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Oscillation by Impulses for a Second-Order Delay Differential Equation

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Abstract—We consider a certain second-order nonlinear delay differential equation and prove that the all solutions oscillate when proper impulse controls are imposed. An example is given. © 2006 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In recent years, there has been an increasing interest on the oscillatory behavior of second-order nonlinear delay differential equation. For example, see the recent papers [1–6]. However, there are only a few papers on second-order nonlinear delay differential equations with impulses. See, for instance, [7,8]. For the general theory of impulsive ordinary differential equations, the reader is referred to the book [9] and to some results on the oscillatory behavior of some second-order nonlinear impulsive ordinary differential equations, please see [10–12].

Some nonimpulsive delay differential equations are nonoscillatory, but they may become oscillatory if some proper impulse controls are added to them. The purpose of this paper is then to study the oscillatory behavior of solutions of a second-order nonlinear delay differential equations with impulses.

In [12], He and Ge study the oscillatory behavior of the following second-order nonlinear impulsive ordinary differential equation:

$$\begin{gathered} \left(r(t)\left(x'(t)\right)^{\sigma}\right)' + f(t,x(t)) = 0, \qquad t \ge t_0, \quad t \ne t_k, \\ x\left(t_k^{+}\right) = I_k(x(t_k)), \quad x'\left({t_k}^{+}\right) = J_k\left(x'(t_k)\right), \qquad k = 1,2\dots, \end{gathered} \tag{1}$$

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where $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim_{k\to\infty} t_k = +\infty$ and σ is any quotient of positive odd integers.

In [7], Peng and Ge prove an oscillation theorem for the second-order delay differential equation with impulses

$$
\begin{aligned}
(r(t)(x'(t))^{\sigma})' + f(t, x(t), x(t-\tau)) = 0, \qquad t \geq t_0, \quad t \neq t_k,\\
x\left(t_k^+\right) = I_k(x(t_k)), \quad x'(t_k{}^+) = J_k(x'(t_k)), \qquad k = 1, 2 \dots,\\
x(t) = \phi(t), \qquad t_0 - \tau \leq t \leq t_0,
\end{aligned}
\tag{2}
$$

where $\tau > 0$, $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim_{k\to\infty} t_k = +\infty$ and $t_{k+1} - t_k > \tau$.

In this paper, we adapt the techniques applied by the authors in [7] and [12] to prove that the equation

$$
\begin{aligned}
x''(t) + f(t, x(t), x'(t)) + g(t, x(t), x(t-\tau)) = 0, \qquad t \geq t_0, \quad t \neq t_k,\\
x(t_k) = I_k(x(t_k{}^-)), \quad x'(t_k) = J_k(x'(t_k{}^-)), \qquad k = 1, 2 \dots,\\
x(t) = \phi(t), \qquad t_0 - \tau \leq t \leq t_0,
\end{aligned}
\tag{3}
$$

oscillates, where $\tau > 0$, $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$ with $\lim_{k\to\infty} t_k = +\infty$ and $t_{k+1} - t_k > \tau$.

While in [7] and [12] the authors prove their results provided a solution exists, we assume that f and g are dominated by continuous functions (see (H_1) and (H_2) below) in order to guarantee the existence of a global (forward) solution of problem (3). The other assumptions are similar to theirs.

Our paper is organized as follows. In Section 2, we present a lemma that plays an important role in the proof of the main result. In Section 3, we obtain the oscillatory behavior of (3) through impulse controls. An example is given in Section 4.

2. PRELIMINARIES

Consider the impulsive differential equation

$$
\begin{aligned}
x''(t) + f(t, x(t), x'(t)) + g(t, x(t), x(t-\tau)) = 0, \qquad t \geq t_0, \quad t \neq t_k,\\
x(t_k) = I_k(x(t_k{}^-)), \quad x'(t_k) = J_k(x'(t_k{}^-)), \qquad k = 1, 2 \dots,
\end{aligned}
\tag{4}
$$

satisfying the initial value condition

$$
x(t) = \phi(t), \qquad t_0 - \tau \leq t \leq t_0, \tag{5}
$$

where $\phi, \phi' : [t_0 - \tau, t_0] \to \mathbb{R}$ have at most a finite number of discontinuities of first kind and are right continuous at these points. We assume that

