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THE POINCARÉ-BENDIXSON THEOREM ON THE KLEIN BOTTLE FOR CONTINUOUS VECTOR FIELDS

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Abstract. We present a version of the Poincaré-Bendixson Theorem on the Klein bottle K^2 for continuous vector fields. As a consequence, we obtain the fact that K^2 does not admit continuous vector fields having a ω-recurrent injective trajectory.

1. **Introduction.** When investigating the asymptotic behavior of dynamical systems, it is essential to know the structure of the sets where the trajectories go to in the future or where they come from in the past. The theory of Poincaré-Bendixson treats the problem of determining the structure of such sets known as "limit sets".

In the end of the 19th century, H. Poincaré presented a theorem in [16], without giving a complete proof, which describes the structure of the limit sets of trajectories in analytic vector fields on the plane. In 1901, I. Bendixson proved in [3] the statement presented by H. Poincaré under the weaker hypothesis that the vector field is of class C^1. The classic version of the Poincaré-Bendixson Theorem states that if a trajectory is bounded and its limit set does not contain any singular point, then it is a periodic orbit.

After the pioneer works of H. Poincaré and I. Bendixson, several generalizations of the Poincaré-Bendixson Theorem were established. For instance, the theorem was generalized for bidimensional manifolds in [9] and [18] (see also [2]). For continuous flows, there is a result by O. Hajek in [10] and for semiflows, we refer to the result by K. Ciesielski in [5]. In the case of impulsive semiflows, we refer to [4]. A version of the Poincaré-Bendixson Theorem for continuous vector fields on the plane can be found in [11]. This last result can be easily extended to the cylinder.

In 1923, H. Kneser proved in [12] that a continuous flow on the Klein bottle without singular points admits a periodic orbit. This result also holds in the presence of singular points. This fact was proved by S. Kh. Aranson in [1] and, independently, by N. G. Markley in [13]. In fact, Aranson and Markley showed that continuous

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flows on the Klein bottle do not admit non-trivial recurrent trajectories. Later, in 1978, a simplified proof of this fact was presented in [8].

In the present paper, we give a version of the Poincaré-Bendixson Theorem for continuous vector fields on the Klein bottle.

We organize the article as follows. In Section 2, we present the basic definitions and terminology of continuous vector fields on compact bidimensional manifolds. Further in this section, we investigate some properties of the limit sets and we introduce the concept of "weak" recurrence for trajectories. In Section 3, we deal with some topological aspects of the Klein bottle which we present in several auxiliary results. Finally, we present a version of the Poincaré-Bendixson Theorem for continuous vector fields on the Klein bottle in two situations. We assume that the continuous vector field has a finite number of singularities and we consider an injective trajectory. Then we describe the limit set of this trajectory in the case it is "weakly" ω-recurrent (Theorem 3.7) and in the case it is not "weakly" ω-recurrent (Theorem 3.8). As a consequence, we conclude that a continuous vector field on the Klein bottle does not have an injective trajectory which is ω-recurrent in the usual sense.

2. **Continuous vector fields.**

This section is devoted to the presentation of the basis of the theory of continuous vector fields on compact bidimensional manifolds. We also investigate some properties of the limit sets and we introduce a weaker notion of recurrence for trajectories.

2.1. **Basic definitions and terminology.**

Let M be a compact bidimensional manifold. By a compact bidimensional manifold we mean a bidimensional manifold which is compact, connected, boundaryless and of class C^∞.

We denote by $\mathfrak{X}(M)$ the space of continuous vector fields on M. Given $X \in \mathfrak{X}(M)$, we say that a curve $\gamma_p : \mathbb{R} \to M$ of class C^1 is a *solution of* X *through* $p \in M$, whenever $\gamma_p(0) = p$ and $\gamma_p'(t) = X(\gamma_p(t))$ for $t \in \mathbb{R}$. By Peano's Theorem, for each point $p \in M$, there exists at least one solution $\gamma_p : \mathbb{R} \to M$. For simplicity, we identify the curve γ_p with its range $\{\gamma_p(t) : t \in \mathbb{R}\}$ which we refer to as a *trajectory* (or orbit) of X through p. The *positive semi-trajectory* (respectively, the *negative semi-trajectory*) of X through p and contained in γ_p is the set

$$\gamma_p^+ := \{\gamma_p(t) : t \geq 0\} \quad \text{(respectively, the set } \gamma_p^- := \{\gamma_p(t) : t \leq 0\}).$$

Thus $\gamma_p = \gamma_p^+ \cup \gamma_p^-$. An *arc of a trajectory* of X is a connected subset of a trajectory.

We say that $p \in M$ is a *singular point* or a *singularity* (respectively a *regular point*) of X, whenever $X(p) = 0$ (respectively $X(p) \neq 0$). A trajectory is called *regular*, if it does not contain singular points. A trajectory γ is *periodic*, if there exists $\tau > 0$ such that $\gamma(t + \tau) = \gamma(t)$ for all $t \in \mathbb{R}$.

Recall that trajectories of continuous vector fields may cross themselves or each other. If a trajectory crosses itself, then it is clear that it contains a periodic orbit. In the present paper, we assume that all trajectories of continuous vector fields are regular, except for the trajectories which are singular points. Indeed this can be done, without loss of generality, since we can consider another continuous vector field with the same set of singularities and the same phase portrait, but having only regular trajectories, except for the trajectories which are singular points (see the Appendix).

We say that any one-dimensional submanifold Σ of M is *transversal* to the vector field X, if it does not contain any singularity of X and it is transversal to each trajectory of X. A transversal is called a *transversal segment*, if it is homeomorphic to a non-degenerate closed subinterval of $\mathbb{R}$. In this manner, any transversal segment admits a total order "$\leq$" induced by the total order of the interval. We say that Σ is a transversal segment through $p \in M$, if p is not an endpoint of Σ. Thus if Σ is a transversal segment through p, then $\Sigma \setminus \{p\}$ has two connected components.

