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# M-deformations of $\mathcal{A}$-simple $\Sigma^{n-p+1}$-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, $n \geq p$

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All $\mathcal{A}$-simple singularities of map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, where $n \geq p$, of minimal corank (i.e. of corank $n-p+1$) have an M-deformation, that is a deformation in which the maximal numbers of isolated stable singular points are simultaneously present in the discriminant. October, 2003 ICMC-USP

## 1. INTRODUCTION

In the present paper we study real deformations of map-germs from $\mathbb{R}^n$ into $\mathbb{R}^p$, where $n \geq p$, for which the maximal numbers of isolated stable singular points are simultaneously present in the discriminant, which we call M-deformations for short (M as in maximal), furthermore we call the maximal numbers of isolated stable singularities 0-stable invariants. (For map-germs of target dimension greater than the source dimension we replace in the definition of a M-deformation discriminant by image.) This terminology is analogous to the concept of a M-morsification of a function-germ, which, for example, exists for singularities of type $A_k$ and $D_k$ [2, 5]). For map-germs very little is known about the existence of M-deformations beyond the classical result by A'Campo [1] and Gusein-Zade [7] that plane curve-germs always have M-deformations, i.e. deformations with $\delta$ real double-points (notice that the $\delta$-number is the only 0-stable invariant in this case). For map-germs $\mathbb{R}^n \to \mathbb{R}^p$, where $n < p$, there is also the notion of a good real perturbation due to Mond for which the homology of the image of a stabilization of a given germ coincides with that of its complexification (again there is an analogous definition for $n \geq p$ with discriminant in place of image). For plane curve-germs this concept coincides with that of an M-deformation, but for map-germs of higher source dimension such good perturbations exist only for a small class of map-germs – e.g. for germs from $\mathbb{R}^2$ to $\mathbb{R}^3$ there is only one series of $\mathcal{A}$-simple corank-1 germs having good perturbations [10]. On the other



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hand, good perturbations are known to exist for all singular map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$ of $\mathcal{A}_e$-codimension 1 and minimal corank (i.e. of corank $\max(1, n-p+1)$), see [4] and [9].

The main result of the present paper is that all $\mathcal{A}$-simple singularities of map-germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, where $n \geq p$, of minimal corank (i.e. of corank $n-p+1$) have an M-deformation. The proof of this result is based on the following property: all $\mathcal{A}$-simple singularities $f$ of minimal corank can be deformed into a germ whose 0-stable invariants differ from those of $f$ by at most one – one can then inductively split-off real stable singular points from 0 one by one. As a corollary we also get lower bounds for the $\mathcal{A}_e$-codimension of $f$ in terms of its 0-stable invariants. The above property does not hold for germs of non-minimal corank nor for germs of positive $\mathcal{A}$-modality. The hypothesis of minimal corank is necessary for the existence of M-deformations (below we give an example of an $\mathcal{A}$-simple corank-2 germ from the plane to the plane that does not have an M-deformation, and that violates the above property). At present we have no example of a germ of minimal corank and positive $\mathcal{A}$-modality without an M-deformation, but we conjecture that the $\mathcal{A}$-simplicity is also a necessary condition.

Finally, looking at the existing classifications of $\mathcal{A}$-simple corank-1 germs from $\mathbb{R}^n$ to $\mathbb{R}^p$, where $n < p$, one checks that these have M-deformations. Hence it is reasonable to conjecture that the existence of M-deformations holds for $\mathcal{A}$-simple singularities of minimal corank for any pair of source and target dimensions.

The plan of this paper is as follows. In Section 2 we introduce some notation and state the main result, and in Section 3 we briefly recall from [16] the definition of certain map-germs $G_{k(s,n)} : \mathbb{K}^{n+s-1} \to \mathbb{K}^{n+s-1}$ associated with $f : \mathbb{K}^n \to \mathbb{K}^n$ (here $k(s,n)$ denotes a partition of $n$ with $s$ summands, and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) whose local multiplicity gives the 0-stable invariants up to an overcount factor. Section 4 contains the proof of the main result, and Section 5 gives lower bounds on the $\mathcal{A}_e$-codimension in terms of the 0-stable invariants and discusses some empirical evidence for the existence of M-deformations for $\mathcal{A}$-simple corank-1 germs $f : \mathbb{R}^n \to \mathbb{R}^p$, for $n < p$.

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## 2. STATEMENT OF MAIN RESULT AND SOME NOTATION

Any $\mathcal{A}$-simple smooth map-germ $f : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$, where $m \geq n$, of rank $n-1$ is given by the pre-normal form

$$(x, y, z) \mapsto (x, g(x, y) + Q(z)),$$

where $(x, y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{m-n}$, $Q(z) = \sum_i \epsilon_i z_i^2$ ($\epsilon_i = \pm 1$) and where $(x, y) \mapsto (x, g(x, y))$ is an $\mathcal{A}$-simple equidimensional corank-1 germ (see Lemma 3.1). Let $\tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_s) : \mathbb{R}^m, S \to \mathbb{R}^n, \tilde{f}(S) =: q$, $\tilde{f}_i(x, y_i) = (x, \tilde{g}_i(x, y_i) + Q_i(z))$, $i = 1, \ldots, s := |S|$, be an $s$-germ appearing in a deformation of $f$ (here $S$ is a finite set of source points being mapped to the point $q$ in the target). The rank $n-1$ $\mathcal{K}$-classes of germs $\mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ are those of $A_k$, with representatives $(x, y^{k+1} + Q(z))$, and the $\mathcal{K}$-classes of $s$-germs $A_{(k_1,\dots,k_s)}$ have an $A_{k_i}$-singularity at the $i$th source point. The stable rank $n-1$ multi-germs are those being transverse to their $\mathcal{K}$-class $A_{(k_1,\dots,k_s)}$, and the isolated stable (or 0-stable) singularities amongst these are those with $\sum_{i=1}^{s} k_i = n$. Let $k(s,n) := (k_1, \dots, k_s)$ be such a partition of $n$ with $s$ summands.

For equidimensional germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^n, 0$ the number of isolated stable $A_{k(s,n)}$-points in a generic deformation of $f$, denoted by $r_{k(s,n)}(f)$, can be calculated by dividing the local multiplicity of a certain map-germ $G_{k(s,n)} : \mathbb{C}^{n+s-1}, 0 \to \mathbb{C}^{n+s-1}$ by some overcount factor (see [16] and Section 3). For rank $n-1$ germs $f : \mathbb{C}^m, 0 \to \mathbb{C}^n, 0$, where $m > n$, of the form $(x, g(x,y) + Q(z))$ the invariants $r_{k(s,n)}(f)$ can simply be calculated from the associated equidimensional germ $(x, g(x,y))$.

