A LOWER BOUND FOR TOPOLOGICAL ENTROPY OF GENERIC NON ANOSOV SYMPLECTIC DIFFEOMORPHISMS

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Abstract. We prove that a $C^1$–generic symplectic diffeomorphism is either Anosov or its topological entropy is bounded from below by the supremum over the smallest positive Lyapunov exponent of its periodic points. We also prove that $C^1$–generic symplectic diffeomorphisms outside the Anosov ones do not admit symbolic extension and finally we give examples of volume preserving surface diffeomorphisms which are not point of upper semicontinuity of entropy function in the $C^1$–topology.

1. Introduction

The topological entropy of a dynamical system is one of the most important numbers which measures the complexity of the system. Informally speaking, the topological entropy calculates the “number of different trajectories” of the dynamics. Formally, we define it in the following way:

$$h(f) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon);$$

where $r(n, \epsilon)$ is the maximum number of $\epsilon$-distinct orbits of length $n$. Two points $x$ and $y$ have $\epsilon$-distinct orbits of length $n$ if there is $0 \leq j \leq n$ such that $d(f^j(x), f^j(y)) > \epsilon$.

For Axiom A diffeomorphisms, Bowen [6] proved that, the entropy of a system determines the rate of growth of the number of periodic points and by a Katok’s result [15], for any $C^{1+\alpha} (\alpha > 0)$ diffeomorphism of a two dimensional manifold, the entropy is bounded from above by such growth rate: $h(f) \leq \limsup_{n \to \infty} \frac{\log P_n(f)}{n}$.

In this paper we prove a new lower bound for the topological entropy of $C^1$–generic symplectic dynamics in terms of the Lyapunov exponents of periodic points of the system, see Theorems A, B.

We also study the regularity of entropy function with respect to the dynamics (see Theorem C) and construct examples of surface diffeomorphisms which are not point of semi continuity of the topological entropy.

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Finally, we prove the non existence of symbolic extensions for $\text{C}^1$–generic symplectic diffeomorphisms outside Anosov systems, see theorem D.

1.1. **Lower estimates for topological entropy.** Newhouse in [22] got some lower bounds for the topological entropy of generic, non-Anosov area preserving surface diffeomorphisms.

Let $M$ be a compact, connected surface with a volume form $m$, and denote by $\text{Diff}^1_{m}(M)$ the set of conservative (volume preserving) $\text{C}^1$ diffeomorphisms. Let

$$s(f) := \sup \left\{ \frac{1}{\tau(p, f)} \log \lambda(p, f) \right\},$$

where the supremum is taken over all hyperbolic periodic points $p$ of $f$, $\tau(p, f)$ denotes the minimum period of $p$ and $\lambda(p, f)$ is the absolute value of the eigenvalue of $Df^\tau(p, f)(p)$ with norm larger than one.

1.1. **Theorem.** ([22]) There exists a residual subset $\mathcal{B} \subset \text{Diff}^1_{m}(M)$ such that if $f \in \mathcal{B}$ is not an Anosov diffeomorphism then

$$h(f) \geq s(f).$$

Here we show that indeed generically the inequality in the opposite direction also holds and consequently we prove the following theorem.

**Theorem A.** There exists a residual subset $\mathcal{B} \subset \text{Diff}^1_{m}(M)$ (volume preserving surface diffeomorphisms) such that if $f \in \mathcal{B}$ is not an Anosov diffeomorphism then

$$h(f) = s(f).$$

Observe that as a corollary of the above theorem and semi continuity of $f \to s(f)$ (see preliminary definitions in the next section) we conclude that “generically” topological entropy is semi continuous in the $\text{C}^1$–topology. However, it is not known whether the semi continuity points of topological entropy form a $\text{C}^1$–generic subset or not.

It is interesting to mention that among Anosov diffeomorphisms, there exists a $\text{C}^1$–open and dense subset $U$ such that any $f \in U$ satisfies $h(f) < s(f)$. Proposition 2.1 gives an upper bound for the entropy of Anosov diffeomorphisms.

In general, there may exist diffeomorphisms where $h(f) > s(f)$. In fact there exist even minimal diffeomorphisms with positive entropy. In the two dimensional case there are minimal homeomorphisms with positive entropy [25]. However, these examples are not volume preserving (and should be outside a $\text{C}^1$– generic subset).

1.2. **Question.** *Is there any example of a conservative $\text{C}^1$ surface diffeomorphism where $h(f) > s(f)$?*
Our next result is a generalization of the Newhouse’s theorem to the higher dimensional symplectic setting. Let \((M, \omega)\) be a compact, connected and smooth Riemannian symplectic manifold with symplectic form \(\omega\). We denote by \(\text{Diff}^1(M)\) the set of diffeomorphisms that preserve \(\omega\), i.e., symplectic diffeomorphisms, and by \(\mathcal{A} \subset \text{Diff}^1(M)\) the subset of symplectic Anosov diffeomorphisms. For any hyperbolic periodic point \(p\) of \(f\), we denote by \(\lambda(p, f)\) the absolute value of the smallest eigenvalue of \(Df^{\tau(p, f)}(p)\) among those with absolute value larger than one. Define
\[
s(f) := \sup_{p} \left\{ \frac{1}{\tau(p, f)} \log |\lambda(p, f)| \right\}
\]
where the supremum is taken over all hyperbolic periodic points \(p\) of \(f\).

**Theorem B.** There exists a residual subset \(\mathcal{B} \subset \text{Diff}^1(M)\), such that if \(f \in \mathcal{B}\) is not an Anosov diffeomorphism then
\[
h(f) \geq s(f).
\]

Let us pose some questions related to the above theorem:

- Is it possible to substitute \(s(f)\) in the above theorem by the supremum over the sum of all positive Lyapunov exponents of periodic points?
- Does the same result hold in the conservative setting?
- Can we prove the same result in the \(C^r\) \((r \geq 2)\) topology? See remark 3.2 for a discussion.

### 1.2. Regularity of Entropy.

An important problem in smooth ergodic theory is the regularity of topological entropy with respect to the dynamics. By a Newhouse result, we know that \(f \rightarrow h(f)\) is upper semicontinuous in the \(C^\infty\) topology for any compact boundaryless manifold. Using Katok’s result, it is indeed continuous for \(C^\infty\) surface diffeomorphisms. Here we prove that:

**Theorem C.** There are examples of surface diffeomorphisms \(f_0 \in \text{Diff}^\infty(M)\) such that \(f \rightarrow h(f)\) is not upper semi continuous at \(f_0\) in the \(C^1\)– topology.

We would like to remark that, Misiurewicz in 1970’s [20] constructed examples of diffeomorphisms defined on manifolds of dimension larger than four which are \(C^r\) discontinuity points of the entropy function.

### 1.3. Symbolic Extensions.

Symbolic dynamics plays a crucial role in the study of ergodic properties of smooth dynamical systems. A dynamical system \((M, f)\) has a *symbolic extension* if there exists a subshift \((Y, \sigma)\) (finite alphabet) and a surjective map \(\pi : Y \rightarrow M\) such that \(\pi \circ \sigma = f \circ \pi\). In this case \((Y, \sigma)\) is called an *extension* of \((M, f)\) and \((M, f)\) a *factor* of \((Y, \sigma)\). Clearly the entropy of \(f\) is smaller or equal to the entropy of \(\sigma\). However in a general case, such estimate for the topological entropy may be too rough.
A symbolic extension is called principal if the map $\pi$ satisfies $h_\nu(\sigma) = h_{\pi_\nu}(f)$ for every $\sigma$-invariant measure $\nu \in \mathcal{M}(\sigma)$ over $Y$. Here $h_\nu(\sigma)$ denotes the metric entropy of $\sigma$ with respect to $\nu$.

Boyle, D. Fiebig, U. Fiebig [7] proved that asymptotically $h$–expansive diffeomorphisms have a principal symbolic extension. By a result of Buzzi [10], all $C^\infty$ diffeomorphisms of compact manifolds are asymptotically entropy expansive and consequently have a principal symbolic extension. Also, recently D. Burguet [8] proved that every $C^2$ surface diffeomorphism has symbolic extensions. These results give positive partial answers to the conjecture of Downarowicz and Newhouse which expects symbolic extension for any $C^r$ ($r \geq 2$) diffeomorphism.

