbb{R}$ into E, and we can define the operator $K : \Sigma(\mathbb{R}; E) \to C(\mathbb{R}; E)$ by the formula

$$(Kf)(t) := A \int_0^t L(t - s)f(s)\, ds, \quad t \in \mathbb{R}, \quad f(\cdot) \in \Sigma(J; E).$$

By H2, for each $g^{\odot}(\cdot) \in \Sigma(\mathbb{R}; D^{\odot})$, we can analogously define $K^{\odot} : \Sigma(\mathbb{R}; D^{\odot}) \to C(\mathbb{R}; E^*)$ by the formula

$$(K^{\odot}g^{\odot})(s) := \int_s^{\infty} L^*(t - s)A^*g^{\odot}(t)\, dt, \quad s \in \mathbb{R}, \quad g^{\odot}(\cdot) \in \Sigma(\mathbb{R}; D^{\odot}).$$

Let $T > 0$ be finite. Then the operators K and $K^{\odot}$ induce the operators

$$(K_T f)(t) := A \int_0^t L(t - s)f(s)\, ds, \quad t \in [0, T], \quad f(\cdot) \in \Sigma(J; E) \tag{9.14}$$

and

$$(K_T^{\odot} g^{\odot})(s) := \int_s^T L^*(t-s) A^* g^{\odot}(s)\, ds, \quad s \in [0,T]; \quad g^{\odot}(\cdot) \in \Sigma(J; D^{\odot}).$$

Let $p, q \in \overline{\mathbb{R}}_+$, and let $\frac{1}{p} + \frac{1}{q} = 1$. For $f(\cdot) \in L^p(J;E)$ and $g^*(\cdot) \in L^q(J;E^*)$, we set

$$\langle\langle f, g \rangle\rangle = \int_J \langle f(s), g^*(s) \rangle\, ds,$$

and by the Fubini theorem, we have

$$\langle\langle K_T f, g^{\odot} \rangle\rangle = \langle\langle f, K_T^{\odot} g^{\odot} \rangle\rangle, \quad f(\cdot) \in \Sigma([0;T];E),\ g^{\odot}(\cdot) \in \Sigma([0;T];D^{\odot}).$$

Now we note that if

$$|||K_T|||_{p_1,p_2} := \sup\left\{ ||K_T f||_{L^{p_2}([0;T];E)} : f(\cdot) \in \Sigma([0;T];E), \|f\|_{L^{p_1}([0,T];E)} = 1 \right\} < \infty,$$

then by the density of $\Sigma([0;T];E)$ in $L^{p_1}([0,T];E)$ and the closedness of the operator A, for any $f(\cdot) \in L^{p_1}([0,T];E)$, the element $K_T f(\cdot) \in L^{p_2}([0,T];E)$ and is defined for almost all $t \in [0,T]$ by expression (9.14). We call attention to that if $p_2 = \infty$, then the domain of K_T, in fact, lies in $C([0,T];E)$, and $K_T : L^{p_1}([0;T];E) \to C([0;T];E)$ is continuous. Analogous arguments lead to the continuity of the mapping $K_T^{\odot} : L^{q_2}([0;T];E^{\odot}) \to C([0;T];E^*)$, where $E^{\odot}$ is the strong closure of $D^{\odot}$. Moreover, using H2 (b), we obtain

$$|||K_T^{\odot}|||_{q_2,q_1} \le |||K_T|||_{p_1,p_2} \le M|||K_T^{\odot}|||_{q_2,q_1}, \tag{9.15}$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1, \quad \frac{1}{p_2} + \frac{1}{q_2} = 1$, and

$$|||K_T^{\odot}|||_{q_2,q_1} := \sup\left\{ ||K_T^{\odot} g^{\odot}||_{L^{q_1}([0;T];E^*)} : g^{\odot} \in \Sigma([0;T];D^{\odot}), \|g^{\odot}\|_{L^{q_2}([0;T];E^*)} = 1 \right\}.$$

9.7 Elevation Theorem for C_0-Groups

Recall that for a generating operator of a C_0-semigroup $\exp(\cdot A)$, the domain $D(A^{\odot})$ is a w^*-dense set in E^*, and $A^{\odot}$ is a closed operator on $E^{\odot} = \overline{D(A^{\odot})}$.

In this section, we consider the operator K_T defined by the formula

$$(K_T f)(t) := A \int_0^t exp((t-s)A) B f(s)\, ds, \quad t \in [0,T], \tag{9.16}$$

where $B \in B(E)$.

Theorem 9.7.1 ([228]). *Let* $A \in \mathbf{GR}(M,\omega)$, *and let* $K_T \in B\Big(L^{p_1}([0,T];E), L^{p_2}([0,T];E)\Big)$ *for certain* $p_1, p_2 \in \overline{\mathbb{R}}_+$ *and* $T \in \mathbb{R}_+$. *Then* $K_T \in B\Big(L^{p_1}([0,T];E), C([0,T];E)\Big)$ *and there exists a constant* $C > 0$ *such that*

$$\|K_T f\|_{C([0,\tau];E)} \le C\|f\|_{L^{p_1}([0,\tau];E)} \quad \textit{for any} \quad \tau \in [0,T]. \tag{9.17}$$

Definition 9.7.1. We say that the operator K_T for the C_0-group $exp(\cdot A)$ satisfies **Condition HG** if there exist constants $C, \tau > 0$ and $T > 0$ such that for any $x \in E$, there exists a function $h_x(\cdot) \in C([-\tau,T];E), Bh_x(t) = \exp(tA)Bx, \quad t \in [-\tau,T]$.

Theorem 9.7.2 ([228]). *Let the conditions of Theorem 9.7.1 hold, and, additionally, let Condition HG hold. Then* $K_T \in B\Big(L^1([0,T];E), C([0,T];E)\Big)$ *and there exists a constants* $C > 0$ *such that* $\|K_T f\|_{C([0,\tau];E)} \le C\|f\|_{L^1([0,\tau];E)}$ *for any* $0 \le \tau \le T$.

Corollary 9.7.1 ([228]). *Let the conditions of Theorem 9.7.1 hold. Then there exist constants* $L, \alpha > 0$ *such that* $SV(K_T,T) \le Le^{\alpha T}T^{1/p_1}, \quad T > 0$.

Corollary 9.7.2 ([228]). *Let* $E = H$ *be Hilbert, and let the conditions of Theorem 9.7.1 hold. Then there exist constants* $L, \alpha > 0$ *such that* $SV(K_T,T) \le Le^{\alpha T}T^{1/2}, \quad T > 0$.

9.8 Elevation Theorem for C_0-Cosine Operator-Valued Functions

Let E be a Banach space, and let $A : D(A) \subset E \to E$ be a generating operator of a C_0-cosine operator-valued function $\{C(\cdot,A)\}$ on E. Also, let us consider the adjoint family $\{C(\cdot,A)^*\}$. It is well known that it is also a cosine-family of linear bounded operators on the dual space, which, however, can be not strongly continuous.

Recall that the space $E^{\odot}$ is defined as

$$E^{\odot} = \{\, x^* \in E^* \,:\, s - \lim_{t\to 0} C(t,A)^* x^* = x^* \,\},$$

where the limit is understood in the strong topology of the space E^*.

On the other hand, cosine operator-valued functions can be studied by using C_0-groups. More precisely, introducing the Kysinski space E^1,, we can

reduce the consideration to a C_0-group. As we know, the family of operators $\{\exp(t\mathbf{A})\}_{t=-\infty}^{+\infty}$ $E^1 \times E$ defined as

$$\exp(t\mathbf{A}) = \begin{pmatrix} C(t,A) & S(t,A) \\ AS(t,A) & C(t,A) \end{pmatrix}, \qquad t \in \mathbb{R}, \tag{9.18}$$

is a C_0-group. Its generating operator is $\mathcal{A} : D(A) \times E^1 \subset E^1 \times E \to E^1 \times E$; it is given as $\mathbf{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$.

Let $B \in B(E)$ be a linear continuous operator on E. It is easy to verify that the family $L(t) := S(t,A)B$, $t \in \mathbb{R}$, satisfies all the conditions presented in Sec. 9.6 with respect to $D^{\odot} = D(A^{\odot})$.

In this section, we consider the continuity of the corresponding convolution operator

$$(K_T f)(t) := A \int_0^t S(t-s,A)Bf(s)\,ds, \qquad t \in [0,T] \subset \overline{\mathbb{R}}_+, \tag{9.19}$$

in $L^p([0,T];E)$ norms, $p \in [1,+\infty]$.

Note that in the case of the cosine operator-valued functions considered, it is not easy to directly prove that $\|K_T\|_{B(L^{p_1}([0,T];E),L^{\infty}([0,T];E))}$ grows exponentially in T. Nevertheless, this property holds, and it is stated in the elevation theorem (Theorem 9.8.1).

Further, it is natural to try to find a new convolution operator $\mathcal{K}_{\mathcal{T}}$ constructed according to the group $\mathcal{U}$ that reduces the study of K_T to that of $\mathcal{K}_{\mathcal{T}}$. Therefore, the results that are true for C_0-groups can be directly extended to cosine operator-valued functions. Consider the following operator bounded on $E^1 \times E$:

$$\mathcal{B} = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$$

and the corresponding convolution operator

$$(\mathcal{K}h)(t) = \mathcal{A} \int_0^t \mathcal{U}(t-s)\mathcal{B}h(s)\,ds, \qquad t \in \mathbb{R}, \tag{9.20}$$

where $h = [g,f]^T \in \Sigma([0,T],E^1 \times E)$.