For real germs $f$ the invariants $r_{k(s,n)}(f)$ are defined by complexifying, but clearly the above geometric interpretation does no longer hold: the number $r^{\mathbb{R}}_{k(s,n)}(f_t)$ of real $A_{k(s,n)}$-points in a deformation $f_t$ of $f$ now depends on the choice of deformation. One only has the obvious inequality $r^{\mathbb{R}}_{k(s,n)}(f_t) \leq r_{k(s,n)}(f)$.

We call a real deformation $f_t$ of $f$ an M-deformation, if the maximal numbers $r_{k(s,n)}(f)$ of 0-stable singularities (for all partitions $k(s,n)$ of $n$) are simultaneously present in the discriminant of $f_t$.

The main result on the existence of M-deformations in the present paper is the following

Theorem 1.1. *All $\mathcal{A}$-simple rank $n-1$ germs $f : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$, where $m \geq n$, have an M-deformation.*

Remark 1.2. The condition on the rank is necessary: $\mathcal{A}$-simple germs of higher corank do in general not have an M-deformation, as the following example shows. For the corank-2 germ $f = (x^2 - y^2 + x^3, xy)$ the invariants $r_{(2)}(f) = 3$ and $r_{(1,1)}(f) = 2$ are the (complex) cusp and double-fold numbers, respectively. But any real stabilization of $f$ has 3 cusps and no double-fold (see [17]). We conjecture that the $\mathcal{A}$-simplicity of $f$ is also necessary, but at present have no example of a rank $n-1$ germ of $\mathcal{A}$-modality 1 without an M-deformation.

We now fix some notation. Let $C_n$ denote the local ring of smooth (or complex-analytic) function germs $f : \mathbb{K}^n, 0 \to \mathbb{K}, 0$ and $\mathcal{M}_n$ its maximal ideal. For the groups $\mathcal{A}$ and $\mathcal{K}$ (of left-right and of contact equivalence, respectively) acting on the space of smooth map-germs and for the tangent spaces to the $\mathcal{A}$- and $\mathcal{K}$-orbits we use the usual notation, such as $T\mathcal{A}\cdot f = tf(\mathcal{M}_n \cdot \theta_n) + wf(\mathcal{M}_p \cdot \theta_p)$ and $T\mathcal{K}\cdot f = tf(\mathcal{M}_n \cdot \theta_n) + f^*\mathcal{M}_p \cdot \theta_f$ (a basic reference for these concepts is the survey on determinacy [19] by Wall). For equidimensional map-germs $f : \mathbb{K}^n, 0 \to \mathbb{K}^n, 0$ of corank 1 we use source coordinates $(x, y) = (x_1, \dots, x_{n-1}, y)$ such that $f(x,y) = (x, g(x,y))$, and target coordinates $(X_1, \dots, X_n)$. In describing elements of $T\mathcal{A} \cdot f$ we sometimes use the shorter notation $e_i$ for the target and source vector fields $\partial/\partial X_i$ and $\partial/\partial x_i$ (where $x_n = y$).

## 3. DEFINING EQUATIONS OF THE 0-STABLE INVARIANTS

In view of Lemma 3.1 we consider in this section equidimensional corank-1 germs $f$ : $\mathbb{K}^n, 0 \to \mathbb{K}^n, 0$ of the form $f(x,y) = (x, g(x,y))$. For such map-germs one can replace the space $(\mathbb{K}^n)^s$ of $s$-fold points in the source (whose $f$-images are a common point in the target) by $\mathbb{K}^{n+s-1}$, with coordinates $(x, y_1, \ldots, y_s) = (x_1, \ldots, x_{n-1}, y_1, \ldots, y_s)$. Recall that $A_{k(s,m)} := A_{(k_1,\ldots,k_s)}$, where $m := \sum_{i=1}^{s} k_i$, denotes the $\mathcal{K}$-class of $s$-germs having an $A_{k_i}$-singularity at the $i$th source point. In [15, 16] the closures of the sets $A_{(k_1,\ldots,k_s)}$ in multi-jet space $J_s^\ell$, $\ell := \sum_{i=1}^{s}(k_i + 1)$, were explicitly defined by iteration for any $s$ and $m \leq n$, and it was shown that these sets are smooth submanifolds of codimension $\sum_{i=1}^{s}(k_i) + s - 1$. (Roughly speaking, the conditions for an $A_{k_j}$ singularity at the $j$th source point, with $f$-image some given point in the target, are reduced modulo the corresponding conditions at the source points 1 to $j-1$ and then divided by a suitable power of $y_j - y_{j-1}$.) Pulling back the ideal defining the closures of these sets by the multi-jet extension of $f$ we get an ideal $(j_s^\ell f)^*(\mathcal{I}(\bar{A}_{k(s,n)}))$ in $C_{n+s-1}$, and for $m = n$ the generators of this ideal define an equidimensional map-germ

$$G_{k(s,n)} = (G_1, \ldots, G_{n+s-1}) : \mathbb{K}^{n+s-1}, 0 \to \mathbb{K}^{n+s-1}$$

whose local multiplicity $m_{G_{k(s,n)}}(0) := \dim_{\mathbb{K}} C_{n+s-1}/G^*_{k(s,n)}\mathcal{M}_{n+s-1}$ is equal to the number $r_{k(s,n)}(f)$ of complex $A_{k(s,n)}$-points appearing in a stabilization of $f$ times an overcount factor $c$ ($c$ is equal to the number of permutations mapping source points of type $A_{k_i}$ to source points of the same type).

It will turn out (see below) that we need the defining equations of the sets $\bar{A}_{k(s,n)}$ only for $s = 1$ and 2, hence we specialize the definitions in [15, 16] to these particular cases. Set $g^{(i)} := \partial^i g/\partial y^i$, then $\bar{A}_{(n)} := \{g^{(1)} = \ldots g^{(n)} = 0\}$. For $s = 2$ we first apply a linear origin-preserving coordinate change $L(x, y_1, y_2) = (x, y_1, y_2 - y_1) =: (x, y, \epsilon)$, and let $g_1^{(i)} := g^{(i)}$. Setting

$$g_2^{(0)} := \sum_{\alpha \geq k_1+1} g_1^{(\alpha)} \epsilon^{\alpha - k_1 - 1}/\alpha!, \qquad g_2^{(i)} := \partial^i g_2^{(0)}/\partial \epsilon^i, \quad i \geq 1,$$

we define

$$\bar{A}_{(k_1, n-k_1)} := \{g_1^{(1)} = \ldots = g_1^{(k_1)} = g_2^{(0)} = \ldots = g_2^{(n-k_1)} = 0\}.$$

Notice that for even $n$ the overcount factor $c$ in $r_{(n/2,n/2)}(f) = c^{-1} \cdot m_{G_{(n/2,n/2)}}(0)$ is 2. For the other 0-stable invariants in the cases $s = 1, 2$ it is one.

