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Existence and impulsive stability for second order retarded differential equations

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We consider certain second order delay differential equations and prove that the solutions can be stabilized by the imposition of proper impulse controls. Our results generalize recent results by Xiang Li and Peixuan Weng. May, 2005 ICMC-USP

Key Words: Delay differential equations; Lyapunov functionals; impulses.

1. INTRODUCTION

One of the most important parts of the qualitative theory in differential equations is the stability of solutions. The problem of stabilizing the solutions by imposing proper impulse controls is very important to many areas of the sciences and engineering. It is important, for instance, in pharmacokinetics, biotechnology, economics, chemical technology and others.

We consider certain second order delay differential equations and prove that the solutions can be stabilized by imposing proper impulse effects. An application of these equations appears, for instance, in impact theory. An impact is a short-time interaction of bodies and can be considered as an impulse action. In this direction we mention systems of billiard type which can be modelled by second order equations with impulses acting on the first derivative only, since positions of the colliding balls do not change at the moment of impulse action (impact) and their velocities acquire finite increments. Equations with impulses and delay are important, for instance, in models describing colliding viscoelastic bodies. See [2].

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The results we prove here generalize recent ones by Xiang Li and Peixuan Weng (see [4]). In [4], the authors prove that the following second order delay differential equations

$$\begin{cases} x''(t) + a(t)x(t-\tau) = 0, \quad t \geq t_0 \\ x(t) = \varphi(t), \quad t_0 - \tau \leq t \leq t_0 \\ x'(t_0) = y_0 \end{cases} \tag{1.1}$$

and

$$\begin{cases} x''(t) + \displaystyle\int_{t-\tau}^{t} b(t-u)x(u)du = 0, \quad t \geq t_0 \\ x(t) = \varphi(t), \quad t_0 - \tau \leq t \leq t_0 \\ x'(t_0) = y_0 \end{cases} \tag{1.2}$$

where $\tau > 0$, $a : [t_0, +\infty) \to \mathbb{R}$ and $b : [0, \tau] \to \mathbb{R}$ are continuous bounded functions, $x : [t_0 - \tau, +\infty) \to \mathbb{R}$, can be exponentially stabilized by fixed moments of impulse effect. This means that, under a finite number of impulsive controls, the solutions become exponentially stable.

In the present paper, we consider the more general equations

$$\begin{cases} x''(t) + \displaystyle\sum_{i=1}^{N} a_i(t)\, x(t-\tau_i) + f(x(t), x'(t)) = 0, \quad t \geq t_0 \\ x(t) = \varphi(t), \quad t_0 - \tau_N \leq t \leq t_0 \\ x'(t_0) = y_0 \end{cases} \tag{1.3}$$

and

$$\begin{cases} x''(t) + \displaystyle\sum_{i=1}^{N} \int_{t-\tau_i}^{t} b_i(t-u)x(u)du + f(x(t), x'(t)) = 0, \quad t \geq t_0 \\ x(t) = \varphi(t), \quad t_0 - \tau_N \leq t \leq t_0 \\ x'(t_0) = y_0 \end{cases} \tag{1.4}$$

where $0 \leq \tau_1 \leq \tau_2 < \ldots < \tau_N$, $a_i : [t_0, +\infty) \to \mathbb{R}$, $i = 1, \ldots, N$, are piecewise continuous bounded functions, $f : \mathbb{R}2 \to \mathbb{R}$ is continuous and bounded, and $b_i : [0, \tau] \to \mathbb{R}$, $i = 1, \ldots, N$, are Lebesgue integrable with

$$\int_0^{\tau_i} |\, b_i(s)|\, ds \leq B, \quad i = 1, \ldots, N$$

and prove that they can be exponentially stabilized by adequate impulse controls.

This paper is organized as follows. In Section 2, we define impulsive stability, that is, exponential stabilization by impulses and exponential stabilization by periodical impulses. In Section 3, we apply Schaefer Fixed Point Theorem to prove the existence of a solution on $[t_0 - \tau_N, T]$ of problems (1.3) and (1.4). In Section 4, by means of Lyapunov methods,

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we obtain the impulsive stability proving that the solutions of problems (1.3) and (1.4) can be exponentially stabilized by impulses.

2. PRELIMINARIES

Given a continuous function $z(t):\mathbb{R}\to\mathbb{R}$, let $z'(t)$ denote its left derivative and $z''(t)=(z'(t))'$. If $z(t)$ is piecewise continuous, then $z(s^-)$ and $z(s^+)$ denote, respectively, its left and right limits as t tends to s.

Let $T,t_0\in\mathbb{R}$ with $T\geq t_0$. By $C([t_0-\tau_N,T],\mathbb{R})$ we denote the Banach space of continuous functions $z:[t_0-\tau_N,T]\to\mathbb{R}$ endowed with the usual supremum norm, $\|z\|_\infty$.

We start by considering the initial value problem

$$\begin{cases} x''(t)+\displaystyle\sum_{i=1}^{N} a_i(t)\,x(t-\tau_i)+f(x(t),x'(t))=0, \quad t\geq t_0 \\ x(t)=\varphi(t), \quad t_0-\tau_N\leq t\leq t_0 \\ x'(t_0)=y_0 \end{cases} \tag{2.1}$$

where $0\leq\tau_1\leq\tau_2<\ldots<\tau_N$, $N\neq 1$, and $\{t_k\}_{k=0}^{\infty}$ is a monotone increasing unbounded sequence of real numbers. We assume that

(H₁) There exists a positive constant A such that for each $i=1,\ldots,N$, $a_i:[t_0,+\infty)\to\mathbb{R}$ is piecewise continuous and

$$|\,a_i(t)|\leq A, \quad i=1,\ldots,N;$$

($\mathbf{H_2}$) $\varphi(t)$ and $\varphi'(t)$ are continuous on $[t_0-\tau_N,t_0]$;
($\mathbf{H_3}$) $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a positive constant $F\geq 1$ such that

$$|\,f(u,v)\,|\leq F, \quad \forall\ \ u,v\in\mathbb{R};$$

($\mathbf{H_4}$) $(T-t_0)^2AN\neq 1$.

We also consider the impulses at time t_k, $k=0,1,2,\ldots$,

$$\begin{array}{c} x(t_k)=I_k(x(t_k{}^-)) \\ x'(t_k)=J_k(x(t_k{}^-)) \end{array} \tag{2.2}$$

subject to the conditions

($\mathbf{H_5}$) I_k, $J_k:\mathbb{R}\to\mathbb{R}$ are continuous and $I_k(0)=J_k(0)=0$, $k\in\mathbb{N}$;
($\mathbf{H_6}$) There exist constants c_k and d_k such that

$$|I_k(x)|\leq c_k \quad \text{and} \quad |J_k(x)|\leq d_k, \quad k\in\mathbb{N},$$

for each $x\in\mathbb{R}$.

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Now we define a solution of the impulsive problem (2.1)-(2.2).

DEFINITION 2.1. A function $x : [t_0 - \tau_N, T) \to \mathbb{R}$, $T \geq t_0$, is a solution of problem (2.1)-(2.2) through (t_0, φ, y_0) if

(i) $x(t)$ and $x'(t)$ are continuous on $[t_0, T] \setminus \{t_k; k \in \mathbb{N}\}$, admit lateral limits at t_k, $k \in \mathbb{N}$, and are right continuous at t_k, $k \in \mathbb{N}$;
(ii) $x(t)$ satisfies (2.1);
(iii) for each $k \in \mathbb{N}$, $x(t_k)$ and $x'(t_k)$ fulfill (2.2).

