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Nonlinear Analysis ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Poisson stability for impulsive semidynamical systems

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ARTICLE INFO

Article history:
Received 14 February 2009
Accepted 5 June 2009

MSC:
34A37
54H20
93D05

Keywords:
Impulsive semidynamical systems
Recursiveness
Poisson stability

ABSTRACT

In this paper, the concept of Poisson stability is investigated for impulsive semidynamical systems. Recursive properties are also investigated.

1. Introduction

Discontinuous semidynamical systems are a natural generalization of classical dynamical systems. Systems which admit abrupt perturbations, called impulses, may present many interesting and unexpected phenomena such as "beating", "dying", "merging" and "noncontinuation of solutions" and many real world problems can be modelled by such systems. For details about this theory, see [1–10], for instance.

Some classical concepts in dynamical systems as recursiveness, Poisson stability, non-wandering points and minimal sets are considered in [11]. Among these concepts, important results are due to Birkhoff [12], Nemyckiĭ and Stepanov [13], Hájek [14], Bhatia and Szegö [11]. On the other hand, many of these concepts are not yet specified for discontinuous semidynamical systems.

In the first part of the present paper, we give a brief overview of the basis of the theory of impulsive semidynamical systems. The second part of the paper concerns the main results. We present the concept of (positive) Poisson stability for impulsive semidynamical systems. Then we give some characterizations of positively Poisson stable points of this system (see Theorem 3.1 in what follows). We also state conditions to a positively Poisson stable point to be periodic (Theorem 3.2). An important result in classical dynamical systems concerns positively Poisson stable points whose orbits differ from their limit sets. We generalize this result for discontinuous systems in Theorem 3.3. In addition, we present the concept of non-wandering points and we establish conditions so that the set of positively Poisson stable points is dense in the space where the impulsive semidynamical system is defined (see Theorem 3.6).

The reader may find another kind of recursive properties for impulsive semidynamical systems in [9].

2. Definitions and notations

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(X, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : X \times \mathbb{R}_+ \longrightarrow X$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in X$ and

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0362-546X/$ – see front matter
doi:10.1016/j.na.2009.06.008

Please cite this article in press as: E.M. Bonotto, M. Federson, Poisson stability for impulsive semidynamical systems, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.06.008

$t, s \in \mathbb{R}_+$. We denote such a system by $(X, \pi, \mathbb{R}_+)$ or simply (X, π). When $\mathbb{R}_+$ is replaced by $\mathbb{R}$, the triple $(X, \pi, \mathbb{R})$ is a *dynamical system*. For every $x \in X$, we consider the continuous function $\pi_x : \mathbb{R}_+ \longrightarrow X$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in X$, the *positive orbit* of x is given by $\pi^+(x) = \{\pi(x, t) : t \in \mathbb{R}_+\}$. For $t \geq 0$ and $x \in X$, we define $F(x, t) = \{y \in X : \pi(y, t) = x\}$ and, for $\triangle \subset [0, +\infty)$ and $D \subset X$, we define

$$F(D, \triangle) = \bigcup \{F(x, t) : x \in D \quad \text{and} \quad t \in \triangle\}.$$

Then a point $x \in X$ is called an *initial point*, whenever $F(x, t) = \emptyset$ for all $t > 0$.

An *impulsive semidynamical system* $(X, \pi; M, I)$ consists of a semidynamical system, (X, π), a non-empty closed subset M of X such that for every $x \in M$, there exists $\varepsilon_x > 0$ such that

$$F(x, (0, \varepsilon_x)) \cap M = \emptyset \qquad \text{and} \qquad \pi(x, (0, \varepsilon_x)) \cap M = \emptyset,$$

and a continuous function $I : M \to X$ which is responsible for the jumps or discontinuities of the trajectories of the semidynamical system. We also define

$$M^+(x) = (\pi^+(x) \cap M) \setminus \{x\},$$

for $x \in X$.

Given an impulsive semidynamical system $(X, \pi; M, I)$ and $x \in X$, we define the function $\phi : X \to (0, +\infty]$ as follows

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x, s) \in M \quad \text{and} \quad \pi(x, t) \notin M \quad \text{for } 0 < t < s, \\ +\infty, & \text{if } M^+(x) = \emptyset. \end{cases}$$

This means that $\phi(x)$ is the least positive time at which the trajectory of x meets M. Thus for each $x \in X$, we call $\pi(x, \phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in $(X, \pi; M, I)$ is an X-valued function $\widetilde{\pi}_x$ defined on the subset $[0, s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such a trajectory follows by the following recurrence

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x, t), & 0 \leq t < s_0 \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = I(x_1)$, $x_1 = \pi(x, \phi(x))$, $\phi(x) = s_0$ $(x = x_0^+)$ and

$$\widetilde{\pi}_x(t) = \begin{cases} \pi(x_n^+, t - t_n), & t_n \leq t < t_{n+1} \\ x_{n+1}^+, & t = t_{n+1}, \end{cases}$$

where $x_{n+1}^+ = I(x_{n+1})$, $x_{n+1} = \pi(x_n^+, \phi(x_n^+))$, $\phi(x_n^+) = s_n$ and $t_{n+1} = \sum_{i=0}^n s_i$, $n = 0, 1, 2, \ldots$.

For details about the structure of these types of impulsive semidynamical systems, the reader may consult [1,2,4,5,8].

2.1. Semicontinuity and continuity of ϕ

The result of this section is borrowed from [1]. It concerns the function ϕ defined previously which indicates the moments of impulse action of a trajectory in an impulsive system.