The following facts will be useful.

Remarks 3.1. (i) Given a pair of $\mathcal{A}_e$-equivalent, equidimensional corank-1 germs $f$ and $f'$, the corresponding pairs of germs $G_{k(s,n)}$ and $G'_{k(s,n)}$ are $\mathcal{K}$-equivalent (see Lemma 2.3 of [16]).

(ii) $r_{k(s,n)}(f) = 0$ for $n + s > m_f(0)$, where $m_f(0)$ denotes the local multiplicity of $f$ at the origin (this follows from the "additivity of the local multiplicities on the diagonal" in the recognition conditions for $\bar{A}_{k(s,n)}$, see [15, 16]). This fact, together with the observation that the $\mathcal{A}$-modality of germs with $m_f(0) \geq n + 3$ is positive (Lemma 3.4 below), implies that we only have to consider 0-stable invariants with $s = 1, 2$.

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## 4. M-DEFORMATIONS AND $\mathcal{A}$-SIMPLICITY

We begin with an outline of the proof of Theorem 2.1. Recall from the introduction that the key property of $\mathcal{A}$-simple singularities of minimal corank, from which an M-deformation can then be obtained inductively, is that such germs $f$ can be deformed into a germ whose 0-stable invariants differ from $f$ by at most one. The proof of this property $(*)$ consists of the following main steps:

(i) Reduction to the equidimensional case $\mathbb{R}^n \to \mathbb{R}^n$.

(ii) For $n \geq 3$ there are no $\mathcal{A}$-simple orbits of local multiplicity $\geq n+3$.

(iii) Germs $f$ of local multiplicity $n+1$ have $r_{k(1,n)}(f) = r_{(n)}(f)$ as the only non-zero 0-stable invariant, and positive $\mathcal{K}$-modality of $G_{(n)}$ implies positive $\mathcal{A}$-modality of $f$. Property $(*)$ then follows from the analogous property for the local multiplicities of $\mathcal{K}$-simple equidimensional map-germs.

(iv) Germs $f$ of local multiplicity $n+2$ have $r_{k(s,n)}(f)$, where $s = 1, 2$, as the only non-zero 0-stable invariants. In this case property $(*)$ follows from a partial classification of $\mathcal{A}$-simple germs listed in Lemma 4.2.2 (we do not know whether all the germs in this list are $\mathcal{A}$-simple, but any $\mathcal{A}$-simple germ of multiplicity $n+2$ is equivalent to some germ in this list). This partial classification is the most unpleasant part of the proof. (Notice that the proofs of the Lemmas 4.2.1 and 4.2.2 merely describe the high-level structure and the cases to be considered, but omit all the routine details, which just require some care due to the fact that the dimension $n$ is not fixed.)

We begin with the reduction to the equidimensional case.

LEMMA 3.1. *Any $\mathcal{A}$-simple smooth map-germ $f : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$, where $m \geq n$, of rank $n-1$ is given by the pre-normal form*

$$(x, y, z) \mapsto (x, g(x, y) + Q(z)),$$

*where $(x, y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{m-n}$, $Q(z) = \sum_i \epsilon_i z_i^2$ ($\epsilon_i = \pm 1$) and where $(x, y) \mapsto (x, g(x, y))$ is an $\mathcal{A}$-simple equidimensional corank-1 germ.*

PROOF. The argument is similar to the one for $n = 2$ (see Lemmas 1.1 and 1.2 in [17]). After a coordinate change we can assume that $f$ is given by

$$h = (x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1}, y_1, \ldots, y_r) + \sum_i \epsilon_i z_i^2),$$

where $i = 1, \ldots, m-n-r+1$ and $g(0, \ldots, 0, y_1, \ldots, y_r) \in \mathcal{M}_r^3$. Two such germs $h = (x, g(x, y) + Q(z))$ and $h' = (x, g'(x, y) + Q(z))$ are $\mathcal{A}$-equivalent if and only if the corresponding germs $(x, g(x, y))$ and $(x, g'(x, y))$ are $\mathcal{A}$-equivalent. We claim that for $r \geq 2$ there are no simple $\mathcal{A}$-orbits over the 2-jet of $(x, g(x, y))$, which for $r = 2$ is $\mathcal{A}^2$-equivalent to

$$(x, a_1 x_1 y_1 + \ldots + a_{n-1} x_{n-1} y_1 + b_1 x_1 y_2 + \ldots + b_{n-1} x_{n-1} y_2).$$

The least degenerate $\mathcal{A}^2$-orbit, corresponding to $a_i \neq 0$ and $a_i b_j \neq a_j b_i$ (for some $i$ and $j \neq i$), has the representative (taking $i = 1$, $j = 2$)

$$\sigma := (x, x_1 y_1 + x_2 y_2).$$

A complete 3-transversal for $\sigma$ is given by

$$t := (0, a y_1^3 + b y_1^2 y_2 + c y_1 y_2^2 + d y_2^3 + e x_3 y_1^2 + \ldots + f x_{n-1} y_2^2).$$

Now we can argue as in Lemma 1.2 of [17] to show that the subspace $\mathbb{K}\{y_1^i y_2^j \cdot e_n : i+j = 3\}$ of $T\mathcal{A}^3 \cdot (\sigma + t)$ is foliated by (at least) a 1-parameter family of orbits. Notice that the more degenerate $\mathcal{A}^2$-orbits and the orbits corresponding to $r > 2$ are all adjacent to $T\mathcal{A}^2 \cdot \sigma$, which implies the claim. □

Remark 3.2 Notice that the discriminants of the germs $(x, g(x,y) + Q(z))$ and $(x, g(x,y))$ coincide.

Theorem 3.3. *All $\mathcal{A}$-simple corank-1 germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ have an M-deformation.*

For $n = 2$ all real types of stabilizations of all simple corank-1 germs are known (see [14]) and amongst these there is always an M-deformation (the result also holds for functions of one variable, $n = 1$), hence we can concentrate on $n \geq 3$. The theorem will follow from Lemmas 3.4, 3.1.1, 3.1.2, 3.2.1, 3.2.2 and 3.2.3 below.