2.2. **Limit sets and weak recurrence.** Let M be a compact bidimensional manifold and $X \in \mathfrak{X}(M)$. The *ω-limit set* of a trajectory γ of X is the set

$$\omega(\gamma) := \{q \in M : \exists\ (t_n)_{n\in\mathbb{N}} \text{ with } t_n \to \infty \text{ and } \gamma(t_n) \to q,\ \text{ as } n \to \infty\}.$$

Analogously, the *α-limit set* of a trajectory γ is defined by

$$\alpha(\gamma) := \{q \in M : \exists\ (t_n)_{n\in\mathbb{N}} \text{ with } t_n \to -\infty \text{ and } \gamma(t_n) \to q,\ \text{ as } n \to \infty\}.$$

The ω-limit set (respectively the α-limit set) of the trajectory γ is also called the ω-limit set of the positive semi-trajectory $\gamma^+ \subset \gamma$ (respectively the α-limit set of the negative semi-trajectory $\gamma^- \subset \gamma$) and it is denoted by $\omega(\gamma^+)$ (respectively by $\alpha(\gamma^-)$).

In the next lines, we present some properties of the ω-limit set of a positive semi-trajectory. An analogous result holds for the α-limit set of a negative semi-trajectory.

Proposition 1. *Let $X \in \mathfrak{X}(M)$ and $\gamma^+ = \{\gamma(t) : t \geq 0\}$ be a positive semi-trajectory of X. Then the following properties hold:*

(*i*) $\omega(\gamma^+) \neq \varnothing$;
(*ii*) $\omega(\gamma^+)$ *is compact;*
(*iii*) $\omega(\gamma^+)$ *is connected.*

Proof. The proof of this proposition follows similarly to the particular case of continuous flows on compact bidimensional manifolds. See [2, Lemma 1.5, p. 48]. □

Definition 2.1. Let γ and $\widetilde{\gamma}$ be trajectories of $X \in \mathfrak{X}(M)$. We say that $\widetilde{\gamma}$ is shadowed by γ (or γ shadows $\widetilde{\gamma}$), whenever the following conditions hold:

(1) there is an increasing sequence $(t_n)_{n\in\mathbb{N}}$ of real numbers, with $\lim\limits_{n\to\infty} t_n = \infty$;
(2) $\widetilde{\gamma}(t) = \lim\limits_{n\to\infty} \gamma(t + t_n),\ t \in \mathbb{R}$, where the limit is uniform on each compact interval of $\mathbb{R}$.

In what follows, we will prove a result for continuous vector fields which guarantees an "invariance" of the ω-limit set. In the case of continuous flows, the invariance of the ω-limit set follows from the continuous dependence of the trajectory on the initial conditions (see [2, Lemma 1.5, p. 48]). In the case of continuous vector fields, we guarantee that, given a point of the ω-limit set, there exists at least one trajectory through this point and entirely contained in the ω-limit set.

Proposition 2. *Let $X \in \mathfrak{X}(M)$ and γ be a trajectory of X. If $p \in \omega(\gamma)$, then there is a trajectory γ_p of X through p and shadowed by γ. Furthermore, $\gamma_p \subset \omega(\gamma)$.*

Proof. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of real numbers, with $t_n \to \infty$ and $\gamma(t_n) \to p$ as $n \to \infty$. Define $p_n = \gamma(t_n)$, for each $n \in \mathbb{N}$. Then $\gamma_{p_n}(t) = \gamma(t+t_n)$ is a trajectory of X through p_n. By Theorem II 3.2 from [11], we can take an increasing subsequence $(t_{n_k})_{k\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ and a trajectory γ_p of X through p satisfying

$$\gamma_p(t) = \lim_{k\to\infty} \gamma(t + t_{n_k}),\ t \in \mathbb{R},$$

where the limit is taken uniformly on each compact interval of $\mathbb{R}$. This completes the proof. $\square$

Corollary 1. *Under the conditions of Proposition 2*, if $\omega(\gamma)$ consists of a single point $p \in M$, then p is a singular point of X and $\lim_{t\to\infty} \gamma(t) = p$.

Definition 2.2. We say that a trajectory γ of $X \in \mathfrak{X}(M)$ is weakly ω-recurrent (respectively weakly α-recurrent), if there exists $p \in \gamma$ such that $p \in \omega(\gamma)$ (respectively $p \in \alpha(\gamma)$). Under these conditions, we also say that γ is weakly ω-recurrent (respectively weakly α-recurrent) at the point p. A trajectory is weakly recurrent if it is weakly ω-recurrent or weakly α-recurrent. If moreover $\gamma \subset \omega(\gamma)$ (respectively $\gamma \subset \alpha(\gamma)$), then we say that γ is ω-recurrent (respectively α-recurrent).

A singular point and a periodic orbit are weakly recurrent trajectories. In each of these cases, we say that the weakly recurrent trajectory is *trivial*.

3. **Main results.**

This section is divided into two parts. In the first part, we deal with two important results for the Klein bottle K^2. One of them says that the complement of a certain kind of closed curve in K^2 is a cylinder. The other result gives us information about simple closed curves in K^2. In the second part of this section, we present a version of the Poincaré-Bendixson Theorem for continuous vector fields in K^2 (see Theorems 3.7 and 3.8).

3.1. **Topological aspects.**

Definition 3.1. Consider a simple closed curve C on a compact bidimensional manifold M. We say that C is two-sided, if it has a neighborhood which is homeomorphic to a cylinder. Otherwise, we say that C is one-sided.

Note that any one-sided curve has a neighborhood which is homeomorphic to a Möbius strip.

Given a two-sided curve C on M and a neighborhood $N(C)$ of C which is homeomorphic to a cylinder, then $N(C) \setminus C$ has two connected components. We define the two sides of C as these connected components.