Also, for $T > 0$, we define the convolution operator $G_T : \Sigma([0,T],E) \to C([0,T],E^1)$ by the relation

$$(G_T f)(t) = \int_0^t C(t-s,A)Bf(s)\,ds, \qquad f \in \Sigma([0,T],E), \quad 0 \le t \le T.$$

Further, we have

$$(\mathcal{K}_T h)(t) = [(G_T f)(t), (K_T f)(t)]^T, \qquad 0 \le t \le T,$$

for $h = [g, f]^T \in \Sigma([0,T], E^1 \times E)$.

Lemma 9.8.1 ([228]). *Let p_1, $p_2 \in [1, +\infty]$, and let $T > 0$.*

(i) $\mathcal{K}_T$ is continuous as a mapping from $L^{p_1}([0,T], E^1 \times E)$ into $L^{p_2}([0,T], E^1 \times E)$ iff G_T is continuous as a mapping from $L^{p_1}([0,T]; E)$ into $L^{p_2}([0,T], E^1)$, and K_T is continuous from $L^{p_1}([0,T]; E)$ into $L^{p_2}([0,T]; E)$.

(ii) Let $T^* > T$. If K_{T^*} is continuous from $L^{p_1}([0,T^*], E)$ into $L^{p_2}([0,T^*], E)$ and if $p_2 = +\infty$, then $\mathcal{K}_T$ is continuous from $L^{p_1}([0,T], E^1 \times E)$ into $L^{p_2}([0,T], E^1 \times E)$.

Assertion (ii) of the lemma with $T = T^*$ and $1 \le p_2 < +\infty$ also holds. This result is a consequence (see Corollary 9.8.2) of the main elevation theorem (Theorem 9.8.1).

Theorem 9.8.1 ([228]). *Assume that there exist T_0 and $p_1, p_2 \in [1, +\infty]$ such that $\|K_{T_0}\| \in B(L^{p_1}([0,T]; E), L^{p_2}([0,T]; E))$. Then there exist constants $L > 0$ and $\alpha > 0$ such that*

$$\|K_T\|_{B(L^{p_1}([0,T];E), C([0,T];E))} \le L e^{\alpha T}, \qquad T > 0. \tag{9.21}$$

Corollary 9.8.1 ([228]). *Assume that the conditions of Theorem 9.8.1 hold. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/p_1}, \qquad T \in \mathbb{R}.$$

Corollary 9.8.2 ([228]). *Let p_1, $p_2 \in [1, +\infty]$, and let $T > 0$. Then $\|K_T\|_{B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))} < +\infty$ iff $\|\mathcal{K}_T\|_{B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))} < +\infty$.*

Corollary 9.8.3 ([228]). *Assume that E is a Hilbert space and that the conditions of Theorem 9.8.1 hold. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/2}, \qquad T > 0.$$

In particular, setting $p_1 = p_2 = +\infty$ in the obtained assertion, we obtain the following interesting consequence.

Corollary 9.8.4. *Assume that E is Hilbert and that $\mathrm{SV}(K_T, T_0) < +\infty$ for a certain $T_0 > 0$. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/2}, \qquad T \in \mathbb{R}_+.$$

In the same way as for C_0-groups, under certain additional conditions, we can strengthen the assertion of Theorem 9.8.1.

Definition 9.8.1. We say that the operator K_T in (9.19) satisfies Condition **HC** if there exist constants $C_0 > 0$ and $T_0 > 0$ such that for any $x \in E$, there exists a function $h_x \in L^\infty([0,T];E)$ having the properties

HC1 $\|h_x\|_{L^\infty([0,T];E)} \leq C\|x\|$

and

HC2 $Bh_x(t) = C(t-T,A)Bx, \qquad 0 \leq t \leq T.$

Note that at least in two cases, Condition **HC** holds; namely

(i) when B commutes with $C(t,A)$; for $t \in \mathbb{R}$, we can set $h_x(t) = C(t-T,A)x$, $0 \leq t \leq T$ and

(ii) when B is invertible; we can set $h_x(t) = B^{-1}C(t-T,A)Bx$, $0 \leq t \leq T$.

Theorem 9.8.2 ([228]). *Assume that there exist $T_0 > 0$ and $p_1, p_2 \in [1,\infty]$ such that*
$\|K_{T_0}\|_{B(L^{p_1}([0,T];E),L^{p_2}([0,T];E))} < \infty$. *Also, assume that condition HC is fulfilled. Then there exist $L > 0, \alpha > 0$ such that $\|K_T\|_{B(L^1([0,T];E),C([0,T];E))} < Le^{\alpha T}$.*

9.9 Perturbation of C_0-Groups of Operators

In this section, we use elevation theorems for studying multiplicative perturbations. Let $\exp(\cdot A)$ be a C_0-group of linear bounded operators on a Banach space E. Let $B \in B(E)$ be a certain linear bounded operator on E. The main problem is to reveal whether the multiplicatively perturbed operator $A_m = A(I+B)$ defined on the natural domain $D(A_m) = \{\, x \in E : (I+B)x \in D(A) \,\}$ is an infinitesimal operator of another C_0-group or not. First, assume that A_m is the infinitesimal operator of the C_0-group $\exp(\cdot A_m)$. As in the case of semigroup, it is easy to verify that for any $x \in E$, the following relation holds:

(9.22)

$$\exp(tA_m)x = \exp(tA)x + A\int_0^t \exp((t-s)A)B\exp(sA_m)x\,ds, \qquad t \in [0,T].$$

Consider the convolution operator $K_T : \Sigma([0,T],E) \to C([0,T];E)$ defined in (9.16). Assume that there exists $T > 0$ such that $\mathrm{SV}(K_T,T) = \|K_T\|_{B(L^\infty([0,T];E),L^\infty([0,T];E))} < +\infty$. Then K_T can be considered as a bounded operator $K_T : C([0,T];E) \to C([0,T];E)$. Relation (9.22) means that for any $x \in E$, the mapping $f_{x,T} : [0,T] \to E$ defined by $f_{x,T}(t) = \exp(tA_m)x$ for $t \in [0,T]$ satisfies the condition

$$f_{x,T}(t) = \exp(tA)x + (K_T f_{x,T})(t), \qquad t \in [0,T].$$

The converse is also true. For a given $x \in E$, we study the solvability in $C([0,T];E)$ of the convolution equation

$$f(t) = \exp(tA)x + (K_T f)(t), \qquad 0 \le t \le T, \tag{9.23}$$

and then show that $f(t)$ can be considered as $\exp(tA_m)x$. As in Sec. 9.1, this approach is divided into two steps:

(Step 1) Prove that there exists $T > 0$ such that K_T maps $C([0,T];E)$ into itself continuously and

(Step 2) prove that $\mathrm{SV}(K_T,T) = \|K_T\|_{B(L^\infty([0,T];E),L^\infty([0,T];E))} < 1$ for a certain sufficiently small $T > 0$.

If (Step 1) and (Step 2) hold, then by the contraction mapping principle Eq. (9.23) is uniquely solvable. Therefore, following Sec. 9.1, we obtain that the perturbed operator A_m is the generating operator of a C_0-semigroup.

Theorem 9.9.1 ([228]). *Assume that there exist $T_0 > 0$, $1 \le p_1 < +\infty$, and $1 \le p_2 \le +\infty$ such that $K_T \in B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))$. Then there exist $L > 0$ and $\alpha > 0$ such that for any $T > 0$,*

$$SV(K_T,T) \le Le^{\alpha T}T^{1/p_1}. \tag{9.24}$$

Moreover, the operator A_m generates a C_0-group and

$$\|\exp(tA) - \exp(tA_m)\| = O(t^{1/p_1}) \quad t \to 0. \tag{9.25}$$

Proof. Estimate (9.24) for $t \in \overline{\mathbb{R}}_+$ was obtained in Corollary 9.7.1. This property implies that assertions (Step 1) and (Step 2) hold. Therefore, as in Sec. 9.1, we obtain that A_m generates a C_0-semigroup $\exp(tA_m)$, $t \in \overline{\mathbb{R}}_+$.

On the other hand, we fix $T > 0$ and set $M = \sup_{-T \le t \le T} \|\exp(tA_m)\|$. Idedentity (9.22), together with (9.24), yields the following for $0 \le t \le T$:

$$\|\exp(tA_m) - \exp(tA)\| \le \|K_t\|_{B(L^\infty([0,T];E),L^\infty([0,T];E))} \le CLe^{\alpha T}|T|^{1/p_1}, \tag{9.26}$$

from which we obtain (9.25) for $t \in \mathbb{R}$.

It remains to show that A_m generates a C_0-group and (9.26) also holds for $t < 0$. Choose $0 < T_0 \le T$ so that $M^2 Le^{\alpha T_0}T_0^{1/p_1} \le 1/2$. Take $0 \le t \le T_0$. Then

$$\exp(tA_m) = \Big(\big(\exp(tA_m) - \exp(tA)\big)\exp(-tA) + I\Big)\exp(tA).$$

Further, by (9.26), for $t > 0$, and by the choice of T_0, we also have

$$\Big\|\Big(\exp(tA_m) - \exp(tA)\Big)\exp(-tA)\Big\| \le 1/2.$$

Therefore, the series expansion shows that the operator $\exp(\cdot A_m)$ is invertible and $\|\exp(tA_m)^{-1}\| \le 2M$. In fact, this proves that A_m generates a C_0-group.