Next we define the exponential stabilization by impulses of the solutions of (2.1).

DEFINITION 2.2. Problem (2.1) is said to be exponentially stabilized by impulses if there exist $\alpha > 0$, a sequence $\{t_k\}_{k \in \mathbb{N}}$ with

$$t_0 < t_1 < t_2 < \ldots < t_k \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty,$$

and sequences of functions, $\{I_k\}$ and $\{J_k\}$, satisfying $(H_5), (H_6)$ such that for all $\varepsilon > 0$, there is $\delta > 0$ such that if a solution $x(t; t_0, \varphi, y_0)$ of (2.1) fulfills

$$\sqrt{\|\varphi\|_\infty{}^2 + y_0{}^2} \leq \delta, \tag{2.3}$$

then

$$\sqrt{x2(t) + x'^2(t)} \leq \varepsilon \exp[-\alpha(t - t_0)], \qquad t \geq t_0. \tag{2.4}$$

When periodic impulses are considered, we consider exponential stabilization by periodic impulses.

DEFINITION 2.3. Problem (2.1) is said to be exponentially stabilized by periodic impulses if there are $\alpha > 0$, a sequence $\{t_k\}_{k \in \mathbb{N}}$ with

$$t_0 < t_1 < t_2 < \ldots < t_k \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty,$$

and $t_k - t_{k-1} = c > 0$, and sequences of functions, $\{I_k\}$ and $\{J_k\}$, satisfying (H_5), (H_6) and

$$I_1(u) = \ldots = I_k(u) = \ldots, \quad k = 1, 2, \ldots, \forall\, u \in \mathbb{R}$$

$$J_1(u) = \ldots = J_k(u) = \ldots, \quad k = 1, 2, \ldots, \forall\, u \in \mathbb{R}$$

such that for all $\varepsilon > 0$, there is $\delta > 0$ such that if a solution $x(t; t_0, \varphi, y_0)$ of (2.1) fulfills (2.3), then we have (2.4).

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Now we consider the initial value problem

$$\begin{cases} x''(t) + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} b_i(t-u)x(u)du + f(x(t), x'(t)) = 0, \quad t \geq t_0, \quad t \neq t_k \\ x(t) = \varphi(t), \quad t_0 - \tau_N \leq t \leq t_0 \\ x'(t_0) = y_0 \end{cases} \tag{2.5}$$

where $0 \leq \tau_1 \leq \tau_2 < \ldots < \tau_N$, $N \neq 1$, and $\{t_k\}_{k=0}^{\infty}$ is a monotone increasing unbounded sequence of real numbers. We assume the hypotheses (H_2) and (H_3) and, instead of (H_1) and (H_4), we consider

$(\mathbf{H_1'})$ There exists a positive constant B such that for each $i = 1, \ldots, N$, $b_i : [0, \tau_N] \to \mathbb{R}$ is Lebesgue integrable and

$$\int_0^{\tau_i} |\, b_i(s)|\, ds \leq B;$$

$(\mathbf{H_4'})$ $(T - t_0)^2 BN \neq 1$.

We also consider the impulses at time t_k, $k = 0, 1, 2, \ldots$,

$$\begin{aligned} x(t_k) &= I_k(x(t_k{}^-)) \\ x'(t_k) &= J_k(x(t_k{}^-)) \end{aligned} \tag{2.6}$$

subject to the conditions (H_5) and (H_6).

The definition of a solution of problem (2.5)-(2.6) is analogous to Definition 2.1. In the same manner, the notions of exponential stability by impulses and exponential stability by periodic impulses hold for (2.5) instead of (2.1).

3. EXISTENCE THEOREMS

The existence of solutions of the problems considered in this paper follow from the general case presented in [1]. However the proofs are more simple, since some of the assumptions can be suppressed and replaced by (H_4).

We start by presenting a result on the existence of a solution on $[t_0 - \tau_N, T]$ of problem (2.1)-(2.2). The idea of the proof is to transform (2.1) into a fixed point problem in order to apply Schaefer Fixed Point Theorem (see [3]) so that there is a solution in each subinterval $[t_0 - \tau_N, t_1)$ and $[t_k, t_{k+1})$, $k \in \mathbb{N}$. The desired solution is therefore obtained from the solutions on the subintervals.

THEOREM 3.1. *Suppose the hypotheses* (H_1) *to* (H_6) *are fulfilled. Then problem* (2.1)-(2.2) *admits a solution on* $[t_0 - \tau_N, T]$.

Proof. Let $C^1([t_0 - \tau_N, T], \mathbb{R})$ denote the Banach space of continuous functions $z : [t_0 - \tau_N, T] \to \mathbb{R}$ of class C^1 endowed with the norm

$$\|z\| = \sup_{t_0 - \tau_N \leq s \leq T} (\, |z(s)| + |z'(s)|\,).$$

Consider the operator

$$N : C^1(\,[t_0 - \tau_N, T], \mathbb{R}\,) \longrightarrow C^1(\,[t_0 - \tau_N, T], \mathbb{R}\,)$$

given by

$$N(x)(t) = \begin{cases} \varphi(t), \qquad t_0 - \tau_N \le t \le t_0, \\ \varphi(t_0) + y_0(t - t_0) - \displaystyle\int_{t_0}^{t} (t-s) \sum_{i=1}^{N} a_i(s) x(s - \tau_i) ds \\ - \displaystyle\int_{t_0}^{t} (t-s)\; f(x(s), x'(s))\, ds, \qquad t_0 \le t \le T. \end{cases}$$

We will prove that N has a fixed point. Notice that φ is a fixed point of the restriction of N to $[t_0 - \tau_N, t_0]$, $N|_{[t_0-\tau_N, t_0]}$. Therefore it remains to prove that $N|_{[t_0,T]}$ admits a fixed point.

We prove the following assertions.

1. N is continuous.

Indeed. Let $\{x_n\}$ be a sequence in $C^1(\,[t_0 - \tau_N, T], \mathbb{R}\,)$ with $x_n \to x$. Then $x_n \to x$ and $x'_n \to x'$ converge uniformly in $C^1(\,[t_0 - \tau_N, T], \mathbb{R}\,)$ and we have

$$\begin{aligned} |N(x_n)(t) - N(x)(t)| &\le (T - t_0)\|x - x_n\| \int_{t_0}^{t} \sum_{i=1}^{N} |a_i(s)|\, ds \\ &\quad + (T - t_0) \int_{t_0}^{t} |\, f(x_n(s), x'_n(s)) - f(x(s), x'(s))|\, ds \\ &\le (T - t_0)^2 AN \|x - x_n\| \\ &\quad + (T - t_0) \int_{t_0}^{t} |\, f(x_n(s), x'_n(s)) - f(x(s), x'(s))|\, ds \end{aligned}$$

which tends to 0 as $n \to +\infty$. Hence

$$\|N(x_n) - N(x)\|_\infty \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

2. N takes bounded sets to bounded sets.

We will prove that given $p \ge 0$, there exists $l \ge 0$ such that

$$x \in B_p = \{y \in C^1(\,[t_0 - \tau_N, T], \mathbb{R}); \|y\| \le p\} \quad \text{implies} \quad \|N(x)\| \le l.$$

Given $t \in [t_0, T]$, we have

$$\begin{aligned} |N(x)(t)| &\le \|\varphi\| + |y_0|(T - t_0) + (T - t_0) \int_{t_0}^{t} \sum_{i=1}^{N} |a_i(s)|\, |x(s - \tau_i)|\; ds \\ &\quad + (T - t_0) \int_{t_0}^{t} |f(x(s), x'(s))| ds \\ &\le \|\varphi\| + |y_0|(T - t_0) + AN(T - t_0)^2 \|x\| + (T - t_0)^2 F. \end{aligned}$$

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The assertion follows by taking $l = \|\varphi\| + |y_0|(T-t_0) + AN(T-t_0)^2 p + (T-t_0)^2 F$.