Let (X, π) be a semidynamical system. Any closed set $S \subset X$ containing x $(x \in X)$ is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set $L \subset X$ such that

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$, for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube* and the set L is called a *bar*. Let (X, π) be a semidynamical system. We now present the conditions TC and STC for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write (TC), if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have an *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists an STC-tube $F(L, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ which is accomplished outside M for M satisfying the condition TC. See [1], Theorem 3.8.

Theorem 2.1. *Consider an impulsive semidynamical system* $(X, \pi; M, I)$. *Assume that no initial point in* (X, π) *belongs to the impulsive set* M *and that each element of* M *satisfies the condition* (TC). *Then* ϕ *is continuous at* x *if and only if* $x \notin M$.

2.2. Additional definitions

Let us consider a metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in X$ and ratio δ. Given $A \subset X$, $diam(A) = \sup\{\rho(a, b) : a, b \in A\}$.

In what follows, $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is an impulsive semidynamical system and $x \in \mathrm{X}$.

We define the *limit set* of x in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ by

$$\widetilde{\mathrm{L}}^{+}(x) = \{y \in \mathrm{X} : \widetilde{\pi}(x, t_n) \stackrel{n\to+\infty}{\longrightarrow} y, \text{ for some sequence } t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty\}.$$

For each $x \in \mathrm{X}$, the set $\widetilde{\mathrm{L}}^{+}(x)$ is closed [8, Remark 1].

The *prolongational limit set* of x in $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ is given by

$$\widetilde{\mathrm{J}}^{+}(x) = \{y \in \mathrm{X} : \widetilde{\pi}(x_n, t_n) \stackrel{n\to+\infty}{\longrightarrow} y, \text{ for } x_n \stackrel{n\to+\infty}{\longrightarrow} x \text{ and } t_n \stackrel{n\to+\infty}{\longrightarrow} +\infty\}.$$

If $\widetilde{\pi}^{+}(A) \subset A$, we say that A is *positively* $\widetilde{\pi}$*-invariant*.

A point $x \in \mathrm{X}$ is called *stationary* or *rest point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for all $t \geq 0$, it is a *periodic point* with respect to $\widetilde{\pi}$, if $\widetilde{\pi}(x, t) = x$ for some $t > 0$ and x is not stationary, and x is an *eventually periodic point* with respect to $\widetilde{\pi}$, if a point $\widetilde{\pi}(x, t)$ is periodic for some $t \geq 0$. We say that $\widetilde{\pi}^{+}(x)$, $x \in \mathrm{X}$, is eventually periodic if $\widetilde{\pi}(x, t)$ is an eventually periodic point for each $t \geq 0$.

Given $x \in \mathrm{X}$, we denote $T(x)$ by $\sum_{i=0}^{\infty} \phi(x_i^{+})$.

3. The main results

In this section, we present some results which concern recursive motions. We shall consider an impulsive semidynamical system $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$, where the following additional hypotheses hold:

- No initial point in (X, π) belongs to the impulsive set M, that is, given $x \in \mathrm{M}$ there are $y \in \mathrm{X}$ and $t \in \mathbb{R}_{+}$ such that $\pi(y, t) = x$.
- Each element of M satisfies the condition (STC) (*consequently, ϕ is continuous on* $\mathrm{X} \setminus \mathrm{M}$, see [1]).
- $\mathrm{M} \cap \mathrm{I}(\mathrm{M}) = \emptyset$.
- For each $x \in \mathrm{X}$, the motion $\widetilde{\pi}(x, t)$ is defined for every $t \geq 0$, i.e. $[0, +\infty)$ denotes the maximal interval of definition of $\widetilde{\pi}_x$. By following [8], the impulsive systems where the motion $\widetilde{\pi}(x, t)$ is defined for all $t \geq 0$ are the most important and interesting, and, moreover, in many cases we may restrict ourselves to such systems (because of the existence of suitable isomorphisms), due to the paper [3].

We start by introducing the concept of recursiveness for an impulsive semidynamical system.

Definition 3.1. Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. A subset $U \subset \mathrm{X}$ is said to be positively $\widetilde{\pi}$-recursive with respect to a subset $V \subset \mathrm{X}$ if for each $T \geq 0$ there are a time $t > T$ and an element $x \in V$ such that $\widetilde{\pi}(x, t) \in U$. We say that U is self-positively $\widetilde{\pi}$-recursive, whenever it is positively $\widetilde{\pi}$-recursive with respect to itself.

Consider the impulsive differential system in $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x_1} = -x_1,\ \dot{x_2} = -x_2,\ (x_1, x_2) \notin \mathrm{M} \\ \mathrm{I} : \mathrm{M} \to \mathrm{N} \end{cases} \tag{1}$$

where the sets $\mathrm{M}, \mathrm{N} \subset \mathbb{R}^2$ are defined by $\mathrm{M} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 4\}$ and $\mathrm{N} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ and the impulse function I assigns to every point $x \in \mathrm{M}$ a point $y \in \mathrm{N}$ which is on the ray joining x to the origin in $\mathbb{R}^2$. The trajectories of this system are presented in Fig. 1. Any neighborhood of the origin is positively $\widetilde{\pi}$-recursive with respect to a given subset V of $\mathbb{R}^2$. In particular, any neighborhood of the origin is self-positively $\widetilde{\pi}$-recursive. Note that the origin is an attractor.

Next, we define (positive) Poisson stability for an impulsive semidynamical system.