Lemma 3.4. *For $n \geq 3$, all $\mathcal{A}$-orbits inside $\mathcal{K}(x_1, \ldots, x_{n-1}, y^{\geq n+3})$ are at least unimodal..*

The proof of the above claim will follow directly from the following

Lemma 3.5. *Let $f : \mathbb{K}^n, 0 \to \mathbb{K}^n, 0$ be a corank 1 $\mathcal{A}$-finitely determined germ. Suppose that the $\mathcal{A}$-orbit of $f$ is open in its $\mathcal{K}$-orbit. Then $m_f(0) \leq n + 2$.*

We shall need the following condition for the openess of the $\mathcal{A}$-orbit within the $\mathcal{K}$-orbit.

Theorem 3.6 ([18], Theorem 5.1 ). *Let $f : \mathbb{K}^n, 0 \to \mathbb{K}^p, 0$ be an $\mathcal{A}$-finitely determined germ and $\{v_1, v_2, \ldots, v_r\}$ a basis for* $\dfrac{\theta_f}{T\mathcal{A}_e \cdot f + f^*\mathcal{M}_p.\theta_f}$.

*The $\mathcal{A}$-orbit of $f$ is open in its $\mathcal{K}$-orbit if and only if $f_i v_j \in T\mathcal{A} \cdot f$, $i = 1, \ldots, p$; $j = 1, \ldots, r$ (Mod $f^*\mathcal{M}_p^2.\theta_f$).*

Proof of Lemma 3.5. When $m_f(0) \leq n + 1$, the only map-germs $f : \mathbb{K}^n, 0 \to \mathbb{K}^n, 0$ with the property that the $\mathcal{A}$-orbit is open in its $\mathcal{K}$-orbit are the infinitesimally stable ones. Then, we can assume that $m_f(0) = n + l$, $l \geq 2$.

Let $f(x,y) = (x, g(x,y))$, where $x = (x_1, \ldots, x_{n-1})$, and $g(x,y) = y^{n+l} + \phi_1(x)y + \ldots + \phi_{n-1}y^{n-1} + \sum_{i=n}^{n+l-2} \phi_i(x) y^i$.

The hypothesis that the $\mathcal{A}$-orbit of $f$ is open in the $\mathcal{K}$-orbit implies that rank $d\phi(0) = n - 1$, where $\phi : \mathbb{K}^{n-1}, 0 \to \mathbb{K}^{n+l-2}, 0$ is defined by $\phi(x) = (\phi_1(x), \ldots, \phi_{n+l-2}(x))$. It also follows that after changing coordinates, we can write $f$ in the form:

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$$(x, g(x,y)) = (x, y^{n+l} + x_1 y + \ldots + x_{n-1}y^{n-1} + \sum_{i=n}^{n+l-2} \phi_i(x)y^i).$$

It follows from the theorem above that we may assume that $\phi_i(x)$ are linear, modulo terms in $f^*\mathcal{M}_n^2.\theta_f$.

Moreover, the $n(n+l-2)$ elements $(0, g(x,y)y^j)$, $(0, x_i y^j)$, $i = 1, \ldots, n-1$ and $j = 1, \ldots, n+l-2$, must be in $T\mathcal{A} \cdot f$, the tangent space to the $\mathcal{A}$-orbit of $f$. The equations relating these elements are:

$$wf(X_n.e_n) + f^*\mathcal{M}_n^2.\theta_f = 0, (\operatorname{Mod} T\mathcal{A} \cdot f)$$

$$tf(y^j.e_n) + f^*\mathcal{M}_n^2.\theta_f = 0, j = 1, \ldots, y^{n+l-2} \, (\operatorname{Mod} T\mathcal{A} \cdot f)$$

$$tf(x_i.e_j) + f^*\mathcal{M}_n^2.\theta_f = 0, i = 1, \ldots, n-1, \; j = 1, \ldots, n-1, \, (\operatorname{Mod} T\mathcal{A} \cdot f)$$

$$tf(g.e_j) + f^*\mathcal{M}_n^2.\theta_f = 0, \; j = 1, \ldots, n-1, \, (\operatorname{Mod} T\mathcal{A} \cdot f)$$

This system has $n(n+l-2)$ variables and $n^2 + l - 1$ equations.

Then we must have $n^2 + l - 1 \geq n(n+l-2)$, which holds if and only if $l \leq \frac{n}{n-1} + 1$. Then, when $n \geq 3$, it follows that $l \leq 2$. □

For $m_f(0) \leq n$ all invariants $r_{k(s,n)}(f)$ are zero, hence we have to consider the cases $m_f(0) = n+1$ and $n+2$.

### 4.1. The case $m_f(0) = n + 1$

It is sufficient to consider germs (see Prop. 4.8 in [13])

$$f = (x, y^{n+1} + P_1(x)y + \ldots + P_{n-1}(x)y^{n-1}),$$

and $m = n$ implies $s = 1$, hence $G_{(n)}$, which is $\mathcal{K}$-equivalent to

$$P := (P_1(x), \ldots, P_{n-1}(x)),$$

is the only relevant germ here.

Lemma 3.1.1. *If $G_{(n)}$ is not $\mathcal{K}$-simple then $f$ is not $\mathcal{A}$-simple.*

Proof. The hypothesis implies that there is a 1-parameter deformation $P^t := (P_1^t(x), \ldots, P_{n-1}^t(x))$ of $P = P^0$ meeting an infinite number of distinct $\mathcal{K}$-orbits. Hence $f_t := (x, y^{n+1} + \sum_{i=1}^{n-1} P_i^t(x)y^i)$ is a deformation of $f = f_0$ meeting an infinite number of $\mathcal{A}$-orbits (by the fact quoted at the end of Section 2). □

Note: if $G_{(n)}$ and $G'_{(n)}$ correspond to $f$ and $f'$, respectively, then the above argument shows that $[G_{(n)}] \to [G'_{(n)}]$ implies $[f] \to [f']$ (i.e. adjacency of $\mathcal{K}$-orbits implies that of

$\mathcal{A}$-orbits). The following claim now implies (by induction) that all $\mathcal{A}$-simple germs of local multiplicity $n+1$ have M-deformations, because $r_{(n)}(f) = m_P(0)$.