3. N takes bounded sets to equicontinuous sets contained in $C^1([t_0-\tau_N, T], \mathbb{R})$.

Let l_1, $l_2 \in [t_0, T]$, with $l_1 < l_2$, and consider $x \in B_p$. Then

$$|N(x)(l_2) - N(x)(l_1)| \le |y_0(l_2 - l_1)|$$

$$+ \left| \int_{t_0}^{l_2} (l_2 - s) \sum_{i=1}^{N} a_i(s) x(s-\tau_i)\, ds - \int_{t_0}^{l_1} (l_1 - s) \sum_{i=1}^{N} a_i(s) x(s-\tau_i)\, ds \right|$$

$$+ \left| \int_{t_0}^{l_2} (l_2 - s) f(x(s), x'(s)) ds - \int_{t_0}^{l_1} (l_1 - s) f(x(s), x'(s)) ds \right| = |y_0(l_2 - l_1)|$$

$$+ \left| \int_{t_0}^{l_1} (l_2 - l_1) \sum_{i=1}^{N} a_i(s) x(s-\tau_i)\, ds + \int_{l_1}^{l_2} (l_2 - s) \sum_{i=1}^{N} a_i(s) x(s-\tau_i)\, ds \right|$$

$$+ \left| \int_{t_0}^{l_1} (l_2 - l_1) f(x(s), x'(s)) ds + \int_{l_1}^{l_2} (l_2 - s) f(x(s), x'(s)) ds \right| \le$$

$$\le |y_0|(l_2 - l_1) + (l_2 - l_1) \int_{t_0}^{l_2} \sum_{i=1}^{N} |a_i(s)|\, |x(s-\tau_i)|\, ds + (l_2 - l_1) \int_{t_0}^{l_2} |f(x(s), x'(s))| ds$$

which tends to 0 as $l_2 \longrightarrow l_1$. The equicontinuity for the cases $l_1 < l_2 \le 0$ and $l_1 \le 0 \le l_2$ follow analogously.

The assertions 1. to 3. above imply that $N(B_p)$ is bounded and equicontinuous for all $p > 0$. Therefore by the Ascoli-Arzelá Theorem, $N(B_p)$ is relatively compact and hence N is compact.

4. The following set is bounded:

$$\Lambda(N) = \{x \in C^1([t_0 - \tau_N, T], \mathbb{R}) : x = \lambda N(x) \textit{ for some } 0 < \lambda < 1\}.$$

Indeed. Let $x \in \Lambda(N)$. Then $x = \lambda N(x)$ for some $0 < \lambda < 1$. Thus for each $t \in [t_0, T]$,

$$x(t) = \lambda N x(t) =$$

$$= \lambda \left\{ \varphi(t_0) + y_0(t - t_0) - \int_{t_0}^{t} (t-s) \sum_{i=1}^{N} a_i(s) x(s-\tau_i)\, ds - \int_{t_0}^{t} (t-s) f(x(s), x'(s)) ds \right\}.$$

¿From assertion 2. and since $0 < \lambda < 1$, we have

$$|x(t)| \;<\; \|\varphi\| + |y_0|(T - t_0) + AN(T-t_0)^2 \|x\| + (T-t_0)^2 F.$$

Therefore

$$\|x(t)\| \leq \frac{\|\varphi\| + |y_0|(T-t_0) + (T-t_0)^2 F}{1 - AN(T-t_0)^2}$$

and hence $\Lambda(N)$ is bounded.

¿From assertions 1. to 4. and Schaefer Fixed Point Theorem, N has a fixed point, say $x(t)$, which is a solution of (2.1). Let us denote such solution by $x_1(t)$. Then for $t \neq t_k$, $x_1(t)$ is a solution of problem (2.1)-(2.2).

Now we suppose that $t = t_1$ and we consider the impulsive problem (2.1)-(2.2). As before, we transform (2.1)-(2.2) into a fixed point problem. We consider the operator

$$N_2 : C^1(\,[t_1, T], \mathbb{R}\,) \longrightarrow C^1(\,[t_1, T], \mathbb{R}\,)$$

given by

$$\begin{aligned} N_2(x)(t) &= I_1(x_1(t_1)) + J_1(x_1(t_1))(t - t_1) \\ &\quad - \int_{t_1}^{t} (t-s) \sum_{i=1}^{N} a_i(s) x(s - \tau_i)\, ds - \int_{t_1}^{t} (t-s) f(x(s), x'(s))\, ds. \end{aligned}$$

Following the steps of assertions 1. to 4., one can show that N_2 admits a fixed point which is a solution of (2.1)-(2.2) and which we denote by x_2. Then for $t = t_k$, $k \in \mathbb{N}$, we consider the operator

$$N_{k+1} : C^1(\,[t_k, T], \mathbb{R}\,) \longrightarrow C^1(\,[t_k, T], \mathbb{R}\,)$$

given by

$$\begin{aligned} N_{k+1}(x)(t) &= I_k(x_k(t_k)) + J_k(x_k(t_k))(t - t_k) \\ &\quad - \int_{t_k}^{t} (t-s) \sum_{i=1}^{N} a_i(s) x(s - \tau_i)\, ds - \int_{t_k}^{t} (t-s) f(x(s), x'(s))\, ds. \end{aligned}$$

Again, following the steps of assertions 1. to 4., it can be shown that N_{k+1} admits a fixed point which is a solution of (2.1)-(2.2). Let us denote such solution by x_{k+1}. Then

$$x(t) = \begin{cases} \varphi(t),\ t \in [t_0 - \tau_N, t_0] \\ x_1(t),\ t \in (t_0, t_1) \\ x_2(t),\ t \in [t_1, t_2) \\ \cdots \\ x_{k+1}(t),\ t \in [t_k, t_{k+1}) \\ \cdots \\ x_m(t),\ t \in [t_m, T) \end{cases}$$

is a solution of problem (2.1)-(2.2). □

Next we prove a result which guarantees the existence of a solution of problem (2.5)-(2.6). The proof follows the ideas of Theorem 3.1 also based on [1].

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THEOREM 3.2. *Suppose the hypotheses* (H_1'), (H_2), (H_3), (H_4'), (H_5) *and* (H_6) *are fulfilled. Then problem* (2.5)-(2.6) *admits a solution on* $[t_0-\tau_N, T]$.