(H_1) $f : [t_0 - \tau, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, nonnegative and $f(t, u, v) \leq z(t)$, for all $u, v \in \mathbb{R}$, where $z(t)$ is continuous in $[t_0 - \tau, \infty)$;

(H_2) $g : [t_0 - \tau, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $ug(t, u, v) > 0$, for all $uv > 0$ and

$$
\frac{g(t, u, v)}{\varphi(v)} \geq p(t) \qquad \text{and} \qquad \frac{g(t, u, v)}{v} \leq q(t),
$$

for all $v \neq 0$, where $p(t)$ and $q(t)$ are continuous in $[t_0 - \tau, \infty)$, $p(t) \geq 0$, $x\varphi(x) > 0$, for all $x \neq 0$ and $\varphi'(x) \geq 0$;

(H_3) I_k, $J_k : \mathbb{R} \to \mathbb{R}$ are continuous, $I_k(0) = J_k(0) = 0$, $k \in \mathbb{N}$ and there exist positive numbers a_k, b_k, c_k and d_k such that

$$
a_k \leq \frac{I_k(x)}{x} \leq b_k, \quad c_k \leq \frac{J_k(x)}{x} \leq d_k, \qquad x \neq 0, \quad k = 1, 2, \dots;
$$

(H$_4$)

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{c_k}{b_k}\,ds=+\infty.$$

Now we define a solution of the impulsive problem (4),(5).

DEFINITION 2.1. *A function $x(t):[t_0-\tau,+\infty)\to\mathbb{R}$ is a solution of problem (4),(5) if*

(i) *$x(t)$ and $x'(t)$ are continuous on $[t_0,+\infty)\setminus\{t_k;k\in\mathbb{N}\}$, there exist lateral limits $x(t_k^-)$, $x'(t_k^-)$, $x(t_k^+)$, $x'(t_k^+)$ with $x(t_k^+)=x(t_k)$ and $x'(t_k^+)=x'(t_k)$, $k\in\mathbb{N}$;*
(ii) *$x(t)$ fulfills (4),(5);*
(iii) *$x(t_k)$ and $x'(t_k)$ fulfill (4), for each $k\in\mathbb{N}$.*

By PC$([a,b],\mathbb{R}^n)$ we mean the Banach space of piecewise right continuous functions $\psi:[a,b]\to\mathbb{R}^n$ with the usual supremum norm. If $x\in$ PC$([t_0-\tau,\sigma],\mathbb{R}^n)$, where $t_0\in\mathbb{R}$, $\sigma\ge t_0$, then for each $t\in[t_0,\sigma]$ we define $x_t\in$ PC$([-\tau,0],\mathbb{R}^n)$ by $x_t(s)=x(t+s)$ for $-\tau\le s\le 0$. We denote by $C([a,b],\mathbb{R}^n)$ the subspace of PC$([a,b],\mathbb{R}^n)$ of continuous functions with the induced norm.

REMARK 2.1. By using the transformation $y(t)=x'(t)$, the nonimpulsive equation in (4) can be transformed into the following system:

$$\begin{aligned}&x'(t)=y(t),\\&y'(t)=-f(t,x(t),y(t))-g(t,x(t),x(t-\tau)),\qquad t\ge t_0.\end{aligned}\tag{6}$$

Consider the function $F:[t_0,+\infty)\times\mathbb{R}^3\to\mathbb{R}$ given by

$$F(t,x_0,x_1,x_2)=f(t,x_0,x_2)+g(t,x_0,x_1).$$

If $\psi\in$ PC$([-\tau,0],\mathbb{R}^2)$, $\psi=(\psi_1,\psi_2)$, we define

$$h(t,\psi)=(\psi_2(0),-F(t,\psi_1(0),\psi_1(-\tau),\psi_2(0))).$$

Then system (6) with the impulsive conditions can be reduced to the system

$$\begin{aligned}&z'(t)=h(t,z_t),\qquad t\ge t_0,\quad t\ne t_k,\\&z(t_k)=H_k(z(t_k{}^-)),\end{aligned}\tag{7}$$

where $z(t)=(x(t),y(t))$, $z_t=(x_t,y_t)$ and $H_k(z(t_k{}^-))=(I_k(x(t_k{}^-),J_k(x'(t_k{}^-))$.

