REAL INTEGRAL CLOSURE AND MILNOR FIBRATIONS

TERENCE T. GAFFNEY AND RAIMUNDO ARAÚJO DOS SANTOS

Abstract. We give a condition to guarantee the existence of a Milnor fibration for real map germs of corank 1, which include cases that are not L-maps in the sense of Massey. Our approach exploits the structure of a family of functions.

1. Introduction

The existence of Milnor fibrations for non-isolated singularities has been studied by many mathematicians using different approaches, for example, in [PS] the authors studied the existence of real Milnor fibrations in the Milnor tube and in spheres where the projection map is given by $\frac{f}{\|f\|}$, or the **strong Milnor fibration** (for further details see [RSV, Se1, AT]). In [Se2] Seade presented a beautiful survey about the existence of real Milnor fibrations for non-isolated singularities as well as interesting new results. In [AT] the authors studied the existence of strong Milnor fibrations for non-isolated singularities using the idea of open book structures in higher dimensions.

In [DM], Massey studies the existence of real Milnor fibrations on the Milnor tube involving the singular zero level of map, as approached by Lê D. Tráng in [LD] and H. Hamm and Lê in [HL].

He considers two conditions called **Milnor condition (a)** and **Milnor condition (b)**, which are sufficient to guarantee the existence of Milnor fibrations for real analytic map germs $f\colon \mathbb{R}^n, 0 \to \mathbb{R}^k, 0$ for $n \geq k \geq 2$. In what follows we will give a short description of these conditions:

Let $f\colon \mathbb{R}^n, 0 \to \mathbb{R}^k, 0, f(x) = (f_1(x), f_2(x), \dots, f_k(x))$ a real analytic map germ. Define the following analytic sets:

The variety of f, denoted $V(f)$, and defined as $V(f) := \{x \in \mathbb{R}^n, 0 : f_1(x) = f_2(x) = \cdots = f_k(x) = 0\}$. Let $\mathcal{U}$ denote the critical set of f, the points where $\{x \in \mathbb{R}^n, 0 : \nabla f_1(x), \dots, \nabla f_k(x)$, are linearly dependent$\}$ (The critical set is also denoted by $\sum(f)$ or $S(f)$ in sequence). Now let r be the function given by the square of distance from the

---

Date: February 10, 2009.

2000 *Mathematics Subject Classification.* Primary 32S55, 32C18, 14P25; Secondary 32S05, 32S10, 14P05.

Key words and phrases. real singularities, real integral closure, Milnor fibrations, Thom's Af-condition, stratification theory.

origin and define $\mathcal{B} = \{x \in \mathbb{R}^n, 0 : \nabla f_1(x), \dots, \nabla f_k(x), \nabla r(x)$, are linearly dependent$\}$. It is clear that $\mathcal{U} \subseteq \mathcal{B}$. Following Massey, we make the following definitions.

Definition 1.1. We say that a map germ f satisfies **Milnor's condition (a) at 0** if and only if $0 \notin \overline{\mathcal{U} - V(f)}$, i.e., $\sum(f) \subseteq V(f)$.

Definition 1.2. We say that the map germ f satisfies **Milnor's condition (b) at 0**, if and only if, **0** is an isolated point of $V(f) \cap \overline{\mathcal{B} - V(f)}$.

We say that $\epsilon > 0$ is a **Milnor radius for f at origin 0**, provided that $\overline{B_\epsilon} \cap (\overline{\mathcal{U} - V(f)}) = \varnothing$ and $\overline{B_\epsilon} \cap V(f) \cap \overline{\mathcal{B} - V(f)} \subseteq \{0\}$, where B_ϵ denotes the open ball in $\mathbb{R}^n$ with radius ϵ and $\overline{B_\epsilon}$ its topological closure in $\mathbb{R}^n$.

Theorem 1.3 ([DM], Theorem 4.3.)**.** *Suppose that f satisfies Milnor's conditions (a) and (b) at origin, and let $\epsilon_0 > 0$ be a Milnor radius for f at* **0***.*

Then, for all $0 < \epsilon \leq \epsilon_0$, there exists $\delta > 0$, $0 < \delta \ll \epsilon$, such that $f_| : \overline{B_\epsilon} \cap f^{-1}(B^*_\delta) \to B^*_\delta$ is a surjective, smooth, proper, stratified submersion and, hence, a locally-trivial fibration.