Lemma 3.1.2. *Let $P : \mathbb{R}^{n-1}, 0 \to \mathbb{R}^{n-1}, 0$ be $\mathcal{K}$-simple germ then there exists a germ $P'$, to which $P$ is $\mathcal{K}$-adjacent to, such that $m_P(0) - m_{P'}(0) \leq 1$.*

Proof. Use classification of $\mathcal{K}$-simple equidimensional real germs. (Note: the lower indices in complex classification [6] denote the Milnor numbers, and for weighted homogeneous $P$ we have $\mu(P) = m_P(0) - 1$. In the real case there is no complete published reference for the classification of the $\mathcal{K}$-simple equidimensional germs and their adjacencies, but at least the classification is well-known. The preprint version of [17] contains a table comparing notation for real and complex orbits which, together with the determination of some partial adjacencies of real orbits, gives the desired result). □

### 4.2. The case $m_f(0) = n+2$

Here we consider the prenormal form

$$f = (x, y^{n+2} + P_1(x)y + \ldots + P_n(x)y^n),$$

and $m = n$ implies $s = 1$ or 2. Hence $G_{(n)}$ and $G_{(n-l,l)}$ $(l = 1, \ldots, [n/2])$ are the only germs corresponding to non-zero 0-stable invariants of $f$.

Lemma 3.2.1. *Any $\mathcal{A}$-simple germ of local multiplicity $n+2$ has one of the following prenormal forms:*

$$f_n = (x, y^{n+2} + x_1 y + \ldots + x_{n-1} y^{n-1}),$$

*or*

$$f_{n-1} = (x, y^{n+2} + x_1 y + \ldots + x_{n-2} y^{n-2} + P_{n-1}(x_{n-1}) y^{n-1} + x_{n-1} y^n),$$

*where $P_{n-1}$ belongs to the square of the maximal ideal.*

Proof. We divide the proof in several steps.

**Step 1**. Any $\mathcal{A}$-simple germ of local multiplicity $n+2$ has the prenormal form

$$f_j = (x, y^{n+2} + x_1 y + \ldots + x_{j-1} y^{j-1} + P_j y^j + x_j y^{j+1} + \ldots + x_{n-1} y^n),$$

where $P_j(x_j, \ldots, x_{n-1})$, $1 \leq j \leq n$, is in the square of the maximal ideal.

Consider the prenormal form $f = (x, y^{n+2} + P_1(x)y + \ldots + P_n(x)y^n)$: suppose the differential of $(P_1, \ldots, P_n)$ has rank $r \leq n-2$ at the origin, by a coordinate change in the $x_i$ we have that $r$ of the $P_j$ are equal to (different) $x_i$. Let $f' : \mathbb{R}^{n-1}, 0 \to \mathbb{R}^{n-1}, 0$ be the restriction of $f$ to $x_i = 0$, for some $x_i$ not appearing linearly in some $P_j$, then $f'$ is non-simple by Lemma 3.4. Split $(C_n)^p$ into a direct sum of two parts, $\oplus_{j \neq i} \mathcal{M}_{x_i} \cdot C_n \cdot \partial/\partial X_j$ and $B$, then there are no more generators for the subspace $B \cap T\mathcal{A} \cdot f$ than there are generators for $T\mathcal{A} \cdot f'$. From the proof of Lemma 3.4 it now follows that $f$ is at least $\mathcal{A}$-unimodal.

Hence, for simple germs $f$, the differential of $(P_1, \ldots, P_n)$ has rank $n-1$ at the origin. Let $P_j$ be the first $P_\ell$ such that the rank of the differential of $(P_1, \ldots, P_\ell)$, for increasing

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$\ell \geq 1$, is less than $\ell$. By direct coordinate changes we can assume that $P_i = x_i$, for $i < j$, $P_j \in \mathcal{M}_{x_j,\ldots,x_{n-1}}$, and $P_{i+1} = x_i$, for $i \geq j$.

**Step 2**. Any germ of type $f_j$ is non-simple if all germs of type $f_{j+1}$ are non-simple.

We will show that any $\mathcal{A}$-orbit in $\mathcal{K}(x, y^{n+2})$ of type $f_j$ is adjacent to some orbit of type $f_{j+1}$. Take a deformation of

$$\begin{aligned} f_j = \ & (x, y^{n+2} + x_1 y + .. + x_l y^l + P(x_{l+1}, .., x_{n-1}) y^{l+1} + x_{l+2} y^{l+2} + \ldots \\ & + x_{n-1} y^{n-1} + x_{l+1} y^n), \end{aligned}$$

with $l = j - 1$ and $P$ in the square of the maximal ideal, by $t.(0, x_{l+2} y^{l+1})$. For non-zero $t$ we apply successive coordinate changes

$$x_{l-2} \mapsto t^{-1}(x_{l-2} - Q(x_{l+1}, .., x_{n-1})), \quad Q \in \mathcal{M}^2$$

$$x_{l-2} \mapsto x_{l-2} - t^{-1} x_{l+2} y, \quad \text{etc.}$$

and obtain

$$(x, y^{n+2} + x_1 y + .. + x_l y^l + x_{l+2} y^{l+1} + Q'(x_{l+1}, .., x_{n-1}) y^{l+2} + x_{l+3} y^{l+3} + \ldots$$

$$+ x_{n-1} y^{n-1} + x_{l+1} y^n),$$

where $Q' \in \mathcal{M}^2$, which is of type $f_{l+2} = f_{j+1}$.

**Step 3**. All $\mathcal{A}$-orbits in $\mathcal{K}(x, y^{n+2})$ of type $f_{n-2}$ have modality at least one.

Set $s := (x, x_1 y + \ldots. + x_{n-3} y^{n-3})$ and consider a general $n$-jet

$$f = s + (0, (a x_{n-2}^2 + b x_{n-2} x_{n-1} + c x_{n-1}^2) y^{n-2} + (d x_{n-2} + e x_{n-1}) y^{n-1})$$

over $s$. We have the following three cases:

*Case 1.* $d = e = 0$: not all $x_i$ appear linearly in some $P_j$, where $(x, y^{n+2} + \sum P_j(x) y^j)$. This leads to non-simple orbits (see Step 1).

*Case 2.* $e$ and $d$ are not both 0, hence we can take (after a suitable coordinate change) $e = 1$, $d = 0$ in the $n$-jet $f$ above. The least degenerate $\mathcal{A}^n$-orbit is then given by $a \neq 0$ (for $a = 0$ see Case 3. below) with representative $f = s + (0, x_{n-2}^2 y^{n-2} + x_{n-1} y^{n-1})$. A complete $(n+1)$-transversal for this $f$ is given by $(0, a' x_{n-2} y^n + b' y^{n+1})$.
There are three cases to be considered (at the $(n+1)$-jet level):

2.1 $b' \neq 0$: leads to $\mathcal{A}$-orbits in $\mathcal{K}(x, y^{n+1})$, see earlier Section 3.1.
2.2 $b' = 0$, $a' \neq 0$: $s' \sim s + (0, x_{n-2}^2 y^{n-2} + x_{n-1} y^{n-1} + x_{n-2} y^n)$.
2.3 $a' = b' = 0$: $(n+1)$-jet $f$.