The next lemma says that if the complement of a two-sided curve on the Klein bottle K^2 is connected, then it is a cylinder. A proof of this fact using the Poincaré-Hopf Index Theorem (see [17]) was presented in [8]. The proof we present here is different. We only use topological properties of K^2 to get the result.

Lemma 3.2. *Let C be a simple closed curve on the Klein bottle K^2. If C is two-sided and $K^2 \setminus C$ is connected, then $K^2 \setminus C$ is a cylinder.*

Proof. Let V_0 and V_1 be neighborhoods of C which are homeomorphic to a cylinder and such that $\overline{V_0} \subset V_1$. By $\chi(M)$ we mean the Euler characteristic of a manifold M. According to [19, p. 205], we have

$$\chi(K^2) = \chi(K^2 \setminus V_0) + \chi(\overline{V_1}) - \chi((K^2 \setminus V_0) \cap \overline{V_1}).$$

Since $\chi(K^2) = 0$, $\chi(\overline{V_1}) = \chi(S^1 \times [0,1]) = 0$ and $\chi((K^2 \setminus V_0) \cap \overline{V_1}) = \chi((S^1 \times [0,1]) \cup (S^1 \times [0,1])) = 0$ hold, $\chi(K^2 \setminus V_0) = 0$ holds. Thus, since $K^2 \setminus V_0$ is a compact manifold whose boundary consists of two circles, it follows from the classification of manifolds theorem that $K^2 \setminus V_0$ is a cylinder and, hence, $K^2 \setminus C$ is a cylinder. This completes the proof. $\square$

Let M be a compact bidimensional manifold with non-empty boundary and assume that the boundary of M is homeomorphic to a circle. Let D be a closed disc (i.e., D is homeomorphic to $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$). By $M \cup_\partial D$ we mean a compact bidimensional manifold obtained by glueing the boundary of M to the boundary of D. This means that if $h : \partial M \to \partial D$ is a homeomorphism from the boundary of M to the boundary of D, then $M \cup_\partial D$ is the quotient space obtained by identifying the points p with $h(p)$ for all $p \in \partial M$. The type of topology in the manifold $M \cup_\partial D$ depends only on the type of topology in M. For further information, the reader may want to consult [14].

The next result says that the Klein bottle does not admit three closed curves which are simple and disjoint and are one-sided.

Lemma 3.3. *If C_1, C_2 and C_3 are three disjoint simple closed curves on the Klein bottle K^2, then at least one of these curves is two-sided.*

Proof. We will prove that if C_1 and C_2 are one-sided curves, then C_3 is necessarily two-sided.

Let $N(C_1)$ be a neighborhood of C_1 in K^2 which is homeomorphic to a Möbius strip and is such that $N(C_1) \cap C_2 = \varnothing$. Let D_1 be a closed disc. By the classification of manifolds theorem, $[K^2 \setminus N(C_1)] \bigcup_\partial D_1$ is the projective plane P^2. Since C_2 is a one-sided curve in P^2, we can consider a neighborhood $N(C_2)$ of C_2 in P^2 which is homeomorphic to a Möbius strip and is such that $N(C_2) \cap C_3 = \varnothing$. Applying the classification of manifolds theorem again, we conclude that $[P^2 \setminus N(C_2)] \bigcup_\partial D_2$ is the sphere S^2, where D_2 denotes a closed disc. Since all closed curves in S^2 are two-sided and C_3 is contained in S^2, it follows that C_3 is two-sided. ☐

3.2. **The Poincaré-Bendixson Theorem.** Let M be a compact bidimensional manifold and let γ be a trajectory of $X \in \mathfrak{X}(M)$. In what follows, $[p, q]_\gamma$ (respectively $(p, q)_\gamma$) denotes the closed oriented arc (respectively the open oriented arc) of trajectory of X contained in γ, with starting point p and ending point q. The orientation of this arc is that induced by X.

Let Σ be a transversal segment to X and let $p, q \in \Sigma$. By $[p, q]_\Sigma$ we mean the subinterval of Σ with endpoints p and q.

For continuous flows, the fact that every ω-recurrent trajectory on the Klein bottle K^2 is trivial was presented in [8], where the following result by M. Peixoto [15] was applied: *Given an ω-recurrent trajectory γ of a continuous flow, there exists a transversal circle through a point of γ.*

In order to treat ω-recurrent trajectories in the case of continuous vector fields, we will introduce the notion of semi-transversal circle. Then we will prove some results concerning the existence of such circles in K^2.

Definition 3.4. Let γ be a trajectory of $X \in \mathfrak{X}(M)$ which intercepts a transversal segment Σ at points p and q. We say that the curve $[p, q]_\gamma \cup [p, q]_\Sigma$ formed by the union of the arc of trajectory $[p, q]_\gamma$ of X with the subinterval $[p, q]_\Sigma$ of Σ is the semi-transversal circle to X through γ, whenever it is a simple closed two-sided curve.

The next result establishes conditions for the existence of a semi-transversal circle through an injective trajectory of $X \in \mathfrak{X}(K^2)$.

Lemma 3.5. *Let γ be an injective trajectory of $X \in \mathfrak{X}(K^2)$. If γ intercepts a transversal segment Σ at four distinct points, then there exists a semi-transversal circle to X through γ.*

Proof. Let p_1, p_2, p_3 and p_4 be distinct consecutive points of the intersection of γ in Σ. Without loss of generality, we can assume that p_1, p_2, p_3 and p_4 form a monotonous sequence in Σ. Otherwise, the construction of a semi-transversal circle through γ is immediate.