Idea of proof. Since f has a Milnor radius $\epsilon_0 > 0$, we have that $\sum(f) \cap \overline{B_{\epsilon_0}} \subset V(f) \cap \overline{B_{\epsilon_0}}$. It means that, for all $0 < \epsilon \leq \epsilon_0$ the map $f_| : B_\epsilon \setminus V(f) \to R^k$ is a smooth submersion. Now from the Milnor condition (b), and the remark above, it follows that: for each ϵ there exists δ, $0 < \delta \ll \epsilon$, such that

$$f_| : S^{n-1}_\epsilon \cap f^{-1}(B_\delta - \{0\}) \to B_\delta - \{0\}$$

is a submersion on the boundary S^{n-1}_ϵ of the closed ball $\overline{B_\epsilon}$. Now, combining these two conditions we have that, for each ϵ, we can choose δ such that

$$f_| : \overline{B_\epsilon} \cap f^{-1}(B_\delta - \{0\}) \to B_\delta - \{0\}$$

is a proper smooth submersion. Applying the version of Ehresmann Fibration Theorem for the manifold with boundary $\overline{B_\epsilon}$, we get that it is a smooth locally trivial fibration. □

Remark 1.4. *Instead of using the Ehresmann theorem for manifold with boundary, we also can apply Thom's 1st isotopy lemma by considering the map $f_|$ as a proper stratified submersion.*

Definition 1.5. We say that the map germ f satisfies the **strong Lojasiewicz inequality at the origin 0** or is an **L-map** if, and only if, there exists an open neighborhood $0 \in U$, and constants $c > 0$, $0 < \theta < 1$, such that for all $x \in U$,

$$|f(x)|^\theta \leq c \cdot \min_{|(a_1,\dots,a_k)|=1} |a_1 \nabla f_1(x) + \cdots + a_k \nabla f_k(x)|$$

In [DM], the author proved that if f is an L-map at origin then the Milnor's conditions (a) and (b) is satisfied.

It is easy to see that pairs of functions which come from a holomorphic function or which come from mixing holomorphic and anti-holomorphic functions are natural candidates to satisfy the L-map properties. (Cf. example 3.15 of [DM].)

The problem is that the class of L-maps may not behave very well for truly real analytic maps, as we now show.

Example 1.6. Consider the maps germs $G(x,y,z) = (x, y(x^2+y^2+z^2))$, $G\colon \mathbb{R}^3 \to \mathbb{R}^2$ and $H(x,y,z) = (x, y^3+x^2y)$, $H\colon \mathbb{R}^2 \to \mathbb{R}^2$. It is easy to see that $V(G) = Oz-$axis, $S(G) = \{(0,0,0)\}$ and $S(H) = \{(0,0)\}$, but $S(H\circ G) = Oz-$axis. The map G is not an L-map at the origin; if it were, the defining inequality would hold along every curve. However, restricting to the curve $\phi(t) = (t,0,0)$, and using $a_1 = 0, a_2 = 1$, if the defining inequality holds, it implies that

$$|t|^\theta \leq c \cdot |(\nabla y(x^2+y^2+z^2)\circ \phi(t)| = c\cdot |t^2|$$

which is impossible. The map G is an L-map at other points of the $Oz-$axis, because G is a submersion there.

Consider now the composition map $H\circ G = (x, (y(x^2+y^2+z^2))^3 + x^2(y(x^2+y^2+z^2)))$.

The map $H\circ G$ is not L analytic along $Oz-$axis. Again, this is easily seen using curves. Suppose $(0,0,z_0)$ a point on the $Oz-$axis different from the origin. Consider the curve $\phi(t) = (t,0,z_0)$. Then $x\circ\phi(t) = t$, while $\nabla((y(x^2+y^2+z^2))^3 + x^2(y(x^2+y^2+z^2))\circ\phi(t) = (0, z_0^2t^2, 0) \mod t^3$. Then the inequality which defines the L analytic condition fails along this curve. The problem in both cases is that the function x is too large in norm compared with the norm of the gradient of the second component function.

We note that the map germ G has an isolated critical point at the origin, so by Milnor's result [Mi] the Milnor fibration exists either in the full or in the hollowed "Milnor tube" involving the zero level, but this map germ does not satisfy the strong Lojasiewicz inequality.

Furthermore, the map germ H is an analytic homeomorphism; since G is an L-map on points of $V(G)$ different from the origin, you might hope that the composition $H\circ G$ would continue to be an L-map at such points, but, as the example shows, this is false.