We have to consider the last two cases further.

Case 2.2: an $(n+2)$-transversal in this case is $(0, ay^{n+2})$. For $f := s' + (0, ay^{n+2})$ we have 3 generators for the $a$-subspace of $T\mathcal{A}^{n+1} \cdot f$ (suppressing terms that are obviously in $T\mathcal{A}^{n+1} \cdot f$):

$$wf(X_n.e_n) = (0, x_{n-2}^2 y^{n-2} + x_{n-2}y^n + ay^{n+2}),$$

$$tf(y.e_n) = (0, (n-2)x_{n-2}^2 y^{n-2} + nx_{n-2}y^n + (n+2)ay^{n+2}),$$

$$tf(x_{n-2}.e_{n-2}) = (0, 2x_{n-2}^2 + x_{n-2}y^n).$$

The resulting 3 by 3 matrix has rank $< 3$, hence $a$ is a modulus:

$$f = (x, ay^{n+2} + x_1 y + ... + x_{n-3}y^{n-3} + x_{n-2}^2 y^{n-2} + x_{n-1}y^{n-1} + x_{n-2}y^n).$$

The least degenerate orbit in Case 2.2 is therefore non-simple.

Case 2.3: the germs with $(n+1)$-jet

$$f = s + (0, x_{n-2}^2 y^{n-2} + x_{n-1}y^{n-1})$$

are adjacent to those in Case 2.2, hence non-simple.

This concludes 2.1 to 2.3 in Case 2. We now come to the last case concerning the general $n$-jet $f$ at the beginning of Step 3.

*Case 3.* $e = 1$ , $a = d = 0$: for $b \neq 0$ the $n$-jet

$$f = s + (0, (bx_{n-2} + cx_{n-1})x_{n-1}y^{n-2} + x_{n-1}y^{n-1})$$

is equivalent to $s' := s + (0, x_{n-2}x_{n-1}y^{n-2} + x_{n-1}y^{n-1})$. A complete $(n+1)$-transversal for $s'$ is given by

$$t := (0, a'x_{n-2}^3 y^{n-2} + b'x_{n-2}y^n + c'y^{n+1}).$$

Setting $f := s' + t$, the subspace of $T\mathcal{A}^{n+1} \cdot f$ spanned by

$$x_{n-2}x_{n-1}y^{n-2}, x_{n-2}^3 y^{n-2}, x_{n-1}y^{n-1}, y^{n+1}, x_{n-2}^2 y^{n-1}, x_{n-2}y^n$$

in the $e_n$-component has the following generators

$$tf(x_{n-1}.e_{n-1}), tf(x_{n-2}.e_{n-2}), wf(X_n.e_n), tf(y.e_n), tf(x_{n-1}.e_n), tf(x_{n-1}^2.e_{n-2}).$$

The resulting 6 by 6 matrix has rank $\leq 5$ (here we work modulo monomials that are outside the subspace in question and are obviously in $T\mathcal{A}^{n+1} \cdot f$). Hence all orbits over the $(n+1)$-jet $f$, and all orbits corresponding to $b = 0$ above (being adjacent to these), are non-simple.

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We can now conclude that all orbits in $\mathcal{K}(x, y^{n+2})$ of type $f_{n-2}$ are non-simple: they either lie in the closure of the non-simple orbits in $\mathcal{K}(x, y^{n+2})$ considered in 2.2 or in the closure of the non-simple orbits in $\mathcal{K}(x, y^{n+1})$ considered in 3.

Steps 1 to 3 imply that the simple $\mathcal{A}$-orbits in $\mathcal{K}(x, y^{n+2})$ must be of type $f_n$ or $f_{n-1}$, and it is clear that we can take $P_n \equiv 0$ in $f_n$. □

LEMMA 3.2.2. *Any $\mathcal{A}$-simple germ of local multiplicity $n+2$ is equivalent to one of the following germs:*

$$\tilde{f} = (x, y^{n+2} + x_1 y + \ldots + x_{n-1} y^{n-1})$$

*or*

$$\tilde{f}_k = (x, y^{n+2} + x_1 y + \ldots + x_{n-2} y^{n-2} + x_{n-1}^k y^{n-1} + x_{n-1} y^n),$$

*where $2 \leq k \leq (n+3)/2$ (for odd $n$) or $2 \leq k$ (for even $n$), or for odd $n$*

$$\tilde{f}_\infty = (x, y^{n+2} + x_1 y + \ldots + x_{n-2} y^{n-2} + x_{n-1} y^n).$$

*(Notice that we do not claim that all these germs are $\mathcal{A}$-simple, just that any $\mathcal{A}$-simple germ of multiplicity $n+2$ must be equivalent to one of these germs.)*

PROOF. From Lemma 3.2.1,

$$f = (x, y^{n+2} + x_1 y + \ldots + x_{n-2} y^{n-2} + p(x_{n-1}) y^{n-1} + q(x_{n-1}) y^n),$$

where $p$ and $q$ do not both belong to the square of the maximal ideal.

When the linear part of $p$ is non-zero, we can use the weighted version of the Complete Transversal Method, as presented in [3] to prove that $f$ is $\mathcal{A}$ equivalent to

$$\tilde{f} = (x, y^{n+2} + x_1 y + \ldots + x_{n-1} y^{n-1})$$

The calculations in the second case, when $p$ belongs to the square of the maximal ideal are harder. Under this assumption, and given the weights $w(x_i) = n+2-i$, for $i = 1, \ldots, n-2$, $w(x_{n-1}) = 2$ and $w(y) = 1$, the weighted homogeneous part of degree $n+2$ of such a germ is

$$f = (x, y^{n+2} + x_1 y + \ldots + x_{n-2} y^{n-2} + x_{n-1} y^n).$$

In what follows we denote by $\mathcal{M}_w^j$ the ideal in $C_n$ generated by all monomials of filtration $j$.