Proof. Consider the operator

$$N : C^1([t_0-\tau_N, T], \mathbb{R}) \longrightarrow C^1([t_0-\tau_N, T], \mathbb{R})$$

given by

$$N(x)(t) = \begin{cases} \varphi(t), \qquad t_0-\tau_N \le t \le t_0, \\ \varphi(t_0) + y_0(t-t_0) - \displaystyle\int_{t_0}^{t} \left[(t-s) \sum_{i=1}^{N} \int_{s-\tau_i}^{s} b_i(s-u) x(u)\, du \right] ds \\ -\displaystyle\int_{t_0}^{t} (t-s) f(x(s), x'(s))\, ds, \qquad t_0 \le t \le T. \end{cases}$$

We will prove that N has a fixed point. As in Theorem 3.1, φ is a fixed point of the restriction of N to $[t_0-\tau_N, t_0]$, $N|_{[t_0-\tau_N,t_0]}$. Therefore it remains to prove that $N|_{[t_0,T]}$ admits a fixed point.

We prove some assertions.

1. N is continuous.

Indeed. Let $\{x_n\}$ be a sequence in $C^1([t_0-\tau_N, T], \mathbb{R})$ with $x_n \to x$. We have

$$\begin{aligned} |N(x_n)(t) - N(x)(t)| &\le (T-t_0)\|x - x_n\| \int_{t_0}^{t} \sum_{i=1}^{N} \int_{s-\tau_i}^{s} |b_i(s-u)|\, du\, ds \\ &\quad + (T-t_0) \int_{t_0}^{t} |f(x_n(s), x'_n(s)) - f(x(s), x'(s))|\, ds \\ &\le (T-t_0)^2 BN \|x - x_n\| \\ &\quad + (T-t_0) \int_{t_0}^{t} |f(x_n(s), x'_n(s)) - f(x(s), x'(s))|\, ds \end{aligned}$$

which tends to 0 as $n \to \infty$. Hence

$$\|N(x_n) - N(x)\| \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

2. N takes bounded sets to bounded sets.

We will prove that given $p \ge 0$, there exists $l \ge 0$ such that

$$x \in B_p = \{ y \in C^1([t_0-\tau_N, T], \mathbb{R}); \|y\| \le p \} \quad \text{implies} \quad \|N(x)\| \le l.$$

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For $t \in [t_0, T]$, we have

$$\begin{aligned} |N(x)(t)| &\leq \|\varphi\| + |y_0|(T-t_0) + (T-t_0)\int_{t_0}^{t}\sum_{i=1}^{N}\int_{s-\tau_i}^{s}|b_i(s-u)|\,|x(u)|du\ ds \\ &\quad + (T-t_0)\int_{t_0}^{t}|f(x(s),x'(s))|ds \\ &\leq \|\varphi\| + |y_0|(T-t_0) + BN(T-t_0)^2\|x\| + (T-t_0)^2F. \end{aligned}$$

The assertion follows by taking $l = \|\varphi\| + |y_0|(T-t_0) + BN(T-t_0)^2 p + (T-t_0)^2 F$.

3. N takes bounded sets to equicontinuous sets contained in $C([t_0-\tau_N, T], \mathbb{R})$.

Let l_1, $l_2 \in [t_0, T]$, with $l_1 < l_2$, and consider $x \in B_p$. Then

$$|N(x)(l_2) - N(x)(l_1)| \leq$$

$$\leq |y_0(l_2-l_1)| + \left|\int_{t_0}^{l_2}\left[(l_2-s)\sum_{i=1}^{N}\int_{s-\tau_i}^{s}b_i(s-u)x(u)\,du\right]ds\right.$$

$$\left. - \int_{t_0}^{l_1}\left[(l_1-s)\sum_{i=1}^{N}\int_{s-\tau_i}^{s}b_i(s-u)x(u)\,du\right]ds\right|$$

$$+\left|\int_{t_0}^{l_2}(l_2-s)f(x(s),x'(s))ds - \int_{t_0}^{l_1}(l_1-s)f(x(s),x'(s))ds\right| =$$

$$= |y_0(l_2-l_1)| + \left|\int_{t_0}^{l_1}\left[(l_2-l_1)\sum_{i=1}^{N}\int_{s-\tau_i}^{s}b_i(s-u)x(u)\,du\right]ds\right.$$

$$\left. + \int_{l_1}^{l_2}\left[(l_2-s)\sum_{i=1}^{N}\int_{s-\tau_i}^{s}b_i(s-u)x(u)\,du\right]ds\right|$$

$$+\left|\int_{t_0}^{l_1}(l_2-l_1)f(x(s),x'(s))ds + \int_{l_1}^{l_2}(l_2-s)f(x(s),x'(s))ds\right| \leq$$

$$\leq |y_0|(l_2-l_1) + (l_2-l_1)\int_{t_0}^{l_2}\sum_{i=1}^{N}\int_{s-\tau_i}^{s}|b_i(s-u)||x(u)|du\,ds + (l_2-l_1)\int_{t_0}^{l_2}|f(x(s),x'(s))|ds$$

which tends to 0 as $l_2 \longrightarrow l_1$. The equicontinuity for the cases $l_1 < l_2 \leq 0$ and $l_1 \leq 0 \leq l_2$ follow analogously.

The assertions 1. to 3. above imply that $N(B_p)$ is bounded and equicontinuous for all $p > 0$. Therefore the Ascoli-Arzelá Theorem implies $N(B_p)$ is relatively compact and hence N is compact.
4. The following set is bounded:

$$\Lambda(N) = \{x \in C^1([t_0 - \tau_N, T], \mathbb{R}) : x = \lambda N(x) \text{ for some } 0 < \lambda < 1\}.$$

Indeed. Let $x \in \Lambda(N)$. Then $x = \lambda N(x)$ for some $0 < \lambda < 1$. Thus for each $t \in [t_0, T]$,

$$x(t) = \lambda N x(t) = \lambda \{\varphi(t_0) + y_0(t - t_0)$$

$$- \int_{t_0}^{t} \left[(t-s) \sum_{i=1}^{N} \int_{s-\tau_i}^{s} b_i(s-u)x(u)\, du \right] ds - \int_{t_0}^{t} (t-s) f(x(s), x'(s))\, ds \Big\}.$$

¿From assertion 2. and since $0 < \lambda < 1$, we have

$$|x(t)| \ < \ \|\varphi\| + |y_0|(T - t_0) + BN(T - t_0)^2 \|x\| + (T - t_0)^2 F.$$

Therefore

$$\|x(t)\| \leq \frac{\|\varphi\| + |y_0|(T - t_0) + (T - t_0)^2 F}{1 - BN(T - t_0)^2}$$

and hence $\Lambda(N)$ is bounded.

¿From assertions 1. to 4. and Schaefer Fixed Point Theorem, N has a fixed point, say $x(t)$, which is a solution of (2.5). Let us denote such solution by x_1. Then for $t \neq t_k$, $x_1(t)$ is a solution of the given problem.