Definition 3.2. Let $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. A point $x \in \mathrm{X}$ is said to be positively Poisson $\widetilde{\pi}$-stable if every neighborhood of x is positively $\widetilde{\pi}$-recursive with respect to $\{x\}$.

In the previous example, the origin (rest point) of system Eq. (1) is the only point in $\mathbb{R}^2$ which is positively Poisson $\widetilde{\pi}$-stable. It is easy to see that rest points and periodic orbits are positively Poisson $\widetilde{\pi}$-stable.

Before giving some characterizations of positively Poisson $\widetilde{\pi}$-stable points, we present an auxiliary result which characterizes the closures of trajectories of an impulsive system.

Lemma 3.1. *Let* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system and* $x \in \mathrm{X}$. *Suppose* $\phi(x_j^{+}) < \infty$ *for every* $j = 0, 1, 2, \ldots$. *Then,*

$$\overline{\widetilde{\pi}^{+}(x)} = \widetilde{\pi}^{+}(x) \cup \widetilde{\mathrm{L}}^{+}(x) \cup \{x_j : j = 1, 2, \ldots\},$$

where $x_j = \pi(x_{j-1}^{+}, \phi(x_{j-1}^{+}))$, $j = 1, 2, \ldots$, *and* $\widetilde{\pi}^{+}(x)$ *is the positive orbit of* x *with respect to* $\widetilde{\pi}$. *Note that if* $\phi(x_j^{+}) < +\infty$, $j = 0, \ldots, k$ *and* $\phi(x_{k+1}^{+}) = +\infty$ *then* $\overline{\widetilde{\pi}^{+}(x)} = \widetilde{\pi}^{+}(x) \cup \widetilde{\mathrm{L}}^{+}(x) \cup \{x_j : j = 1, 2, \ldots, k+1\}$.

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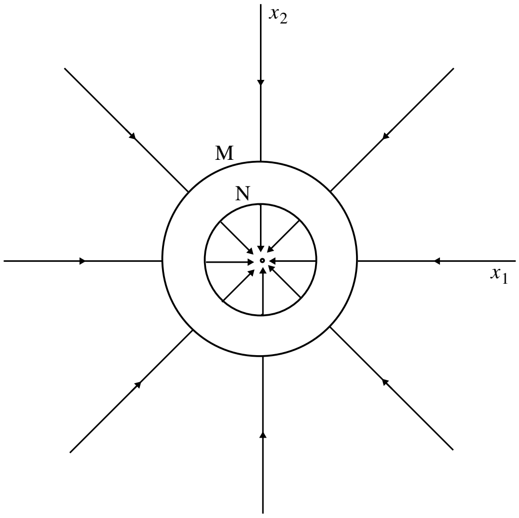

Fig. 1. Any neighborhood of the origin is self-positively $\widetilde{\pi}$-recursive.

Proof. It is enough to show that $\overline{\widetilde{\pi}^+(x)} \subset \left(\widetilde{\pi}^+(x) \cup \widetilde{L}^+(x) \cup \{x_j : j = 1, 2, \ldots\}\right)$.

Let $y \in \overline{\widetilde{\pi}^+(x)}$ be arbitrary. Then there exists a sequence $\{w_n\}_{n\geq 1} \subset \widetilde{\pi}^+(x)$ such that

$$w_n \overset{n\to+\infty}{\longrightarrow} y.$$

Since $\{w_n\}_{n\geq 1} \subset \widetilde{\pi}^+(x)$, we have $w_n = \widetilde{\pi}(x, t_n)$ for each $n = 1, 2, 3, \ldots$, where $\{t_n\}_{n\geq 1}$ is a sequence of positive real numbers. We may assume that

$$\lim_{n\to+\infty} t_n = +\infty \quad \text{or} \quad \lim_{n\to+\infty} t_n = a < +\infty,$$

because $w_n \overset{n\to+\infty}{\longrightarrow} y$.

Suppose $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$. Then $y \in \widetilde{L}^+(x)$ and the inclusion holds.

Now, suppose the sequence $\{t_n\}_{n\geq 1}$ converges to some real number a. We discuss two possibilities: when $a = \sum_{j=0}^{m} \phi(x_j^+)$ for some $m = 0, 1, 2, \ldots$, and when $a \neq \sum_{j=0}^{m} \phi(x_j^+)$ for every $m = 0, 1, 2, \ldots$.

Suppose $a = \sum_{j=0}^{m} \phi(x_j^+)$ for some $m = 0, 1, 2, \ldots$. If there exists a subsequence $\{t_{n_k}\}_{k\geq 1}$ of $\{t_n\}_{n\geq 1}$ such that $t_{n_k} < a$, $k = 1, 2, \ldots$, then there is an integer $N_1 > 0$ such that

$$\widetilde{\pi}(x, t_{n_k}) = \pi(x_m^+, \bar{t}_{n_k}), \quad \text{for } n_k > N_1,$$

where $t_{n_k} = \sum_{j=0}^{m-1} \phi(x_j^+) + \bar{t}_{n_k}$, $0 \leq \bar{t}_{n_k} < \phi(x_m^+)$ and $\bar{t}_{n_k} \overset{n_k\to+\infty}{\longrightarrow} \phi(x_m^+)$. Thus,

$$\widetilde{\pi}(x, t_{n_k}) = \pi(x_m^+, \bar{t}_{n_k}) \overset{n_k\to+\infty}{\longrightarrow} \pi(x_m^+, \phi(x_m^+)) = x_{m+1}$$

which means that $y = x_{m+1}$.