Finally, using the series expansion once again, we obtain $0 \le t \le T_0$,

$$\big\|\exp(-tA_m) - \exp(-tA)\big\| \le \sum_{j=1}^{+\infty} \Big\|\Big(\exp(tA_m) - \exp(tA)\Big)\exp(-tA)\Big\|^j =$$

$$= \sum_{j=1}^{+\infty} \left(M^2 L e^{\alpha t} t^{1/p_1}\right)^j \le 2M^2 L e^{\alpha t} t^{1/p_1},$$

which proves (9.26) for $t < 0$, too.

Note that in this theorem, $1 \le p_1 < +\infty$. For Hilbert spaces the value $p_1 = +\infty$ is also admissible. Indeed, using Corollary 9.7.2, we easily obtain the following theorem.

Theorem 9.9.2 ([228]). *Assume that $E = H$ is Hilbert and there exist $T_0 > 0$ and p_1, $p_2 \in [1, +\infty]$, such that $K_T \in B(L^{p_1}([0,T]; E), L^{p_2}([0,T]; E))$. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le L e^{\alpha T} T^{1/2}, \qquad T \in \mathbb{R}. \tag{9.27}$$

Moreover, the operator A_m generates a C_0-group and

$$\|\exp(tA) - \exp(tA_m)\| = O(t^{1/2}) \text{ as } t \to 0.$$

Finally, under Condition **HG**, we can apply Theorem 9.7.2, which leads to the following theorem, which is true for Banach spaces.

Theorem 9.9.3 ([228]). *Let Condition* **HG** *hold. Also, assume that there exist $T_0 > 0$ and p_1, $p_2 \in [1, +\infty]$, such that $\|K_{T_0}\|_{B(L^{p_1}([0,T_0],E),L^{p_2}([0,T_0],E))} < +\infty$. Then $A_m = A(I + B)$ is a generating operator of a C_0-group, and we have the estimate*

$$\|\exp(tA) - \exp(tA_m)\| = O(t) \quad t \to 0.$$

Moreover, if E is reflexive, then B assumes its values in the domain of A, AB is a bounded operator, and $A_m = A + AB$ is a bounded perturbation of A.

9.10 Perturbations of C_0-Cosine Operator-Valued Functions

Let $C(\cdot, A)$ be a C_0-cosine family of linear bounded operators on a Banach space E with a generating operator $A : D(A) \subseteq E \to E$. Taking another linear bounded operator B on E, we pose the problem: whether the multiplicatively perturbed operator $A_m = A(I + B)$,acting on the domain $D(A_m) = \{ x \in E : (I + B)x \in D(A) \}$ is a generating operator of another C_0-cosine family or not. First, assume that A_m is a generating operator of a C_0-cosine family $C(\cdot, A_m)$. We already Knew that
(9.28)

$$C(t, A_m)x = C(t, A)x + A \int_0^t S(t-s, A)BC(s, A_m)x\, ds, \qquad t \in \mathbb{R}, \quad x \in E.$$

Acting in the same way as in Sec. 9.9, for a fixed $x \in E$, we arrive at the convolution equation

$$f(t) = C(t, A)x + (K_T f)(t), \qquad 0 \le t \le T, \tag{9.29}$$

where K_T is defined in (9.19) and f is sought for in $C([0, T]; E)$. As was shown in Sec. 9.1, if

(i) there exists $T > 0$ such that K_T continuously maps $C([0, T]; E)$ onto itself and

(ii) $\mathrm{SV}(K_T, T) = \|K_T\|_{B(L^\infty([0,T];E), L^\infty([0,T];E))} < 1$ for a sufficiently small $T > 0$,

then by the contraction mapping principle, Eq. (9.29) is uniquely solvable and A_m is a generating operator of a C_0-cosine family.

Theorem 9.10.1 ([228]). *Assume that there exist $T_0 > 0$, $1 \le p_1 < +\infty$, and $1 \le p_2 \le +\infty$ such that $\|K_T\|_{B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))} < \infty$. Then there exist $L > 0$ and $\alpha > 0$ such that for any $T > 0$, the following inequality holds:*

$$\mathrm{SV}(K_T, T) \le Le^{\alpha T} T^{1/p_1}. \tag{9.30}$$

Moreover, the operator A_m generates a C_0-cosine family, and

$$\|C(t, A) - C(t, A_m)\| = O(t^{1/p_1}) \quad t \to 0. \tag{9.31}$$

Note that $1 \le p_1 < +\infty$ in the theorem just presented. Using Corollary 9.8.3, we easily obtain the following theorem.

Theorem 9.10.2 ([228]). *Assume that $E = H$ is Hilbert and that there exist $T_0 > 0$ and $p_1, p_2 \in [1, +\infty]$ such that $\|K_T\|_{B(L^{p_1}([0,T];E), L^{p_2}([0,T];E))} < +\infty$. Then there exist $L > 0$ and $\alpha > 0$ such that*

$$\mathrm{SV}(K_T, T) \le Le^{\alpha T} T^{1/2}, \qquad T > 0. \tag{9.32}$$

Moreover, the operator A_m *generates a* C_0*-cosine family, and*

$$\|C(t,A) - C(t,A_m)\| = O(t^{1/2}) \quad t \to 0.$$

Theorem 9.10.3 ([228]). *Assume that Condition* **HC** *holds. Also, assume that there exist* $T_0 > 0$ *and* $p_1, p_2 \in [1, +\infty]$ *such that* $\|K_{T_0}\|_{B(L^{p_1}([0,T_0];E),L^{p_2}([0,T_0];E))} < +\infty$. *Then* $A_m = A(I+B)$ *is a generating operator of a* C_0*-cosine family, and the following estimate holds:*

$$\|C(t,A) - C(t,A_m)\| = O(t) \quad t \to 0.$$

References

[1] A. Ashiraliev, "On the coercive solvability of parabolic equations in spaces of smooth functions," *Izv. Akad. Nauk TurkSSSR, Ser. Fiz.-Mat, Khim., and Geel. Nauk*, **3**, 3–13 (1989).

[2] G. Babescu, "On a certain connection between second-order linear differential and cosine operator-valued functions on Banach spaces," *Lucr. Semin. Mat. Fiz. Inst. Politehn. Trian Vuia, Timisoara*, **146**, 13–16 (1982).

[3] G. Babescu, "Spectral theorems for functions with values of cosine and sine operators," *Bull. Sci. Tehn. Inst. Politehn. Temisoara. Mat. Fiz.*, **27**, No. 1, 13–20 (1982).

[4] A. G. Baskakov, "Harmonic analysis of cosine and exponential operator-valued functions," *Mat. Sb.*, **124(166)**, No. 1(5), 68–95 (1984).

[5] A. G. Baskakov, *Harmonic Analysis of Linear Operators* [in Russian], Voronezh Univ., Voronezh (1987).

[6] O. V. Besov, V. P. Il'in, and S. M. Nikol'skii, *Integral Representations of Functions and Embedding Theorems* [in Russian], Nauka, Moscow (1996).

[7] L. Berg and I. Lëfstrem, *Interpolation Spaces. An Introduction* [Russian transl.], Mir, Moscow (1980).

[8] G. M. Vainikko and M. Shlapikiene, "To a certain S.G. Krein theorem on perturbation of operators generating analytic semigroups," *Uch. Zap. Tartu Univ.*, **281**, 186–189 (1971).

[9] V. V. Vasil'ev, *Introduction to the Theory of Abstract Differential Equations* [in Russian], BGU, Belgorod (1988).

[10] V. V. Vasil'ev, "On the coercive solvability of abstract parabolic equations with a constant operator," *Differents. Uravn.,* **14**, No. 8, 1507–1510 (1978).

[11] V. V. Vasil'ev and E. V. Emel'yanova, "The coercive solvability of abstract parabolic equations in weighted spaces," *Deposit at VINITI*, No. 6621-V88.

[12] V. V. Vasil'ev, "On the set of equations of thermoelastisity,;; *Deposit. at VINITI*, No. 1681 - V87 (1987).

[13] V. V. Vasil'ev, "Spectral properties of a certain operator matrix," *Deposit at VINITI*, No. 1573-V88 (1988).

[14] V. V. Vasil'ev, S. G. Krein, and S. I. Piskarev, "Operator semigroups, cosine operator=valued functions, and linear differential equations," In: *Progress in Science and Technology, Series on Mathematical Analysis* [in Russian], Vol. 28, All-Union Institute for Scientific and Technical Information (VINITI), Akad. Nauk SSSR, Moscow (1990), pp. 87–202.

[15] V. V. Vasil'ev and S. I. Piskarev, *Differential Equations in a Banach Space, Theory of Operator Semigroups* [in Russian], MGU, Moscow (1996).

[16] V. V. Vasil'ev and S. I. Piskarev, *Bibliographical Index, Differential Equations in Abstract Spaces* [in Russian], NIVTs MGU, Moscow (2001).

[17] V. S. Vladimirov, Yu. N. Drozhzhinov, and B. I. Zav'yalov, Tauberian Theorems of Distributions [in Russian], Nauka, Moscow (1986).

[18] J. Goldstein, *Semigroups of Linear Operators and Their Applications* [Russian transl.], Vishcha Shkola, Kiev (1989).

[19] M. L. Gorbachuk and I. T. Matsishin, "On solutions of evelutionary equations with degeneration in a Banach spaces," In: Spectral Theory of Operator-Differential Equations [in Russian], Kiev (1986), pp. 5–10.