Let us consider, then, the following simple closed curves $[p_1, p_2]_\gamma \cup [p_1, p_2]_\Sigma$, $[p_2, p_3]_\gamma \cup [p_2, p_3]_\Sigma$ and $[p_3, p_4]_\gamma \cup [p_3, p_4]_\Sigma$ (see Figure 1(a)). By slightly perturbing these curves, it is possible to obtain disjoint simple closed curves $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{C}_3$ (see Figure 1(b)). Then, it follows from Lemma 3.3, that at least one of these curves is one-sided. Thus there exists a semi-transversal circle to X through γ and the result follows. □

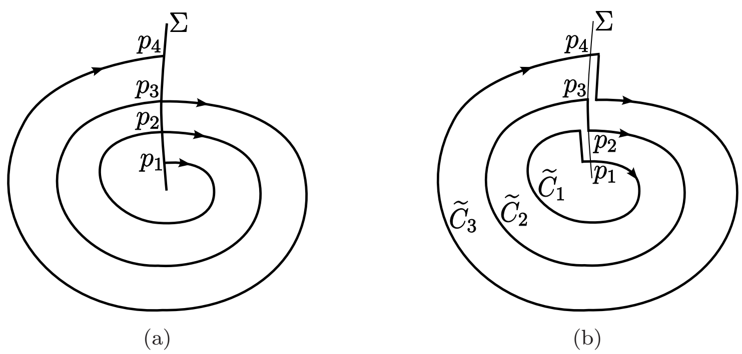

FIGURE 1. Simple closed curves.

Corollary 2. *If γ is a weakly ω-recurrent injective trajectory (respectively weakly α-recurrent) of $X \in \mathfrak{X}(K^2)$, then there exists a semi-transversal circle C to X through γ. Moreover, $K^2 \setminus C$ is connected.*

Proof. Suppose γ is weakly ω-recurrent at $p \in \gamma$ and let Σ be a transversal segment through p. Since $p \in \omega(\gamma)$, the positive semi-trajectory $\gamma_p^+ \subset \gamma$ intercepts Σ at an infinite number of points. Thus, by Lemma 3.5, there exists a semi-transversal circle $C = [p_1, p_2]_\gamma \cup [p_1, p_2]_\Sigma$ to X through γ, where $p_1 \in \gamma_p^+ \subset \gamma$ (see Figure 2).

Suppose p belongs to a connected component of $\Sigma \setminus \{p_2\}$ which contains p_1 (see Figures 2(a) and 2(b)). Let us denote by p_3 the first point at which the positive semi-trajectory $\gamma_{p_2}^+ \subset \gamma$ intercepts this component. Since the arc of trajectory $(p_2, p_3)_\gamma$ is a path joining the two sides of C, without intercepting C, $K^2 \setminus C$ is connected.

In the case where p does not belong to a connected component of $\Sigma \setminus \{p_2\}$ which contains p_1, it is also true that $K^2 \setminus C$ is connected, since the arc of trajectory $(p, p_1)_\gamma$ is a path joining the two sides of C, without intercepting C (see Figure 2(c)). □

Now, we will extend the notion of a graph for continuous vector fields. Then we will present a version of the Poincaré-Bendixson Theorem which says that, in any continuous vector field on the Klein bottle K^2, with a finite number of singularities, the ω-limit set of a weakly ω-recurrent injective trajectory is either a periodic orbit or a graph.

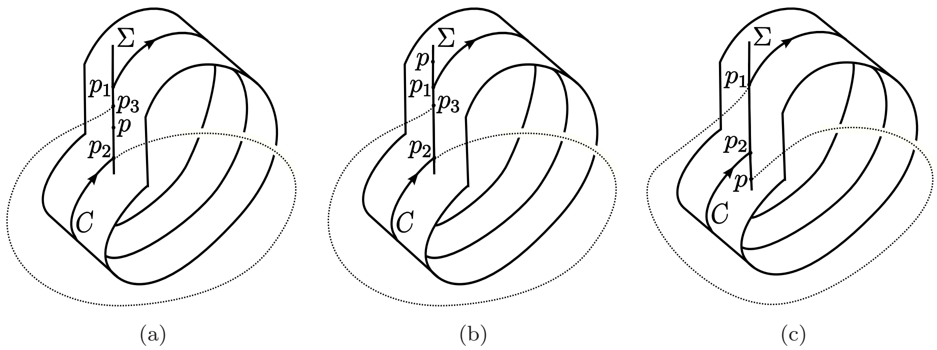

FIGURE 2. The connectivity of $K^2 \setminus C$.

Definition 3.6. A graph for $X \in \mathfrak{X}(M)$ is a connected closed subset of M which has a finite number of singularities and a sequence (finite or infinite) of regular injective trajectories such that the α- and ω-limit sets are singularities.

Theorem 3.7. *Let $X \in \mathfrak{X}(K^2)$ be a continuous vector field with a finite number of singularities and let γ be a weakly ω-recurrent injective trajectory. Then $\omega(\gamma)$ is precisely one of the following sets:*

(*i*) *A periodic orbit;*
(*ii*) *A graph.*

Proof. By Corollary 2, there exists a semi-transversal circle C to X through γ such that $K^2 \setminus C$ is connected. By Lemma 3.2, we can assume that $K^2 \setminus C$ is an annulus, where the boundary circles, which we shall denote by C_1 and C_2, are oriented by the arrows as in Figure 3. Such circles are identified by a homeomorphism $h : C_1 \to C_2$ which preserves orientation.

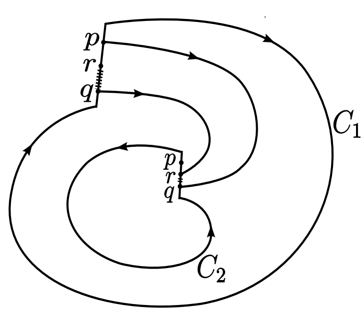

FIGURE 3. Annulus with boundary C_1 and C_2.

We assert that there exists $r \in \gamma$ such that the positive semi-trajectory $\gamma_r^+ \subset \gamma$ stays in the annulus $K^2 \setminus C$ or in a Möbius strip. Indeed, it is enough to prove that if γ intercepts C in three consecutive points p, q and r, then the positive semi-trajectory $\gamma_r^+ \subset \gamma$ stays in a Möbius strip. We denote by $\overline{pq}$ the arc of C with endpoints p and q and containing r.