We divide the calculations in steps, using again the weighted Complete Transversal Method to prove that:

Step 1. All terms of filtration $n+2k$, $k \geq 1$, belong to $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+1} \theta_f$,

Step 2. If fil $v(x, y) = n + 2k + 1$, then $(0, v(x, y)) \equiv (0, x_{n-1}^{k+1} y^{n-1})$, Mod $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+2} \theta_f$, $k \geq 1$.

Step 3. For $n$ odd and $k \geq \frac{n+3}{2}$, the term $(0, x_{n-1}^{k+1} y^{n-1})$ belongs to $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+2} \theta_f$.

Notice that the following elements are in $T\mathcal{A}_1 \cdot f$ :

$$a) \quad wf(X_n e_n) = (0, y^{n+2} + x_1 y + \ldots + x_{n-2}y^{n-2} + x_{n-1}y^n),$$

$$b) \quad tf(\alpha(x).e_n) = (0, \alpha(x)((n+2)y^{n+1} + x_1 + 2x_2 y + \ldots + nx_{n-1}y^{n-1}),$$

$\forall \alpha \in \mathcal{M}_n$, and

$$c) \quad tf(\alpha.e_j) = (0, \alpha y^j), 1 \le j \le n-2; \quad tf(\alpha.e_{n-1}) = (0, \alpha y^n),$$

$\forall \alpha \in \mathcal{M}_x$ or $\alpha = X_n$. Notice also that

$$d) \quad \text{If } \eta(x,y) \in T\mathcal{A}_1 \cdot f, \text{then } \alpha(x)\eta(x,y) \in T\mathcal{A}_1 \cdot f, \quad \forall \alpha \in \mathcal{M}_x.$$

**Step 1**. We use induction. For $k = 2$ it follows easily that all terms of filtration $n+4$ are in $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+5}\theta_f$. By the induction hypothesis, $(0, y^{n+2l})$ and all terms $(0, \alpha(x)y^j)$, with $\mathrm{fil}\, \alpha(x)y^j$ equal to $n+2l$, $1 \le l \le k$, are in $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+1}$.

Let $v(x,y) = \alpha(x)y^j$, $\mathrm{fil}\,(v) = n+2k+2$. If $0 \le j \le n-2$ or $j = n$, it follows from (c) and (d) that $(0, v) \in T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+1}$. Otherwise, there are three possibilities:

(i) $v(x,y) = \alpha(x)y^{n+2l}$, $1 \le l \le k$,
(ii) $v(x,y) = \alpha(x)y^{n+2l+1}$, $\alpha \in \mathcal{M}_x^2$, $1 \le l \le k$,
(iii) $v(x,y) = x_{n-2(k-l)-1}y^{n+2l-1}$, $1 \le l \le k$,

Case (i) now follows easily from the induction hypothesis and equation (d).

In case (ii), we can assume $\alpha(x)$ is a monomial of filtration $2k-2l+1$. Since $n+2l+1$ and $n+2k+2$ have different parities, there is an index $j$ such that $\alpha(x) = x_j\beta(x)$ and $\mathrm{fil}\,(x_j y^{n+2l+1}) = n+2k'+2$, for some $k' < k$. Then the result follows again from the induction hypothesis and (d).

In case (iii), we first use equation (b) to write:

$$(0, v) \equiv 0, \mathrm{Mod}\, T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2l},$$

where

$$v(x,y) = (n+2)y^{n+2l-1} + x_1 y^{2l-2} + \ldots + jx_j y^{j+2l-3} + \ldots + nx_{n-1}y^{n+2l-3}$$

and $\mathrm{fil}\,(x_j y^{j+2l-3}) = n+2l-1$.

If the parity of $j+2l-3$ is equal to the parity of $n+2l-1$ then $\mathrm{fil}\,(x_{n-2(k-l)-1}y^{j+2l-3}) = n+2l'+2$ for some $l' < k$, and we can apply the induction hypothesis. The other possibility is $\mathrm{fil}\,(y^{j+2l-3}) = n+2l'+2$ for some $l' < k$, and we again get the result.

**Step 2**. One can easily check the statement for $k = 1$. Let $v(x,y) = \alpha(x)y^j, \mathrm{fil}\, v = n+2l+1, 1 \le l \le k$.

By the induction hypothesis, $(0, v(x,y)) \equiv (0, x_{n-1}^{l+1} y^{n-1}), \mathrm{Mod}\mathcal{M}_w^{n+2k+2}\theta_f$. From Step 1 it follows that, when $j$ is odd, the element $(0, x_{n-2k+j} y^{n+j-1})$ belongs to $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+l+2}\theta_f$. Then, using equation (b), we can write:

$$(0, v) \equiv 0, \mathrm{Mod}\, T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+2}\theta_f,$$

where

$$v(x,y) = (n+2)y^{n+2k+1} + (n-2k)y^{n-1} + \ldots +$$

$$(n-2k+2l)x_{n-2k+2l}y^{n+2l-1} + \ldots + (n-2)x_{n-2}y^{n+2k-3} + nx_{n-1}y^{n+2k-1},$$

for $0 \leq l \leq k-1$.

Moreover,

$$(0, x_{n-2k+2l}y^{n+2l-1}) \equiv (0, x_{n-2k+2l}x_{n-1}^{l}y^{n-1}) \equiv$$

$$\equiv (0, x_{n-1}^{l}(x_{n-2k+2l}y^{n-1})) \equiv (0, x_{n-1}^{k+1}y^{n-1}).$$

And this proves that $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{n+2k+2}\theta_f$ contains all terms of filtration $n+2k+1$, $k \geq 1$, Mod $(0, x_{n-1}^{k+1}y^{n-1})$.

Let $n$ be odd, $k = \frac{n+3}{2}$. Then $n+2k+1 = 2n+4$, and from equations (b) and (a), we can write the following two linearly independent equations :

$$tf(y^{n+3}.e_n) = (0, (n+2)y^{2n+4} + x_1y^{n+3} + \ldots + nx_{n-1}y^{2n+2}) \equiv 0,$$

modulo $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{2n+5}\theta_f$, and

$$wf(X_n^2 e_n) = (0, y^{2n+4} + 2x_1y^{n+3} + \ldots + 2x_{n-1}y^{2n+2} + x_1^2y^2 + \ldots),$$

modulo $T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{2n+5}\theta_f$.

**Step 3.** It follows from Step 2 that the above system reduces to:

$$(0, (n+2)y^{2n+4} + Ax_{n-1}^{\frac{k+1}{2}}y^{n-1}) \equiv 0, \quad \mathrm{Mod}\, T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{2n+5}\theta_f,$$

$$(0, y^{2n+4} + Bx_{n-1}^{\frac{k+1}{2}}y^{n-1}) \equiv 0, \quad k = \frac{n+3}{2}, \quad \mathrm{Mod}\, T\mathcal{A}_1 \cdot f + \mathcal{M}_w^{2n+5}\theta_f.$$

and it is now easy to make an inductive procedure to conclude the proof of Step 3.