Let us mention that Diaz, Fisher, Pacifico and Vieitez [12] proved that every $C^1$ partially hyperbolic diffeomorphism with a nonhyperbolic central bundle that splits in a dominated way into 1-dimensional sub-bundles is asymptotically $h$–expansive and therefore has a principal symbolic extension. See also [11], [9].

On the other hand, Downarowicz and Newhouse, using Theorem 1.1 proved in [21] that far from Anosov diffeomorphisms, generic area preserving diffeomorphisms in the $C^1$ topology admits no symbolic extensions. We extend this result to the symplectic diffeomorphisms of higher dimensional manifolds.

**Theorem D.** There exists a residual subset $\mathcal{B} \subset \text{Diff}_{1,\omega}(\mathcal{M})$, such that if $f \in \mathcal{B}$ and $f$ is not Anosov, then $f$ has no symbolic extension.

Using this result we are able to give a short proof of the stability conjecture in symplectic setting. We say that a symplectic diffeomorphism is structurally stable if there is some neighborhood $\mathcal{U}$ of $f$ in $\text{Diff}_{1,\omega}(\mathcal{M})$ such that every diffeomorphism $g \in \mathcal{U}$ is topologically conjugate to $f$.

**1.3. Corollary.** A diffeomorphism $f \in \text{Diff}_{1,\omega}(\mathcal{M})$ is structurally stable if and only if, $f$ is Anosov.

**Proof.** Suppose $f \in \text{Diff}_{1,\omega}(\mathcal{M})$ is structurally stable. It is clear that having symbolic extension is invariant under topological conjugacy. By Zehnder [29], smooth diffeomorphisms are dense among the symplectic ones and since $C^\infty$ diffeomorphisms have a principal symbolic extension, every diffeomorphism in some neighborhood of $f$ also has a principal symbolic extension. So, according to Theorem D this is only possible if $f$ is Anosov.

This paper is organized as following: In section 2 we will prove some generic results about topological entropy and the Lyapunov exponents of periodic points. In particular, we prove Theorem A. In section 3, using a main technical proposition (Proposition 3.1) we prove Theorems B and C. In section 4, we prove the main technical proposition and finally in section 5 we construct examples of (upper semi) discontinuity points of topological entropy.
2. Entropy and Lyapunov exponents of periodic points

In this section, firstly we review some background definitions and prove an strict upper bound for the entropy of dynamics in a $C^1$–open and dense subset of Anosov volume preserving diffeomorphisms. After that, using a result of Abdenur, Bonatti and Crovisier [1] we prove an upper bound for the entropy of $C^1$–generic volume preserving diffeomorphisms of compact manifolds (of any dimension) and apply it to prove Theorem A.

2.1. Preliminary definitions. Given $f \in \text{Diff}^1(M)$ and a hyperbolic periodic point $p$ of $f$, we denote by $\chi(p, f)$ the smallest positive Lyapunov exponent of $p$, i.e., $\chi(p, f) = 1/\tau(p, f) \log \lambda(p, f)$ where $\lambda(p, f) = (\sigma(Df^{-\tau(p, f)}|E^u))^{-1}$ and $\sigma$ denotes the spectral radius of the map. As before $\tau(p, f)$ is the minimum period of $p$. In fact, defining

$$\mu_p = \frac{1}{\tau(p, f)} \sum_{i=0}^{\tau(p)-1} \delta_{f^i(p)}$$

as the periodic measure for $p$, where $\delta_{f^i(p)}$ is the dirac mass concentrated on $f^i(p)$, we have that $\chi(p, f)$ is the smallest positive Lyapunov exponent of the ergodic measure $\mu_p$.

Now, given $n \in \mathbb{N}$ we consider $s_n(f) := \max \{\chi(p, f); \ p \in H_n(f)\}$, where $H_n(f)$ denotes the set of hyperbolic periodic points of period smaller or equal than $n$. Since $H_n(f) \subset H_{n+1}(f)$, we have $s_n(f) \leq s_{n+1}(f)$, and then $s(f) := \lim_{n \to \infty} s_n(f)$ is well defined. From robustness of the hyperbolic periodic points we conclude that the functional $s_n$ is continuous for every $n \in \mathbb{N}$, which implies that $s(f)$ is lower semicontinuous.

2.2. Upper bound for the entropy of generic Anosov diffeomorphisms. In the setting of volume preserving Anosov diffeomorphisms, there is an upper bound for the topological entropy of generic diffeomorphisms. More precisely, if we denote by $\mathcal{A}_m$ the set of $C^1$–volume preserving Anosov diffeomorphisms on $M$, then we have the following result:

2.1. Proposition. There exists a $C^1$–open and dense subset $\mathcal{F} \subset \mathcal{A}_m$ of volume preserving Anosov diffeomorphisms such that for any $f \in \mathcal{F}$ with $\dim(E^u) = u$ we have

$$h(f) < \sup_{p \in \text{Per}(f)} \sum_{i=1}^{u} \chi^+_i(p)$$

where the sum is over all positive Lyapunov exponents of $p$.

Proof. Let $f$ be a $C^1$ volume preserving Anosov diffeomorphism. After a small $C^1$–perturbation, if necessary, we can assume that $f$ is $C^2$. This regularization result is due to Avila [5]. We know that for volume preserving Anosov diffeomorphisms the Lebesgue measure $m$ is the unique ergodic equilibrium state
for the potential $\phi^u(\cdot) = -\log J^u(f)$ where $J^u(f) := |det Df|^E(u(f))$. Recall that the entropy maximizing measure is the equilibrium state for the identically zero potential. Using Bowen’s result, it is clear that the entropy maximizing measure $\mu$, coincides with Lebesgue if and only if the potential $\phi^u$ is cohomologous to a constant function. So, perturbing $f$ in the $C^1$–topology we can assume that $\mu$ is singular with respect to the Lebesgue measure.

Now recall that in Bowen’s approach, the entropy maximizing measures are obtained as the limit of periodic distributions. That is

$$\mu_n := \frac{\sum_{p \in \text{Per}_n(f)} \delta_p}{\#\text{Per}_n(f)} \rightarrow \mu$$

Since $\phi^u(\cdot)$ is a continuous function, we have $\int -\phi^u(x) d\mu(x) = \lim_{n \to \infty} \int -\phi^u(x) d\mu_n(x)$ and by definition we conclude that $\int -\phi^u d\mu_n \leq \sup_{p \in \text{Per}_n(f)} \sum_{i=1}^u \chi^+_i(p)$. Then,

$$\int -\phi^u d\mu \leq \sup_{p \in \text{Per}(f)} \sum_{i=1}^u \chi^+_i(p).$$

Now, as $f$ is Anosov the pressure of $\phi^u(\cdot)$ is zero. Hence,

$$0 = P_f(\phi^u) = h_m(f) + \int \phi^u d\mu$$

$$> h_\mu(f) + \int \phi^u d\mu$$

$$= h(f) + \int \phi^u d\mu.$$

So it comes out out that

$$h(f) < \int -\phi^u d\mu \leq \sup_{p \in \text{Per}(f)} \sum_{i=1}^u \chi^+_i(p).$$

Finally, we claim that any $C^1$–perturbation of $f$ also satisfies a similar inequality. Indeed, as entropy is locally constant (by structural stability of Anosov diffeomorphisms) and $f \rightarrow \sup_{p \in \text{Per}(f)} \sum_{i=1}^u \chi^+_i(p)$ is a lower semi continuous function, we conclude that for any $g$, $C^1$–close enough to $f$ we have

$$h(g) < \sup_{p \in \text{Per}(g)} \sum_{i=1}^u \chi^+_i(p).$$

\[\square\]

2.3. **Proof of Theorem A.** Let $f$ be a $C^{1+\alpha}$ diffeomorphism on a compact manifold and $\mu$ be an ergodic hyperbolic measure. By Katok’s result ([15]), there exists a sequence of periodic points $p_n$ such that the dirac measures on the orbit of $p_n$ converge to $\mu$. Moreover, the Lyapunov exponents of $p_n$ converge to the
Lyapunov exponents of \( \mu \). By variational principle \( h(f) = \sup_\mu h_\mu(f) \), where the supremum is over all \( f \)-invariant ergodic probability measures. By Ruelle’s inequality \( h_\mu(f) \leq \sum \lambda_i^+ \), where the sum is over all positive Lyapunov exponents of \( \mu \). If we suppose that the supremum in the variational principle can be taken over (only) hyperbolic measures, then we conclude that
\[
h(f) \leq \sup_{p \in \text{Per}(f)} \sum \lambda_i^+(p).
\]
Using Abdenur, Bonatti and Crovisier’s ideas in [1] we show that for \( C^1 \)-generic volume preserving diffeomorphisms, the above inequality holds. Here we denote by \( \text{Diff}^1_m(M) \) the set of \( C^1 \) volume preserving diffeomorphisms.