If $t_{n_k} = a$ for infinitely many k, it is obvious.

On the other hand, if there is no such t_{n_k} that $t_{n_k} \leq a$ for each $k = 1, 2, \ldots$, then there is an integer $N_2 > 0$ such that

$$\widetilde{\pi}(x, t_{n_k}) = \pi(x_{m+1}^+, \bar{\bar{t}}_{n_k}), \quad \text{for } n_k > N_2,$$

where $0 \leq \bar{\bar{t}}_{n_k} < \phi(x_{m+1}^+)$ and $\bar{\bar{t}}_{n_k} \overset{n_k\to+\infty}{\longrightarrow} 0$. Consequently,

$$\widetilde{\pi}(x, t_{n_k}) \overset{n\to+\infty}{\longrightarrow} x_{m+1}^+,$$

that is

$$y = x_{m+1}^+ \in \widetilde{\pi}^+(x).$$

But, if $a \neq \sum_{j=0}^{m} \phi(x_j^+)$ for every $m = 0, 1, 2, \ldots$, then $0 \leq a < \phi(x)$ or there is an integer $\ell \geq 0$ such that $\sum_{j=0}^{\ell} \phi(x_j^+) < a < \sum_{j=0}^{\ell+1} \phi(x_j^+)$. Let us assume the latter. Then there is an integer $N_3 > 0$ such that

$$\widetilde{\pi}(x, t_n) = \pi(x_{\ell+1}^+, \widehat{t}_n), \quad \text{for } n > N_3,$$

where $0 \leq \widehat{t}_n < \phi(x_{\ell+1}^+)$ and $\widehat{t}_n \overset{n\to+\infty}{\longrightarrow} a - \sum_{j=0}^{\ell} \phi(x_j^+) := \widehat{a} < \phi(x_{\ell+1}^+)$.

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Since

$$\pi(x^+_{\ell+1}, \widehat{t}_n) \stackrel{n\to+\infty}{\longrightarrow} \pi(x^+_{\ell+1}, \widehat{a}) \in \widetilde{\pi}^+(x),$$

we conclude that $y = \pi(x^+_{\ell+1}, \widehat{a}) \in \widetilde{\pi}^+(x)$ and the theorem is proved. □

Remark 3.1. In [10], Kaul considers an impulsive semidynamical system $(\Omega, \widetilde{\pi})$, where $\Omega \subset \mathrm{X}$ is an open set in a metric space X and the continuous impulse operator I is defined on the boundary $\partial\Omega$ of Ω in X and takes values in Ω. He proves that $\overline{\widetilde{\pi}^+(x)} = \widetilde{\pi}^+(x) \cup \widetilde{\mathrm{L}}^+(x)$ for each $x \in \Omega$ (see [10, Lemma 2.10]). This equality does not involve the elements $x_j, j = 1, 2, \ldots,$ because the phase space of $(\Omega, \widetilde{\pi})$ does not contain the points $\{x_j : j = 1, 2, 3, \ldots\}$.

Now, we mention an important lemma that will be very useful later in this paper. This lemma was proved by Kaul in [10] for impulsive semidynamical systems of type $(\Omega, \widetilde{\pi})$ (see Lemma 2.3 there). However, this result still holds for impulsive systems of the form $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$. See [5, Lemma 3.2].

Lemma 3.2. *Given an impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$, *where* X *is a metric space, suppose* $w \in \mathrm{X} \setminus M$ *and* $\{z_n\}_{n\geq 1}$ *is a sequence in* X *which converges to the point* w. *Then, for any* $t \in [0, T(w))$, *there exists a sequence of real numbers* $\{\varepsilon_n\}_{n\geq 1}$, *with* $\varepsilon_n \stackrel{n\to+\infty}{\longrightarrow} 0$, *such that* $t + \varepsilon_n < T(z_n)$ *and* $\widetilde{\pi}(z_n, t + \varepsilon_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w, t)$.

In Lemma 3.2, when $\widetilde{\pi}(w, t) \neq w^+_j$ for every $j = 1, 2, 3, \ldots,$ the convergence $\widetilde{\pi}(z_n, t + \varepsilon_n) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w, t)$ does not depend on the sequence $\{\varepsilon_n\}_{n\geq 1}$, that is, $\widetilde{\pi}(z_n, t) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w, t)$, whenever $t \neq \sum_{j=0}^{k} \phi(w^+_j)$ for every $k = 0, 1, 2, \ldots.$ We show this fact in the next lemma.

Lemma 3.3. *Given an impulsive semidynamical system* $(\mathrm{X}, \pi; \mathrm{M}, \mathrm{I})$, *where* X *is a metric space, suppose* $w \in \mathrm{X} \setminus M$ *and* $\{z_n\}_{n\geq 1}$ *is a sequence in* X *which converges to* w. *Then, for any* t *such that* $t \neq \sum_{j=0}^{k} \phi(w^+_j)$, $k = 0, 1, 2, \ldots,$ *we have* $\widetilde{\pi}(z_n, t) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w, t)$.