Note that these examples can be thought of as families of functions parameterized by x. These examples suggest that better results can be obtained by studying the maps as families of functions.

In what follows we will show how to use the real integral closure tools for modules, as defined by the first author in [G], to prove that under certain natural conditions, these kind of map germs, some with non-isolated singular set, have Thom's A_f-condition and consequently, have a Milnor's fibration.

We also introduce the notion of the uniform Lojasiewicz inequality after proposition 3.2. This inequality plays the role of Massey's strong Lojasiewicz inequality in this paper. It is essentially a Lojasiewicz Inequality, but at the level of a family of functions.

The existence of Thom's A_f-condition for non isolated complex complete intersection singularities have been studied by the first author in [G1] as well. Here we are concerned with the real case.

2. Notations and setup

Denote by $(\mathcal{A}_n, \mathfrak{m}_\mathfrak{n})$ the local ring of real analytic function germs at the origin in $\mathbb{R}^n$ and by $\mathcal{A}_n^p$, the $\mathcal{A}_n$ free module of rank p. If (X, x) is the germ of a real analytic set at x, denote by $\mathcal{A}_{X,x}$ the local ring of real analytic function germs on (X, x), and by $\mathcal{A}_{X,x}^p$ the corresponding free module of rank p.

Definition 2.1. [G, GTW]
Suppose $(X, 0)$ is a real analytic set germ in $\mathbb{R}^n$, M a submodule of $\mathcal{A}_{X,x}^p$. Then:

1) $h \in \mathcal{A}_{X,x}^p$ is in the real integral closure of M, denoted by $\overline{M}$, iff for all analytic paths $\phi : (\mathbb{R}, 0) \to (X, 0)$, $h \circ \phi \in (\phi^* M)\mathcal{A}_1$;

2) $h \in \mathcal{A}_{X,x}^p$ is strictly dependent on M, iff for all analytic paths $\phi : (\mathbb{R}, 0) \to (X, 0)$, $h \circ \phi \in \mathfrak{m}_1(\phi^* M)\mathcal{A}_1$. We denote $M^\dagger$ the set of elements strictly dependent on M.

The definition of the real integral closure of a module is equivalent to the following formulation using analytic inequalities.

Proposition 2.2. *([G], p318) Suppose $h \in \mathcal{A}_{X,x}^p$, M a submodule of $\mathcal{A}_{X,x}^p$. Then $h \in \overline{M}$ if, and only if, for each choice of generators $\{s_i\}$ of M, there exists a constant $C > 0$ and a neighborhood U of x such that for all $\psi \in \Gamma(Hom(\mathbb{R}^p, \mathbb{R}))$,*

$$||\psi(z) \cdot h(z)|| \leq C||\psi(z) \cdot s_i(z)||$$

for all $z \in U$.

If you alter the form of Massey's inequality allowing $\theta = 1$, then it is equivalent to asking that $I(f)\mathcal{A}_n^2$ is in the integral closure of the jacobian module of f, which is the submodule of $\mathcal{A}_n^2$ generated by the partial derivatives of f. The Lojasiewicz inequality in the complex analytic case is equivalent to asking that f is strictly dependent on the jacobian ideal at all points where the jacobian ideal is zero. We conjecture that Massey's inequality is equivalent to $I(f)\mathcal{A}_n^2$ being strictly dependent on the jacobian module of f at all critical points of f. If Massey's inequality holds at the origin on some neighborhood U, then $I(f)\mathcal{A}_n^2$ is strictly dependent on the Jacobian module of f at all critical points of f on U.

If we assume θ is a rational number p/q then more can be said. In this case, Massey's inequality takes the form

$$|f(x)|^p \leq c \cdot \min_{|(a_1,\dots,a_k)|=1} |(a_1 \nabla f_1(x) + \cdots + a_k \nabla f_k(x))|^q$$

.

If f is a function germ then this amounts to saying that $I(f)^p$ is in the real integral closure of $J^q(f)$. The module case works as follows. Given the jacobian module of f, $JM(f) \subset \mathcal{A}_n^2$, we can view $JM(f)$ and $I(f)\mathcal{A}_n^2$ as ideals $\mathcal{M}(f)$ and $\mathcal{I}(f)$ in the ring $A_n[T_1, T_2]$, then the inequality says that the real integral closure of $(\mathcal{M}(f))^q$ contains $(\mathcal{I}(f))^p$.