[20] R. Ya. Grabovskaya and V. I. Kononenko, "On the cosine operator-valued function generated by the sum of two commuting operators," *Deposit. at VINITI*, No. 783-84 (1984).

[21] R. Ya. Grabovskaya and V. I. Kononenko, "On therepresentation of a cosine function whose generator is the sum of two commuting operators," In: *VII Schools on Operator Theory in Function Spaces* [in Russian], Part !, Riga (1983), pp. 63–64.

[22] R. Ya. Grabovskaya, V. I. Kononenko, and S. D. Shmulevich, "On the connection between solutions of certain linear diffeffential equations in Banach spaces," *Deposit. at VINITI*, No. 2479-85 (1985).

[23] V. P. Gudkin, D. A. Dyment, and V. A. Matveev, "Coercive solvability of abstract parabolic equations in Holder spaces with weight," In: Proc KhABIZHT Conference [in Russian]

[24] I. N. Gurova, "On a certain topological method for studying difference schemes," *Dokl. Akad. Nauk SSSR*, **248**, No. 25–28 (1979).

[25] Yu. L. Daletskii and M. G. Krein, *Stability of solutions of differential equations in a Banach space* [in Russian], Nauka, Moscow (1970).

[26] Yu. N. Drozhzhinov and B. I. Zav'yalov, "Quasiasymptotics of distributions and Tauberian theorems in complex domain," *Mat. Sb.*, **102(144)**, No. 3, 372–390 (1977).

[27] Yu. V. Egorov, *Linear Differential Equations of Principal Type* [in Russian], Nauka, Moscow (1984).

[28] V. V. Ivanov, I. V. Mel'nikova, and F. M. Filinkov, *Differential-Operator Equatons and Ill-Posed problems* [in Russian], Nauka, Moscow (1995).

[29] T. Kato, *Perturbation Theory of Linear Operators* [Russian transl.], Mir, Moscow (1972).

[30] P. A. Kirichyk, "On perturbation of an operator-valued function of the Mittag-Leffler type," *Dokl. Akad. Nauk USSR*, No. 11, 9–12 (1990).

[31] Ph. Clement et al, *One-Parameter Semigroups* [Russian transl.[, Mir. Moscow (1992).

[32] V. A. Kostin, "Abstract strongly continuous pairs of trigonometrical transformation groups," *Deposit at VINITI*, No.829-80 (1980).

[33] V. A. Kostin, "To the Cauchy problem for ordinary differential equations in a Banach space," In: *Partial Differential Equations* [in Russian], Vol. 99, Novosibirsk (1989), pp..93–116.

[34] V. A. Kostin, "On the exact uniformly well-posed solvability of the Cauchy problem," *Dokl. Akad. Nauk SSSR*, **319**, No. 1, 38–41 (1991).

[35] V. A. Kostin, "To the solution of a certain problem related to the abstract cosine function," *Dokl. Ross. Akad. Nauk*, **336**, No. 5 (1994).

[36] V. A. Kostin, "On the Cauchy problem for an abstract hyperbolic equation," In: *Int. Sci. Conf. on Differ. and Integr. Equat, Abstracts of Reports*, Samara (1995), p. 135.

[37] V. A. Kostin. "To the Cauchy problem for abstract differential equations with fractional derivatives," *Dokl. Ross. Akad. Nauk* , **326**, No. 4, 597–600 (1992).

[38] V. A. Kostin, "On analytic semigroups and strongly continuous cosine functions," *Dokl. Akad. Nauk SSSR*, **307**, No. 4, 796–799 (1989).

[39] V. A. Kostin, "On a certain criterion of a strongly continuous cosine function," In: *XIV School on Operator Theory, Abstract of Reports* [in Russian], Novgorod (1989), p. 32.

[40] V. A. Kostin, "On a certain representation of a strongly continuous cosine function," *Deposit. at VINITI*, No. 2959-83 (1983).

[41] V. A. Kostin, "To the Solomyak–Iosida theorem on analytic semigroups," *Algebra Analiz*, **11**, No. 1, 91–106 (1999).

[42] A. N. Kochubei, "Cauchy problem for evolutionary equation of fractional order," *Differents. Uravn.*, **25**, No 8, 1359–1368 (1989).

[43] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometric Methods in Nonlinear Analysis* [in Russian], Nauka, Moscow (1975).

[44] M. A. Krasnosel'skii et all, *Integral Operators in Spaces of Lebesgue Integrable Functions* [in Russian], Nauka, Moscow (1966).

[45] S. G. Krein, *Linear Differential Equations in a Banach Space* [in Russian], Nauka, Moscow (1967).

[46] S. G. Krein and M. I. Khazan, "Differential equations in a Banach space," In: *Progress in Science and Technology, Series on Mathematical Analysis* [in Russian], Vol. 21, All-Union Institute for Scientific and Technical Information (VINITI), Akad. Nauk SSSR, Moscow (1983), pp. 130–264.

[47] S. G. Krein, Yu. M. Petunin, and E. I. Semenov, *Interpolation of Linear Operators* [in Russian], Nauka, Moscow (1978).

[48] I. V. Mel'nikova, "Well-posedness of the Cauchy problem for a second-order equation, and properties of the generalized resolvent," *Deposit. at VINITI*, No. 5523-82 (1982).

[49] I. V. Melnikova, "On the Cauchy problem for second-order equations," *Differents. Uravn.,* **19**, No. 3, 537–538 (1983).

[50] I. V. Mel'nikova, "Operator-valued functions $E(t) and ES(t)$, and the well-posedness of the Cauchy problem for second-order equations," *Deposit. at VINITI*, No. 4537-82 (1982).

[51] I. V. Melnikova, "Family M, N of operator-valued functions and second-order equations in a Banach space," *Izv. Vuzov, Mat.* No. 2, 45–52 (1985).

[52] I. V. Mel'nikova, "Myadera–Feller–Philips-type theorem for a complete second-order equation in a Banach space," *Izv. Vuzov, Mat.*, No. 4, 34–40 (1985).

[53] I. V. Mel'nikova and A. I. Filinkov, "Classification and well-posedness of the Cauchy problem for second-order equations in a Banach space," *Dokl. Akad. Nauk SSSR*, **276**, No. 5, 1066–1071 (1984).

[54] I. V. Mel'nikova and A. I. Filinkov, "On the principal with respect to the Cauchy problem operator for a complete second-order equation in a Banach space," *Deposit. at VINITI*, No. 3719-V86 (1986).

[55] I. V. Mel'nikova and A. I. Filinkov, "Connection of the well-posedness of the Cauchy problem for an equation and a system in a Banach space," *Dokl. Akad. Nauk SSSR*, **300**, No. 2, 280–284 (1988).

[56] I. P. Natanson, *Theory of Functionsof a Real Variable* [in Russian], Nauka, Moscow (1974)..

[57] Yu. M. Petunin and A. N. Plichko, *Theory of Characteristics of Spaces and Its Applications* [in Russian], Visshaya Shkola, Kiev (1980).

[58] S. Piskarev, "Convergence of difference schemes for solving nonlinear parabolic equations," *Math. Zametki*, **44**, No. 1, 112–123 (1988).

[59] S. I. Piskarev, "Compact convergence in approximation of differential equations in a Banach space," *Uch. Zap. Tartus. Univ.,* **633**, 11–18 (1982). .

[60] S. I. Piskarev, "Periodic and almost-periodic cosine operator-valued functions," *Mat. Sb.,* **118,** No. 3, 386–398 (1982).

[61] S. I. Piskarev, "Almost-periodic solutions of second-order equations," *Sib. Mat. Zh.* **25**, No. 3, 137–147 (1984).

[62] S. I.Piskarev, "Compactness properties in the theory of cosine operator-valued functions," *Mat. Zametki*, **51**, No. 5, 151–153 (1992).

[63] E. A. Polyanskii, *Method for Correction of a Solution of a Parabolic Equation in an Inhomogeneous Wave Guide* [in Russian]. Nauka, Moscow (1985).

[64] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. 1, Functional Analysis* [Russian transl.], Mir, Moscow (1977). .

[65] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Vol. 2, Harmonic Analysis, Self-Adjoiness* [Russian transl.], Mir, Moscow (1978).

[66] T. I. Smagina, "On a certain theorem of Phillips", In: *Operator Equations in Function Spaces* [in Russian], Voronezh (1986), pp. 91–93.

[67] V. I. Sobolev, *Lectures on Additional Chapters of Mathematical Analysis* [in Russian], Nauka, Moscow (1968).

[68] P. E. Sobolevskii, "Coercive inequalities for abstract parabolic equations," *Dokl. Akad. Nauk SSSR*, **157**, No. 1 (1964).

[69] P. E. Sobolevskii, "Semigroup theory and stability of difference schemas," In: *Operator Theory on Functional Spaces* [in Russian], Novosibirsk (1977), pp. 304–337.

[70] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators* [Russian transl.], Mir, Moscow (1980).

[71] D. Henry, *Geometric Theory of Semilinear Parabolic Equations* [Russian transl.], Mir, Moscow (1985).

[72] P. Hess, *Periodic-Parabolic Boundary-Value Problems and Positivity* [Russian transl], Mir, Moscow (2001).

[73] E. Hille and P. Phillips, *Functional Analysis and Semigroups* [Russian transl.], IL, Moscow (1962.)

[74] Amann H. Linear and quasilinear parabolic problems. Volume 1, Abstract linear theory . Monographs in Mathematics, Vol. 87, Birkhauser Verlag, 1995.

[75] Amerio M., Prouse G. Almost-periodic functions and functional equations. N.Y., 1971.

