Rigidity of bi-Lipschitz equivalence of weighted homogeneous function-germs in the plane

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Introduction

Main problems in Singularity Theory:

- classification of singular points of mappings
- description of the topology of singular sets defined as zero sets of these mappings.

The first results on singularities of differentiable mappings go back to the 50's and 60's, in the last century, with the pionner works of Hassler Whitney and René Thom.

The foundations of the theory of singularities of *complex hypersurfaces* have been set by Zariski, Milnor, Lê Dung Trang and Teissier.

The aim is to discuss the classification of (real or complex) analytic functions in a neighborhood of 0.

Let

$$\mathcal{O}_n := \{f : \mathbb{K}^n, 0 \to \mathbb{K},\ f \text{ analytic function },\ \mathbb{K} = \mathbb{R} \text{ or } \mathbb{C}\},$$

be the local ring of germs of analytic functions at the origin.

Assume f has isolated singularity at 0.

The classical equivalence relation is the *analytic equivalence*, and many results are known in this case.

In this lecture we discuss the *bi-Lipschitz classification* of function germs. Not many results are known in this case.

The goal is to present the following result:

Theorem: (A.Fernandes, —, 2012, [2])

If two weighted homogeneous (but not homogeneous) function-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ are strongly bi-Lipschitz equivalent then they are analytically equivalent.

Summary

- Analytic equivalence
- Bi-Lipschitz equivalence
- Henry and Parusinski's example
- The main result and idea of the proof
- Open problems

Analytic equivalence

$$\mathcal{R} = \{h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0), \text{ germs of analytic diffeomorphisms}\}$$

is the group of right equivalences.

The group $\mathcal{R}$ acts on $\mathcal{O}_n$

$f \sim g$ if $\exists$ a germ of analytic diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $g = f \circ h^{-1}$.

If f and g are analytically equivalent then $h(\{g = c\}) = \{f = c\}$.

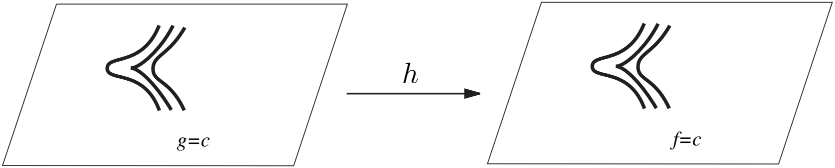


Figure: Analytic Equivalence

The classification of function germs with respect to analytic equivalence has moduli. The classical example is due to H. Whitney.

(H. Whitney, 1965)

$$f_t(x, y) = xy(x - y)(x - ty), \quad 0 < |t| < 1.$$

For each t, $X_t = f_t^{-1}(0)$ is the set of 4 lines through the origin in $\mathbb{K}^2$.

The invariant is the cross ratio

$$j = \frac{CA}{CB} / \frac{DA}{DB}$$

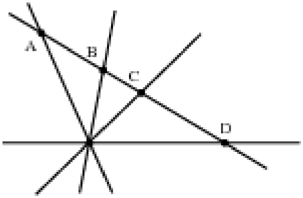


Figure: $j = \frac{CA}{CB} / \frac{DA}{DB}$

Thom Levine Theorem ([3])

Let U be a domain in $\mathbb{K}$, W a neighborhood of 0 in $\mathbb{K}^n$ and $F : W \times U \to \mathbb{K}$, such that $F(0,t) = 0$, F analytic. Let $f_t(x) = F(x,t)$, $\forall t \in U$, $\forall x \in W$.

The following conditions are equivalent

(i) There exists a family of analytic diffeomorphisms $H : W \times U \to W$, $h_t(0) = 0, \forall t \;\; h_0(x) = x$, such that

$$f_t \circ h_t = f_0$$

(ii) There exists a family of analytic vector fields $v : W \times U \to \mathbb{K}^n$, $v(0,t) = 0 \;\forall t \in U$, such that

$$\frac{\partial f_t}{\partial t}(x) = df_t(x)(v(x,t)), \;\forall t \in U, \;\forall x \in W$$