In this way, under Hypotheses (H$_1$) to (H$_3$), in particular the dominance of f and g, imply the global existence of solutions of (7) by [13, Theorem 3.1]. Therefore, we can guarantee that there is a solution of (4) in $[t_0,+\infty)$.

Now we define an oscillatory solution of the impulsive problem (4),(5).

DEFINITION 2.2. *A solution of (4),(5) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it is called oscillatory.*

Now we present a lemma which is a version of Theorem 1.4.1 in [9] replacing the left continuity by the right continuity of $m(t)$ and $m'(t)$ at t_k, $k\in\mathbb{N}$.

LEMMA 2.1. *Suppose*

(i) *the sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies $0\le t_0<t_1<\cdots<t_k<\cdots$ with $\lim_{k\to\infty}t_k=+\infty$,*
(ii) *$m,m':\mathbb{R}_+\to\mathbb{R}$ are continuous on $\mathbb{R}_+\setminus\{t_k;k\in\mathbb{N}\}$, there exist the lateral limits $m(t_k^-)$, $m'(t_k^-)$, $m(t_k^+)$, $m'(t_k^+)$ and $m(t_k^+)=m(t_k)$, $k=1,2,\ldots$,*
(iii) *for $k=1,2,\ldots$ and $t\ge t_0$, we have*

$$m'(t)\le p(t)m(t)+q(t),\qquad t\ne t_k,\tag{8}$$

$$m(t_k)\le d_k m(t_k{}^-)+b_k,\tag{9}$$

where $p, q \in C(\mathbb{R}_+, \mathbb{R})$, d_k and b_k are real constants with $d_k \geq 0$. Then the following inequality holds:

$$m(t) \leq m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)\,ds\right) + \int_{t_0}^{t} \prod_{s<t_k<t} d_k \exp\left(\int_{s}^{t} p(u)\,du\right) q(s)\,ds$$
$$+ \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)\,ds\right) b_k, \qquad t \geq t_0. \tag{10}$$

REMARK 2.2. If inequalities (8) and (9) are reversed, then inequality (10) is also reversed.

3. MAIN RESULT

In this section, we will show that every solution of (4),(5) is oscillatory under hypotheses (H_1) to (H_4).

In the sequel, let $x(t)$ be a solution of (4),(5).

LEMMA 3.1. *Suppose (H_1) to (H_4) are fulfilled and there exists $T \geq t_0$ such that $x(t) > 0$ for $t \geq T - \tau$. Then $x'(t_k) \geq 0$ and $x'(t) \geq 0$ for $t \in [t_k, t_{k+1})$, where $t_k \geq T$.*

PROOF. Suppose $x(t) > 0$, for $t \geq T - \tau$. Then $x(t - \tau) > 0$, $t \geq T$. At first, we prove that $x'(t_k^-) \geq 0$, $t_k \geq T$. If otherwise, there exists some $t_j \geq T$ such that $x'(t_j^-) < 0$. From (H_3) and (4), we obtain

$$x'(t_j) = J_j(x'(t_j^-)) \leq c_j x'(t_j^-) < 0.$$

Let $x'(t_j) = -\alpha$, $\alpha > 0$. By (H_1) and (H_2), given $t \in [t_j, t_{j+1})$, we have

$$x''(t) \leq -g(t, x(t), x(t-\tau)) \leq -p(t)\varphi(x(t-\tau)) \leq 0.$$

Then $x'(t)$ is nondecreasing in $t \in [t_j, t_{j+1})$. Moreover,

$$\begin{aligned}
&x'(t_{j+1}^-) \leq x'(t_j) = -\alpha < 0,\\
&x'(t_{j+2}^-) \leq x'(t_{j+1}) = J_{j+1}(x'(t_{j+1}^-)) \leq c_{j+1} x'(t_{j+1}^-) \leq c_{j+1}(-\alpha) < 0,\\
&x'(t_{j+3}^-) \leq x'(t_{j+2}) = J_{j+2}(x'(t_{j+2}^-)) \leq c_{j+2} x'(t_{j+2}^-) \leq c_{j+2} c_{j+1}(-\alpha) < 0,
\end{aligned}$$

and by induction one can prove that

$$x'(t_{j+n}^-) \leq -\prod_{i=1}^{n-1} c_{j+i}\alpha < 0. \tag{11}$$

Hence, $x'(t)$ is decreasing in $[t_j, +\infty)$.