[76] Agase S.B., Raghavendra V. Existence of mild solutions of semilinear differential equations in Banach spaces. Indiana J. Pure Appl. Math., 1990, Vol. 21, 9, p. 813-821.

[77] W. Arendt, O. El-Mennaoui, and V. Kéyantuo. Local integrated semigroups: evolution with jumps of regularity. *J. Math. Anal. Appl.*, 186(2):572–595, 1994.

[78] Arendt W., Pruss J. Vector valued Tauberian theorems and asymptotic behaviour of linear Volterra equations // SIAM .- Appl. Math.- 1992.- 23 .- p.412-448.

[79] Ashyralyev A., Sobolevskii P.E. Well-posedness of parabolic difference equations. Birkhäuser Verlag. 1994. 349 p.

[80] Ashyralyev A., Piskarev S. and Weis L. On well-posedness of difference schemes for abstract parabolic equations in $L^p([0,T];E)$ spaces. Subm. to Numerical Functional Analysis and Optimization.

[81] Autret L. Entire vectors and time reversible Cauchy problems. Semigroup Forum.- 1993.- 46.- p.347-351.

[82] Babesku Gh. Bromwich's type representations for cosine and sine functions // Semin. Equat. Funct. Univ. Timisoara.- 1987.- 85.- p. 1-7. (, 1988, 12897).

[83] Babesku Gh. Regularity and uniform continuity properties of cosine and sine class of operators // Lucr. Semin. Math. Fis. Inst. politehn. Timisoara.- 1983.- November.- p. 47-50. (, 1985, 101203).

[84] Babesku Gh. Uber kosinus familien von liniaren stark stetigen operatoren und ihren infinitesimalen generatoren // Bul. Sti. Tehn. Inst. politehn. Timisoara. Ser. Math.- Fiz.- 1980.- 25, 1.- p. 12-17. (, 1982,5904).

[85] Baillon J.B. Ceracter borne de certains generatours de semigroupes lineares dans les espaces de Banach // C.R.Acad. Sc. Paris. 1980. V.290. P. 757-760.

[86] Balakrishnan A.V. An operational calculus for infinitesimal generators of semigroups. Trans. Amer. Math. Soc., 1959, 91, p.330-353.

[87] Bales L.A. Higher order single step fully discrete approximations for second order hyperbolic equations with time depedent coefficients // SIAM J. Numer. Anal.- 1986.- 23, 1.- p. 27-43. (, 1986, 111160).

[88] Barbu V. Nonlinear semigroups and differential equations in Banach spases // Noordhoff.- Leyden.-1976.

[89] Bart H. Periodic strongly continuous semigroups // Ann. Mat. Pura ed Appl.- 1977.- 115.- p. 311-318. (, 1978, 111029).

[90] Bart H., Golberg S. Characterization of almost periodic strongly continuous groups and semigroups // Math. Ann.- 1978.- 236, 2.- C. 105-116. (, 1979, 3736).

[91] Benilan Ph., Crandall M., Pazy A. Nonlinear evolution equations in Banach spaces // .

[92] Bobylev N.A. et al. Semilinear approximation of semilinear periodic problems in Banach spaces. Nonlinear Analisys, 1988.- 33.- p.473-482.

[93] Brezis H. Operateurs maximaux monotones et semigroups de contractions dans les espaces de Hilbert . North-Holland. 1973.

[94] Butzer P.L., Berens H. Semigroups of operators and approximation. New York. 1967. - 318 c.

[95] Chander R., Buche A.B. On some analytic properties of the modified sine operator // Indian J. Pure Appl. Math.-1981.- 12.-p.585-593.

[96] Chyan D.-K. Shaw S.-Y., Piskarev S. On maximal regularity and semivariation of cosine operator functions. (English. English summary) J. London Math. Soc. (2) 59 (1999), no. 3, 1023–1032.

[97] Chang J.-Ch., Shaw S.-Y. Rates of approximation and egrodic limits of resolvent families // Arch. Math.- 1996.- 66.- p.320-330.

[98] Chander R., Singh H. On the measurability and continuity properties of modified exponential cosine operator // J. Indiana Math. Soc. - 1980.- 44. - p. 365-368. (, 1981, 9784).

[99] Cioranescu I. Characterizations of almost periodic strongly continuous cosine operator functions // J. Math. Anal.- 1986.- 57, 1.- p. 222-229. (, 1986, 91294).

[100] Cioranescu I., Keyantuo V. On operator cosine functions in UMD spaces // Semigroup Forum. - 2001.- 63, - p. 429 - 440.

[101] Cioranescu I., Lizama C. Spectral properties of cosine operator functions // Aequat. Math.- 1988.- 36, 1.- p. 80-98.

[102] Cioranescu I., Lizama C. Some applications of Fejer's theorem to operator cosine functions in Banach spaces // Proc. Amer. Math. Soc.- 1997.- 125.- 8.- p. 2353-2362.

[103] Cioranescu I., Neumann U. Stielties vectors and cosine functions generators // Stud. math. (PPL).- 1987.- 1.- p. 1-7. (, 1988, 7922).

[104] Cioranescu I. , Ubilla P. A characterization of uniformly bounded cosine functions generators // Integral Equat. Oper. Theory.- 1989.- 12, 1.- p. 1-11.

[105] Choinacki W. Group representations of bounded cosine functions// J. Rein. Angew. Math., 1996, 478, p.61-84.

[106] Crandall M.G., A remark on semilinear perturbations of abstract parabolic equations. Nonlinear Anal., Theory, Meth. & Appl., 1985, Vol. 9, 12, p. 1331-1335.

[107] Cuthbert J.R. On semigroups such that $T(t) - I$ is compact for some $t > 0$ // Z. Wars.- 1971.- 18, 1-2.- p. 9-16. (, 1971, 882).

[108] Da Prato G. Abstract differential equations, maximal regularity and linearisation // Proc. Symp. Pure Math.- 1986.- 45.-1.- p.359-370.

[109] Da Prato G., Giusti E. A characterization on generators of abstract cosine functions // Boll. Del. Un. Mat.- 1967.- 22.- p. 367-362. (, 1968, 5635).

[110] Da Prato G., Ianelli M. Linear integrodifferential equations in Banach spaces // Rend. Sem. Math. Padova.- 1980.- 62.- p.207-219.

[111] Daners D., Medina P.K. Abstract evolution equations. Periodic Problems and applications. Pitman Res. Notes in Math. Ser 279, Longman, New-York,1992.

[112] R. deLaubenfels, Z. Huang, S. Wang, and Y. Wang. Laplace transforms of polynomially bounded vector-valued functions and semigroups of operators. *Israel J. Math.*, 98:189–207, 1997.

[113] de Laubenfels R. Existence families, functional calculi and evolution equations // Springer-Verlag. 1994. Lecture Notes Math. V.1570.

[114] de Laubenfels R. Integrated semigroupes and integro-differential equations // Math. Z. - 1990.- 204 .- p.501-514.

[115] de Laubenfels R., Piskarev S. The growth rate of cosine families // J. Math. Anal. Appl. - 1995.- 196 .- p. 442-451.

[116] Desch, W., and W. Schappacher, Some generation results for perturbed semigroups, in Trends in semigroup theory and applications, Ph. Clement et al (Eds.), Marcel Dekker, New York, 1989, 125-152

[117] Dettman J.W. Related semigroups and the abstract Hilbert transform // Appl. Anal.- 1977.- 79A.- p. 173-182. (, 1974, 9380).

[118] Dettman J.W. Saturations theorems connected with the abstract wave equation // SIAM J. Math. Anal.- 9, 1.- p. 59-64, 1978. (, 1978, 121368).

[119] Diekmann O., Gyllenberg M., Thieme H. Pertubing semigroups by solving Stieltjes renewal equations // Differential and integral equations. 1993. V.6. P. 155-181.

[120] Eberhard B., Greiner G. Baillon's theorem on maximal regularity // Acta Appl. Math. 27.- 1992. - p. 47-54.

[121] El-Mennaoni O. Traces de semi-groupes holomorphes singuliers Í'origine et comportement asymptotique // Thise, Besanson, 1992.

[122] El-Mennaoni O., Keyantuo V. Trace theorems for holomorphic semigroups and the second order Cauchy problems // Proc. Amer. Math. Soc., 1996, 124, 5, p. 1445-1458.

[123] Emel'anova E.V., Vasilev V. The coercive solvability of abstract parabolic equations in Banach spaces // Lecture notes in pure and applied mathematics.- 1993 .- v.155 .- p. 167-171.

[124] Engel K.-J. Operator Matrices and Systems of evolution equations. 1999.

[125] Engel K.-J., Nagel R. One parameter semigroups for linear evolution equations. Graduate Texts in Math.- Berlin.: Springer.- 2000.

[126] Ergens T., Karasözen B., Piskarev S. Approximation for semilinear Cauchy problems involving second order equations in separable Banach spaces. Nonlin. Anal., 1997, 28, p.1157-1165.

[127] Fattorini H.O. Second order linear differential equations in Banach spaces. North Holland. Amsterdam. 1985. 314 P.

[128] Fattorini H.O. A note of franctional derivatives of semigroups and cosine functions // Pacif. J. Math.- 1983.- 109, 2.- p. 335-347. (, 1984, 81121).

[129] Fattorini H.O. On the growth of solutions to second order differential equations in Banach spaces // Proc. Roy. Soc. Edinburgh.- 1985.- 101A, 3-4.- p. 237-252. (, 1986, 81022).