2.2. Theorem. There exists a residual subset \( \mathcal{R} \subset \text{Diff}^1_m(M) \) (\( M \) of any dimension) such that for any \( f \in \mathcal{R} \)
\[
h(f) \leq \sup_{p \in \text{Per}(f)} \sum_{i=1}^{\mu_p} \lambda_i^+(p)
\]
where the sum is over all positive Lyapunov exponents of the periodic point \( p \), counting with multiplicity.

The proof of Theorem A follows from Theorem 1.1 and the previous theorem.

Proof of Theorem A: Let \( \mathcal{B} \) be the residual subset which is the intersection of residual subsets given by theorems 1.1 and 2.2. If the supremum in Theorem 2.2 was taken over hyperbolic periodic points then this supremum in dimension two would be equal to \( s(f) \) and then Theorem A was proved.

Hence, we divide the proof into two cases. First suppose \( f \in \mathcal{B} \) is such that \( h(f) = 0 \), then the equality, \( h(f) = s(f) \), comes directly from Theorem 1.1 since \( s(f) \geq 0 \). In the second case, \( h(f) > 0 \). In Theorem 2.2, the supremum could be taken over periodic points having positive Lyapunov exponents. Indeed, in dimension two a periodic point of a conservative diffeomorphism has positive Lyapunov exponent iff it is a hyperbolic periodic point. So, we have equality between \( h(f) \) and \( s(f) \).

In what follows we prove Theorem 2.2. Using Abdenur, Bonatti and Crovisier [1], we can prove the following Proposition.

2.3. Proposition. There is a residual subset \( \mathcal{R} \subset \text{Diff}^1_m(M) \) such that if \( f \in \mathcal{R} \) and \( \mu \) is an ergodic measure for \( f \), then there are periodic measures \( \mu_p \) converging to \( \mu \) in the weak topology, and moreover the vectors formed by the Lyapunov exponents of \( \mu_p \), \( L(\mu_p) \in \mathbb{R}^d \), also converge to the Lyapunov vector \( L(\mu) \in \mathbb{R}^d \).

In fact, they proved this result for dissipative diffeomorphisms, Theorem 3.8 in [1]. But, unless generic arguments, their theorem is a consequence of Proposition 6.1 there, which we state here for simplicity.
2.4. Proposition. Let $\mu$ be an ergodic invariant probability measure of a diffeomorphism $f$ of a compact manifold $M$. Fix a $C^1$-neighborhood $U$ of $f$, a neighborhood $V$ of $\mu$ in the space of probability measures with the weak topology, a Hausdorff neighborhood $K$ of the support of $\mu$, and a neighborhood $O$ of $L(\mu)$ in $\mathbb{R}^d$. Then there is $g \in U$ and a periodic point $p$ of $g$ such that the Dirac measure $\mu_p$ associated to $p$ belongs to $V$, its support belongs to $K$, and its Lyapunov vector $L(\mu_p)$ belongs to $O$.

They divided the proof of this proposition in two lemmas, Lemma 6.2 and Lemma 6.3 there. In the first one, given an ergodic measure $\mu$ for $f$ they found strategic periodic points $p_n$ for diffeomorphisms $f_n \in U$ with good properties. Then, in Lemma 6.3, they proved that the Lyapunov vectors $L(\mu_p)$ converge to $L(\mu)$. However, by the ergodic closing lemma (see [4]) and Frank’s lemma (see [16] and [2]) the proof of Proposition 2.4 in the volume preserving case is exactly the same as Proposition 6.1 in [1]. Now, using Proposition 2.3 we prove Theorem 2.2.

Proof of Theorem 2.2: Let $f \in \mathcal{R}$, where $\mathcal{R}$ is as residual subset as in Proposition 2.3. Given any $\varepsilon > 0$, by variational principle there is an ergodic measure $\mu \in M(f)$ such that

$$h(f) < h_{\mu}(f) + \varepsilon.$$ 

By Ruelle’s inequality $h_{\mu}(f) \leq \sum \chi_i^+(\mu)$, where the sum is over all positive Lyapunov exponents of $\mu$. Now, by Proposition 2.3 there is a periodic point $p$ of $f$ such that $\sum \chi_i^+(\mu) < \sum \chi_i^+(\mu_p) + \varepsilon$. And then, we have

$$h(f) < \sup_{p \in \text{Per}(f)} \sum_i^+ \chi_i(p) + 2\varepsilon.$$ 

Therefore, since $\varepsilon$ is arbitrarily small the theorem is proved.

\[\square\]

3. Entropy estimates for symplectomorphisms

Let $p$ be a hyperbolic periodic point of $f : M \to M$. We denote by $H(p, f)$ the set of transversal homoclinic points of $p$.

Zhuhong Xia in [27] proved that there exists a residual subset $\mathcal{H} \subset \text{Diff}_c^1(M)$ such that if $f \in \mathcal{H}$ and $p$ is a hyperbolic periodic point of $f$ then the transversal homoclinic points are dense on stable and unstable manifolds: $W^s(o(p), f) \cup W^u(o(p), f) \subset \text{cl}(H(p, f))$. Let us now state the main proposition and prove Theorems B and D. We postpone the proof of this proposition to the next section. Recall that $\chi(p, f) = 1/\tau(p, f) \log \lambda(p, f)$ is the smallest positive Lyapunov exponent for a hyperbolic periodic point $p$ of $f$.

3.1. Proposition. (Main Technical Proposition) Let $p$ be a hyperbolic periodic point of a non Anosov diffeomorphism $f \in \mathcal{H} \subset \text{Diff}_c^1(M)$. Given a positive integer $n > 0$
and any neighborhood \( N \subset \text{Diff}^1(M) \) of \( f \), there exists an open set \( \mathcal{U} \subset N \) such that if \( g \in \mathcal{U} \), then \( g \) has a basic hyperbolic set \( \Lambda(p(g), n) \subset \text{cl}(H(p(g), g)) \), where \( p(g) \) is the continuation of the hyperbolic periodic point \( p \) of \( f \) for \( g \), such that the following properties are satisfied:

a) \( h(g|_{\Lambda(p(g), n)}) > \chi(p(g), g) - \frac{1}{n} \),

b) There exists an ergodic measure \( \mu \in \mathcal{M}(\Lambda(p(g), n)) \) such that

\[
h_{\mu}(g) > \chi(p(g), g) - \frac{1}{n},
\]

c) For every ergodic measure \( \mu \in \mathcal{M}(\Lambda(p(g), n)) \), we have

\[
\rho(\mu, \mu_{p(g)}) < \frac{1}{n},
\]

where \( \rho \) is a metric which generates the weak topology and

d) For every periodic point \( q \in \Lambda(p(g), n) \), we have

\[
\chi(q, g) > \chi(p(g), g) - \frac{1}{n}.
\]

3.2. Remark. Let us mention that in the proof of the above Proposition, one needs to perturb \( f \) and create a tangency and then an interval of tangencies. If we desire to do this perturbation in the \( C^r \) topology for \( r > 1 \), we need higher order tangencies. Using results of Gonchenko-Shil’nikov-Turaev in [17], [18] or V. Kaloshin [14], it is possible to perform such perturbation in the two dimensional case (dissipative and conservative). However, in the higher dimensional case, it is not clear how to create high order tangencies.

3.1. Proof of Theorem B. First we consider \( \mathcal{D} = \text{Diff}^1(M) \setminus \text{cl}(\mathcal{A}) \), which is the complement of the closure of \( C^1 \) symplectic Anosov diffeomorphisms.

For positive integers \( n \) and \( m \), let \( B_{n,m} \) be the set of diffeomorphisms \( f \) in \( \mathcal{D} \) such that there are \( p \in H_n(f) \) and a hyperbolic basic set \( \Lambda \subset \text{cl}(H(p, f)) \), satisfying

\[
h(f|_{\Lambda}) > s_n(f) - \frac{1}{m}.
\]

Theorem B follows immediately from the next claim.

**Claim:** \( B_{n,m} \) is an open and dense subset of \( \mathcal{D} \), for every positive integers \( n \) and \( m \).