Since $K^2 \setminus C$ is an annulus and γ is injective, the semi-trajectory $\gamma_r^+ \subset \gamma$ can intercept C only once more in the arc $\overline{rq} \subset \overline{pq}$ with endpoints r and q (see Figure 3). Therefore the semi-trajectory $\gamma_r^+ \subset \gamma$ stays in the Möbius strip (see Figure 4) and the assertion follows.

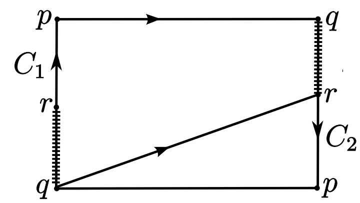

FIGURE 4. Möbius strip.

Since the Poincaré-Bendixson Theorem holds for continuous vector fields on the cylinder (see [11, Theorem 4.2, p. 154]), the above assertion and the fact that the cylinder is an orientable double covering of the Möbius strip imply that $\omega(\gamma)$ is a periodic orbit or a graph and the proof is complete. □

As an immediate consequence of the proof of Theorem 3.7, we obtain a version of Theorem 2 from [8] (see also [1] and [13]) for continuous vector fields. We present such result in the next corollary.

Corollary 3. *Any continuous vector field $X \in \mathfrak{X}(K^2)$ has no ω-recurrent injective trajectory.*

Now, we proceed so as to present a version of the Poincaré-Bendixson Theorem on the Klein bottle K^2 in the case where no injective trajectory in $X \in \mathfrak{X}(K^2)$ is weakly ω-recurrent. The next two results will be used in the proof of this theorem.

Proposition 3. *Let $X \in \mathfrak{X}(M)$ be a continuous vector field on an orientable compact bidimensional manifold M and let γ be an injective trajectory which is not weakly ω-recurrent. If $\widetilde{\gamma}$ is a periodic orbit which is shadowed by γ, then $\omega(\gamma) = \widetilde{\gamma}$.*

Proof. The main ideas of this proof were borrowed from [2, Lemma 1.6, p. 49].

Since M is orientable, $\widetilde{\gamma}$ has a neighborhood U which is homeomorphic to an annulus such that $\widetilde{\gamma}$ separates U into two connected components. Then it follows from the continuity of X that the set of singular points of X is closed and, hence, disjoint from $\widetilde{\gamma}$. Thus we can consider that the neighborhood U is sufficiently small such that is does not contain any singular point of X.

Let Σ be a transversal segment through a point $p \in \widetilde{\gamma}$ such that $\Sigma \subset U$ and $\widetilde{\gamma}$ intercepts Σ only at p. Since $p \in \omega(\gamma)$, γ intercepts Σ infinitely many times. Let p_1 and p_2 be consecutive points of intersection of γ and Σ at increasing times (see Figure 5). Since $\widetilde{\gamma}$ is shadowed by γ, we can take p_1 sufficiently close to p so that the arc of trajectory $[p_1, p_2]_\gamma$ is contained in U. Then, since $\widetilde{\gamma}$ separates U and $\widetilde{\gamma} \cap \gamma = \varnothing$, $[p_1, p_2]_\gamma$ belongs to a unique component of $U \setminus \widetilde{\gamma}$. Therefore the simple closed curve $\Gamma = [p_1, p_2]_\gamma \cup [p_1, p_2]_\Sigma$ does not intercept $\widetilde{\gamma}$ and it is contained in U.

If Γ bounds a simply connected domain D in U, then $D \subset U$ contains a singular point. This fact follows from two known results. See, for instance, [11, Theorem 4.3, p. 155] and [11, Theorem 3.1, p. 150]. But this contradicts the choice of the neighborhood U. Thus the curve Γ is not homotopic to a point in U. Moreover, the union of Γ and $\widetilde{\gamma}$ bounds an annular domain $K \subset U$ (see Figure 5).

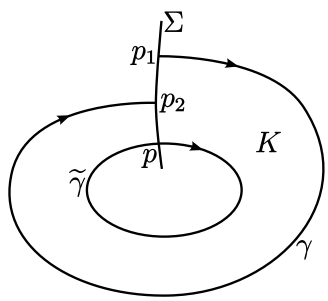

FIGURE 5. The annulus K.

Let Σ_R and Σ_L denote the connected components of $\Sigma \setminus \{p\}$. In order to fix ideas, suppose p_1, $p_2 \in \Sigma_R$. Since p belongs to the ω-limit set of γ, one and only one of the following situations holds:

(a) the positive semi-trajectory $\gamma_{p_2}^+ \subset \gamma$ does not intercept Σ_L;

(b) $\gamma_{p_2}^+ \subset \gamma$ intercepts Σ_L at points which are arbitrarily close to p.

In case (a), we will show that p_2 belongs to Σ and is between p_1 and p. Suppose the contrary, that is, suppose p_1 belongs to Σ and is between p_2 and p. Then $\gamma_{p_2}^+ \subset \gamma$ leaves the annulus K and no longer intercepts it, since in $[p_1, p_2]_\Sigma$ all semi-trajectories leave K and γ is injective (see Figure 6). But in such a case, this means that $\gamma_{p_2}^+ \subset \gamma$ does not intercept Σ in any neighborhood of p, contradicting the fact that $p \in \omega(\gamma)$.

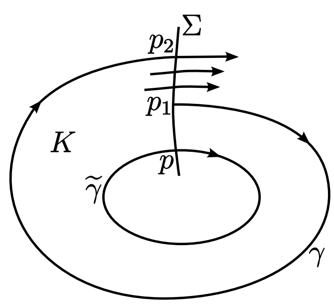

FIGURE 6. p_1 between p_2 and p.