[130] Fattorini H.O. Ordinary differential equations in linear topological spaces, I // J. Different. Equat.- 1969.- 5, 1.- p. 72-105. (, 1969, 10539).

[131] Fattorini H.O. Ordinary differential equations in linear topological spaces, II // J. Different. Equat.- 1969.- 6.- p. 50-70.

[132] Fattorini H.O. Second order linear differential equations in Banach spaces. North Holland. Amsterdam.- 1985. 314 p. (, 1986, 6928).

[133] Fattorini H.O. Two points boundary value problems for operational differential equations. Annali Scuola Normale Superiore, 1974, Ser IV , Vol.I, n.1-2, p.63-79.

[134] Fattorini H.O. The underdetermined Cauchy problem in Banach spaces // Math. Ann. - 1973.- 200.- p. 103-112. (, 1973, 7727).

[135] Fattorini H. O. Un teorema de perturbacion para generatores de functiones coseno, Revista de la Union Mathematica Argentina 25, 1971, p. 199-211.

[136] Fattorini H.O. Uniformly bounded cosine functions in Hilbert spaces // Indiana Univ. Math. J.- 1970.- 20.- p. 411-425. (, 1971, 9522).

[137] Fattorini H.O., Radnitz A. The Cauchy problem with incomplete initial data in Banach spaces // Michigan Math. J.- 1971.- 18.- p. 291-320. (, 1972, 6742).

[138] Giusti E. Funzioni coseno periodische // Boll. Union. Mat. Ital.- 1967.- 22.- p. 478-485.

[139] Goldstein J.A. On the connection between first and second order differential equations in Banach spaces // J. Math. Anal. Appl.- 1970.- 30, 2.- p. 246-251. (, 1970, 11690).

[140] Goldstein J.A. Semigroups and second-order differential equations // J. Funct. Anal.- 1969.- 4.- p. 50-70. (, 1970, 5652).

[141] Goldstein J.A. The universal addability problem for generators of cosine functions and operator groups // Houston J. Math.- 1980.- 6, 3.- p. 365-373. (, 1981, 9785).

[142] Goldstein J.A., Oharu S. A remark on nonlinear cosine functions // Semigroup Forum.- 1987.- 34, 3.- p. 359-366. (, 1987, 101138).

[143] Goldstein J.A., Radin C., Schowalter R.E. Convergence rates of ergodic limits for semigroups and cosine functions // Semigroup Forum.- 1978.- 16.- p. 89-95. (, 1979, 2644).

[144] Gowers W. T., Maurey B. Banach spaces with small spaces of operators. Math. ann., 1997, 307, p. 543-568.

[145] Grabmüller H. Cosine families of unbounded operators // Appl. Anal.- 1985.- 19.- p. 1-38.

[146] Grabmüller H. Cosinus-Familien unbeschrankter operatoren // ZAMM Angew. Math. Mech.- 1985.- 65, 5.- p. 338-340. (, 1985, 101202).

[147] Grimmer R., Prüss J. On linear Volterra equations in Banach spaces // Comp. and Math. and Appl. - 1985.- 11.- p. 189-205.

[148] B.Z. Guo, On the exponential stability of C_0-semigroups on Banach spaces with compact perturbations, *Semigroup Forum,* 59(2) (1999), 190-196.

[149] Guidetti D. Strongly continuous semigroups, weak solutions and the variation of parameter formula in locally convex space // Boll.Union. Mat. Ital.- 1985.- C4, 1.- p. 431-440. (, 1986, 81017).

[150] D. Guidetti. The parabolic mixed Cauchy-Dirichlet problem in spaces of functions which are Hölder continuous with respect to space variables. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 7(3):161–168, 1996.

[151] Guidetti D., Karasozen B. and Piskarev S. Approximation of abstract differential equations. 2002, VINITI, p.

[152] D. Guidetti and S. Piskarev. On maximal regularity of difference schemes for parabolic equations in $C^\theta(\overline{\Omega})$ spaces. 8, Dipartimento di Matematica, Universitaá degli Studi di Bologna, 1997. Preprint.

[153] D. Guidetti and S. Piskarev. On maximal regularity of Crank-Nicholson schemes for parabolic equations in $C^\theta(\overline{\Omega})$ space. In: *Proccedings of International Conference "Function spaces. Differential operators. Problems in mathematical education*, volume 2, pages 2231–2237. RUDN, Moscow, 1998.

[154] D. Guidetti and S. Piskarev. Stability of Rothe's scheme and maximal regularity for parabolic equations in $C^\theta(\overline{\omega})$. In *Progress in partial differential equations, Vol. 1 (Pont-à-Mousson, 1997)*, pages 167–180. Longman, Harlow, 1998.

[155] Gustavson K., Lumer G. Multiplicative pertubations of semigroup generators // Pacif. J. Math.- 1972.- 41, 3.- p. 731-742. (, 1973, 2810).

[156] Heikkila S., Lakshmikantham V. Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations. Marcel Dekker, New York - Basel, 1994.

[157] Henriques H.R. Cosine operator families such that $C(t) - I$ is compact for all $t > 0$ // Indian J. Pure and Appl. Math.- 1985.- 16, 2.- p. 143-152. (, 1985, 9967).

[158] Henriquez, Hernn R. On Stepanov-almost periodic semigroups and cosine functions of operators // J. Math. Anal. Appl. – 146 (1990), no. 2, – p. 420–433.

[159] Hersh R. Explicit solutions of a class of hihger order abstract Cauchy problem // J. Different. Equat.- 1970.- 8, 3.- p. 570-579. (, 1971, 9235).

[160] Hieber M. Integrated semigroups and differential operators on $L^p(\mathbb{R}^N)$-spaces. Math. Ann., 1991, 291, p.1-16.

[161] Hieber M., Prüss J. Heat cernels and maximal $L^p - L^q$ estimates for parabolic evolution equations // Preprint.- 1998.

[162] Honig C.S. Volterra Stiltjes-Integral Equations. North-Holland, Amsterdam. 1975.

[163] Hoppe R.H.W. Discrete approximations of cosine operator functions. I // SIAM J. Numer. Anal., 19(6): p.1110–1128, 1982.

[164] Hoppe R.H.W. Interpolation of cosine operator functions // Ann. Mat. Pura Appl. (4), 136: p.183–212, 1984.

[165] Hörmander L. Estimates for translation invariant operators in L^p spaces // Acta Math., 1960, 104, p.93-140.

[166] Jefferies B. and Piskarev S. Tauberian theorems for C_0-semigroups. *Proceedings of 4th International Conference on Functional Analysis and Approximation Theory, Maratea, Italy, 2000.*, .

[167] Jefferies B., Piskarev S. Tauberian theorems for cosine operator functions. Proceedings of V.A. Steklov Institute, 2002, 236.

[168] Jung Chan Chang, Shaw S.-Y. Perturbation theory of abstract Cauchy problems and Volterra equations // Nonlin. Anal. Theory Meth. and Appl. - 1997.- 30.- 6.- p.3521-3528.

[169] N.J Kalton and G. Lancien. A solution to the problem of L^p-maximal regularity. *Math. Z.*, 235 (3): 559-568, 2000.

[170] N. Kalton and L. Weis. The H^∞ calculus and sums of closed operators. *submitted*.

[171] Kantorovitz S. Analytic families of semigroups // Semigroup Forum.- 1997.- 54.- p.356-363.

[172] Kantorovitz S. C^n-operational calculus, non-commutative Teylor formula and perturbation of semigroups // J. Funct. Anal.- 1993.- 113.- 1.- p. 139-152.

[173] Kantorovitz S. Semigroups of operators and spectral theory. Pitman Reseach Notes in Math. Series.- 330.- Longman.- 1995.- 135p.

[174] Kantorovitz S., Piskarev S. Mean stability of semigroups and cosine operator functions // Taiwaness Journal of Mathematics - to appear.

[175] Keyantuo V. Intagrated semigroups and related partial differential equations // J. Math. Anal. Appl., 1997, 212, p.135-153.

[176] Keyantuo V. The Laplace transform and the ascent method for abstract wave equations. J. Diff. Equations, 1995, 122,p.27-47.

[177] Kisynski J. On cosine operator functions and one parameter groups of operators // Studia Match.-1972.-44.-p. 93-105. (,1973,1528).

[178] Kozlov V., Maz'ya V. Differential equations with operator coeffitients with applications to boundary value problems for partial differential equations. Springer Monogr. in Math. 1999, XX.

[179] Kurepa S. A cosine functional eqation in Hilbert space // Can. J. Match.-1960.-12.-p. 45-49.

[180] Kurepa S. A cosine functional eqation in n-dimentional vector space // Glas. mat.-1958.-13.-p. 169-189.

[181] Kurepa S. A cosine functional equation in Banach algebras // Acta. Sci. Match. Szeged.-1962.-23.-p. 255-267.

[182] Kurepa S. A weakly measurable selfajoint cosine functions // Glas. mat.-1973.-8,1.-p. 73-79. (, 1973, 10718).

[183] Kurepa S. Decomposition of weakly measurable semigroups and cosine operatos functions // Glas. mat.-1976.-3, 11(31).-p.91-95. (, 1977, 12866).

[184] Kurepa S. Semigroups and cosine functions // Func. Anal. Lect. Notes Match.-1982.-948.-p. 47-72. (, 1983, 6797).

[185] Kusano T., Oharu S. Semilinear evolution equations with singularities in ordered Banach spaces. Diff. and Integral equations, 1992, V.5, 6, p.1383-1405.