To prove the claim, take an arbitrary diffeomorphism \( f \in \mathcal{D} \cap \mathcal{H} \), and arbitrary positive integers \( n \) and \( m \).

By definition of \( s_n \), there exists \( p_0 \in H_n(f) \) such that

\[
s_n(f) = \chi(p_0, f).
\]
Using Proposition 3.1, we can find $f_1 \in C^1$ close to $f$ such that $f_1$ has a hyperbolic basic set $\Lambda \subset cl(H(p_0(f_1), f_1))$, and

$$h(f_1|\Lambda) > \chi(p_0(f_1), f_1) - \frac{1}{3m}.$$  

Now, by robustness of $\Lambda$ and $p_0$, the invariance of topological entropy and that $s_n$ is continuous, for any $g \in C^1$ near $f_1$ we have that

$$h(g) \geq h(g|\Lambda(g)) = h(f_1|\Lambda) > \chi(p_0(f_1), f_1) - \frac{1}{3m} \geq \chi(p_0, f) - \frac{2}{3m} = s_n(f) - \frac{2}{3m} > s_n(g) - \frac{1}{m},$$

which proves the claim since $\mathcal{H}$ is a residual subset in $\text{Diff}^1_0(M)$.

3.2. Proof of Theorem D. Recall that $(Y, \sigma)$ is a symbolic extension of $(M, f)$ if there exists a continuous surjective map $\pi : Y \to M$ such that $\pi \circ \sigma = f \circ \pi$. As we have mentioned, it may happen that symbolic extensions of a system have larger entropy than the system.

Hence, let

$$h_{ext}^\pi(\mu) = \sup\{h_{\nu}(\sigma|Y) : \pi_\nu = \mu\}, \quad \text{for } \mu \in M(f),$$

and observe that principal symbolic extensions minimize these functions.

Let $S(f)$ be the set of all possible symbolic extensions $(Y, \sigma, \pi)$ of $(M, f)$. We say that $S(f) = \emptyset$ if there is no symbolic extension of $(M, f)$. We define the residual entropy of the system by

$$h_{res}(f) = h_{sec}(f) - h(f),$$

where

$$h_{sec}(f) = \begin{cases} 
\inf_{(Y, \sigma, \pi) \in S(f)} [h(\sigma) : (Y, \sigma, \pi) \in S(f)] & \text{if } S(f) \neq \emptyset \\
\infty & \text{if } S(f) = \emptyset 
\end{cases}$$

So to prove Theorem D we need to show that $h_{sec}(f) = \infty$ for all non Anosov symplectic diffeomorphism $f$ in a residual subset $\mathcal{B} \subset \text{Diff}^1_0(M)$.

Let $f : M \to M$ be a homeomorphism in a compact metric space $M$. An increasing sequence $\alpha_1 \leq \alpha_2 \leq \ldots$ of partitions of $M$ is called essential for $f$ if

1. $\text{diam}(\alpha_k) \to 0$ when $k \to \infty$, and
2. $\mu(\partial \alpha_k) = 0$ for every $\mu \in \mathcal{M}(f)$. Where $\partial \alpha_k$ denotes the union of boundaries of all elements of the partition $\alpha_k$.

A sequence of simplicial partitions is a nested sequence $S = \{\alpha_1, \alpha_2, \ldots\}$ of partitions whose diameters go to zero, and each $\alpha_k$ is given by some smooth triangulation of $M$. By Proposition 4.1 in [21] there is a residual subset $\mathcal{R}_S \subset \text{Diff}^1(M)$ such that if $f \in \mathcal{R}_S$ then $S$ is an essential sequence of partitions for $f$.

Hence, for every fixed $k$, the function

$$h_k(\mu) = h_\mu(\alpha_k),$$

is the infimum of continuous functions over $\mathcal{M}(f)$, and hence upper semicontinuous. Here $h_\mu(\alpha_k)$ is the entropy of the partition $\alpha_k$ for $f$. The following proposition provides a very useful way to prove non existence of symbolic extensions. It was also proved in [21].

3.3. Proposition. Let $f \in \mathcal{R}_S$ and suppose $E$ be some compact subset in $\mathcal{M}(f)$ such that there exists a positive real number $\rho_0$ such that for every $\mu \in E$ and $k > 0$,

$$\limsup_{v \in E, v \to \mu} [h_v(f) - h_k(v)] > \rho_0.$$

Then,

$$h_{\text{sex}}(f) = \infty.$$

Recall $H_n(f)$ denotes the set of hyperbolic periodic points with period smaller or equal than $n$, and $H(f) = \bigcup_{n \geq 1} H_n(f)$. We denote by $\mathcal{R}_1$ the set of diffeomorphisms $f$ such that $H(f) \neq \emptyset$, which is an open and dense subset of $\text{Diff}^1(M)$ by Pugh’s closing lemma. Hence, for every $f \in \mathcal{R}_1$, $\tau(f)$ as the smallest period of the elements in $H(f)$ is well defined. Thus, if we define $\mathcal{R}_{1,m} \subset \mathcal{R}_1$ as the set of diffeomorphisms $f$ such that $\tau(f) = m$, then $\mathcal{R}_1$ is a disjoint union of $\mathcal{R}_{1,m}$’s for $m \geq 1$.

Now, for each $f \in \mathcal{R}_1$ we define

$$\chi(f) = \sup\{\chi(p, f) : p \in H(f) \text{ and } \tau(p, f) = \tau(f)\}.$$

Then, $\chi(f) > 0$ and depends continuously on $f \in \mathcal{R}_1$.

Recalling that $\mathcal{A} \subset \text{Diff}^1(M)$ is the set of Anosov diffeomorphisms, let $\mathcal{R}_{2,m} = \mathcal{R}_{1,m} \setminus \overline{cl(\mathcal{A})}$, which implies $\mathcal{R}_1 \setminus \overline{cl(\mathcal{A})} = \bigcup_m \mathcal{R}_{2,m}$.

Suppose now that $\Lambda$ is an $f$-invariant periodic set with basis $\Lambda_1$ and $\alpha = A_1, A_2, \ldots, A_s$ is a finite partition of $M$. We say that $\Lambda$ is subordinate to $\alpha$ if for each positive integer $n$, there exists an element $A_{i_n} \in \alpha$ such that $f^n(\Lambda_1) \subset A_{i_n}$. Hence, if $\mu \in \mathcal{M}(f/\Lambda)$, then $h_\mu(\alpha) = 0$.

Now, for a positive integer $n$, we say that a diffeomorphism $f$ satisfies property $S_n$ if for every $p \in H_n(f)$ with $\chi(p, f) > \frac{\chi(f)}{2}$,
1. There exists a hyperbolic basic set of zero dimension $\Lambda(p,n)$ for $f$ such that $\Lambda(p,n) \cap \partial\alpha_n = \emptyset$ and $\Lambda(p,n)$ is subordinate to $\alpha_n$.

3. There exists an ergodic measure $\mu \in \mathcal{M}(\Lambda(p,n))$ such that $h_\mu(f) > \chi(p,f) - \frac{1}{n}$.

4. For every ergodic measure $\mu \in \mathcal{M}(\Lambda(p,n))$, we have 
$$\rho(\mu, \mu_p) < \frac{1}{n}.$$ 

5. For every periodic point $q \in \Lambda(p,n)$, we have 
$$\chi(q, f) > \chi(p, f) - \frac{1}{n}.$$ 

Given positive integers $m \leq n$, we denote by $D_{m,n} \subset \mathcal{R}_{2,m}$ the subset of diffeomorphisms $f$ satisfying property $S_n$.

Since there are finitely many periodic points in $H_n(f)$ with smallest positive Lyapunov exponent larger than $\chi(f)/2$, by Proposition 3.1 we conclude that conditions (3), (4) and (5) above are satisfied for diffeomorphisms in an open and dense subset of $\mathcal{R}_{2,m}$.

Now, given a partition $\alpha_n$, we see in the next section, in the proof of Proposition 3.1, that the hyperbolic set $\Lambda(p,n)$ can be constructed subordinate to $\alpha_n$. In fact, this hyperbolic set could be contained in an open set $U$ with arbitrary diameter. Therefore, since this is a robust property we have proved the following lemma.

3.4. Lemma. For positive integers $m \leq n$, $D_{m,n}$ is open and dense in $\mathcal{R}_{2,m}$.

Using property $S_n$ and the above lemma, the proof of Theorem D is similar to the proof of Theorem 1.3 in [21]. For sake of completeness we will reproduce it here.