Proof. Since $w \in \mathrm{X} \setminus M$, there is $\eta > 0$ such that $B(w, \eta) \cap \mathrm{M} = \emptyset$. By the convergence of the sequence $\{z_n\}_{n\geq 1}$, there is an integer $N > 0$ such that $z_n \in B(w, \eta)$ for all $n > N$. From the continuity of ϕ on $\mathrm{X} \setminus M$, we have

$$\phi(z_n) \stackrel{n\to+\infty}{\longrightarrow} \phi(w).$$

Suppose $0 \leq t < \phi(w)$. Then given $\epsilon < \phi(w) - t$, there exists $N_1 > N$ such that $|\phi(z_n) - \phi(w)| < \epsilon$, for all $n > N_1$, that is,

$$\phi(z_n) > -\epsilon + \phi(w) > t, \quad \text{for } n > N_1.$$

Then, since $\widetilde{\pi}(z_n, t) = \pi(z_n, t)$ for $n > N_1$, $\widetilde{\pi}(w, t) = \pi(w, t)$ and $\pi(z_n, t) \stackrel{n\to+\infty}{\longrightarrow} \pi(w, t)$, we have

$$\widetilde{\pi}(z_n, t) \stackrel{n\to+\infty}{\longrightarrow} \widetilde{\pi}(w, t).$$

Now, suppose $\phi(w) < t < \phi(w) + \phi(w^+_1)$. Since $t - \phi(w) > 0$, there is $N_2 > N$ such that $|\phi(z_n) - \phi(w)| < t - \phi(w)$ for all $n > N_2$. Then

$$\phi(z_n) < t, \quad \text{for all } n > N_2.$$

We claim that $t - \phi(z_n) < \phi((z_n)^+_1)$ for a sufficiently large n. Indeed. Since $0 < t - \phi(w) < \phi(w^+_1)$, there is $\varepsilon_1 > 0$ such that

$$]t - \phi(w) - \varepsilon_1, t - \phi(w) + \varepsilon_1[\cap]\phi(w^+_1) - \varepsilon_1, \phi(w^+_1) + \varepsilon_1[= \emptyset.$$

Also, since $\phi(z_n) \stackrel{n\to+\infty}{\longrightarrow} \phi(w)$, there exists $N_3 > N$ such that

$$t - \phi(z_n) \in]t - \phi(w) - \varepsilon_1, t - \phi(w) + \varepsilon_1[, \quad \text{for } n > N_3.$$

On the other hand, since $w^+_1 \notin \mathrm{M}$ $(\mathrm{M} \cap \mathrm{I}(\mathrm{M}) = \emptyset)$, there is an $\eta_1 > 0$ such that $B(w^+_1, \eta_1) \cap \mathrm{M} = \emptyset$. Because I is a continuous function, we have $(z_n)^+_1 \stackrel{n\to+\infty}{\longrightarrow} w^+_1$. Then, by the continuity of ϕ on $B(w^+_1, \eta_1)$, we also have $\phi((z_n)^+_1) \stackrel{n\to+\infty}{\longrightarrow} \phi(w^+_1)$. Thus there exists $N_4 > N$ such that

$$\phi((z_n)^+_1) \in]\phi(w^+_1) - \varepsilon_1, \phi(w^+_1) + \varepsilon_1[, \quad \text{for } n > N_4.$$

Consequently

$$t - \phi(z_n) < \phi((z_n)^+_1),$$

for $n > \max\{N_3, N_4\}$. Then for $n > \max\{N_2, N_3, N_4\}$, we have

$$\phi(z_n) < t \quad \text{and} \quad t - \phi(z_n) < \phi((z_n)_1^+).$$

Hence, since $\tilde{\pi}(z_n, t) = \pi((z_n)_1^+, t - \phi(z_n))$ for $n > \max\{N_2, N_3, N_4\}$, $\tilde{\pi}(w, t) = \pi(w_1^+, t - \phi(w))$ and $\pi((z_n)_1^+, t - \phi(z_n)) \overset{n\to+\infty}{\longrightarrow} \pi(w_1^+, t - \phi(w))$, we obtain

$$\tilde{\pi}(z_n, t) \overset{n\to+\infty}{\longrightarrow} \tilde{\pi}(w, t).$$

The above process can be continued by induction and the result follows. $\square$

Remark 3.2. The general assumption that the motion $\tilde{\pi}(x, t)$, $x \in X$, is defined for all $t \geq 0$ is not important here and the proof in more general case is the same as above.

Remark 3.3. In Lemma 3.3, the convergence $\tilde{\pi}(z_n, t) \overset{n\to+\infty}{\longrightarrow} \tilde{\pi}(w, t)$ where $t \neq \sum_{j=0}^{k} \phi(w_j^+)$, $k = 0, 1, 2, \ldots$, is equivalent to $\pi((z_n)_j^+, \lambda) \overset{n\to+\infty}{\longrightarrow} \pi(w_j^+, \lambda)$ for each $j = 0, 1, 2, \ldots$ and $0 \leq \lambda < \phi(w_j^+)$.

The next theorem characterizes positively Poisson $\tilde{\pi}$-stable points.

Theorem 3.1. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system and* $x \in X \setminus M$. *Then the following statements are equivalent:*

1. *x is positively Poisson $\tilde{\pi}$-stable;*
2. *Given a neighborhood U of x and $T > 0$, then $\tilde{\pi}(x, t) \in U$ for some $t > T$;*
3. *$x \in \tilde{L}^+(x)$;*
4. *$\overline{\tilde{\pi}^+(x)} = \tilde{L}^+(x)$;*
5. *$\tilde{\pi}^+(x) \subset \tilde{L}^+(x)$;*
6. *For every $\epsilon > 0$, there exists $t \geq 1$ such that $\tilde{\pi}(x, t) \in B(x, \epsilon)$.*

Proof. It is easy to see that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (1). Let us show that (3) $\Rightarrow$ (4).