[186] Le Merdy Ch. The semilarity problem for bounded analytic semigroups on Hilbert space// Semigroup Forum, 1998, V.56, p. 205-224.

[187] C. Le Merdy. Counterexamples on L_p-maximal regularity. *Math. Z.*, 230:47–62, 1999.

[188] Li M., Gu X.H. and Huang F.L., On unbounded perturbations of semigroups: compactness and norm continuity, *Semigroup Forum*, 2001, to appear.

[189] Li Miao , Guo B.-Z. and Piskarev S. Compactness and norm continuity of the difference of two cosine functions. Subm. to Taiwaness Journal of Mathematics.

[190] Li, Y.-C., and S.-Y. Shaw, Integrated C-cosine functions and the abstract Cauchy problem, 1991, preprint

[191] Liang Jin, Xiao Tijun. Wellposedness results for certain classes of higher order abstract Cauchy problems connected with integrated semigroups // Semigroup forum.- 1998.- 56.- p.84-103.

[192] Liang Jin, Xiao Tijun. Norm continuity (for $t > 0$) of propagators of arbitary order abstract differential equations in Hilbert spaces // Journal of Math. Anal. and Appl.- 1996.- 204.- p.124-137.

[193] Liu Zh., Yong J. Qualitative properties of certain C_0-semigroupes arising in elastic systems with various damping // Advances in Diff. Equat., 1998, 3, 5, p.643-686.

[194] Lizama C. A characterization of periodic resolvent operators. Results in Math., 1990, 18, p.93-105.

[195] Lizama C. On positivity strongly continuous cosine functions // Math. Balcanica.- 1990.- 4.- p.43-49.

[196] Lizama C. Uniform continuity and compactness for resolvent families of operators. Seminar Notes in Func. Anal. and Partial Diff. Equations, Baton Rouge, USA, 1992-1993, p.29-43.

[197] Gunter Lumer and Luc Paquet. Semi-groupes holomorphes, produit tensoriel de semi-groupes et équations d'évolution. In *Séminaire de Théorie du Potentiel, No. 4 (Paris, 1977/1978)*, pages 156–177. Springer, Berlin, 1979.

[198] Lumer G., Paquet L. Semi-groupes holomorphes et équations d'évolution // C.R. Acad. Sci. Paris Sér. A-B, 284(4): p. 237–240, 1977.

[199] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1995.

[200] Z.H. Luo, B.Z. Guo and O. Morgul, Stability and Stabilization of Infinite Dimensional Systems with Applications, *Springer-Verlag, London Berlin Heidelberg,* 1999.

[201] Lutz D. Compactness properties of operator cosine functions // C. r. math. Rept. Acad. sci. Canada.- 1980.- 2.- p. 277-280. (, 1981, 10808).

[202] Lutz D. On bounded time-dependent perturbations of operator cosine functions // Aequat. math.- 1981.- 23, 2-3.- p. 197-203. (, 1983, 7858).

[203] Lutz D. Periodische operatorwertige cosinusfunktionen // Result. Math.- 1981.- 4, 1.- p. 75-83. (, 1981, 11954).

[204] Lutz D. Some spectral properties of bounded operator cosine functions // Math. Repts. Acad. Sci. Can.- 1982.- 4, 2.- p. 81-85. (, 1982, 10819).

[205] Lutz D. Strongly continuous operator cosine functions // Functional Analysis, Lect. Notes Math.- 1982.- 948.- p. 73-97. (, 1983, 6798).

[206] Lutz D. Uber der konvergenz operatorwertiger cosinusfunctionen mit gestortenz infinitesmalen erzeuger // Period. math. hung.- 1983.- 14, 1.- p. 101-105. (, 1983, 111221).

[207] Lutz D. Uber operatorwertiger losungen der functionalgleichung der cosinus // Math. Z.- 1980.- 171.- p. 233-245. (, 1980, 11995).

[208] Lutz D. Which operators generate cosine functions ? // Acad. Naz. Lincei. Rend. Cl. Fiz. Mat. Natur.- 1977.- 63, 5.- p. 314-317. (, 1979, 7916).

[209] Martinez C., Sanz M., Marco L. Fractional powers of operators. J. Math. Soc. Japan, 1988, 40, 2, p. 331-347.

[210] Mel'nikova I.V. Well-posedness of differential-operator problems. I. The Cauchy problem in spaces of distributions. *J. Math. Sci. (New York)*, 93(1):1–21, 1999. Functional analysis, 2.

[211] Mel'nikova I.V. Properties of the Lions d-semigroups, and generalized well-posedness of the Cauchy problem// *Funktsional. Anal. i Prilozhen.*, 31(3), p. 23–34, 95, 1997.

[212] Mel'nikova I.V., Al'shansky M.A. Well-posedness of Cauchy problem in Banach space: regular and degenerate cases // Journal Math. Science, 1997, Vol. 87, 4, p. 3732-3780.

[213] Mel'nikova I.V., Anufrieva U.A. and Ushkov V. Yu. Degenerate distribution semigroups and well-posedness of the Cauchy problem. *Integral Transform. Spec. Funct.*, 6(1-4):247–256, 1998. Generalized functions—linear and nonlinear problems (Novi Sad, 1996).

[214] Miletta P.D. Approximation of solutions to Evolution Equations. Mathematical Methods in the Applied Sciences, 1994, 17, p. 753-763.

[215] Muramatu T. Besov spaces and analytic semigroups of linear operators. J. Math. Soc. Japan, 1990, 42, 1 p.133-146.

[216] Nagy B. Approximation theorems for cosine operator functions // Acta math. Acad. sci Hung.- 1977.- 29, 1-2.- p. 69-76. (, 1977, 12783).

[217] Nagy B. Cosine operator functions and the abstract Cauchy problem // Period. Math. Hung.- 1976.- 7, 3-4.- p. 213-217. (, 1978, 2862).

[218] Nagy B. On cosine operator functions in Banach spaces // Acta sci. math.- 1974.- 36, 3-4.- p. 281-289. (, 1975, 9694).

[219] Nagy B. On the generation of cosine operator functions // Publs. math.- 1974.- 21, 1-2.- p. 151-154. (, 1976, 3863).

[220] Naito T., Nguyen Van Minh. Evolution semigroups and spectral criteria for almost periodic solutions of periodic evolution equations. J. Diff. Equat., 1999, 152, p. 358-376.

[221] Nelson S., Triggiani R. Analytic properties of cosine operators // Proc. Amer. Math. Soc.- 1979.- 74.- p. 101-104. (, 1979, 12899).

[222] Neubrander F. On the relation between the semigroup and its infinitesimal generator // Proc. Amer. Math. Soc. 1987. V.100. 1. P. 104-108.

[223] Nguyen Thanh Lan. On nonautonomous higher order Cauchy problems. Preprint, 1998.

[224] Nguyen Thanh Lan. On the wellposedness of nonautonomous second order Cauchy problems. Arbeitsberich Functionalanalysis,1998.

[225] Niechwiej Y. Cosine functions of unbounded operators and second order differential equations // RADOVI Matematicki.- 6.- 1990.- p.195-204.

[226] Oka H. A class of complete second order linear differential equations. Proc. Amer. Math. Soc., 124, 10, p. 3143 - 3150, 1996.

[227] Paquet L. Equations d'evolution pour operatours locaus et equations aus derives partilles. C.R. AB, 1978, 286, 4, p. A215–A218.

[228] Palencia C., Piskarev S. On the multiplicative perturbations of C_0-groups and C_0-cosine operator functions // Semigroup Forum, 2001, Vol. 63, 2, p. 127-152.

[229] Pazy A. Semigroups of linear operators and applications to partial differential equations . Springer-Verlag, New York. 1983. 280 p.

[230] J. Peng, M. Wang and G. Zhu, The spectrum of the generator of a class of perturbed semigroups with applications, *Acta Math. Appl. Sinica*, 20(1997), 107-113.(in Chinese)

[231] Petryshyn W.V. Approximation-solvability of nonlinear functional and differential equations. Marcel Dekker, New York - Basel - Hong Kong. 1993.

[232] Piskarev S., Shaw S.-Y. Multiplicative perturbations of C_0-semigroups and some applications to step responses and cumulative outputs // J. Func. Anal. 1995. V.128. N. 2. P. 315-340.

[233] Piskarev S., Shaw S.-Y. On certain operator families related to cosine operator functions // Taiwanese J. Math.- 1997.- 1.- p.527-546.

[234] Piskarev S., Shaw S.-Y. Perturbation and comparison of cosine operator functions // Semigroup Forum.-1995.-51.- p.225-246.

[235] Piskarev S., Shaw S.-Y. Perturbation of cosine operator functions by step response and cumulative output // Proceedings of the conference on evolution equations.- University of Strathclyde, July, 25-29, 1994.

[236] Popescu N. Equations differentielles de deuxieme ordre et operateurs hermitien-equivalents dans les espaces de Banach // Proc. 4-th Conf. Oper. Theory, Timisoara.- 1979.- p. 12-22. (, 1982, 11207).

[237] Prüss J. Evolutionary, integral equations and applications. Birkháuser, Basel, 1993.

[238] Prüss J. Positivity and regularity of hyperbolic Volterra equations in Banach spaces // Math. Ann.- 1987.- 279.- p.317-344.

[239] Radic M.V., Vajzovic F. On the bounded cosine operator function in Banach space // Mat. .- 1987.- 39, 2.- p. 187-204. (, 1988, 21152).