Proof of Theorem D: Let
$$\mathcal{R}_2 = \bigcup_{m \geq 1} \bigcap_{n \geq m} D_{m,n}.$$ 

By Lemma 3.4, $\mathcal{B} := \mathcal{R}_S \cap (\mathcal{R}_2 \cup \mathcal{A})$ is a residual subset in $\text{Diff}_0^3(M)$.

Hence, the proof is finished if we show that all non Anosov diffeomorphism $f \in \mathcal{B}$ has no symbolic extensions, i.e., $h_{scw}(f) = \infty$.

Let $f \in \mathcal{B}$ be a non Anosov diffeomorphism. Now, we define
$$\mathcal{E}_1 = \left\{ \mu_p; \text{ such that } p \in H(f) \text{ and } \chi(p, f) > \frac{\chi(f)}{2} \right\},$$ 

and we denote by $\mathcal{E}$ the closure of $\mathcal{E}_1$ in $\mathcal{M}(f)$.
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Using property $S_n$, we verify that the hypothesis of Proposition 3.3 are satisfied for this set $E$ and $\rho_0 = \frac{\chi(f)}{2}$. Note that, by construction of $E$, it is enough to verify the hypothesis only for measures $\mu_p \in E_1$.

Hence, let us fix $\mu_p \in E_1$ and $k \in \mathbb{N}$. Now, for any $n \in \mathbb{N}$ large enough, since $f \in B$, there exists a hyperbolic basic set $\Lambda(p,n)$ of $f$ subordinate to $\alpha_n$, and an ergodic measure $\nu_n \in \mathcal{M}(\Lambda(p,n))$ such that $\rho(\nu_n, \mu_p) < 1/n$ and

\[ h_{\nu_n}(f) > \chi(p, f) - \frac{1}{n}. \]

Also note that if $n > k$, $\alpha_n$ is a finer partition than $\alpha_k$, which implies $\Lambda(p,n)$ is also subordinated to $\alpha_k$. Hence, as $\nu_n \in \mathcal{M}(\Lambda(p,n))$

\[ h_k(\nu_n) = 0. \]

Therefore, we have that $\nu_n \to \mu_p$, when $n \to \infty$, and moreover for $n$ large enough

\[ |h_{\nu_n}(f) - h_k(\nu_n)| = h_{\nu_n}(f) > \chi(p, f) - \frac{1}{n} > \rho_0, \]

where the last inequality is satisfied since $\mu_p \in E_1$.

To complete the proof we need to show that $\nu_n$ is in $E$, for every $n$. To see this, we recall first that $\nu_n$ is approximated by periodic measures since it is an ergodic measure supported on a hyperbolic basic set. That is, there exist $q_{m,n} \in \Lambda(p,n)$, hyperbolic periodic points of $f$, such that $\mu_{q_{m,n}}$ converges to $\nu_n$ in the weak topology. Now, by item 5 of property $S_n$ we have that $\mu_{q_{m,n}} \in E_1$, which implies $\nu_n$ belongs to $E$. And then, the proof of theorem D is complete. □

4. Symplectic Perturbations: proof of Proposition 3.1

Before going into the proof of the proposition, let us recall some basic facts about symplectic structure. Let $(V, \omega)$ be a symplectic vector space of dimension $2n$. For any subspace $W \subset V$, its *symplectic orthogonal* is defined as

\[ W^\omega = \{ v \in V; \omega(v, w) = 0 \quad \text{for all} \quad w \in W \}. \]

The subspace $W$ is called *symplectic* if $W^\omega \cap W = \{0\}$. $W$ is called *isotropic* if $W \subset W^\omega$, that is $\omega|W \times W = 0$. A special case of isotropic subspace is a *Lagrangian* subspace, i.e., when $W = W^\omega$. For a symplectic manifold $(M, \omega)$ and a symplectic diffeomorphism $f$ it is easy to see that for any point on an unstable (stable) manifold of a hyperbolic periodic point, the tangent space to unstable (stable) manifold is a Lagrangian subspace.

The proof of the main proposition is done in three steps where the second and third are the main ones and use the symplectic structures.

Let $f \in \mathcal{H}$ be a non Anosov diffeomorphisms.
Step 1- We find $g_1$, $C^1$-close to $f$ such that $p$ is still a hyperbolic periodic point of $g_1$ and $g_1$ exhibits a homoclinic tangency between $W^s(o(p), g_1)$ and $W^u(o(p), g_1)$. Moreover, $g_1 = Df_p$ in a small neighborhood of the orbit of $p$ (in local symplectic coordinates).

Step 2- We find $g_2$, $C^1$-close to $g_1$ where $g_2$ has an interval of homoclinic tangencies. For that, we should perform careful symplectic perturbations.

Step 3- Finally, we perturb $g_2$ to obtain $g$ with a hyperbolic invariant set satisfying the properties required by the proposition. All $C^1$-perturbations of $g$ also share the same properties for the corresponding hyperbolic set.

Proof of step 1: The idea of creating tangency in the absence of hyperbolicit y (even in the symplectic case) goes back to Newhouse (step 6, Theorem 1.1 in [23]). Let us briefly comment on his argument and show that we can perturb $f$ to obtain $g_1$ with homoclinic tangency and $g_1$ being linear in a small neighborhood of the orbit of $p$ as required. In order to obtain tangency, firstly Newhouse found a transversal homoclinic point $q$ of $p$ such that the angle between stable and unstable directions at $q$ is very small. Hence, since compact parts of stable and unstable manifolds vary continuously with the di ffeomorphism, we can firstly use a pasting lemma of Arbieto-Matheus [3] to linearize the di ffeomorphism in a neighborhood of the periodic point $p$. Then after a perturbation, we can find a homoclinic tangency of $p$ such that the di ffeomorphism is linearized in a neighborhood of $p$, as we wanted.

We remark that by homoclinic tangency we mean the existence of at least one (it can be unique) common direction between the tangent spaces of stable and unstable manifolds at the point of tangency.

Proof of step 2: For simplicity we suppose $p$ is a hyperbolic fixed point of $g_1$. Now, using step 1, let $V$ be a neighborhood of $p$ where $g_1$ is linear. Suppose that $E^s_p = \mathbb{R}^n \times \{0\}^n$ and $E^u_p = \{0\}^n \times \mathbb{R}^n$. Moreover, by Darboux’s Theorem, we can also suppose that in $V$, $\omega$ is the standard symplectic form of $\mathbb{R}^{2n}$ i.e, $\omega = \sum dx_i \wedge dy_i$.

Let $q$ be a point of homoclinic tangency between $W^s_{\text{loc}}(p, g_1)$ and $W^u(p, g_1)$, such that $q \in V$ and $g_1^{-1}(q) \notin V$. Hence, let $U \subset V$ be a small enough neighborhood of $q$ such that $g_1^{-1}(U) \cap V = \emptyset$. We denote by $D$ the connected component of $W^u(p, g_1) \cap U$ that contains $q$.

Now we perturb $g_1$ in order to get an interval of homoclinic tangencies. Since stable (unstable) manifold is locally a graph, it’s not difficult to do this in the conservative scenario using the point of homoclinic tangency $q$. In the symplectic setting this may be done using the fact that stable (unstable) manifold is a lagrangian manifold as we explain it below.

First, we can consider another symplectic coordinate on $U$ to simplify the notation. More precisely, this new symplectic coordinate is such that $q$ is the
origin and the following is satisfied:

\[ W^s_{\text{loc}}(p, g_1) \cap U = \{ y_1 = y_2 = \ldots = y_n = 0 \} \cap U, \]

\[ T_q D = \{ y_1 = x_2 = \ldots = x_n = 0 \}, \]

which implies

\[ W^s_{\text{loc}}(p, g_1) \cap U \cap T_q D = \langle e_1 \rangle, \]

where \{e_1, \ldots, e_n, \ldots, e_{2n}\} is the canonical basis of \( \mathbb{R}^{2n} \). Note that, we are using \( \text{dim}(T_q W^s(p, g_1) + T_q W^u(p, g_1)) = 2n - 1 \), which we can suppose after a perturbation, if necessary.

The following lemma is a technical lemma that allows us to build the interval of homoclinic tangencies.