Let consider $f : \mathbb{R}^n, 0 \to \mathbb{R}^2, 0$, $f(x, y_1, \dots, y_{n-1}) = (x, g(x, y_1, \dots, y_{n-1}))$, where sometimes we let y denote $(y_1, \dots, y_{n-1})$. We assume $f(S(f)) = 0$ or equivalently $S(f) \subseteq V(f)$

(or Milnor condition (a)), where $V(f)$ is the variety of f, and let $X_k := V(g - s^k) \subset \mathbb{R}^{n+1}$, $F_k := (s,x)|_{X_k}$, where now we are considering $g - s^k \colon \mathbb{R}^{n+1} \to \mathbb{R}, (s,x,y) \mapsto g(x,y) - s^k$ and $F\colon \mathbb{R}^{n+1} \to \mathbb{R}^2, F(s,x,y) = (s,x)$. So, $F_k = F|_{X_k}$. Also consider $\Pi\colon \mathbb{R}^{n+1} \to \mathbb{R}, \Pi(s,x,y) = x$ and $\Pi_k := \Pi|_{X_k}$.

3. Some results

The first result is:

Lemma 3.1. *In the setup above we have* $S(F_k) = S(\Pi|_{X_k}) = S(f) \times \{0\}$.

Proof. Consider the jacobian matrices of the respective maps.

$$J(F_k) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ -ks^{k-1} & g_x & g_{y_1} & \dots & g_{y_{n-1}} \end{pmatrix}, \; J(\Pi_k) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -ks^{k-1} & g_x & g_{y_1} & \dots & g_{y_{n-1}} \end{pmatrix}$$

and $J(f) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ g_x & g_{y_1} & g_{y_2} & \dots & g_{y_{n-1}} \end{pmatrix}$.

It is easy to see that the critical locus of f is given by $S(f) = V(g_{y_1}, ..., g_{y_{n-1}}) \subset \mathbb{R}^n \times \{0\}$. In the other two cases $(g_{y_1}, ..., g_{y_{n-1}})$ also vanish on the respective singular sets, and since we assume Milnor condition (a), this implies x and g are zero as well. In turn, on X_k, this implies $s = 0$, which shows that the singular locus of Π_k is the same as F_k. □

Note that the singular locus of X_k is given by $V(J(g(x,y))) \cap \{s = 0\}$, and this is contained in $S(f) \times \{0\}$. Assume now that we have a stratification of X_k which satisfies $A_{\Pi_k}-$condition. Since, by the Milnor condition (a), $S(f) \times \{0\}$ lies in $x = 0$, it follows that the strata of $S(f) \times \{0\}$ satisfy the Whitney A condition. Assume also that there exists a neighborhood $0 \in U \subset \mathbb{R}^n$, such that

$$|g(x,y)|^\theta \leq c \sup_{i=1,\dots,n-1} \{|g_{y_i}(x,y)|\}, \quad 0 < \theta < 1$$

holds for some neighborhood V_z of any $z = (x,y) \in U \setminus \{0\}$, where the constant c may depend on z, but the constant θ does not. This inequality plays the role in this paper that Massey's inequality does in the definition of L-maps. If we think of $g(x,y)$ as a family of functions parameterized by x, then our inequality is a **uniform Lojasiewicz inequality** and we denote it by such in the following, as it relates the values of g to the values of the partials of g in the y variables, which are the "state space" variables.

Notice that the above inequality implies that $S(f) \subset V(g)$, so if we assume this inequality we need only assume $S(f) \subset V(x)$, and this coupled with the inequality implies Milnor condition (a). Thinking of f as a family of functions, the Milnor condition (a) means that $S(g_x)$ is non-empty only for $x = 0$.

Proposition 3.2. *Suppose that in the setup above we have $\theta < \dfrac{k-1}{k}$. Then the A_{Π_k} stratification of X_k is also an A_{F_k} stratification, perhaps adding the origin as a stratum.*

Proof. We have the singular set of F_k is $S(f) \times \{0\}$, and this lies in $F_k = 0$, and the stratification of $S(f) \times \{0\}$ satisfies Whitney A condition. So it suffices to consider the pairs $(X_k \backslash S(F_k), S_{(p,0)})$, $(p,0) \in S(F_k), p \neq 0$, $S_{(p,0)}$ is the stratum of $S(F_k)$ which contains the point $(p,0)$.