We consider the case when $\phi(x_j^+) < +\infty$ for each $j = 0, 1, 2, \ldots$. By Lemma 3.1, $\overline{\tilde{\pi}^+(x)} \supseteq \tilde{L}^+(x)$. Then it is enough to prove the reverse inclusion.

Let $y \in \tilde{\pi}^+(x)$. Then $\tilde{\pi}(x, \lambda) = y$ for some $\lambda \geq 0$. Since $x \in \tilde{L}^+(x)$, there is a sequence $\{t_n\}_{n\geq 1}$, with $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$, such that $\tilde{\pi}(x,\ t_n) \overset{n\to+\infty}{\longrightarrow} x$. Since $x \notin M$, by Lemma 3.2, there exists a sequence of real numbers $\{\varepsilon_n\}_{n\geq 1}$, $\varepsilon_n \overset{n\to+\infty}{\longrightarrow} 0$, such that

$$\tilde{\pi}(x, t_n + \varepsilon_n + \lambda) \overset{n\to+\infty}{\longrightarrow} \tilde{\pi}(x, \lambda) = y.$$

Thus $y \in \tilde{L}^+(x)$ because $t_n + \varepsilon_n + \lambda \overset{n\to+\infty}{\longrightarrow} +\infty$. Hence $\tilde{\pi}^+(x) \subset \tilde{L}^+(x)$ and since $\tilde{L}^+(x)$ is closed it follows that $\overline{\tilde{\pi}^+(x)} \subset \tilde{L}^+(x)$, which completes the proof. $\square$

Using Lemma 3.2, the next result follows straightforwardly.

Lemma 3.4. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system and* $x \in X \setminus M$. *If x is positively Poisson $\tilde{\pi}$-stable, then $\tilde{\pi}(x, t)$ is positively Poisson $\tilde{\pi}$-stable for every $t \geq 0$.*

Theorem 3.2 below establishes necessary conditions for a positively Poisson $\tilde{\pi}$-stable point to be a periodic point.

Theorem 3.2. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system and* $x \in X$. *Suppose* $\tilde{\pi}^+(x) = \cup\{\pi(x_j^+, [0, \phi(x_j^+))) : j = 0, \ldots, k\}$, $\phi(x_k^+) < +\infty$, *and* $\tilde{\pi}(x_j^+, [0, \phi(x_j^+))) \cap \tilde{\pi}(x_i^+, [0, \phi(x_i^+))) = \emptyset$ *for* $i \neq j$, $i, j = 1, \ldots, k$. *If x is positively Poisson $\tilde{\pi}$-stable, then x is a periodic point.*

Proof. At first, let us note that

$$\tilde{\pi}(x_k^+, \phi(x_k^+)) = x_{k+1}^+ \in \tilde{\pi}^+(x).$$

Since $\tilde{\pi}^+(x) = \cup\{\pi(x_j^+, [0, \phi(x_j^+))) : j = 0, \ldots, k\}$ is an invariant set, $\phi(x_k^+) < +\infty$ and $T(x) = +\infty$, we conclude that x_{k+1}^+ is a periodic orbit. Suppose $x_{k+1}^+ \neq x$, then

$$\tilde{\pi}^+(x) = \tilde{\pi}(x, [0, \nu)) \cup \tilde{\pi}^+(x_{k+1}^+) \quad \text{for some } 0 < \nu < \phi(x)\ (x_{k+1}^+ = \pi(x, \nu))$$

or

$$\tilde{\pi}^+(x) = \tilde{\pi}(x, [0, \phi(x) + \cdots + \phi(x_{\ell-1}^+) + \lambda)) \cup \tilde{\pi}^+(x_{k+1}^+)$$

for some $0 \leq \lambda < \phi(x_\ell^+)$ where ℓ is some integer in $\{1, 2, \ldots, k\}$ $(x_{k+1}^+ = \pi(x_\ell^+, \lambda))$.

Therefore $x \notin \tilde{L}^+(x)$ which is a contradiction because x is positively Poisson $\tilde{\pi}$-stable. Thus $x_{k+1}^+ = x$ and x is periodic. $\square$

Please cite this article in press as: E.M. Bonotto, M. Federson, Poisson stability for impulsive semidynamical systems, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.06.008

In [13], Nemyckiĭ and Stepanov established an important result about positively Poisson stable points $x \in X$ such that $\pi^+(x) \neq L^+(x)$ (see also [11]). We extend this result for the impulsive case in the next theorem.

Theorem 3.3. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system and suppose* X *is a complete metric space. Let* $x \in X \setminus M$ *be positively Poisson* $\tilde{\pi}$*-stable and not an eventually periodic point. Then the set* $\tilde{L}^+(x) - \tilde{\pi}^+(x)$ *is dense in* $\tilde{L}^+(x)$.