4.1. Lemma. There exists a symplectic diffeomorphism \( \phi : U \to \mathbb{R}^{2n} \) over its image, \( C^1 \) close to identity map in a small neighborhood of \( q \), such that \( \phi(D) \cap W^s_{\text{loc}}(p, g_1) \cap U \) contains one segment of line.

Proof. Here we will use coordinates \((x, y)\) with respect to the following decomposition of the space \( \mathbb{R}^{2n} = E \oplus F \), where \( E \) and \( F \) are generated by \{e_1, e_{n+2}, \ldots, e_{2n}\} and \{e_2, \ldots, e_{n+1}\}, respectively. Observe that by the choice of the symplectic coordinates in \( U \) we have that \( E = T_q D \), and \( q = (0, 0) \).

Since the unstable manifold is locally a graph (of some function) and \( D \) is a small disk inside \( W^u(p, g_1) \), there exists a \( C^1 \) map \( j : B \subset \mathbb{R}^n \to \mathbb{R}^{2n} \), \( j(x) = (x, r(x)) \), such that \( j(B) = D \). Moreover, \( j \) is such that \( Dr(0) = 0 \), and as \( D \subset W^u(p, g_1) \) is a lagrangian submanifold, we have \( j^* \omega = 0 \), where \( j^* \omega \) is the pull-back of the form \( \omega \) by \( j \). Analogously, if \( i : \mathbb{R}^n \to \mathbb{R}^{2n} \) is the natural inclusion, \( i(x) = (x, 0) \), we have \( i^* \omega = 0 \) (recall \( \omega \) in \( U \) is the standard symplectic form on \( \mathbb{R}^{2n} \)).

Let us define \( \phi : U \to \mathbb{R}^{2n} \) by \( \phi(x, y) = (x, y - r(x)) \). Taking \( U \) smaller, if necessary, \( \phi \) is a diffeomorphism from \( U \) into its image and \( C^1 \) close to identity, since \( Dr(0) = 0 \). Hence, to conclude the lemma we show that \( \phi \) is also a symplectic map. Denote by \( \pi : \mathbb{R}^{2n} \to \mathbb{R}^n \) the projection \( \pi(x, y) = x \). We can rewrite \( \phi \) as \( \phi = Id + i \circ \pi - j \circ \pi \). Then,

\[ \phi^* \omega = \omega + \pi^* i^* \omega - \pi^* j^* \omega = \omega, \]

where we use that \( i^* \omega = j^* \omega = 0 \) in the second equality. \( \square \)

Using the pasting lemma of Arbieto-Matheus [3] and the map \( \phi \) given by Lemma 4.1, we can find \( R : U \to U \), \( C^1 \)-close to identity with \( R = \phi \) in a small neighborhood of \( q \) and \( R = Id \) outside another small neighborhood containing the last one. Hence, considering \( \tilde{R} : M \to M \) with \( \tilde{R} = Id \) in \( U^c \) and \( \tilde{R} = R \) in \( U \), and taking \( g_2 = \tilde{R} \circ g_1 \) we have a \( C^1 \) symplectic perturbation of \( g_1 \) that coincides with \( g_1 \) in \((g_1^{-1}(U))^c\). Moreover, this perturbation exhibits an interval of homoclinic tangencies as we wanted. More precisely, there is a segment of
line $I \subset W^s_{loc}(p, g_2) \cap W^u(p, g_2) \cap U$. Note that $I$ is in the space generated by the unit vector $e_1$, and thus $I$ contains an interval $\{(x_1, 0, \ldots, 0), -2a \leq x_1 \leq 2a\}$, for some $a > 0$ small enough.

**Proof of Step 3:** Now we will use this interval of homoclinic tangencies to create nice hyperbolic sets. Let $N$ be a large positive integer and $\delta > 0$ an arbitrary small real number. As before we can find a symplectic diffeomorphism $\Theta : M \to M$, $\delta - C^1$ close to $Id$ such that $\Theta = Id$ in $U^c$ and

$$\Theta(x, y) = \left(x_1, \ldots, x_n, y_1 + A \cos \frac{\pi x_1 N}{2a}, y_2, \ldots, y_n\right), \text{ for } (x, y) \in B(0, r) \subset U,$$

for $A = \frac{2Ka\delta}{\pi N}$, $r > 0$ small enough. Here $K$ is a constant depending only on the symplectic coordinates on $U$. Hence, $g = \Theta \circ g_2$ is $\delta - C^1$ close to $g_2$ and moreover $g = g_2$ in the complement of $g_2^{-1}(U)$. Note that $g$ depends on $N$, however to simplify our notations we will denote it here by $g$ independent of $N$. We also would like to observe that, this perturbation is an adaptation of Newhouse’s snake perturbation for higher dimensions, i.e., it destroys the interval of homoclinic tangencies and creates $N$ transversal homoclinic points for $p$ inside $U$.

Using $\Theta$, we choose two points in the unstable manifold of $p$ for $g$, $z_1 = \Theta(-a, 0, \ldots, 0)$ and $z_2 = \Theta(a, 0, \ldots, 0)$. Consider $\gamma_1$ and $\gamma_2$ two disks transverse to unstable manifold $W^u(p, g)$ at $z_1$ and $z_2$, respectively.

From now on we use the fixed symplectic coordinates on $V$. Recall that $g$ is equal to $g_1$ in $V$ and thus $g$ is linear on $V$.

For any $E$, by $C(E, x)$ denote the connected component of $E$ containing $x$. By $\lambda$–Lema and the choice of $\gamma_1$ and $\gamma_2$, $C(g^{-j}(\gamma_1) \cap V, g^{-j}(z_1))$ and $C(g^{-j}(\gamma_2) \cap V, g^{-j}(z_2))$ accumulate on $W^s_{loc}(p, g)$ for large values of $j > 0$. Thus defining $D_j' := W^s_{loc}(p, g) \cap U_j$ for any large enough $j$, it is possible to define a rectangle $D_j = D_j' \times D_j^u$ as the cartesian product between $D_j'$ and $D_j^u$, where $D_j^u$ is the smallest possible disk in $\{(0, \ldots, 0, y_1, \ldots, y_n), y_i \in \mathbb{R}\}$ such that $\pi_2(C(g^{-j}(\gamma_i) \cap V, g^{-j}(z_i))) \subset D_j^u$, for $i = 1, 2$. Here $\pi_2(x, y) = y$ stands for the projection on the second $n$th-coordinate of $\mathbb{R}^{2n}$.

Let $J \subset U$ be a small enough disk in the unstable manifold $W^u(p, g)$ containing the $N$ transversal homoclinic points built before. Let $T >> 0$ be such that $g^{-T}(J) \subset V$, and moreover $g^{-T}(\gamma_i)$, $i = 1, 2$, is $C^1$–close to $W^s_{loc}(p, g)$. We denote by $\tilde{I}$ the $A/2$–neighborhood of $J$, and define $\Gamma = g^{-T}(\tilde{I})$. See figure 1.

Now, let $t_0$ be the smallest positive integer such that $C(g^{-t_0}(\gamma_i), g^{-t_0}(z_i))$ is $A/2 - C^1$ close to $W^s_{loc}(p, g)$, $i = 1, 2$. Note that if $t' \geq t_0$ and $g^{t'-T}(D_{r'}) \subset \Gamma$, then $g^{t'}(D_{r'}) \cap (D_r)$ contains $N$ disjoint connected components. Hence, we consider $z_3 = (b, 0, \ldots, 0)$ and $z_4 = (b', 0, \ldots, 0)$ two points in the local stable manifold of $p$, where $b$ and $b'$ are the left and right boundary points in the first coordinate of
$W_{loc}^s(p, g) \cap U$. Also, let $\gamma_3$ and $\gamma_4$ be two disks transverse to $W_{loc}^u(p, g)$ at $z_3$ and $z_4$, respectively. By $\lambda$–lemma again we can define $t_1$ as the smallest possible positive integer such that

$$C(g^t(\gamma_i), g^t(z_i)) \cap C(g^{-T}(\gamma_j), g^{-T}(z_j)) \cap \Gamma \neq \emptyset,$$

for $j = 1, 2$ and $i = 3, 4$.

Finally, we define $t = \max\{t_0, t_1 + T\}$, see figure 1. Note that $t$ depends on $N$ and goes to infinity whenever $N \to \infty$.