Now we suppose that $t = t_1$, that is, t is the first instant of impulse action and we consider the impulsive problem (2.5)-(2.6). Again as in Theorem 3.1, we transform (2.5)-(2.6) into a fixed point problem. We consider the operator

$$N_2 : C([t_1, T], \mathbb{R}) \longrightarrow C([t_1, T], \mathbb{R})$$

given by

$$\begin{aligned} N_2(x)(t) \ &= \ I_1(x_1(t_1)) + J_1(x_1(t_1))(t - t_1) \\ &\quad - \int_{t_1}^{t} \left[(t-s) \sum_{i=1}^{N} \int_{s-\tau_i}^{s} b_i(s-u)x(u)\, du \right] ds - \int_{t_1}^{t} (t-s) f(x(s), x'(s)) ds. \end{aligned}$$

Following the steps of assertions 1. to 4., it can be shown that N_2 admits a fixed point which is a solution of (2.5)-(2.6). We denote such solution by x_2.

When $t = t_k$, for each $k \in \mathbb{N}$, we consider the operator

$$N_{k+1} : C([t_k, T], \mathbb{R}) \longrightarrow C([t_k, T], \mathbb{R})$$

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given by

$$N_{k+1}(x)(t) = I_k(x_k(t_k)) + J_k(x_k(t_k))(t - t_k) - \int_{t_k}^{t} \left[(t-s) \sum_{i=1}^{N} \int_{s-\tau_i}^{s} b_i(s-u)x(u)\, du \right] ds - \int_{t_k}^{t} (t-s) f(x(s), x'(s)) ds$$

and then one can show, as in assertions 1. to 4., that N_{k+1} admits a fixed point which is a solution of (2.5)-(2.6). Let us denote such solution by x_{k+1}.

Then a solution given by

$$x(t) = \begin{cases} \varphi(t), \ t \in [t_0 - \tau_N, t_0] \\ x_1(t), \ t \in (t_0, t_1) \\ x_2(t), \ t \in [t_1, t_2) \\ \cdots \\ x_{k+1}(t), \ t \in [t_k, t_{k+1}) \\ \cdots \\ x_m(t), \ t \in [t_m, T) \end{cases}$$

satisfies problem (2.5)-(2.6). □

4. STABILITY THEOREMS

In this section, we prove that problems (2.1) and (2.5) can be exponentially stabilized by impulses.

THEOREM 4.1. *Suppose hypotheses* (H_1) *to* (H_4) *are fulfilled and* $f(u,v)v \geq 0$, *for all* $u, v \in \mathbb{R}$. *If* $(A+F)\tau < \exp\{-[1+N(A+F)]\tau\}$, *where* $\tau = \sum_{i=1}^{N} \tau_i$, *then problem* (2.1) *can be exponentially stabilized by impulses.*

Proof. Suppose $(A+F)\tau < \exp\{-[1+N(A+F)]\tau\}$, with $\tau = \sum_{i=1}^{N} \tau_i$. Then there exist $\alpha > 0$ and $l \geq \tau$ such that

$$(A+F)\tau \leq \exp[-2\alpha(l+\tau)] \exp\{-[1+(A+F)N]l\}. \tag{4.1}$$

Let α and l be as in (4.1). For every sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_0 < t_1 < \ldots < t_k < \ldots$ and $\lim_{k\to+\infty} t_k = +\infty$, with $\tau_N \leq t_k - t_{k-1} \leq l$, let

$$I_k(u) = J_k(u) = d_k u, \ k = 1, 2, \ldots,$$

where

$$d_k = \sqrt{\frac{p_k - (A+F)\tau}{2}}$$

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and

$$p_k = \exp[-2\alpha(t_{k+1} - t_k + \tau)] \exp\{-[1 + N(A+F)](t_{k+1} - t_k)\}.$$

Then d_k is a non-negative real number, since $p_k \geq (A+F)\tau$ from (4.1).

For every $\varepsilon > 0$, let

$$\delta = \frac{\varepsilon}{\sqrt{1 + (A+F)\tau}} \exp[-\alpha(t_1 - t_0)] \exp\left[-\frac{1}{2}[1 + (A+F)N](t_1 - t_0)\right]. \qquad (4.2)$$

We will show that, for each solution $x(t; t_0, \varphi, y_0)$ of (2.1), we have

$$\sqrt{\|\varphi\|_\infty{}^2 + y_0{}^2} \leq \delta \quad \text{implies} \quad \sqrt{x^2(t) + x'^2(t)} \leq \varepsilon \exp[-\alpha(t - t_0)], \qquad t \geq t_0.$$

Let $t \in [t_0, t_1)$ and consider the Lyapunov functional

$$V(t) = x^2(t) + x'^2(t) + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |a_i(s+\tau_i)| x^2(s) ds + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |f(x(s+\tau_i), x'(s+\tau_i))| x^2(s) ds$$

which satisfies

(i) $V(t) \geq x^2(t) + x'^2(t)$.

(ii) $V(t) \leq [(1 + (A+F)\tau] [\, \|x_t\|^2 + x'^2(t)]$, where $\|x_t\| := \sup\limits_{t-\tau_N \leq s \leq t} |x(s)|$,

since

$$V(t) \leq x^2(t) + x'^2(t) + \|x_t\|^2 \sum_{i=1}^{N} \int_{t}^{t+\tau_i} |a_i(s)| ds + \|x_t\|^2 \sum_{i=1}^{N} \int_{t}^{t+\tau_i} |f(x(s), x'(s))| ds \leq$$

$$\leq x^2(t) + x'^2(t) + \|x_t\|^2 A\tau + \|x_t\|^2 F\tau \leq [1 + (A+F)\tau][\, \|x_t\|^2 + x'^2(t)].$$

$V(t)$ also satisfies

(iii) $V'(t) \leq [1 + N(A+F)]V(t)$, for $t \in (t_0, t_1)$,

since $x(t)$ is a solution of (2.1) and

$$V'(t) = 2x(t)x'(t) + 2x'(t)x''(t) + \sum_{i=1}^{N} |a_i(t+\tau_i)| x^2(t) - \sum_{i=1}^{N} |a_i(t)| x^2(t-\tau_i)$$

$$+ \sum_{i=1}^{N} |f(x(t+\tau_i), x'(t+\tau_i))| x^2(t) - \sum_{i=1}^{N} |f(x(t), x'(t))| x^2(t-\tau_i) \leq$$

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$$= 2x(t)x'(t) + 2\sum_{i=1}^{N}|a_i(t)|x(t-\tau_i)x'(t) - 2f(x(t),x'(t))x'(t) + \sum_{i=1}^{N}|a_i(t+\tau_i)|x^2(t)$$

$$-\sum_{i=1}^{N}|a_i(t)|x^2(t-\tau_i) + \sum_{i=1}^{N}|f(x(t+\tau_i),x'(t+\tau_i))|x^2(t) - \sum_{i=1}^{N}|f(x(t),x'(t))|x^2(t-\tau_i) \leq$$

$$\leq x^2(t) + {x'}^2(t) + \sum_{i=1}^{N}|a_i(t)|{x'}^2(t) + \sum_{i=1}^{N}|a_i(t)|x^2(t-\tau_i)$$

$$-2f(x(t),x'(t))x'(t) + \sum_{i=1}^{N}|a_i(t+\tau_i)|x^2(t) - \sum_{i=1}^{N}|a_i(t)|x^2(t-\tau_i)$$