We now consider the impulsive differential inequalities

$$\begin{aligned}
x''(t) &\leq 0, && t > t_j, \quad t \neq t_k, \quad k = j+1, j+2, \dots,\\
x'(t_k) &\leq c_k x'(t_k^-), && k = j+1, j+2, \dots.
\end{aligned}$$

By Lemma 2.1 with $m(t) = x'(t)$, we have

$$m(t) \leq m(t_j^-) \prod_{t_j<t_k<t} c_k,$$

that is,

$$x'(t) \leq x'(t_j^-) \prod_{t_j<t_k<t} c_k. \tag{12}$$

Now considering (12) and knowing that $x(t_k) = I_k(x(t_k^-)) \le b_k x(t_k^-)$, $k = j+1, j+2, \dots$, by Lemma 2.1, we conclude that

$$x(t) \le \prod_{t_j<t_k<t} b_k \left[x(t_j^-) + x'(t_j^-) \int_{t_j}^{t} \prod_{t_j<t_k<s} \frac{c_k}{b_k} \, ds \right]. \tag{13}$$

By (H_4) and taking j sufficiently large, we find $x(t) \le 0$. But this is a contradiction, since $x(t) > 0$, for $t \ge T - \tau$. Therefore, $x'(t_k^-) \ge 0$, $t_k \ge T$.

It follows from (H_3) that $x'(t_k) \ge c_k x'(t_k^-) \ge 0$ for any $t_k \ge T$. Because $x'(t)$ is decreasing in $[t_k, t_{k+1})$, then $x'(t) \ge x'(t_k) \ge 0$, $t \in [t_k, t_{k+1})$, $t_k \ge T$ and the proof is complete. ∎

Remark 3.1. When $x(t)$ is eventually negative, under hypotheses (H_1) to (H_4), one can prove in a similar way that $x'(t_k) \le 0$ and $x'(t) \le 0$ for $t \in [t_k, t_{k+1})$, where $t_k \ge T$.

Theorem 3.1. *Suppose (H_1) to (H_4) are fulfilled and there exists a positive integer k_0 such that $a_k \ge 1$, for all $k \ge k_0$. If*

$$\sum_{k=0}^{+\infty} \int_{t_k}^{t_{k+1}} \prod_{t_0<t_k<u} \frac{1}{d_k} p(u) \, du = +\infty, \tag{14}$$

then all solutions of (4),(5) oscillate.

Proof. We suppose, without loss of generality, that $k_0 = 1$. Let $x(t)$ be a nonoscillatory solution of (4),(5). We can assume that $x(t) > 0$, $t \ge t_0$. By Lemma 3.1, $x'(t) \ge 0$ and $x'(t_k) \ge 0$, $t \in [t_k, t_{k+1})$, where $t_k \ge t_0$.

By (H_3) and the fact that $a_k \ge 1$, $k = 1, 2, \dots$, we obtain

$$x(t_0) < x(t_1^-) \le x(t_1) \le x(t_2^-) \le \cdots .$$

It follows that $x(t)$ is nondecreasing in $[t_0, +\infty)$.

Now let

$$m(t) = \frac{x'(t)}{\varphi(x(t-\tau))}. \tag{15}$$

Then $m(t_k) \ge 0$ and $m(t) \ge 0$, $t \ge t_0$. By (H_1) and equation (4), we have

$$\begin{aligned} m'(t) &= \frac{-f(t, x(t), x'(t)) - g(t, x(t), x(t-\tau))}{\varphi(x(t-\tau))} - \frac{x'(t)\varphi'(x(t-\tau))x'(t-\tau)}{\varphi^2(x(t-\tau))} \\ &\le -p(t), \qquad t \ge t_0, \quad t \ne t_k, \quad t_k + \tau. \end{aligned}$$

It follows from (H_3), equation (4), $a_k \ge 1$ and $\varphi'(x) \ge 0$ that

$$m(t_k) = \frac{x'(t_k)}{\varphi(x(t_k - \tau))} \le \frac{d_k x'(t_k^-)}{\varphi(x(t_k^- - \tau))} = d_k m(t_k^-) \tag{16}$$

and

$$m(t_k + \tau) = \frac{x'(t_k + \tau)}{\varphi(x(t_k))} \le \frac{x'(t_k^- + \tau)}{\varphi(a_k x(t_k^-))} \le \frac{x'(t_k^- + \tau))}{\varphi(x(t_k^-))} = m(t_k^- + \tau). \tag{17}$$