By the choice of $t$, we have that $g^t(D_t) \cap D_t$ has $N$ disjoint connected components, and $t$ is the smallest possible number such that $D_t$ is $A/2 - C^1$ close to $W_{loc}^s(p, g)$ and $g^t(D_t)$ is $A/2 - C^1$ close to $J \subset W^u(p, g)$. Therefore, we have a horseshoe with $N$ legs, which implies that the $g^t$–maximal invariant set in $D_t$

$$\tilde{\Lambda}(p, N) := \bigcap_{j \in \mathbb{Z}} g^j(D_t)$$

is a hyperbolic set conjugated to a shift of $N$ symbols. Thus the topological entropy $h(g^t|\tilde{\Lambda}(p, N)) = \log N$, and taking

$$\Lambda(p, N) = \bigcup_{j=1}^t g^j(\tilde{\Lambda}(p, N))$$

we have $h(g^t|\Lambda(p, N)) = \frac{1}{t} \log N$.

The following lemma is the main point in this step.

---

**Figure 1**

---
4.2. Lemma. For $A$ and $t$ defined as before, there exists a positive integer $K_1$ independent of $A$, such that

$$A < K_1 \max \{ \| Dg_p^{-1} \|, \| Dg_p^t \| \}.$$ 

Proof. Since $V$ is a neighborhood of $p$ where $g$ is linear, if $m$ is the largest number such that $g^j(x) \in V$ for $0 \leq j \leq m$, there exist constants $K_2$ and $K_3$ depending on the symplectic coordinate on $V$ such that

$$K_2 \| Dg^m \|^{-1} \leq d(x, W^s_{loc}(p, g)) \leq K_3 \| Dg^m \|,$$

for $x \in V$. Analogously, if $m$ is the largest number such that $g^{-j}(x) \in V$ for $0 \leq j \leq m$, then there exist constants $K_4$ and $K_5$ such that

$$K_4 \| Dg^{-m} \|^{-1} \leq d(x, W^u_{loc}(p, g)) \leq K_5 \| Dg^{-m} \|.$$

Now, by choice of $t$, either there exists $z \in D_t$ such that

$$d(g(z), W^u_{loc}(p, g)) \geq A/2,$$

or there exists $z \in g^t(D_t)$ such that

$$d(g^{-1}(z), J) \geq A/2.$$

Suppose the first case. Recall that for $j > T$ the rectangle $D_j$ is defined and moreover $D_j \subset V$, which implies that $g(z), g(g(z)), ..., g^{j-1-T}(g(z)) \in V$. Hence, using inequality (4) we have

$$\frac{A}{2} \leq K_3 \| Dg^{-t+T+1} \|.$$

On the other hand, using inequality (5) and the neighborhood $\Gamma$, we can do the same thing for the second case, obtaining

$$\frac{A}{2} \leq K_5 \| Dg^{t-1-T} \|.$$

And then, since $Dg$ is bounded and $T$ is independent of $A$ we can find $K_1$ as we claimed.

From now on we fix a large positive integer $n$. As $A = \frac{2K_2\delta}{\pi N}$, for $N$ large enough, by Lemma 4.2 we have

$$\frac{1}{t} \log N > \min \left\{ \frac{1}{t} \log \| Dg_p^{-j} \|^{-1}, \frac{1}{t} \log \| Dg_p^j \|^{-1} \right\} = \frac{1}{2n}.$$

Observe that when $t$ goes to infinity the above minimum converges to the minimum between the smallest positive Lyapunov exponent (which is in fact $\chi(p, g)$), and the absolute value of the largest negative Lyapunov exponent of
However, since we are in the symplectic setting these two numbers coincide. Therefore, we can find a positive integer $N_1$ such that
\[
\frac{1}{t} \log N_1 > \chi(p, g) - \frac{1}{n},
\]
which implies that it is possible to find a $C^1$-perturbation $g$ of $f$ such that
\[
h(g|\Lambda(p, N_1)) > \chi(p, g) - \frac{1}{n}.
\]

For general case, when $p$ is not a fixed point of $g_1$, i.e., $\tau(p, g_1) > 1$, we have that $q \in W^s_{loc}(p, g_1) \cap W^u(f(p), g_1)$, for some $0 \leq j < \tau(p, g_1)$. Then as we did before, we can find a perturbation $g$ of $g_1$ and $t = \tau(p, g)\hat{t} + j$ such that $g^t$ has a hyperbolic basic set $\tilde{\Lambda}(p, N)$. Moreover, there is a relation between the norm of $Dg\tau(p, g)$ and $A$ as in Lemma 4.2, changing $t$ by $\hat{t}$. Hence, we can find $N_1$ such that
\[
\tag{8} h(g|\Lambda(p, N)) > \chi(p, g) - \frac{1}{n}, \text{ for } N \geq N_1.
\]

So, we can suppose that $p$ is fixed. Next, we find a positive integer $N_2$ such that if $\mu \in \mathcal{M}(f|\Lambda(g, N_2))$ is ergodic then $\rho(\mu, \mu_p) < 1/n$. Indeed, given $\zeta > 0$ arbitrary small it is enough to find $N = N(\zeta)$ such that (the orbit of) any point of $\Lambda(p, N)$ visits very frequently the ball of radius $\zeta$ and center $p$.

As $p$ is a hyperbolic fixed point we have
\[
\bigcap_{i \in \mathbb{Z}} g^i(V) = \{p\}.
\]

Hence, given $\zeta > 0$ arbitrary small, there exists a positive integer $n_1 \geq T$, depending on $\zeta$, such that for every $n_2 \geq n_1$
\[
\text{diam} \bigcap_{-n_2 \leq i \leq 0} g^i(V) < \zeta.
\]

Now, if $\overline{V} = \bigcap_{i=0}^{n_1} g^{-i}(V)$ and $z \in \overline{V}$, then for every $r \in [n_1, (l-1)n_1)$ we have that
\[
g^r(z) \in \bigcap_{|i| < n_1} g^i(V) \subset B_{\zeta}(p).
\]

So, the fraction of time in $[0, ln_1)$ that the orbit of $z$ stay in $B_{\zeta}(p)$ is $\frac{l - 2}{l}$. 


Recalling that $t$ is the period of the periodic set $\Lambda(p, N)$ of $g$, we define $k = t - T$. Given $N$ large enough, let $l \in \mathbb{N}$ be such that $(l + 1)n_1 \geq k > ln_1$. Since for every $z \in \Lambda(p, N)$ there exists $r \in [0, t)$ such that $g^r(z) \in \mathcal{V}$, the frequency of the time that the orbit of $z$ passes through $B_z(p)$ is larger than

$$\frac{(l - 2)n_1}{(l + 1)n_1 + T}.$$  

As $l \to \infty$ when $N \to \infty$; given $\zeta_1 > 0$ we can choose $N_2$ such that the frequency of visit of $z \in \Lambda(p, N)$ through $B_z(p)$ is larger than $1 - \zeta_1$, and then choosing $\zeta_1$ smaller if necessary, we have

$$d(\mu, \mu_p) < \frac{1}{n^l},$$

for every ergodic measure $\mu \in \mathcal{M}(\Lambda(p, N)), N \geq N_2$.

Finally, we find $N_3$ in order to obtain property (d) for $\Lambda(p, N), N \geq N_3$. Let us define

$$V^p_k = V \cap g(V) \cap \ldots \cap g^k(V), \text{ and}$$

$$V^s_k = V \cap g^{-1}(V) \cap \ldots \cap g^{-k}(V).$$

Given vectors $v, w \in \mathbb{R}^{2n}$ and subspaces $E, F \subset \mathbb{R}^{2n}$, we define

$$\text{ang}(v, w) = \left| \tan \left( \arccos \left( \frac{\langle v, w \rangle}{||v|| ||w||} \right) \right) \right|,$$  

$$\text{ang}(v, E) = \min_{w \in E, ||w|| = 1} \text{ang}(v, w) \quad \text{and} \quad \text{ang}(E, F) = \min_{w \in E, ||w|| = 1} \text{ang}(w, F).$$

### 4.3. Remark

Another definition of the angle between two subspaces in literature is the following: If $\mathbb{R}^n = E \oplus F$ is some decomposition, let $L : E^\perp \to E$ be the linear map such that $F = \{w + Lw; \ w \in E^\perp\}$, and then some authors define the angle between $E$ and $F$ as $\|L\|^{-1}$. Nevertheless, there is an equivalence between this definition and the one presented here.

We need the following lemma.