Proof. Let $z \in \tilde{L}^+(x)$ and $\epsilon > 0$ be arbitrary. Then there is a monotone sequence $\{t_n\}_{n\geq 1}$, $t_n \xrightarrow{n\to+\infty} +\infty$ such that

$$\tilde{\pi}(x, t_n) \xrightarrow{n\to+\infty} z.$$

Choose $\tau_1 > t_1$ such that $\tilde{\pi}(x, \tau_1) \in B(z, \epsilon)$ (which exists as x is positively Poisson $\tilde{\pi}$-stable). Then $\tilde{\pi}(x, \tau_1) \notin \tilde{\pi}(x, [0, t_1])$ (otherwise either x would be an eventually periodic point), consequently $\tilde{\pi}(x, \tau_1) \notin \overline{\tilde{\pi}(x, [0, t_1])}$, according to the assumption that M is disjoint from I(M). Thus, there exists $\epsilon_1 > 0$, with $\epsilon_1 < \frac{\epsilon}{2}$, such that

$$B(\tilde{\pi}(x, \tau_1), \epsilon_1) \subset B(z, \epsilon) \quad \text{and} \quad B(\tilde{\pi}(x, \tau_1), \epsilon_1) \cap \overline{\tilde{\pi}(x, [0, t_1])} = \emptyset.$$

By Lemma 3.4, $\tilde{\pi}(x, \tau_1)$ is positively Poisson $\tilde{\pi}$-stable. Thus we can choose $\tau_2 > t_2$ such that

$$\tilde{\pi}(x, \tau_2) \in B(\tilde{\pi}(x, \tau_1), \epsilon_1).$$

Since $\tilde{\pi}(x, \tau_2) \notin \overline{\tilde{\pi}(x, [0, t_2])}$ (we use the same argument done above), there exists $\epsilon_2 > 0$, with $\epsilon_2 < \frac{\epsilon_1}{2}$, such that

$$B(\tilde{\pi}(x, \tau_2), \epsilon_2) \subset B(\tilde{\pi}(x, \tau_1), \epsilon_1) \quad \text{and} \quad B(\tilde{\pi}(x, \tau_2), \epsilon_2) \cap \overline{\tilde{\pi}(x, [0, t_2])} = \emptyset.$$

Continuing with this process, we obtain $\tau_n > t_n$ and $0 < \epsilon_n < \frac{\epsilon_{n-1}}{2}$, for $n = 3, 4, \ldots$, such that

$$B(\tilde{\pi}(x, \tau_n), \epsilon_n) \subset B(\tilde{\pi}(x, \tau_{n-1}), \epsilon_{n-1}) \quad \text{and} \quad B(\tilde{\pi}(x, \tau_n), \epsilon_n) \cap \overline{\tilde{\pi}(x, [0, t_n])} = \emptyset.$$

The sequence $\{\tilde{\pi}(x, \tau_n)\}_{n\geq 1}$ has the property that

$$d(\tilde{\pi}(x, \tau_n), \tilde{\pi}(x, \tau_{n-1})) < \epsilon_{n-1} \leq \frac{\epsilon}{2^{n-1}}$$

for $n = 2, 3, \ldots$. Thus $\{\tilde{\pi}(x, \tau_n)\}_{n\geq 1}$ is a Cauchy sequence which converges to a point w in X. Since $\tau_n \xrightarrow{n\to+\infty} +\infty$, we have $w \in \tilde{L}^+(x)$. Note also that $w \in B(z, \epsilon)$.

Now, we claim that $w \notin \tilde{\pi}^+(x)$. Suppose the contrary, that is, $w = \tilde{\pi}(x, \lambda)$ for some $\lambda \geq 0$. Note that there is a natural number n such that $t_n > \lambda$ and

$$w \in \tilde{\pi}(x, [0, t_n]).$$

However, by construction $w \in B(\tilde{\pi}(x, \tau_n), \epsilon_n)$, for all $n = 1, 2, \ldots$, and $B(\tilde{\pi}(x, \tau_n), \epsilon_n) \cap \overline{\tilde{\pi}(x, [0, t_n])} = \emptyset$. Thus we have a contradiction. Hence $w \notin \tilde{\pi}^+(x)$ and the theorem is proved. □

The next result follows straightforward from Theorem 3.3.

Theorem 3.4. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system, where* X *is a complete metric space and* $x \in X \setminus M$. *If* $\tilde{\pi}^+(x) = \tilde{L}^+(x)$ *then* $\tilde{\pi}^+(x)$ *is eventually periodic.*

Now, we introduce the concept of non-wandering points for impulsive systems. This concept for dynamical systems is due to Birkhoff, [12].

Definition 3.3. Let $(X, \pi; M, I)$ be an impulsive semidynamical system. A point $x \in X$ is said to be non-wandering, if every neighborhood U of x is self-positively $\tilde{\pi}$-recursive.

Theorem 3.5 below presents some properties of non-wandering points. The proof is analogous to the continuous case. See, for instance, Theorems III.2.12, III.2.13 and III.2.14 from [11].

Theorem 3.5. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system and* $x \in X$.

(1) *The following statements are equivalent:*
  (a) *x is non-wandering;*
  b) *$x \in \tilde{J}^+(x)$.*
(2) *Every $y \in \tilde{L}^+(x)$ is non-wandering.*
(3) *Let $P \subset X$ be such that every $y \in P$ is positively Poisson $\tilde{\pi}$-stable. Then every $y \in \overline{P}$ is non-wandering.*

Given an impulsive semidynamical system $(X, \pi; M, I)$, the set of positively Poisson $\tilde{\pi}$-stable points can be dense in X, when some conditions are fulfilled. This result is presented next.

Theorem 3.6. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system, where* X *is a complete metric space. Suppose the following conditions hold:*

(a) *Every* $x \in X$ *is non-wandering;*
(b) *If* $\widetilde{\pi}(U, t) \cap U \neq \emptyset$, *with* $U \subset X$ *open and* $t \geq 0$, *then* $\widetilde{\pi}(U, t) \cap U$ *contains an open subset.*
(c) *For* $V \subset X$, $\operatorname{diam}(\widetilde{\pi}(V, t)) \leq \operatorname{diam}(V)$ *whenever* $t \geq 0$.