Then using (16) and (17), by Lemma 2.1, we obtain

$$m(t) \le m(s) \prod_{s<t_k<t} d_k - \int_s^t \prod_{u<t_k<t} d_k \, p(u) \, du, \qquad t_0 \le s \le t. \tag{18}$$

Let $s \to t_0$ and $t \to t_1^-$. It follows from (16) and (18) that

$$m(t_1) \leq d_1 m(t_1^-) \leq d_1 \left[m(t_0) - \int_{t_0}^{t_1} p(u)\, du \right] = d_1 m(t_0) - d_1 \int_{t_0}^{t_1} p(u)\, du.$$

Similarly, knowing that $t_2 - t_1 > \tau$ and using (17) and the above inequality, we get

$$\begin{aligned} m(t_2) \leq d_2 m(t_2^-) &\leq d_2 \left[m(t_1+\tau) - \int_{t_1+\tau}^{t_2} p(u)\, du \right] \\ &\leq d_2 \left[m(t_1^- +\tau) - \int_{t_1+\tau}^{t_2} p(u)\, du \right] \\ &\leq d_2 \left[m(t_1) - \int_{t_1}^{t_2} p(u)\, du \right] \\ &\leq d_2 d_1 m(t_0) - d_2 d_1 \int_{t_0}^{t_1} p(u)\, du - d_2 \int_{t_1}^{t_2} p(u)\, du. \end{aligned}$$

By induction, we obtain

$$\begin{aligned} m(t_n) &\leq d_1 d_2 \cdots d_n m(t_0) - d_1 d_2 \cdots d_n \int_{t_0}^{t_1} p(u)\, du - d_2 \cdots d_n \int_{t_1}^{t_2} p(u)\, du \\ &\quad - \cdots - d_{n-1} d_n \int_{t_{n-2}}^{t_{n-1}} p(u)\, du - d_n \int_{t_{n-1}}^{t_n} p(u)\, du \\ &= \prod_{t_0 < t_k < t_{n+1}} d_k \left[m(t_0) - \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \prod_{t_0 < t_k < u} \frac{1}{d_k} p(u)\, du \right]. \end{aligned}$$

Then in view of (14) and $m(t_n) \geq 0$, we find a contradiction as $n \to +\infty$, and the proof is finished. ∎

With the next corollaries, we intend to show that inequality (14) is fulfilled. We use Theorem 3.1 to conclude the results.

Corollary 3.1. *Suppose* (H_1) *to* (H_4) *are fulfilled and there exists a positive integer* k_0 *such that* $a_k \geq 1$ *and* $d_k \leq 1$, *for all* $k \geq k_0$. *If*

$$\int_{t_0}^{+\infty} p(u)\, du = +\infty,$$

then all solutions of (4),(5) oscillate.

Proof. Suppose, without loss of generality, that $k_0 = 1$. Since $1/d_k \geq 1$, we have

$$\begin{aligned} \sum_{k=0}^{+\infty} \int_{t_k}^{t_{k+1}} \prod_{t_0 < t_k < u} \frac{1}{d_k} p(u)\, du &= \lim_{n \to +\infty} \left(\sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \prod_{t_0 < t_k < u} \frac{1}{d_k} p(u)\, du \right) \\ &= \lim_{n \to +\infty} \left(\int_{t_0}^{t_1} p(u)\, du + \int_{t_1}^{t_2} \frac{1}{d_1} p(u)\, du + \int_{t_2}^{t_3} \frac{1}{d_1 d_2} p(u)\, du + \cdots \right. \\ &\quad \left. + \int_{t_{n-1}}^{t_n} \frac{1}{d_1 d_2 \cdots d_{n-1}} p(u)\, du + \int_{t_n}^{t_{n+1}} \frac{1}{d_1 d_2 \cdots d_n} p(u)\, du \right) \\ &\geq \lim_{n \to +\infty} \left(\int_{t_0}^{t_1} p(u)\, du + \int_{t_1}^{t_2} p(u)\, du + \int_{t_2}^{t_3} p(u)\, du + \cdots \right. \\ &\quad \left. + \int_{t_{n-1}}^{t_n} p(u)\, du + \int_{t_n}^{t_{n+1}} p(u)\, du \right) \\ &= \lim_{n \to +\infty} \left(\int_{t_0}^{t_{n+1}} p(u) du \right) = +\infty. \end{aligned}$$