Let us prove, then, the proposition for case (a). By the previous paragraph, we know that p_2 is in Σ between p_1 and p. Let p_3 denote the first point at which $\gamma_{p_2}^+ \subset \gamma$ intercepts Σ. We assert that p_3 is in Σ between p_2 and p. Indeed, since the semi-trajectory $\gamma_{p_2}^+ \subset \gamma$ enters K, it cannot leave K because in $[p_1, p_2]_\Sigma$ all semi-trajectories enter K and γ is injective (see Figure 5). For this reason, $p_3 \notin [p_1, p_2]_\Sigma$ and, hence, $p_3 \in [p_2, p]_\Sigma$. Repeating this procedure several times, one can obtain a sequence $(p_n)_{n \in \mathbb{N}}$ in Σ with the following properties:

(a_1) $p_{n+1} \in \gamma_{p_n}^+ \cap \Sigma$, where $\gamma_{p_n}^+ \subset \gamma$;

(a_2) the point p_{n+1} is in Σ between p_n and p (monotonicity);

(a_3) $\gamma^+_{p_1} \subset \gamma$ does not intercept Σ at any point, except for the points p_n, $n \in \mathbb{N}$ (see Figure 7).

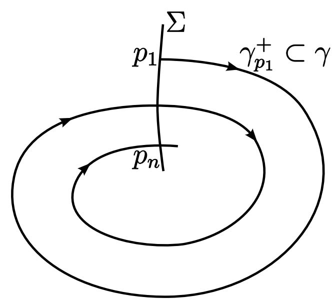

FIGURE 7. The monotone sequence $(p_n)_{n\in\mathbb{N}}$.

It follows from property (a_2) that the sequence $(p_n)_{n\in\mathbb{N}}$ has an accumulation point $q \in \Sigma$. Since $p \in \omega(\gamma)$, $q = p$ holds. Besides, (a_2) and (a_3) imply $\omega(\gamma) \cap \Sigma = \{p\}$.

Since the semi-trajectory $\gamma^+_{p_2} \subset \gamma$ is contained in K, the argument above can be employed and therefore one can conclude that:

(a_4) any transversal segment through a point $\widetilde{p} \in \widetilde{\gamma}$ intercepts $\omega(\gamma)$ only at $\widetilde{p}$.

Using property (a_4), let us suppose $\omega(\gamma) \setminus \widetilde{\gamma} \neq \varnothing$. Then $\omega(\gamma) \setminus \widetilde{\gamma}$ has an accumulation point $\widetilde{p} \in \widetilde{\gamma}$, since $\omega(\gamma)$ is connected. Let $\Sigma_{\widetilde{p}}$ be a transversal segment through $\widetilde{p}$. Using Proposition 2 and the fact that every neighborhood of $\widetilde{p}$ contains points of $\omega(\gamma) \setminus \widetilde{\gamma}$, then there exists a trajectory $\gamma_{p_*} \subset \omega(\gamma) \subset \overline{K}$ through a point $p_* \in \omega(\gamma) \setminus \widetilde{\gamma}$ and intercepting $\Sigma_{\widetilde{p}}$. The point of interception is necessarily $\widetilde{p}$, due to (a_4).

Let $\gamma_{p_*}(\tau) \in \widetilde{\gamma}$, whenever $\gamma_{p_*}(t) \notin \widetilde{\gamma}$ for t between 0 and τ. Consider a transversal segment Σ_* through $\gamma_{p_*}(\tau)$. Then, a small translation of Σ_* to a convenient direction is a transversal segment which intercepts $\widetilde{\gamma}$ and γ_{p_*} at two distinct points (see Figure 8). This contradicts (a_4) and implies the result in case (a).

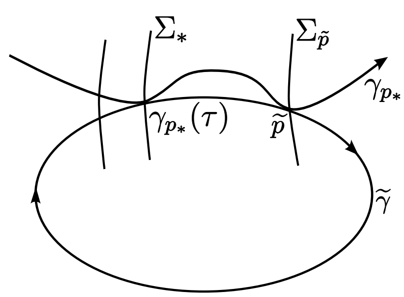

FIGURE 8. The transversal segments $\Sigma_{\widetilde{p}}$ and Σ_*.

Now, we consider case (b). It follows from the previous arguments that, in this case, the semi-trajectory $\gamma^+_{p_2} \subset \gamma$ leaves K and it no longer intercepts K. Similarly to the construction of K, one can construct an annulus $\check{K} \subset U$ bounded by $\widetilde{\gamma}$ and by a simple closed curve formed by the union of an arc $[\check{p}_1, \check{p}_2]_\gamma$ of the semi-trajectory $\gamma^+_{p_2} \subset \gamma$ and the subinterval $[\check{p}_1, \check{p}_2]_\Sigma$ of Σ (see Figure 9). Since $\widetilde{\gamma} \subset \omega(\gamma)$, the semi-trajectory $\gamma^+_{p_2} \subset \gamma$ enters the annulus $\check{K}$ and, therefore, $\check{p}_2$ is between $\check{p}_1$ and p necessarily.

The rest of the proof follows analogously to case (a), with K, p_1 and p_2 replaced by $\check{K}$, $\check{p}_1$ and $\check{p}_2$ respectively. □

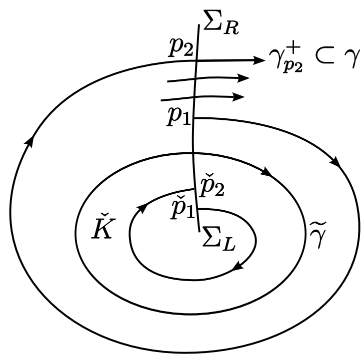

FIGURE 9. The annulus $\check{K}$.