Then the set of positively Poisson $\widetilde{\pi}$*-stable points is dense in* X.

Proof. Let $x \in X$ and $\epsilon > 0$. At first, let us suppose that $x \notin M$. We can assume, without loss of generality, that $B(x, \epsilon) \cap M = \emptyset$. Let $U_0 = B(x, \epsilon)$. Since x is a non-wandering point, there exists $t_1 > 1$ such that

$$\widetilde{\pi}(U_0, t_1) \cap U_0 \neq \emptyset.$$

By condition (b), $\widetilde{\pi}(U_0, t_1) \cap U_0$ contains an open subset. So let $z_1 \in \widetilde{\pi}(U_0, t_1) \cap U_0$ and choose $\epsilon_1 < \frac{1}{2}$ such that

$$U_1 = B(z_1, \epsilon_1) \subset \widetilde{\pi}(U_0, t_1) \cap U_0 \quad \text{and} \quad \overline{U_1} \subset U_0.$$

Since z_1 is a non-wandering point, there exists $t_2 > 2$ such that

$$\widetilde{\pi}(U_1, t_2) \cap U_1 \neq \emptyset.$$

As above, we can take $z_2 \in \widetilde{\pi}(U_1, t_2) \cap U_1$ and $\epsilon_2 < \frac{1}{2^2}$ such that

$$U_2 = B(z_2, \epsilon_2) \subset \widetilde{\pi}(U_1, t_2) \cap U_1.$$

Proceeding as before, given $n \in \mathbb{N}^*$, there are $t_n > n$ and $z_n \in \widetilde{\pi}(U_{n-1}, t_n) \cap U_{n-1}$ such that $U_n = B(z_n, \epsilon_n) \subset \widetilde{\pi}(U_{n-1}, t_n) \cap U_{n-1}$, with $\varepsilon_n < \frac{1}{2^n}$. Since $U_n \supset U_{n+1}$, $n = 0, 1, 2, \ldots$, we have $\overline{U}_n \supset \overline{U}_{n+1}$, $n = 0, 1, 2, \ldots$. Moreover, since *diam*$(\overline{U}_n) \overset{n\to+\infty}{\longrightarrow} 0$ and X is complete, we have $\cap\{\overline{U}_n : n = 0, 1, 2, \ldots\}$ is a singleton $\{y\}$ with $y \in \overline{U}_1 \subset U_0$.

Now, we claim that y is a positively Poisson $\widetilde{\pi}$-stable point. Indeed. Since $\widetilde{\pi}(\overline{U}_{n-1}, t_n) \cap \overline{U}_{n-1}$ is nonempty for each n (because $\widetilde{\pi}(U_{n-1}, t_n) \cap U_{n-1}$ is nonempty for each n) and $y \in \overline{U}_n$ for all $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned}\rho(\widetilde{\pi}(y, t_n), y) &\leq \operatorname{diam}(\widetilde{\pi}(\overline{U}_{n-1}, t_n)) + \operatorname{diam}(\overline{U}_{n-1}) \\ &\leq 2\operatorname{diam}(\overline{U}_{n-1}) \overset{n\to+\infty}{\longrightarrow} 0,\end{aligned}$$

and since $t_n \overset{n\to+\infty}{\longrightarrow} +\infty$, it follows that $y \in \widetilde{L}^+(y)$. Hence, there is $y \in B(x, \epsilon)$ such that $y \in \widetilde{L}^+(y)$.

Now, suppose $x \in M$ and let $\epsilon > 0$ be given. Let $\lambda > 0$ be such that $\widetilde{\pi}(x, \lambda) = \pi(x, \lambda) \in B(x, \epsilon)$. Let $y = \widetilde{\pi}(x, \lambda)$ and $\epsilon_1 > 0$ be such that $B(y, \epsilon_1) \subset B(x, \epsilon)$ and $B(y, \epsilon_1) \cap M = \emptyset$. By the previous case, there exists $z \in B(y, \epsilon_1)$ such that $z \in \widetilde{L}^+(z)$. Consequently, $z \in B(x, \epsilon)$ with $z \in \widetilde{L}^+(z)$.

Then the set of positively Poisson $\widetilde{\pi}$-stable points is dense in X. □

Let X be a topological space, $(X, \pi; M, I)$ be an impulsive semidynamical system and μ be a real-valued measure on X. We say that μ is invariant with respect to $\widetilde{\pi}$ if for every measurable subset $A \subset X$ we have $\mu(A) = \mu(\widetilde{\pi}(A, t))$ for all $t \geq 0$. We say that μ is a positive measure if $\mu(A) \geq 0$ for every measurable set $A \subset X$. And a positive measure μ is normalized if $\mu(X) = 1$. The proof of the next theorem follows as Theorem IX.1 in [15].

Theorem 3.7. *Let* $(X, \pi; M, I)$ *be an impulsive semidynamical system, where* X *is a topological space which has a countable basis of neighborhoods. Let* μ *be a normalized invariant measure on* X. *Then almost every point in* X *is positively Poisson* $\widetilde{\pi}$*-stable, that is if* E *denotes the set of points in* X *that are not positively Poisson* $\widetilde{\pi}$*-stable, then* $\mu(E) = 0$.

Acknowledgements

We thank the anonymous referee for the careful correction and useful suggestions.
The first author was supported by FAPESP 2008/03680-4 and the second author was supported by FAPESP 2008/02879-1.

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