[240] Rankin III S.M. A remark on cosine families // Proc. Amer. Math. Soc.- 1980.- 79.- p. 376-378. (, 1981, 2831).

[241] B.P. Rao, Uniform stabilization of a hybrid system of elasticity, *SIAM J. Control & Optim.*, 33(1995), 440-445.

[242] Rabiger R., Ricker W. C_0-semigroups and cosine families of linear operators in hereditarily indecomposable Banach spaces // Acta Sci. Math., - 1998, 64, 3-4, p. 697-706.

[243] Reghis M., Babescu Gh. Cosine and sine functions valued in Banach algebras (II): some spectral properties // Ann. Univ. Timisoara. Ser. sti. mat.-fiz.- 1988.- 26, 1.- p. 93-96.

[244] Ruess W.M., Vu Q.P. Asymptotically almost periodic solutions of evolution equations in Banach spaces. J. Diff. Equat., 1995, 122, p. 282-301.

[245] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, *SIAM Review*, 20(1978), 639-739.

[246] Rassias J. M. Six-dimensional Landau inequalities // Bull. Calcutta Math. Soc., 1997, 87, 6, p. 473-494.

[247] Rassias J. M. Multi-dimensional Landau extremum problems // J.R. Acad. Bulg. Sci., 1997, 50, 2, p. 5-9.

[248] Sandefur J.T.Jr., Existence and uniqueness of solutions of second order nonlinear differential equations. SIAM J. Math. Anal., 1984, Vol. 14, 3, p. 477-487.

[249] Serisawa H., Watanabe M. Perturbations for cosine families in Banach spaces // Houston J. Math.- 1986.- 12, 1.- p. 117-124. (, 1987, 5991).

[250] Serisawa H., Watanabe M. Time dependent perturbations for cosine families // Houston J. Math.- 1986.- 12, 4.- p. 579-586. (, 1987, 111122).

[251] Shaw S.-Y. Mean and pointwise ergodic theorems for cosine operator functions // Math. J. Okajama Univ.- 1985.- 27, Dec.- c. 197-203. (, 1986, 81013).

[252] Shaw S.-Y. Mean ergodic theorems and functional equations // J. Funct. Anal. 1989. V.87. p. 428-441.

[253] Shaw S.-Y. On w^*-continuous cosine operator functions // J. Funct. Anal.- 1986.- 66, 1.- p. 73-95. (, 1986, 81014).

[254] Shaw S.-Y. , Lee C.-S., Chiou W.-L. Representation formulas for cosine and sine functions of operators, II // Aequat. Math.- 1986.- 31, 1.- p. 64-75. (, 1987, 1913).

[255] Shaw S.-Y., Piskarev S. Asymptotic behaviour of semigroup-related functions // Saitama Math. J. 1994. V.12. P. 7-12.

[256] Shimizu M., Miyadera I. Perturbation theory for cosine families on Banach spaces // Tokyo J. Math.- 1978.- 1, 2.- p. 333-343. (, 1979, 9813).

[257] Shklyar A.Ya. Complete second order linear differential equations in Hilbert spaces // Operator theory Advanced and Appl., V.92, Biruhäuser, 1997.

[258] Sova M. Cosine operator functions // Rozpr. mat.- 1966.- 49.- p. 1-47.

[259] Sova M. Linear differential equations in Banach spaces // Rozpr. CSAV MPV.- 1975.- 85, 6.- p. 1-86. (, 1975, 12967).

[260] Sova M. On exponential growth of solutions of abstract initial value problems // Cas. pestov. mat.- 1981.- 106, 1.- p. 1-30. (, 1981, 9790).

[261] Stewart H.B. Generation of analytic semogroups by strongly elliptic operators // Trans. Amer. Math. Soc., 1974, 199, p. 141-162.

[262] Stewart H.B. Generation of analytic semigroups by strongly elliptic operators under general boundary conditions // Trans. Amer. Math. Soc., 1980, 259, p. 299-310.

[263] Takenaka T. A note on the generation theorem of cosine families in Banach spaces // : Met. Coll. Eng., Chubu Univ.- 1986.- 22.- p. 71-76. (, 1987, 111123).

[264] Takenaka T., Okazawa N. A Phillips-Miyadera type perturbation theorem for cosine functions of operators // Tohoku Math. J.- 1978.- 30.- p. 107-115. (, 1978, 121356).

[265] Travis C.C. Differentiability of weak solutions to an abstract inhomogenious differential equation // Proc. Amer. Math. Soc. 1981. V.82. P. 425-430.

[266] Travis C. C., Webb G. F. Second order differential equations in Banach space // Nonlin. equat. in abstract space.- 1978.- p. 331-361. (, 1980, 10795).

[267] Travis C. C., Webb G. F. Compactness, regularity and uniform continuity properties of strongly continuous cosine families // Houston J. Math.- 1977.- 3, 4.- p. 555-567. (, 1978, 111032).

[268] Travis C. C., Webb G. F. Cosine families and abstract non-linear second order differential equations // Acta math. Acad. Sci. hung.- 1978.- 32, 3.- 4.- p. 75-96. (, 1979, 6737).

[269] Travis C. C., Webb G. F. Perturbation of strongly continuous cosine family generators // Colloq. math.- 1981.- 45, 2.- p. 277-285.

[270] R. Triggiani, Lack of uniform stabilization for noncontractive semigroups under compact perturbation, *Proc. Amer. Math. Soc.,* 105(1989), 375-383.

[271] Tymovski S. On the analogue of the formula $cos(t) = (e^t + e^{-t})/2$ for operator cosine function // Rocz. Ptm.- 1983.- Ser. 1.- 23, 1.- p. 173-182. (, 1984, 11099).

[272] Vainikko G. Approximative methods for nonlinear equations (two approaches to the convergence problem. Nonlinear Anal. Theory Meth. and Appl., 2, (1978), 647-687.

[273] Vajzovic F. On the cosine operator function // Glas. mat. 1987.- 22, 2.- p. 381-406. (, 1989, 11237).

[274] Vajzovic F. On cosine operator functions in Banach spaces // Rad. Acad. Nauka i Umjetn. bih od. prirod. i mat. Nauka.- 1982.- 62, 20.- p. 23-30. (, 1983, 7856).

[275] van Neerven J. The asymptotic behaviour of semigroups of linear operators// Birkhauser Verlag.- 1996.

[276] van Neerven J. Hahn-Banach type theorem for adjoint semigroups. // Math. Ann., 1990, 28, p. 63-71.

[277] Vidav I., Spectral of perturbed semigroups with applications to transport theory, *J. Math. Anal. Appl.*, 30(1970), 264-279.

[278] Voigt J. On convex compactness property for strong operator topology // Notes di Matematika 12(1992), p. 259-269.

[279] Warga J., Optimal Control of Differential Equations, Academic Press, New York, 1972.

[280] Watanabe M. A new proof of the generation theorem of cosine families in Banach spaces // Houston J. Math.- 1984.- 10, 2.- p. 285-290. (, 1985, 101204).

[281] Watanabe M. A perturbation theory for abstract evolution equations of second order // Proc. Jap. Acad.- 1982.- 58.- p. 143-146. (, 1982, 11973).

[282] Watanabe M. Cosine families of operators and applications // Different. Equat. in Banach spaces: Lect. Notes Math.- 1986.- 1223.- p. 278-292. (, 1987, 111121).

[283] Webb G.F. A representation formula for strongly continuous cosine families // Aequat. math.- 1980.- 20, 1.- p. 115-116. (, 1980, 10796).

[284] Webb G.F., Compactness of bounded trajectories of dynamic systems in infinite dimensional systems, *Proc. of the Royal Soc. of Edinburg,* 84A (1979), p. 19 - 33.

[285] Webb G.F. An operator-theoretic formulation of asynchronic exponential grouth // Trans. Amer. Math. Soc. 1987. V.303. p. 751-763.

[286] L. Weis. A new approach to maximal L_p regularity. *Proc of the 6th Internat. Conf on Evolution Equations*, Marcel Dekker 2000.

[287] L. Weis. Operator-valued multiplier theorems and maximal L_p regularity. *Mathematische Annalen*, **319**, pp. 735 -758, 2001.

[288] Xiao T., Liang J. Analyticity of the propagators of second order linear differential equations in Banach spaces. Semigroups Forum, 1992, V.44, p.356-363.

[289] Xiao T., Liang J. Entire solutions of higher order abstract Cauchy problems // J. Math. Anal. and Appl., 1997, 208, p.298-310.

[290] Xiao T., Liang J. Differential operators and C-welposedness of complete second order abstract Cauchy problems // Pacific J. Math., 1998, V.186, 1, p. 167-200.

[291] Xiao T., Liang J. The Cauchy problem for higher order abstract differential equations. Lect. Notes in Math. 1701, 1998.

[292] You P.H. Characteristic conditions for a C_0-semigroups with continuity in the uniform operator topology for $t > 0$ in Hilbert space // Proc. Amer. Math. Soc.- 1992.- 116.- p.991-997.

[293] Zaidman S. Topics in abstract differential equations. Pitman, V.304, 1994.

[294] Zaidman S. Topics in abstract differential equations II. Pitman, V.321, 1994.

[295] Zaidman S. Functional analysis and differential equations in abstract spaces. Chapman & Hall/CRC, Monographs and Surveys in Pure and Applied Mathematics, V. 100, 1999.

[296] Q.C. Zhang and F.L. Huang, On compact perturbations of C_0-semigroups, *J. Sichuan Univ.* 35(1998), 829-833. (in Chinese)