Suppose we have such a point $(p,0) \in S(F_k), p \neq 0$, where the $A_{F_k}-$condition fails for the pair $(X_k \setminus S(F_k), S_{(p,0)})$, where $S_{(p,0)}$ is the stratum of $S(F_k)$ which contains the point $(p,0)$. By assumption we know that the $A_{\Pi_k}-$condition holds at $(p,0)$, and we know that the tangent vectors to the stratum $S_{(p,0)}$ at $(p,0)$ are in $\ker(F(p,0))$. The last follows because $S(F_k) = S(f) \times 0 \subset V(x) \times 0$.

Now the $A_{\Pi_k}-$condition holds iff the column vector $J(\Pi_k) \cdot v$ is strictly dependent on the module generated by the partial derivatives with respect to variables s, y at $(p,0)$, where v is any tangent vector to $S_{(p,0)}$ at $(p,0)$. This follows because it is known that the strict dependence condition must hold for the module generated by all the partial derivatives; but since $v \in V(x,s)$, $J(\Pi_k) \cdot v$ lies in $(0, I)$ where I is the ideal generated by $(g_{y_1}, ..., g_{y_{n-1}})$, it follows that only the partials with respect to s, y can be used, as only they give elements with a zero as first entry.

The $A_{F_k}-$condition holds iff $J(s, x, g(x,y) - s^k) \cdot v$ is strictly dependent on the module generated by the partial derivatives with respect to y by the reasoning of the previous paragraph.

If A_{F_k} fails and A_{Π_k} holds we have that there exists a curve $\varphi \colon \mathbb{R}, 0 \to X_k, (p,0)$ such that while $\nabla_y g(x,y).v \in (s^{k-1}, g_{y_1}, ..., g_{y_{n-1}})^{\dagger}$, for all v tangent to $S_{(p,0)}$, $(\nabla_y g \cdot v \circ \varphi) \notin m_1 \varphi^*(g_{y_1}, ..., g_{y_{n-1}})$, for some v, where m_1 is the maximal ideal in the ring of germs of analytic functions of one variable.

Claim: These two facts imply that $(s^{k-1}) \notin \overline{(g_{y_1}, ..., g_{y_{n-1}})}$. Further, if $s^{k-1} \in \overline{(g_{y_1}, ..., g_{y_{n-1}})}$, then the $A_{F_k}-$condition holds.

Proof of claim. If $(s^{k-1}) \in \overline{(g_{y_1}, ..., g_{y_{n-1}})}$, then

$$(s^{k-1}, g_{y_1}, ..., g_{y_{n-1}})^{\dagger} = (g_{y_1}, ..., g_{y_{n-1}})^{\dagger}.$$

However, the first fact implies that $\nabla_y g \cdot v$ is in the left hand side of the above equation, while the second fact implies it is not in the right, so $(s^{k-1}) \notin \overline{(g_{y_1}, ..., g_{y_{n-1}})}$.

If $(s^{k-1}) \in \overline{(g_{y_1}, ..., g_{y_{n-1}})}$, then the $A_{F_k}-$condition holds because then $(g_{y_1}, ..., g_{y_{n-1}})^{\dagger}$ will contain $\nabla_y g(x,y) \cdot v$, for all v tangent to $S_{(p,0)}$.

So we may suppose $(s^{k-1}) \notin \overline{(g_{y_1}, ..., g_{y_{n-1}})}$. This implies there exists a curve $\varphi : \mathbb{R}, 0 \to X_k, (p,0)$ such that the $ord(s^{k-1} \circ \varphi) < \min\{ord(g_{y_i} \circ \varphi)\}$. **(1)**

Now as $s^{k-1} = (s^k)^{\frac{k-1}{k}}$ and using the fact that under X_k we have $g(x,y) = s^k$, so we have $(s^k)^{\frac{k-1}{k}} \circ \varphi = g^{\frac{k-1}{k}} \circ \varphi$.

Now $|g^{\frac{k-1}{k}} \circ \varphi(t)| < C \min_{i=1,\ldots,n-1}\{|g_{y_i} \circ \varphi(t)|\}$, for all t small enough by the uniform Lojasiewicz inequality. This implies that $ord(g^{\frac{k-1}{k}} \circ \varphi) > \min\{ord(g_{y_i} \circ \varphi)\}$. Then, $ord(s^{k-1} \circ \varphi) > \min\{ord(g_{y_i} \circ \varphi)\}$. This contradicts (1) above.