Proposition 4. *Let $X \in \mathfrak{X}(K^2)$ and γ be an injective trajectory which is not weakly ω-recurrent. Suppose the vector field X has a finite number of singularities in $\omega(\gamma)$. If $\widetilde{\gamma}$ is an injective trajectory shadowed by γ, then $\omega(\widetilde{\gamma})$ and $\alpha(\widetilde{\gamma})$ are singularities.*

Proof. Suppose $\omega(\widetilde{\gamma})$ contains a regular point p. Let Σ be a transversal segment through p. Since $p \in \omega(\widetilde{\gamma})$, the positive semi-trajectory $\widetilde{\gamma}^+ \subset \widetilde{\gamma}$ intercepts Σ infinitely many times. Then, by Lemma 3.5, there exists a semi-transversal circle to X through $\widetilde{\gamma}$, say $\widetilde{C} = [p_1, p_2]_{\widetilde{\gamma}} \cup [p_1, p_2]_\Sigma$. Since $\widetilde{\gamma}$ is shadowed by γ, we can construct a semi-transversal circle C to X through γ which is sufficiently close to $\widetilde{C}$, with p_1 or p_2 in C (see Figure 10).

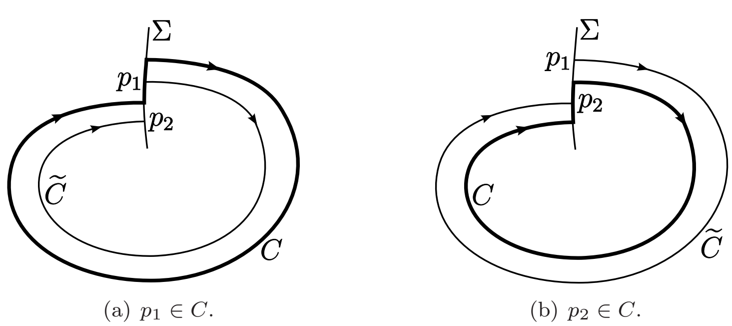

FIGURE 10. The semi-transversal circles C and $\widetilde{C}$.

In order to fix ideas, let us suppose p_1 belongs to C (Figure 10(a)). Since $p_1 \in \omega(\gamma)$, there exist arcs of trajectories of X which are contained in γ and join the two sides of C without intercepting C. Thus $K^2 \setminus C$ is connected.

Now, the argument we used to prove Theorem 3.7 allows us to conclude that $\omega(\gamma)$ is a graph, since $\widetilde{\gamma}$ is a non-periodic trajectory contained in $\omega(\gamma)$. But this contradicts the existence of a regular point p in $\omega(\widetilde{\gamma})$. Therefore, $\omega(\widetilde{\gamma})$ is a singular point.

Analogously, one can prove that $\alpha(\widetilde{\gamma})$ is a singular point which belongs to $\omega(\gamma)$. □

Theorem 3.8. *Let $X \in \mathfrak{X}(K^2)$ be a continuous vector field with a finite number of singularities. If γ is an injective trajectory which is not weakly ω-recurrent, then $\omega(\gamma)$ is exactly one of the following sets:*

(i) *A singularity;*
(ii) *A periodic orbit;*
(iii) *A graph.*

Proof. If $\omega(\gamma)$ does not contain regular points, then $\omega(\gamma)$ is a singleton which is a singularity, since X has a finite number of singularities and $\omega(\gamma)$ is connected.

Now, we suppose $\omega(\gamma)$ contains a regular point p. Consider a transversal segment Σ to X through p. By Lemma 3.5, there exists a semi-transversal circle to X through γ, say $C = [p_1, p_2]_\gamma \cup [p_1, p_2]_\Sigma$.

If C is a curve which is not homotopic to a point in K^2, then we know that $K^2 \setminus C$ is a cylinder (whenever $K^2 \setminus C$ is connected, by Lemma 3.2) or it consists of two Möbius strips (see [6, p. 139]). Then, there exists $r \in \gamma$ such that the positive semi-trajectory $\gamma_r^+ \subset \gamma$ remains in a cylinder or in a Möbius strip. Hence $\omega(\gamma)$ is a periodic orbit or a graph.

If C is a curve which is homotopic to a point in K^2, then C bounds an open disc D in K^2. Note that, in this case, $p \notin C$. If $p \in D$, then $\omega(\gamma)$ is a periodic orbit or a graph (see [11, Theorem 4.2, p. 154]).

In what follows, we can assume that all semi-transversal circles to X through γ are homotopic to a point in K^2 and that p does not belong to any of the open discs that these circles bound. Hence $p \notin D$ (see Figure 11).

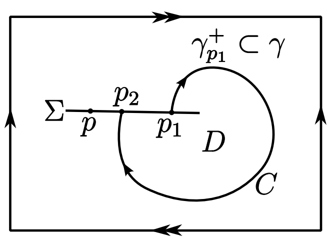

FIGURE 11. The circle C homotopic to a point in K^2.

Since $p \in \omega(\gamma)$, then the positive semi-trajectory $\gamma_{p_1}^+ \subset \gamma$ intercepts Σ at points forming a sequence $(p_n)_{n\in\mathbb{N}}$, where $p_n = \gamma(t_n)$, $(t_n)_{n\in\mathbb{N}}$ is an increasing sequence of real numbers and $p_n \to p$ as $n \to \infty$.

Consider the closed curves

$$C_n = [p_n, p_{n+1}]_\gamma \cup [p_n, p_{n+1}]_\Sigma,\ n = 1, 2, \ldots .$$

We assert that C_n is two-sided for each $n \in \mathbb{N}$. Indeed, suppose, by contradiction, there exists $n_0 \geq 2$ such that C_{n_0} is a one-sided curve. Then we have the following possibilities:

(a) there exists $n > n_0$ such that C_n is a two-sided curve;
(b) C_n is a one-sided curve for all $n \geq n_0$.

Note that case (a) cannot occur, because otherwise C_n would be a semi-transversal circle non-homotopic to a point in K^2 and this contradicts our assumption.