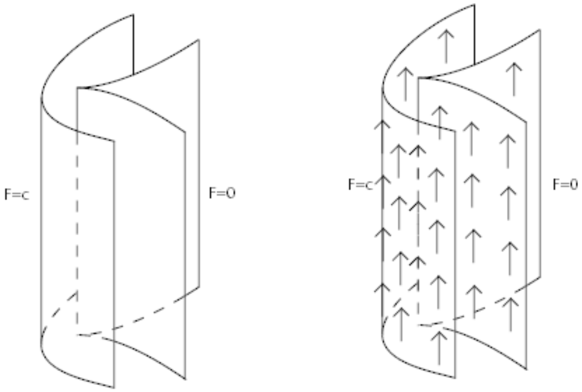
Condition (i) holds: The family F is analytically trivial

Condition (ii) holds: The vector field

$$V(x,t) = V(x,t) = \frac{\partial}{\partial t} - v(x,t)$$

is tangent to the level sets of F.

$$v(x,t) = \Sigma_1^n v_i(x,t)\frac{\partial}{\partial x_i},\ \ v_i(0,t) = 0$$



(a) $F = c$ (b) $V(x,t) = \frac{\partial}{\partial t} - v(x,t)$

Bi-Lipschitz equivalence

Definition

A germ of homeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ is bi-Lipschitz if there exists a constant $C > 0$ such that

$$\frac{1}{C}|x - y| \leq |h(x) - h(y)| \leq C|x - y|$$

for all x, y sufficiently close to 0.

Definition

$f \sim_{bi-Lipschitz} g$ if there exists a bi-Lipschitz homeomorphism h such that $g = f \circ h$.

The conditions corresponding to (i) and (ii) in Thom-Levine's theorem are not equivalent for the bi-Lipschitz equivalence. Clearly

$$(ii) \Longrightarrow (i)$$

But, it is only known that the derivative of a bi-Lipschitz homeomorphism is bounded and exists almost everywhere.

Definition

The family $F : W \times U \to \mathbb{C}$, $F(0,t) = 0$, $f_t(x) = F(x,t)$ is *strongly bi-Lipschitz trivial* when there exists a continuous family of Lipschitz vector fields $v_t : W \to \mathbb{C}^n$, $v(0,t) = 0$ such that

$$\frac{\partial f_t}{\partial t}(x) = df_t(x)(v(x,t)), \ \forall t \in U, \ \forall x \in W$$

Remark

If f_t is strongly bi-Lipschitz trivial, then for all $t \neq t' \in U$,

$$f_t \sim_{bi-Lipschitz} f_{t'}$$

Lipschitz stratification, Mostowski, 1985

It follows from Mostowski's work that the bi-Lipschitz classification of analytic sets is tame.

(See recent work of Birbrair, Neumann and Pichon on the metric structure of normal surfaces singularities)

Risler and Trotman asked in 1997 :
$f^{-1}(0) \sim_{bi-Lipschitz} g^{-1}(0) \Longrightarrow f \sim_{bi-Lipschitz} g$?

Example: Henry-Parusinski, Comp. Math. 136, 2003,[4]

$$f_t(x, y) = x^3 + y^6 - 3t^2xy^4 \; ; \; 0 < |t| < \frac{1}{2}$$

Parusinski and Henry proved that given $t \neq s$, there is no $\phi \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ germ of bi-Lipschitz homeomorphism such that $f_t = f_s \circ \phi$, i.e. f_t is not bi-Lipschitz equivalent to f_s.

This shows in particular that the bi-Lipschitz classification of function germs has modality.

Theorem, Henry-Parusinski

There is no bi-Lipschitz vector field

$$V(x,y,t) = \frac{\partial}{\partial t} + v_1(x,y,t)\frac{\partial}{\partial x} + v_2(x,y,t)\frac{\partial}{\partial y}, \;\; v_1(0,0,t) = v_2(0,0,t) = 0,$$

defined in a neighborhood of $(0,0,t_0)$ and tangent to the levels of

$$F(x,y,t) = x^3 - 3t^2xy^4 + y^6.$$

Corollary

The family f_t is not strongly bi-Lipschitz trivial

Proof of the Theorem:

Let us suppose V does exist. Then $\frac{\partial F}{\partial v} = 0$, that is,

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial t} + v_1(x, y, t)\frac{\partial F}{\partial x} + v_2(x, y, t)\frac{\partial F}{\partial y} = 0$$

Now, let $\Gamma(x, y, t)$ be the family of polar curves of F :

$$\Gamma(x, y, t) = \{(x, y, t) \,|\, \frac{\partial F}{\partial x} = 3(x^2 - t^2y^4) = 0\}$$

Γ consists of 2 branches, $x = \pm ty^2$.