$$+\sum_{i=1}^{N}|f(x(t+\tau_i),x'(t+\tau_i))|x^2(t) - \sum_{i=1}^{N}|f(x(t),x'(t))|x^2(t-\tau_i) \leq$$

$$\leq x^2(t)+{x'}^2(t)+\sum_{i=1}^{N}|a_i(t)|{x'}^2(t)+\sum_{i=1}^{N}|a_i(t+\tau_i)|x^2(t)+\sum_{i=1}^{N}|f(x(t+\tau_i),x'(t+\tau_i))|x^2(t) \leq$$

$$\leq x^2(t) + {x'}^2(t) + ANx'^2(t) + ANx^2(t) + FNx^2(t) \leq$$

$$\leq [1+N(A+F)][x^2(t)+{x'}^2(t)] \leq [1+N(A+F)]V(t).$$

Solving $V'(t) \leq [1+N(A+F)]V(t)$, we obtain

$$V(t) \leq V(t_0)\exp\{[1+N(A+F)](t-t_0)\}. \tag{4.3}$$

Therefore

$$x^2(t) + {x'}^2(t) \leq V(t) \leq V(t_0)\exp\{[1+N(A+F)](t-t_0)\} \leq$$

$$\leq [1+(A+F)\tau][\,\|x_{t_0}\|^2 + {x'}^2(t_0)\,]\exp\{[1+N(A+F)](t_1-t_0)\} \leq$$

$$\leq [1+(A+F)\tau]\delta^2\exp\{[1+N(A+F)](t_1-t_0)\} =$$

$$= \varepsilon^2\exp[-2\alpha(t_1-t_0)] \leq \varepsilon^2\exp[-2\alpha(t-t_0)]$$

and hence

$$\sqrt{x^2(t)+{x'}^2(t)} \leq \varepsilon\exp[-\alpha(t-t_0)]. \tag{4.4}$$

But from the right continuity of $x(t)$ and $x'(t)$, (4.4) also holds on $[t_0,t_1)$. Therefore

$$\sqrt{x^2(t)+x'^2(t)} \leq \varepsilon\exp[-\alpha(t-t_0)], \qquad t\in[t_0,t_1).$$

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It follows that

$$\sup_{t_1-\tau_i\leq t\leq t_1}[x^2(t)+{x'}^2(t)]\leq \varepsilon^2\exp\left[-2\alpha(t_1-t_0-\tau)\right]. \tag{4.5}$$

Now, we repeat the procedure above for $t\in(t_1,t_2)$. Analogous to (4.3), we obtain

$$V(t)\leq V(t_1^+)\exp\{[1+N(A+F)](t_2-t_1)\}=$$

$$=\left\{x^2(t_1^+)+{x'}^2(t_1^+)+\sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|a_i(s+\tau_i)|x^2(s)ds\right.$$

$$\left.+\sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|f(x(s+\tau_i),x'(s+\tau_i))|x^2(s)\,ds\right\}\exp\{[1+N(A+F)](t_2-t_1)\}=$$

$$=\left\{x^2(t_1)+{x'}^2(t_1)+\sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|a_i(s+\tau_i)|x^2(s)ds\right.$$

$$\left.+\sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|f(x(s+\tau_i),x'(s+\tau_i))|x^2(s)\,ds\right\}\exp\{[1+N(A+F)](t_2-t_1)\}=$$

$$=\left\{I_1(x^2({t_1}^-))+J_1({x'}^2({t_1}^-))+\sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|a_i(s+\tau_i)|x^2(s)ds\right.$$

$$\left.+\sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|f(x(s+\tau_i),x'(s+\tau_i))|x^2(s)\,ds\right\}\exp\{[1+N(A+F)](t_2-t_1)\}=$$

$$=\left\{{d_1}^2[x^2({t_1}^-)+{x'}^2({t_1}^-)]+\sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|a_i(s+\tau_i)|x^2(s)ds\right.$$

$$\left.+\sum_{i=1}^{N}\int_{t_1-\tau_i}^{t_1}|f(x(s+\tau_i),x'(s+\tau_i))|x^2(s)\,ds\right\}\exp\{[1+N(A+F)](t_2-t_1)\}\leq$$

$$\leq {d_1}^2\sup_{t_1-\tau_i\leq t\leq t_1}[x^2(t)+{x'}^2(t)]\exp\{[1+N(A+F)](t_2-t_1)\}$$

$$+\sup_{t_1-\tau_i\leq t\leq t_1}x^2(t)A\tau\exp\{[1+N(A+F)](t_2-t_1)\}$$

$$+\sup_{t_1-\tau_i\leq t\leq t_1}x^2(t)F\,\tau\exp\{[1+N(A+F)](t_2-t_1)\}\leq$$

$$\leq [{d_1}^2 + (A+F)\tau] \sup_{t_1-\tau_i\leq t\leq t_1} [x^2(t) + {x'}^2(t)] \exp\{[1+N(A+F)](t_2-t_1)\} \leq$$

$$\leq [{d_1}^2 + (A+F)\tau]\varepsilon^2 \exp[-2\alpha(t_1-t_0-\tau)] \exp\{[1+N(A+F)](t_2-t_1)\}.$$

¿From the definitions of d_1 and p_1, we have

$$x^2(t) + {x'}^2(t) \leq V(t) \leq$$

$$\leq \varepsilon^2[{d_1}^2 + (A+F)\tau] \exp[-2\alpha(t_1-t_0-\tau)] \exp\{[1+N(A+F)](t_2-t_1)\} =$$

$$= \varepsilon^2 \left(\frac{p_1 + (A+F)\tau}{2}\right) \exp[-2\alpha(t_1-t_0-\tau)] \exp\{[1+N(A+F)](t_2-t_1)\} \leq$$

$$\leq \varepsilon^2 p_1 \exp[-2\alpha(t_1-t_0-\tau)] \exp\{[1+N(A+F)](t_2-t_1)\} =$$

$$= \varepsilon^2 \exp[-2\alpha(t_2-t_0)] \leq \varepsilon^2 \exp[-2\alpha(t-t_0)].$$

Hence

$$\sqrt{x^2(t) + x'^2(t)} \leq \varepsilon \exp[-\alpha(t-t_0)]. \tag{4.6}$$

In fact we have from the right continuity of $x(t)$ and $x'(t)$ that (4.6) holds for $t \in [t_1, t_2)$. Thus

$$\sqrt{x^2(t) + x'^2(t)} \leq \varepsilon \exp[-\alpha(t-t_0)], \qquad t \in [t_1, t_2).$$

It follows that

$$\sup_{t_2-\tau_i\leq t\leq t_2} [x^2(t) + {x'}^2(t)] \leq \varepsilon^2 \exp[-2\alpha(t_2-t_0-\tau)]. \tag{4.7}$$

With analogous arguments, it follows that for all $k \in \mathbb{N}$

$$\sqrt{x^2(t) + x'^2(t)} \leq \varepsilon \exp[-\alpha(t-t_0)], \qquad t \in [t_{k-1}, t_k).$$

Hence

$$\sqrt{x^2(t) + x'^2(t)} \leq \varepsilon \exp[-\alpha(t-t_0)], \qquad t \geq t_0,$$

and the proof is complete. □

Now we prove that problem (2.5) can be exponentially stabilized by impulses. The proof follows the ideas of Theorem 4.1.