It is a simple calculation to verify that, in any case, for all $j \geq 1$, the $n+2k+1+j$-transversal over the weighted $n+2k+1$ jet of $f$ is empty and this completes the proof of Lemma 3.2.2. □

Lemma 3.2.3. *For the germs in the previous lemma we have:*

$$r_{(n)}(\tilde{f}) = r_{(n-l,l)}(\tilde{f}) = 2, \quad 1 \leq l < n/2$$

$r_{(n/2,n/2)}(\tilde{f}) = 1$ *for even* $n$

*and*

$r_{(n)}(\tilde{f}_k) = r_{(n-l,l)}(\tilde{f}_k) = 3, \quad 1 \leq l < n/2$
$r_{(n/2,n/2)}(\tilde{f}_k) = k$ *for even* $n$

*and*

$r_{(n)}(\tilde{f}_\infty) = r_{(n-l,l)}(\tilde{f}_\infty) = 3, \quad 1 \leq l < n/2.$

PROOF. For $\tilde{f}$ one calculates up to $\mathcal{K}$-equivalence:

$$G_{(n)} \sim (x_1, \ldots, x_{n-1}, (n+2)!y^2/2)$$

and

$$G_{(n-l,l)} \sim (x_1, \ldots, x_{n-1}, cy^2, \epsilon),$$

where $c = (n+2)(ln+l-1)/(l+1) > 0$ and $y := y_1$, $\epsilon := y_2 - y_1$ in the defining equations of $\bar{A}_{(k-l,l)}$.

For $\tilde{f}_\infty$:

$$G_{(n)} \sim (x_1, \ldots, x_{n-2}, -(n+2)!y^3/3, x_{n-1})$$

and

$$G_{(n-l,l)} \sim (x_1, \ldots, x_{n-2}, cy^3, x_{n-1}, \epsilon),$$

where $c = 2(2l-n)(1+n-l)$, which is zero for $l = n/2$ and non-zero for $l < n/2$.

For $\tilde{f}_k$ we get, except for $l = n/2$, the same $G_{(n)}$ and $G_{(n-l,l)}$ as for $\tilde{f}_\infty$ and for even $n$ in addition:

$$G_{(n/2,n/2)} \sim (x_1, \ldots, x_{n-2}, cy^{2k}, x_{n-1}, \epsilon),$$

where $c = (l-2)![(n+2)(l-n-1)/(2(l+1))]^k \neq 0.$ □

PROOF OF THEOREM 3.3 (CONCLUSION). Now one easily constructs an M-deformation of $\tilde{f}$. Using the adjacencies $[\tilde{f}_2] \to [\tilde{f}]$, $[\tilde{f}_{k+1}] \to [\tilde{f}_k]$ and $[\tilde{f}_\infty] \to [\tilde{f}_{(n+3)/2}]$ and the fact that the corresponding 0-stable invariants differ by at most one, we see by induction that all $\tilde{f}_k$ and also $\tilde{f}_\infty$ have M-deformations, because we can split off the real $A_{k(s,n)}$-points (in the target) from the origin one by one. (For $k(s,n) = (n/2, n/2)$ an origin-preserving deformation of $\tilde{f}_{k+1}$ to $\tilde{f}_k$ induces a deformation

$$G^t_{(n/2,n/2)} \sim (x_1, \ldots, x_{n-2}, ay^{2k}(y^2+bt), x_{n-1}, \epsilon),$$

where $t$ is the deformation parameter and $a, b$ are non-zero constants. Thus, for appropriate $t$, we have a pair of real $A_{(n/2,n/2)}$ source points that are mapped to the same target point. For the other $k(s,n)$ we have a single real $A_{k(s,n)}$ in the source, defined by a linear equation, that splits off the origin for $t \neq 0$.) □

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## 5. CONCLUDING REMARKS

Looking at the proof of our main theorem on M-deformations we observe the following: given any $\mathcal{A}$-simple germ $f : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ of rank $n-1$ and $m \geq n$, there exists a germ $f'$ such that $[f] \to [f']$ and $r_{k(s,n)}(f) - r_{k(s,n)}(f') \leq 1$, for all partitions $k(s,n)$ of $n$. From this property we have the following lower bound on the $\mathcal{A}_e$-codimension.

COROLLARY 4.1. *Let $f : \mathbb{R}^m, 0 \to \mathbb{R}^n, 0$ be an $\mathcal{A}$-simple germ of rank $n-1$, then*

$$\operatorname{cod}(\mathcal{A}_e, f) \geq r_{k(s,n)}(f) - 1.$$

If one could show that the above property also holds for $\mathcal{A}$-simple corank-1 germs $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, where $n < p$, then one could show that these also have M-deformations as in the conclusion of the proof of Theorem 3.3. For particular pairs of dimensions $(n,p)$ such germs indeed have M-deformations. For $n < p$, $A_{k(s,m)}$ is an isolated and stable $s$-germ if $k(s,m) = (k_1, \ldots, k_s)$, where $m = \sum_i k_i$ and $k_i \geq 0$, satisfies the equality $(m+s-1)(p-n+1) = n+s-1$, and $r_{k(s,m)}(f)$ again denotes the number of these concentrated at the origin in the source of $f_{\mathbb{C}}$ (see [16]).

For $(n,p) = (1,2)$ it is known by classical results of A'Campo and Gusein-Zade that any germ $f$ has an M-deformation with $r_{(0,0)}(f) = \delta(f)$ real double points. For corank-1 germs in dimension $(2,3)$ Mond [12] states that there are real deformations with $r_{(1)}(f) = C(f)$ cross-caps and (without proof) that there are real deformations with $r_{(0,0,0)}(f) = T(f)$ triple-points. We do not know whether there is always an M-deformation in which $C(f)$ cross-caps and $T(f)$ triple-points appear *simultaneously*. For corank-1 germs in dimension $(3,4)$ there are two 0-stable invariants, namely $r_{(1,0)}(f)$ and $r_{(0,0,0,0)}(f)$. The latter invariant is 0 for all $\mathcal{A}$-simple germs in the classification of Houston and Kirk [8], and one can easily show that there are deformations with $r_{(1,0)}(f)$ real $A_{(1,0)}$-points for all simple corank-1 germs listed in [8]. And these deformations are M-deformations.

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