Evaluating $v_2(x, y, t)$ along the two branches of the polar, we get

$$v_2(\pm ty^2, y, t) = \frac{\pm t^2 y}{1 \mp 2t^3}$$

Comparing v_2 on the two branches of the polar curve Γ, we get

$$(I): \ v_2(ty^2, y, t) - v_2(-ty^2, y, t) = \frac{t^2 y}{1 - 2t^3} - \frac{-t^2 y}{1 + 2t^3} \sim y$$

On the other hand, since v_2 is bi-Lipschitz, then

$$(II): \ |v_2(ty^2, y, t) - v_2(-ty^2, y, t)| \leq C|ty^2|$$

(II) contradicts (I)

Parusinski-Henry's Invariant

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of analytic function with Taylor expansion:

$$f(x, y) = H_k(x, y) + H_{k+1}(x, y) + \dots .$$

Assume $H_k(1, 0) \neq 0$, i.e., f is *mini-regular.*

Let γ be a polar arc, that is, $x = \gamma(y)$ is a branch of the polar curve

$$\Gamma : \frac{\partial f}{\partial x}.$$

Let $h_0 = h_0(\gamma) \in \mathbb{Q}_+$ and $c_0 = c_0(\gamma) \in \mathbb{C}^*$ be given by the expansion:

$$f(\gamma(y), y) = c_0 y^{h_0} + \dots, \; c_0 \neq 0.$$

Let l be a line in the tangent cone $C_0(X) = \{H_k(x, y) = 0\}$, where $X = f^{-1}(0)$. Let $\Gamma(l)$ denote the set of all polar arcs tangent to l at 0.

Definition

Let

$$\mathcal{I}(l) = \{c_0(\gamma) y^{h_0(\gamma)} | \gamma \in \Gamma(l)/\mathbb{C}^*\},$$

where $c \in \mathbb{C}^*$ acts by multiplication on y.

The invariant of f, $Inv(f)$ is the set of all $\mathcal{I}(l)$, where l runs over all lines in $Sing(C_0(X))$.

Theorem

Let f_1 and f_2 be two analytic function germs $(\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ mini-regular in x. If f_1 and f_2 are bi-Lipschitz equivalent then $Inv(f_1) = Inv(f_2)$.

The result

Definition

Let $w = (w_1, \ldots, w_n)$ be a n-tuple of positive integers. We say that a polynomial function $f(x_1, \ldots, x_n)$ is w-homogeneous of degree d if

$$f(s^{w_1}x_1, \ldots, s^{w_n}x_n) = s^d f(x_1, \ldots, x_n),$$

for every $s \in \mathbb{C}^*$.

We denote by $H_w^d(n, 1)$ the space of w-homogeneous polynomials in n-variables of degree d.

Theorem

Let $f(x, y, t)$ be a polynomial function such that for every $t \in U$, the function $f_t(x, y) = F(x, y, t)$ is w-homogeneous ($w_1 > w_2$) with isolated singularity in $(0, 0) \in \mathbb{C}^2$. If $\{f_t : t \in U\}$ as a family of function germs in $(0, 0) \in \mathbb{C}^2$, is strongly bi-Lipschitz trivial, then f_{t_1} is analytically equivalent to f_{t_2} $\forall t_1, t_2 \in U$.

Proof.

From the hypothesis, there exists a Lipschitz vector field

$$V(x,y,t) = \frac{\partial}{\partial t} + v_1(x,y,t)\frac{\partial}{\partial x} + v_2(x,y,t)\frac{\partial}{\partial y}, v_i(0,0,t) = 0,\ i = 1,2.$$

This vector field is tangent to the level sets of F, that is,

$$\frac{\partial F}{\partial V} = \frac{\partial F}{\partial t} + v_1(x,y,t)\frac{\partial F}{\partial x} + v_2(x,y,t)\frac{\partial F}{\partial y} = 0$$

Let

$$\Gamma_t = \{(x,y,t)\ :\ \frac{\partial F}{\partial x} = 0\}$$

be the family of polar curves of the family F.

□

The polar is a family of algebraic curves and may have multiple components. The proof of the theorem in this general case is the most difficult part of the theorem.

We assume Γ_t is *reduced* and $y = 0$ is not a factor of $\frac{\partial F}{\partial x} = 0$.