□

Lemma 3.3. *Suppose we have an A_f stratification on $\mathbb{R}^n$, then for all k we have an A_F stratification on X_k(or $A_{F_k}-$condition holds), in which the strata on $X_k \cap (\mathbb{R}^n \times \{0\})$ are the same as for the A_f stratification, and the union of the open strata is the complement of $S(F_k) = S(f) \times 0$.*

Proof. Note that the A_f-condition implies the Whitney A condition on strata of $X_k \cap (\mathbb{R}^n \times \{0\}) = V(g)$. So, we only have to check for the open strata. Suppose we have a curve φ such that $ord((D(g) \cdot v) \circ \varphi)$ fails to be greater than the $ord(J(g) \circ \varphi)$ where v is a tangent vector to a stratum of $X_k \cap (\mathbb{R}^n \times \{0\})$ at $\varphi(0)$. Then using $\Pi_{\mathbb{R}^n} \circ \varphi$ (where $\Pi_{\mathbb{R}^n}$ stands for the projection of $\mathbb{R}^{n+1} \to \mathbb{R}^n$), we get a curve where their order inequality again holds. This implies A_f-condition fails. □

Lemma 3.4. *Suppose X_k has an A_{F_k} stratification for some stratification $\mathcal{S}$, where $S(F_k) = S(f) \times 0$ is a union of strata, and the complement of $S(F_k)$ is a union of open strata, Then, this induces an A_f stratification of $\mathbb{R}^n$.*

Proof. Again we know that the Whitney A condition holds between the strata of $S(f)$, so we just need to consider the open strata. Suppose A_f fails at some point p, where A_F-condition holds for X_k at $(p, 0)$, then there exists a curve φ, and functions $\psi_1(t)$, $\psi_2(t)$ such that

$\lim_{t\to 0} \frac{1}{t^p}[(\psi_1(t), \psi_2(t)) \cdot J(f)(\varphi(t))]$, which is a limiting conormal vector to the fibers of f, fails to contain $T_p(S_p)$. Now we re-parameterize φ. Suppose $g(\varphi(t))$ has order r, and use $t = \tau^k$. So, $g(\varphi(\tau^k))$ has order kr. This implies that $g \circ \varphi(\tau^k) = \tau^{kr}h$, for some h, with $h(0) \neq 0$. Consider $g \circ \varphi(\tau^k) = [\tau^r h^{\frac{1}{k}}]^k$.

Let $s \circ \widehat{\varphi} = \tau^r h^{\frac{1}{k}}$ define an extension of φ, then $\widehat{\varphi}(\tau) = (\varphi(\tau^k), \tau^r h^{\frac{1}{k}})$ lies on X_k over p.
Now set $\psi_0 = -\psi_2(\tau^k)s^{k-1} \circ \widehat{\varphi}(\tau)$, then

$$\lim_{\tau\to 0} \frac{1}{\tau^{pk}}\langle\psi_0(\tau), \psi_1(\tau), \psi_2(\tau)\rangle. \begin{pmatrix} 1 & 0 & ..0 \\ 0 & 1 & ..0 \\ s^{k-1} & g_x & g_y \end{pmatrix} \circ \varphi(\tau^k) =$$

$= \lim_{\tau\to 0} \frac{1}{\tau^{pk}}\langle\psi_1(\tau), \psi_2(\tau)\rangle. \begin{pmatrix} 1 & ..0 \\ g_x & g_y \end{pmatrix} \circ \varphi(\tau^k)$, so this limit is the same and A_F-condition fails.

□

Theorem 3.5. *Suppose f as in our setup, and the uniform Lojasiewicz inequality holds, then there exists an A_f stratification of $\mathbb{R}^n$ in which $S(f)$ is a union of strata, and the open strata are the complement of $S(f)$.*

Proof. Choose k as in the proof of proposition 3.2, then we know that since Π_k is an analytic function, there exists an A_{Π_k} stratification. (For the complex analytic case this is proved in [H], and the complex stratification can be used to give a refinement of a real Whitney stratification of X_k in which $S(f)\times\{0\}$ is a stratified set, which is an A_{Π_k} stratification.) Since $S(\Pi_k)=S(f)\times 0$ by proposition 3.1, $S(f)\times 0$ is a union of strata, and its complement is a union of open strata. Now by proposition 3.2, this stratification is also an A_{F_k} stratification, and by lemma 3.3, induces the desired A_f stratification. □

Example 3.6. It is useful to see how the above proof breaks down for the map $f=(x,xy)$ which is the basic example of a map which does not have an A_f stratification. Here the singular locus is the Oy-axis, so the map satisfies Milnor condition (a). Further, X_k has equation $xy-s^k=0$, and a A_{Π_k} stratification is given by $\{\{0\},\{Oy-0\},\{X_k-Oy\}$. However, no uniform Lojasiewicz inequality holds, for in a neighborhood of points $(0,y)$, $y\neq 0$, $C|xy|>|g_y|=|x|$ for $C=2/|y|$.