THEOREM 4.2. *Suppose the hypotheses* (H_1'), (H_2), (H_3) *and* (H_4') *are fulfilled and* $f(u,v)v \geq 0$, *for all* $u, v \in \mathbb{R}$. *If* $(B+F)\tau < \exp\{-[1+N(B+F)]\tau\}$, *where* $\tau = \sum_{i=1}^{N} \tau_i$, *then problem* (2.5) *can be exponentially stabilized by impulses.*

Proof. If $(B+F)\tau < \exp\{-[1+N(B+F)]\tau\}$, with $\tau = \sum_{i=1}^{N} \tau_i$, then there exist $\alpha > 0$ and $l \geq \tau$ such that

$$(B+F)\tau \leq \exp[-2\alpha(l+\tau)] \exp\{-[1+(B+F)N]l\}. \tag{4.8}$$

Let α and l be as in (4.8). For every sequence $\{t_k\}_{k\in\mathbb{N}}$ such that $t_0 < t_1 < \ldots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = \infty$, with $\tau_N \leq t_k - t_{k-1} \leq l$, let

$$I_k(u) = J_k(u) = d_k u, \; k = 1, 2, \ldots,$$

where

$$d_k = \sqrt{\frac{p_k - (B+F)\tau}{2}}$$

and

$$p_k = \exp[-2\alpha(t_{k+1} - t_k + \tau)] \exp\{-[1+N(B+F)](t_{k+1} - t_k)\}.$$

Then d_k is a non-negative real number, since $p_k \geq (B+F)\tau$ from (4.8).

For every $\varepsilon > 0$, let

$$\delta = \frac{\varepsilon}{\sqrt{1+(B+F)\tau}} \exp[-\alpha(t_1 - t_0)] \exp\left[-\frac{1}{2}[1+(B+F)N](t_1 - t_0)\right]. \tag{4.9}$$

We have to show that for each solution $x(t; t_0, \varphi, y_0)$ of (2.5),

$$\sqrt{\|\varphi\|_\infty{}^2 + y_0{}^2} \leq \delta \quad \text{implies} \quad \sqrt{x^2(t) + x'^2(t)} \leq \varepsilon \exp[-\alpha(t - t_0)], \qquad t \geq t_0.$$

Given $t \in [t_0, t_1)$, we define the Lyapunov functional

$$V(t) = x^2(t) + x'^2(t) + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} \left[\int_u^t |b_i(u - s + \tau_i)| x^2(s) ds\right] du$$

$$+ \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |f(x(s+\tau_i), x'(s+\tau_i))| \, x^2(s) ds.$$

which satisfies the properties:

(i) $V(t) \geq x^2(t) + {x'}^2(t)$.

(ii) $V(t) \leq [(1+(B+F)\tau]\,[\,\|x_t\|^2 + {x'}^2(t)]$, where $\|x_t\| := \sup\limits_{t-\tau_N \leq s \leq t} |x(s)|$.

Indeed. We have

$$\begin{aligned} V(t) \quad &\leq \quad x^2(t) + {x'}^2(t) + \|x_t\|^2 \sum_{i=1}^{N} \int_{t-\tau_i}^{t} \int_0^{\tau_i} |b_i(s)| ds\, du \\ &\quad + \ \|x_t\|^2 \sum_{i=1}^{N} \int_t^{t+\tau_i} |f(x(s), x'(s))|\, ds \\ &\leq \quad x^2(t) + {x'}^2(t) + \|x_t\|^2 B\tau + \|x_t\|^2 F\tau \leq [1+(B+F)\tau][\,\|x_t\|^2 + {x'}^2(t)]. \end{aligned}$$

We also have

(iii) $V'(t) \leq [1 + N(B+F)]V(t)$, for $t \in (t_0, t_1)$,

since $x(t)$ is a solution of (1.4) and

$$V'(t) = 2x(t)x'(t) + 2x'(t)x''(t) + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(u-t+\tau_i)| x^2(t) du$$

$$-\sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(t-s)| x^2(s) ds + \sum_{i=1}^{N} |f(x(t+\tau_i), x'(t+\tau_i))| x^2(t) - \sum_{i=1}^{N} |f(x(t), x'(t))| x^2(t-\tau_i) \leq$$

$$\leq 2x(t)x'(t) + 2\sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(t-s)| x(s) x'(t) ds - 2f(x(t), x'(t))x'(t)$$

$$+\sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(u-t+\tau_i)| x^2(t) du - \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(t-s)| x^2(s) ds$$

$$+\sum_{i=1}^{N} |f(x(t+\tau_i), x'(t+\tau_i))| x^2(t) - \sum_{i=1}^{N} |f(x(t), x'(t))| x^2(t-\tau_i) \leq$$

$$\leq x^2(t) + {x'}^2(t) + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(t-s)| x^2(s) ds + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(t-s)| {x'}^2(t) ds$$

$$-2f(x(t), x'(t))x'(t) + \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(u-t+\tau_i)| x^2(t) du - \sum_{i=1}^{N} \int_{t-\tau_i}^{t} |b_i(t-s)| x^2(s) ds$$

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$$+\sum_{i=1}^{N}|f(x(t+\tau_i),x'(t+\tau_i))|x^2(t)-\sum_{i=1}^{N}|f(x(t),x'(t))|x^2(t-\tau_i)\le$$

$$\le x^2(t)+{x'}^2(t)+\sum_{i=1}^{N}\int_{t-\tau_i}^{t}|b_i(t-s)|{x'}^2(t)ds+\sum_{i=1}^{N}\int_{t-\tau_i}^{t}|b_i(u-t+\tau_i)|x^2(t)du$$

$$+\sum_{i=1}^{N}|f(x(t+\tau_i),x'(t+\tau_i))|x^2(t)\le$$

$$\le x^2(t)+{x'}^2(t)+\sum_{i=1}^{N}\int_{0}^{\tau_i}|b_i(s)|{x'}^2(t)ds+\sum_{i=1}^{N}\int_{0}^{\tau_i}|b_i(s)|x^2(t)ds+FNx^2(t)=$$

$$=\left[1+\sum_{i=1}^{N}\int_{0}^{\tau_i}|b_i(s)|ds+NF\right]x^2(t)+\left[1+\sum_{i=1}^{N}\int_{0}^{\tau_i}|b_i(s)|ds\right]{x'}^2(t)\le$$

$$\le(1+NB+NF)\,x^2(t)+(1+NB)\,{x'}^2(t)\le$$

$$\le[1+N(B+F)][x^2(t)+{x'}^2(t)]\le[1+N(B+F)]V(t).$$

Solving $V'(t)\le[1+N(B+F)]V(t)$, we obtain

$$V(t)\le V(t_0)\exp\{[1+N(B+F)](t-t_0)\}. \tag{4.10}$$

Therefore

$$x^2(t)+{x'}^2(t)\le V(t)\le V(t_0)\exp\{[1+N(B+F)](t-t_0)\}\le$$

$$\le[1+(B+F)\tau][\,\|x_{t_0}\|^2+{x'}^2(t_0)\,]\exp\{[1+N(B+F)](t_1-t_0)\}\le$$

$$\le[1+(B+F)\tau]\delta^2\exp\{[1+N(B+F)](t_1-t_0)\}=$$

$$=\varepsilon^2\exp[-2\alpha(t_1-t_0)]\le\varepsilon^2\exp[-2\alpha(t-t_0)].$$

Hence

$$\sqrt{x^2(t)+{x'}^2(t)}\le\varepsilon\exp[-\alpha(t-t_0)]. \tag{4.11}$$

But from the right continuity of $x(t)$ and $x'(t)$, (4.11) also holds on $[t_0,t_1)$. Therefore

$$\sqrt{x^2(t)+{x'}^2(t)}\le\varepsilon\exp[-\alpha(t-t_0)],\qquad t\in[t_0,t_1).$$