### 4.4. Lemma

There exists constant $K_6$ such that if $z \in V^s_k, v \in \mathbb{R}^{2n} \setminus E^s_p$ and $\text{ang}(g^k(v), E^s_p) \geq 1$, then

$$|Dg^k(z)(v)| \geq K_6 ||Dg^{-k}_{p}||^{-1} ||v|| \min \{\text{ang}(v, E^s_p), 1\}.$$  

**Proof.** Let $v = (v^s, v^u), v^u(u) \in E^u_p$, for every $v \in \mathbb{R}^{2n}$. Let $||v'|| = \max(||v^s||, ||v^u||)$ be the maximum norm.

Since $E^s_p = E^s_p$ and $Dg^k(z) = Dg^k_p$ if $z \in V$, we have

$$\text{ang}(v, E^s_p) = \frac{|v^u|}{|v'|} \quad \text{and} \quad 1 \leq \text{ang}(Dg^k_v(z)(v), E^s_p) = \frac{|Dg^k_v(z)(v^u)|}{|Dg^k_v(z)(v^u)|}.$$  

Then,
which implies

\[
|Dg^k(z)(v)|' = |Dg_p^k(v^u)| \\
\geq ||Dg_p^{-k}|E_p^u||^{-1}|v^u|, \\
= ||Dg_p^{-k}|E_p^u||^{-1}|v^u| \text{ang}(v, E^u);
\]

which implies

(12) \[|Dg^k(z)(v)|' \geq ||Dg_p^{-k}|E^u_p||^{-1}|v'| \min\{\text{ang}(v, E^u), 1\}.
\]

Therefore, by the equivalence between norms, the result follows. \(\square\)

Recall that

\[\Lambda(p, N) = \bigcup_{i=0}^{l-1} g^i(\tilde{\Lambda}(p, N))\]

is a hyperbolic set for \(g\), with \(\tilde{\Lambda}(p, N) \subset V\), and \(t = k + T\) where \(g^i(\tilde{\Lambda}(p, N)) \subset V\) for \(0 \leq i \leq k\). Moreover, by construction of \(\tilde{\Lambda}(p, N)\) we know that the hyperbolic decomposition \(T_{\Lambda(p, N)}M = \tilde{E}^s \oplus \tilde{E}^u\) is such that \(\tilde{E}^s(z)\) and \(\tilde{E}^u(g^k(z))\) are close to \(E^s(p)\) and \(E^u(p)\), respectively, for every \(z \in \tilde{\Lambda}(p, N)\). In particular, \(\text{ang}(Dg^k(z)(v), E^s(p)) > 1\) for any \(v \in \tilde{E}^u(z)\).

Hence, we can use Lemma 4.4 to find a constant \(K_6\), such that for every \(z \in \tilde{\Lambda}(p, N)\) and \(v \in \tilde{E}^u(z)\),

(13) \[|Dg^r(z)(v)| \geq (C K_6)^{l} ||Dg_p^{-k}||^{-1}|v|\], \(r = l(k + T), l \in \mathbb{N}\).

where

\[C = \inf_{z \in V \setminus g^{-1}(V), \|v\|=1} ||Dg^r(z)(v)||\].

Therefore, it is not difficult to see that for \(N\) large enough, all points in \(\tilde{\Lambda}(p, N)\) have positive Lyapunov exponents larger than \(\chi(p, g) - 1/n\). In particular, we can choose \(N_3\) in order to get \(k >> T\), such that for any periodic point \(q \in \Lambda(p, N)\), \(N > N_3\),

\[\chi(q, g) \geq \chi(p, g) - \frac{1}{n}\].

Hence, if we take \(\Lambda(p, n) = \Lambda(p, N)\) for \(N = \max\{N_1, N_2, N_3\}\), the properties of proposition are satisfied for the perturbation \(g\) of \(f\).

Now, by robustness of the hyperbolic periodic point \(p\) and the set \(\Lambda(p, n)\), properties (a) and (b) are also satisfied for diffeomorphisms close to \(g\), since the ergodic measure \(\mu_n\) is the one that maximize topological entropy of \(\Lambda(p, n)\).

In order to prove properties (c) and (d) for \(g\) we have concerned with a neighborhood of \(\Lambda(p, n)\), which implies these properties are also true for the continuations of this hyperbolic set. So, we conclude the proof of proposition. \(\square\)
5. Example of discontinuity point for topological entropy

In order to prove Theorem C, we construct an example \( f \) of a \( C^\infty \) area preserving diffeomorphism over \( S^2 \) such that the entropy function from \( \text{Diff}^1_\omega(S^2) \) to \( \mathbb{R} \) is not upper semi-continuous at \( f \).

Let \( S \) be a surface different from \( \mathbb{T}^2 \). As \( S \) does not accept Anosov diffeomorphism, by Theorem A, for generic volume preserving diffeomorphisms of \( S \) we have

\[
h(f) = s(f).
\]

Hence, using that \( s(.) \) is a lower semi-continuous function, if we find a diffeomorphism \( f \in \text{Diff}^1_\omega(S^2) \) such that \( h(f) < s(f) \), then this is an example where topological entropy is not upper semi-continuous.

In order to find such diffeomorphism, we use the following result of Lai-Sang Young [28].

5.1. Theorem. Let \( \phi : \mathbb{R} \times M \to M \) be a flow in a 2-dimensional manifold \( M \). Then, the diffeomorphism \( \phi_t = \phi(t,.) \) over \( M \) has zero topological entropy, i.e, \( h(\phi_t) = 0 \), for every \( t \).

By this result and the above discussion, to prove Theorem C it is enough to find a Hamiltonian flow in \( S^2 \) with a hyperbolic periodic orbit.

In what follows we describe the construction of such example which uses the well known mathematical pendulum, see [24]. Recall that a vector field \( X_H \) over a compact symplectic manifold \((M, \omega)\) is Hamiltonian iff there exists a smooth map \( H : M \to \mathbb{R} \) such that

\[
\omega(X_H,.) = dH.
\]

Also, recall \( \phi_t = \phi(t,.) : M \to M \) is a symplectic diffeomorphism, for every \( t \in \mathbb{R} \), where \( \phi \) is the flow generated by \( X_H \). Note that in dimension two the space of conservative diffeomorphisms coincides with the symplectic ones.

From now on we consider the symplectic manifold \((S^2, \omega)\) the two dimensional sphere, with \( \omega(x) = \langle x, u \times v \rangle \), for \( x \in S^2 \) and \( u, v \in T_x S^2 \), some induced area form over \( S^2 \). If we consider \( S^2 \) with cylindrical polar coordinates \((\theta, z)\), \( 0 \leq \theta < 2\pi \) and \(-1 < z < 1\), away from its poles, we can verify that \( \omega = d\theta \wedge dz \).

Let \( H_1(\theta, z) = z \) be the height function over the sphere, and \( X_{H_1} \) be the Hamiltonian vector field generated by \( H_1 \). Note the flow generated by \( X_{H_1} \) has no hyperbolic periodic orbits, more precisely the poles are non-hyperbolic singularities, and the flow far from them is \( \phi(t,(\theta, z)) = (\theta + t, z) \), i.e., rotations.

On the other hand, we can use the famous mathematical pendulum on \( S^1 \times \mathbb{R} \) to build hyperbolic periodic orbits in the previous flow. Let \( H_2 : S^1 \times \mathbb{R} \to \mathbb{R} \) be the total energy of the pendulum, \( H_2(\theta, z) = \frac{1}{2} z^2 - \cos \theta \), then the Hamiltonian vector field \( X_{H_2} \) on the cylinder gives us the phase portrait of the pendulum. We observe that the flow generated by \( X_{H_2} \) has an unstable equilibrium at \( p = (\pi, 0) \). Now, considering \( \beta : (-1, 1) \to \mathbb{R} \) the \( C^\infty \) bump function such that
\( \beta(x) = 1 \) if \( |x| < 1/2 \) and \( \beta(x) = 0 \) if \( |x| > 2/3 \), we define \( H : S^1 \times (-1,1) \to \mathbb{R} \) as follows

\[
H(\theta, z) = \beta(|z|)H_2(\theta, z) + (1 - \beta(|z|))H_1(\theta, z).
\]

Hence, after some coordinate change we can look for this function over \( S^2 \).

In fact, what we did was just to carry the pendulum flow to the sphere by changing the height function on some strip, see figure 2. And finally, \( X_H \) is a Hamiltonian vector field on \( S^2 \) and the flow generated by it has a hyperbolic singularity as we wanted.

**Figure 2. Phase portrait of \( X_H \)**

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