Example 3.7. We wish to show that the key inequality applies to the examples we started with. We first consider $G(x,y,z)=(x,y(x^2+y^2+z^2))=(x,g_1(x,y,z))$, and show

$$|g_1(x,y,z)|^{2/3}\leq c\sup\{|g_{1,y}(x,y,z),g_{1,z}(x,y,z)|\}.$$

This will follow if we show that

$$|y(x^2+y^2+z^2)|^2\leq c\sup|x^2+3y^2+z^2|^3.$$

But, this is obvious.

Now consider $H\circ G=(x,g(x,y,z))$, where $H(x,y)=(x,y^3+x^2y)=(x,h(x,y))$.

As before we know that

$$|h(x,y)|^2\leq C|h_y(x,y)|^3$$

at all points. Now we want to show that the composition $H\circ G$ also satisfies a uniform Lojasiewicz inequality.

We have $h^p\circ G=g_1^p(g_1^2+x^2)^p$, while $g_y^{p+1}=(h_y(G)g_{1,y})^{p+1}$. Now $y^2+x^2\leq 3y^2+x^2=h_y(x,y)$, so

$$|h^p\circ G(x,y,z)|\leq|g_1^p(x,y,z)||h_y^p(G(x,y,z))|$$

We want to show

$$|h^p\circ G(x,y,z)|\leq|h_y(G(x,y,z))|^{p+1}|g_{1,y}(x,y,z)|^{p+1}$$

It suffices to show that

$$|g_1^p(x,y,z)|\leq|h_y(G(x,y,z))||g_{1,y}(x,y,z)|^{p+1}$$

From the form of g_1 we see that

$$|g_1^p(x,y,z)|=|y^{p-2}|(|y^2||x^2+y^2+z^2|^p)$$

Now $|y^2||x^2+y^2+z^2|^p\leq|g_{1,y}(x,y,z)|^{p+1}$, so it suffices to prove that there exists p such that $|y^{p-2}|\leq|h_y(G(x,y,z))|$. Looking at the form of $h_y(G(x,y,z))$ we see that $p=6$ suffices.

For our final result we show that the property of satisfying a uniform Lojasiewicz inequality is an analytic invariant of a family. In general, it remains open as to whether or not the families obtained by the composition of two maps whose associated families satisfy a uniform Lojasiewicz inequality also satisfy a uniform Lojasiewicz inequality.

Proposition 3.8. *Suppose* $R(x, y_1, \ldots, y_{n-1}) = (x, r(x, y_1, \ldots, y_{n-1}))$ *and* $L(x, Y) = (x, l(x, Y))$ *are bi-analytic map germs at the origin of* $\mathbb{R}^n$ *and* $\mathbb{R}^2$ *respectively,* $F(x, y) = (x, f(x, y_1, \ldots, y_{n-1}))$ *as analytic germ at the origin ,* F *satisfies a uniform Lojasiewicz inequality with rational* θ*. Assume* L *preserves the* y*-axis.Then* $L \circ F \circ R$ *satisfies a uniform Lojasiewicz inequality with the same* θ *as for* F*.*

Proof. Notice that it suffices to prove the proposition for R and L separately. Consider $L \circ F$. We know that

$$|f(x, y_1, \ldots, y_{n-1}))|^{\theta} \leq C \sup_{i=1,\ldots,n-1} \{|f_{y_i}(x, y_1, \ldots, y_{n-1})|\}$$

Since $l(x, y) = y(l_0(x, y)$ with $l_0(0, 0) \neq 0$, and $(l \circ F)_{y_i}(x, y) = l_y(x, f(x, y))f_{y_i}(x, y)$, with $l_y(0, f(0, 0)) \neq 0$, it follows that

$$|l(f(x, y_1, \ldots, y_{n-1}))|^{\theta} \leq C' \sup_{i=1,\ldots,n-1} \{|(l \circ F)y_i(x, y_1, \ldots, y_{n-1}))|\}.$$

For the case $F \circ R$ the result follows from the fact that the set $\{(f \circ R)_{y_i}\}$ generate the same ideal as $\{R^*(f_{y_i})\}$, so that the existing inequality pulls back under R to give the desired new inequality, with perhaps a different constant C.