It follows that

$$\sup_{t_1-\tau_i\le t\le t_1}[x^2(t)+{x'}^2(t)]\le \varepsilon^2\exp\left[-2\alpha(t_1-t_0-\tau)\right]. \tag{4.12}$$

Now, we repeat the procedure above for $t\in(t_1,t_2)$. Analogous (4.10), we obtain

$$V(t)\le V(t_1^+)\exp\{[1+N(B+F)](t_2-t_1)\}=$$

$$=\left\{x^2(t_1^+)+{x'}^2(t_1^+)+\sum_{i=1}^N\int_{t_1-\tau_i}^{t_1}\left[\int_u^{t_1}|b_i(u-s+\tau_i)|x^2(s)ds\right]du\right.$$

$$\left.+\sum_{i=1}^N\int_{t_1-\tau_i}^{t_1}|f(x(s+\tau_i),x'(s+\tau_i))|x^2(s)\,ds\right\}\exp\{[1+N(B+F)](t_2-t_1)\}=$$

$$=\left\{x^2(t_1)+{x'}^2(t_1)+\sum_{i=1}^N\int_{t_1-\tau_i}^{t_1}\left[\int_u^{t_1}|b_i(u-s+\tau_i)|x^2(s)ds\right]du\right.$$

$$\left.+\sum_{i=1}^N\int_{t_1-\tau_i}^{t_1}|f(x(s+\tau_i),x'(s+\tau_i))|x^2(s)\,ds\right\}\exp\{[1+N(B+F)](t_2-t_1)\}=$$

$$=\left\{I_1(x^2(t_1{}^-))+J_1({x'}^2(t_1{}^-))+\sum_{i=1}^N\int_{t_1-\tau_i}^{t_1}\left[\int_u^{t_1}|b_i(u-s+\tau_i)|x^2(s)ds\right]du\right.$$

$$\left.+\sum_{i=1}^N\int_{t_1-\tau_i}^{t_1}|f(x(s+\tau_i),x'(s+\tau_i))|x^2(s)\,ds\right\}\exp\{[1+N(B+F)](t_2-t_1)\}=$$

$$=\left\{d_1{}^2[x^2(t_1{}^-)+{x'}^2(t_1{}^-)]+\sum_{i=1}^N\int_{t_1-\tau_i}^{t_1}\left[\int_u^{t_1}|b_i(u-s+\tau_i)|x^2(s)ds\right]du\right.$$

$$\left.+\sum_{i=1}^N\int_{t_1-\tau_i}^{t_1}|f(x(s+\tau_i),x'(s+\tau_i))|x^2(s)\,ds\right\}\exp\{[1+N(B+F)](t_2-t_1)\}\le$$

$$\le d_1{}^2\sup_{t_1-\tau_i\le t\le t_1}[x^2(t)+{x'}^2(t)]\exp\{[1+N(B+F)](t_2-t_1)\}$$

$$+\sup_{t_1-\tau_i\le t\le t_1}x^2(t)B\tau\exp\{[1+N(B+F)](t_2-t_1)\}$$

$$+\sup_{t_1-\tau_i\le t\le t_1}x^2(t)F\tau\exp\{[1+N(B+F)](t_2-t_1)\}\le$$

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$$\leq [{d_1}^2 + (B+F)\tau] \sup_{t_1-\tau_i \leq t \leq t_1} [x^2(t) + {x'}^2(t)] \exp\{[1+N(B+F)](t_2-t_1)\} \leq$$

$$\leq [{d_1}^2 + (B+F)\tau]\varepsilon^2 \exp[-2\alpha(t_1-t_0-\tau)] \exp\{[1+N(B+F)](t_2-t_1)\}.$$

¿From the definitions of d_1 and p_1, we have

$$x^2(t) + {x'}^2(t) \leq V(t) \leq$$

$$\leq \varepsilon^2[{d_1}^2 + (B+F)\tau] \exp[-2\alpha(t_1-t_0-\tau)] \exp\{[1+N(B+F)](t_2-t_1)\} =$$

$$= \varepsilon^2 \left(\frac{p_1 + (B+F)\tau}{2}\right) \exp[-2\alpha(t_1-t_0-\tau)] \exp\{[1+N(B+F)](t_2-t_1)\} \leq$$

$$\leq \varepsilon^2 p_1 \exp[-2\alpha(t_1-t_0-\tau)] \exp\{[1+N(B+F)](t_2-t_1)\} =$$

$$= \varepsilon^2 \exp[-2\alpha(t_2-t_0)] \leq \varepsilon^2 \exp[-2\alpha(t-t_0)].$$

Hence

$$\sqrt{x^2(t) + {x'}^2(t)} \leq \varepsilon \exp[-\alpha(t-t_0)]. \tag{4.13}$$

In fact we have from the right continuity of $x(t)$ and $x'(t)$ that (4.13) holds for $t \in [t_1, t_2)$. Thus

$$\sqrt{x^2(t) + {x'}^2(t)} \leq \varepsilon \exp[-\alpha(t-t_0)], \qquad t \in [t_1, t_2).$$

It follows that

$$\sup_{t_2-\tau_i \leq t \leq t_2} [x^2(t) + {x'}^2(t)] \leq \varepsilon^2 \exp[-2\alpha(t_2-t_0-\tau)]. \tag{4.14}$$

With analogous arguments, it follows that for all $k \in \mathbb{N}$,

$$\sqrt{x^2(t) + {x'}^2(t)} \leq \varepsilon \exp[-\alpha(t-t_0)], \qquad t \in [t_{k-1}, t_k).$$

Hence

$$\sqrt{x^2(t) + {x'}^2(t)} \leq \varepsilon \exp[-\alpha(t-t_0)], \qquad t \geq t_0,$$

and the proof is complete. □

Remark 4. 1. If in addition to the hypotheses of Theorem 4.1 (respectively Theorem 4.2), t_k, I_k and J_k also satisfy

$$t_k - t_{k-1} = l \quad \text{and} \quad I_k(u) = J_k(u) = du, \quad k = 1, 2, \ldots, \tag{4.15}$$

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with

$$d = \sqrt{\frac{p - (* + F)\tau}{2}} \quad \text{and} \quad p = \exp[-2\alpha(l + \tau)] \exp\{-[1 + N(* + F)]l\}, \tag{4.16}$$

where $* = A$ (resp. $* = B$), then problem (2.1) (resp. problem (2.5)) can be exponentially stabilized by periodical impulses.

Remark 4. 2. By the proofs of Theorem 4.1 and Theorem 4.2, one can see that all solutions of problems (2.1) and (2.5) converge exponentially to zero under impulsive controls. Thus our theorems actually prove the global stabilization for (2.1) and (2.5).

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