We will show that (b) does not hold as well. If $(p_n)_{n\in\mathbb{N}}$ is not a monotone sequence in Σ, then $\gamma^+_{p_1} \subset \gamma$ intercepts Σ at three points which are non-consecutive. Then we can construct a semi-transversal circle to X through γ at these points, which is not homotopic to a point in K^2, and this contradicts our assumption. Thus $(p_n)_{n\in\mathbb{N}}$ is a sequence which is necessarily monotone. On the other hand, by Lemma 3.3, there are only two disjoint one-sided curves on the Klein bottle K^2. Hence there is no $n_0 \geq 2$ such that C_{n_0} is one-sided.

By the assertion we just proved and from the assumption that all semi-transversal circles to X through γ are homotopic to a point in K^2, it follows that $(p_n)_{n\in\mathbb{N}}$ is a monotone sequence in Σ. Consequently, $\omega(\gamma) \cap \Sigma = p$ and then, by the arbitrary choice of $p \in \omega(\gamma)$, this implies there exists a unique trajectory of X through p and contained in $\omega(\gamma)$, since otherwise there would exist two distinct points of $\omega(\gamma)$ in the same transversal segment and, as a consequence, we would be able to construct a semi-transversal circle to X through γ and non-homotopic to a point in K^2.

Now, if $\omega(\gamma)$ contains a periodic orbit, it follows from Proposition 3 and from the fact that the torus is an orientable double covering of the Klein bottle that $\omega(\gamma)$ is a periodic orbit.

Suppose $\omega(\gamma)$ does not contain any periodic orbit. Then all trajectories of X in $\omega(\gamma)$ are injective. Then, by Proposition 4, $\omega(\gamma)$ is a graph and the proof is complete. $\square$

4. **Appendix.** Let M be a bidimensional compact manifold and let $X \in \mathfrak{X}(M)$ be a continuous vector field. By F we mean the set of all singularities of X, where F may be finte or infinite. In this setting, we have the following result.

Proposition 5. *Under the above conditions, there exists a continuous vector field $Y \in \mathfrak{X}(M)$ whose set of singular points is F and such that:*

(i) *Y and X have the same phase portrait;*
(ii) *all trajectories of Y are regular except for the trajectories which are singular points.*

Proof. By Proposition 3.2 of [7], there exists a family $\{M_n\}_{n\in\mathbb{N}}$ of compact subsets of M such that $\bigcap_{n=1}^{\infty} M_n = F$, where $M_1 = M$ and $M_{n+1} \subset \mathrm{Int}(M_n)$, for all $n \in \mathbb{N}$. Let us define $V_n = \mathrm{Int}(M_n) \setminus M_{n+2}$, $n \in \mathbb{N}$. Clearly $\{V_n\}_{n\in\mathbb{N}}$ is a locally finite open cover of $M \setminus F$. Let $\{\psi_n : M \setminus F \to [0,1] : n \in \mathbb{N}\}$ be a partition of unity strictly subordinate to this cover. For each $n \in \mathbb{N}$, we extend ψ_n to a continuous map $\widetilde{\psi}_n$ in M identifying $\widetilde{\psi}_n(p) = 0$ for every $p \in F$. In this manner, each $\widetilde{\psi}_n X$ belongs to $\mathfrak{X}(M)$.

Let $\rho_0 = 1$, $\rho_1 = 1$ and

$$\rho_n = \mathrm{dist}(M_{n+2}, M \setminus M_n) := \inf\{d(p,q) : p \in M_{n+2},\ q \in M \setminus M_n\},$$

for $n = 2, 3, \ldots$, where $d(.,.)$ denotes the Riemannian metric in M. Let $\|.\|$ denote the norm defined by this metric. For each $n \in \mathbb{N}$, we choose a positive real number c_n such that

$$\|c_n\widetilde{\psi}_n X\| < \min\Big\{\frac{1}{2^n}, \frac{\rho_{n-1}}{3}, \frac{\rho_n}{3}, \frac{\rho_{n+1}}{3}\Big\}.$$

Since

$$\sum_{n=1}^{\infty}\|c_n\widetilde{\psi}_n X\| \leq \sum_{n=1}^{\infty}\frac{1}{2^n} < \infty,$$

the series $\sum_{n=1}^{\infty} c_n\widetilde{\psi}_n X$ converges to a continuous vector field $Y \in \mathfrak{X}(M)$ as required in (i).

It remains to prove that Y satisfies (ii). But, for each $n = 2, 3, \ldots$, we have

$$\begin{aligned}\left\|\left(\sum_{n=1}^{\infty} c_n\widetilde{\psi}_n X\right)\Big|_{V_n}\right\| &= \left\|\Big(c_{n-1}\widetilde{\psi}_{n-1}X + c_n\widetilde{\psi}_n X + c_{n+1}\widetilde{\psi}_{n+1}X\Big)\Big|_{V_n}\right\| \\ &\leq \|c_{n-1}\widetilde{\psi}_{n-1}X\| + \|c_n\widetilde{\psi}_n X\| + \|c_{n+1}\widetilde{\psi}_{n+1}X\| \\ &\leq \rho_n.\end{aligned}$$

And this implies $\|Y|_{V_n}\| \leq \rho_n$.

Let γ be a trajectory of Y such that $\gamma(0) \in \overline{M \setminus M_n}$ and $\gamma(\tau) \in M_{n+2}$, where $\tau > 0$ and $n = 2, 3, \ldots$. Then

$$\rho_n \leq \|\gamma(\tau) - \gamma(0)\| = \left\|\int_0^{\tau}\gamma'(s)ds\right\| = \left\|\int_0^{\tau} Y(\gamma(s))ds\right\| \leq \tau\rho_n.$$

Thus $\tau \geq 1$, whenever $\gamma(0) \in \overline{M \setminus M_n}$ and $\gamma(\tau) \in M_{n+2}$, for each $n = 2, 3, \ldots$. Therefore we can conclude that γ attains a singularity only at infinity by the definition of F, since $M_{n+1} \subset \mathrm{Int}(M_n)$, for all $n \in \mathbb{N}$. This proves (ii). $\square$

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