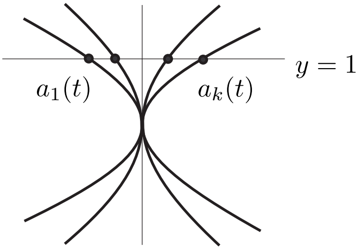
In this case, let $a_1(t), \ldots, a_k(t)$ be the roots of

$$\frac{\partial F}{\partial x}(x, 1, t) = 0,$$

(as the degree of $\frac{\partial F}{\partial x}(x, 1, t) = 0$ does not depend on t, the functions $a_i(t)$ are continuous.

Since $w_1 > w_2$, the parametrization of the branch γ_i of Γ is

$$\gamma_i(s) = (a_i(t)s^{w_1}, s^{w_2}, t), \;\; i = 1, \ldots, k.$$



We define the functions $k_1(t), \ldots, k_r(t)$ as

$$k_i(t) = \frac{\frac{\partial F}{\partial t}(a_i(t), 1, t)}{\frac{\partial F}{\partial y}(a_i(t), 1, t)}$$

and prove the following proposition:

Proposition

With the above notation $k_i(t) = k_j(t), \ \forall i, j = 1, \ldots, r$

Proof of the proposition

Notice that on the polar set, we have

$$v_2(x, y, t) = -\frac{\frac{\partial F}{\partial t}(x, y, t)}{\frac{\partial F}{\partial y}(x, y, t)}$$

Comparing v_2 on two branches of the polar we get

$$\left|\frac{\frac{\partial F}{\partial t}(a_i(t)s^{w_1}, s^{w_2}, t)}{\frac{\partial F}{\partial y}(a_i(t)s^{w_1}, s^{w_2}, t)} - \frac{\frac{\partial F}{\partial t}(a_j(t)s^{w_1}, s^{w_2}, t)}{\frac{\partial F}{\partial y}(a_j(t)s^{w_1}, s^{w_2}, t)}\right| = |k_i(t) - k_j(t)||s|^{w_2}$$

On the other hand, since $v_2(x, y, t)$ is Lipschitz, then

$$|v_2(a_i(t)s^{w_1}, s^{w_2}, t) - v_2(a_j(t)s^{w_1}, s^{w_2}, t)| \leq C|s|^{w_1}$$

But $w_1 > w_2$, then we must have $k_i(t) = k_j(t) = k(t)$, and this proves the proposition.

Proof of the theorem (when the family of polar curves is reduced)

Now, for fixed t, the function

$$\frac{\partial F}{\partial t}(x, y, t) - k(t)y\frac{\partial F}{\partial y}(x, y, t)$$

is analytic and identically zero on each branch of the polar.

If $\frac{\partial F}{\partial x}$ is a reduced family of plane curves (without repeated branches) then there exists $b(x, y, t)$, analytic function such that

$$\frac{\partial F}{\partial t}(x, y, t) = k(t)y\frac{\partial F}{\partial y} + b(x, y, t)\frac{\partial F}{\partial x}.$$

We now use the weighted homogeneity of F to get that the family is analytically trivial.

Problems

We first recall the following result and example:

Theorem

A. Fernandes,—, 2004, [1]: Let $f : \mathbb{K}^n, 0 \to \mathbb{K}, 0$ *be the germ of a weighted homogeneous polynomial function of type* $(w_1, \dots, w_n : d)$, $w_n \leq \dots \leq w_1$ *with isolated singularity. Let* $f_t(x) = f(x) + t\theta(x, t)$, $t \in [0, 1]$, *be a deformation of* f. *If* $fil(\theta) \geq d + w_1 - w_n$, *then* f_t *is strongly bi-Lipschitz trivial.*

Example

The family

$$f_t(x, y) = x^3 - 3t^2xy^{3n-2} + y^{3n}$$

is not strongly bi-Lipschitz trivial. Moreover, Parusinski and Henry's invariant does not distinguish the elements of the family f_t.

Problems

(1) Prove that the rigidity theorem also holds for deformations
$f_t(x,y) = f(x,y) + t\theta(x,y),$ with $d < \text{fil}(\theta) < d + w_1 - w_n$.

(2) Investigate the bi-Lipschitz invariance of higher order terms of the asymptotic expansion of f over the branches of the polar curves.

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Adam Parusinski and Jean-Pierre Henry. *Existence of moduli for bilipschitz equivalence of analytic functions.* Compositio Math. **136** (2003), pp 217–235.