□

Acknowledgements

The authors thank David Massey for sharing earlier drafts of [DM] with them, and acknowledge the inspiration received from this paper.

The second author thanks the first author, Terence Gaffney, for all his support and attention given during the development of the post-doc at NEU/USA under his supervision. He thanks Mrs. Mary Gaffney for all the generosity, kindness and assistance in all moments of his stay and to all the NEU staff for making the visiting period pleasant and productive. Thanks a lot !

Also, he would like to thank the Brazilian agencies FAPESP/São Paulo and the CNPq, grant PDE, number: 200643/2007-0. The former for supporting his first four month stay in U.S. in 2006, when the post-doc project started, and the latter for supporting his stay in U.S. from October/2007 to July/2008 during the post-doc. Thanks !

References

[AR] R. Araújo dos Santos, *Uniform (m)-condition and strong Milnor fibrations,* in *Singularities II: Geometric and Topological aspects*, *Proc. of the Conference in honour of Lê Dũng Tráng*, eds. J. P. Brasselet et al., (Cuernavaca, Mexico, 2007), *Contemporary Mathematics*, **475** (2008) 43–59.

[AT] R. Araújo dos Santos; M. Tibar, *Real map germs and higher open book structures,* available on arXiv.org, arXiv: 0801.3328.(Submitted for publication)

[G] T. Gaffney, *Integral closure of modules and Whitney equisingularity,* Invent. Math., 107, 1992, no. 2, 301–322.

[G1] T. Gaffney, *Non-isolated Complete Intersection Singularities and the A_f Condition, Singularities I: Algebraic and Analytic aspects*, *Proc. of the Conference in honour of Lê Dũng Tráng*, eds. J. P. Brasselet et al., (Cuernavaca, Mexico, 2007), *Contemporary Mathematics*, **474** (2008), 85–94.

[G2] T. Gaffney, *The multiplicity of pairs of modules and hypersurface singularities*, Real and complex singularities, 143–168, Trends Math., Birkhäuser, Basel, 2007.

[GTW] T. Gaffney, D. Trotman and L. Wilson, *Equisingularity of sections, (t^r) condition, and the integral closure of modules*, to appear Journal of Algebraic Geometry.

[HL] H. A. Hamm; Lê D. Tráng *Un théorème de Zariski du type de Lefschetz.* Ann. Sci. École Norm. Sup. (4) 6 (1973), 317–355.

[H] H. Hironaka, *Stratification and flatness,* Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 199–265. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[Ja] A.Jaquemard, *Fibrations de Milnor pour des applications réelles,* Boll. Un. Mat. Ital., vol.37, 1, pp.45–62, 3-B,1989.

[LD] Lê D. Tráng, *Some remarks on relative monodromy.* Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pp. 397–403. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977

[DM] D. Massey, *Real Analytic Milnor Fibrations and a Strong Lojasiewicz Inequality,* available on arXiv.org, math/0703613.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton 1968.

[PS] A. Pichon and J. Seade, *Fibred Multilinks and singularities $f\overline{g}$,* Mathematische Annalen, Vol. 342, no.3, 487–514, 2008.

[RSV] M. A. Ruas; J. Seade; A. Verjovsky, *On real singularities with a Milnor fibration.* Trends in singularities, 191–213, Trends Math., Birkhäuser, Basel, 2002.

[Se1] J. Seade, *On the topology of isolated singularities in analytic spaces.* Progress in Mathematics, 241. Birkhäuser Verlag, Basel, 2006. xiv+238 pp. ISBN: 978-3-7643-7322-1; 3-7643-7322-9

[Se2] J. Seade, *On Milnor's fibration theorem for real and complex singularities.* Singularities in geometry and topology, 127–158, World Sci. Publ., Hackensack, NJ, 2007

Northeastern University, 360 Huntington Avenue, Boston, MA 02115-5000, USA.
E-mail address: t.gaffney@neu.edu

Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Av. Trabalhador São-Carlense, 400 - Centro Postal Box 668, São Carlos - São Paulo - Brazil Postal Code 13560-970, São Carlos, SP, Brazil.
E-mail address: rnonato@icmc.usp.br