Then condition (14) is satisfied. Hence, by Theorem 3.1, all solutions of the impulsive system (4),(5) oscillate. ∎

COROLLARY 3.2. *Suppose* (H_1) *to* (H_4) *are fulfilled and there exist a positive integer* k_0 *and a constant* $\alpha > 0$ *such that* $a_k \geq 1$ *and* $1/d_k \geq t_{k+1}^\alpha$, *for all* $k \geq k_0$. *If*

$$\int_{t_1}^{+\infty} t^\alpha p(t)\,dt = +\infty,$$

then all solutions of (4),(5) oscillate.

PROOF. Suppose, without loss of generality, that $k_0 = 1$ and $t_1 \geq 1$. Since $1/d_k \geq t_{k+1}^\alpha$, for all $k \geq k_0$, we obtain

$$1 \leq t_1 < \cdots < t_k < t_{k+1} < \cdots$$

and

$$\begin{aligned} \frac{1}{d_1} &\geq t_2^\alpha, \\ \frac{1}{d_1}\frac{1}{d_2} &\geq t_2^\alpha\, t_3^\alpha \geq t_3^\alpha, \ldots, \\ \frac{1}{d_1}\frac{1}{d_2}\cdots\frac{1}{d_n} &\geq t_2^\alpha\, t_3^\alpha \cdots t_{n+1}^\alpha \geq t_{n+1}^\alpha, \ldots. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{k=0}^{+\infty}\int_{t_k}^{t_{k+1}} \prod_{t_0<t_k<u} \frac{1}{d_k} p(u)\,du &= \lim_{n\to+\infty}\left(\sum_{k=0}^{n}\int_{t_k}^{t_{k+1}} \prod_{t_0<t_k<u}\frac{1}{d_k}p(u)\,du\right) \\ &= \lim_{n\to+\infty}\left(\int_{t_0}^{t_1} p(u)\,du + \int_{t_1}^{t_2}\frac{1}{d_1}p(u)\,du + \int_{t_2}^{t_3}\frac{1}{d_1 d_2}p(u)\,du + \cdots\right. \\ &\quad \left. + \int_{t_{n-1}}^{t_n}\frac{1}{d_1 d_2\cdots d_{n-1}}p(u)\,du + \int_{t_n}^{t_{n+1}}\frac{1}{d_1 d_2\cdots d_n}p(u)\,du\right) \\ &\geq \lim_{n\to+\infty}\left(\int_{t_0}^{t_1} p(u)\,du + \int_{t_1}^{t_2} t_2^\alpha p(u)\,du + \int_{t_2}^{t_3} t_3^\alpha p(u)\,du + \cdots\right. \\ &\quad \left. + \int_{t_{n-1}}^{t_n} t_n^\alpha p(u)\,du + \int_{t_n}^{t_{n+1}} t_{n+1}^\alpha p(u)\,du\right) \\ &\geq \lim_{n\to+\infty}\left(\int_{t_1}^{t_2} u^\alpha p(u)\,du + \int_{t_2}^{t_3} u^\alpha p(u)\,du + \cdots\right. \\ &\quad \left. + \int_{t_{n-1}}^{t_n} u^\alpha p(u)\,du + \int_{t_n}^{t_{n+1}} u^\alpha p(u)\,du\right) \\ &= \lim_{n\to+\infty}\left(\int_{t_1}^{t_{n+1}} u^\alpha p(u)\,du\right) = \int_{t_1}^{+\infty} u^\alpha p(u)\,du = +\infty. \end{aligned}$$

Hence, condition (14) is satisfied and Theorem 3.1 implies all solutions of (4),(5) oscillate. ∎

THEOREM 3.2. *Suppose* (H_1) *to* (H_4) *are fulfilled and* $\varphi(ab) \geq \varphi(a)\varphi(b)$, *for any* $ab \neq 0$. *If*

$$\sum_{k=0}^{+\infty}\int_{t_k}^{t_{k+1}} \prod_{t_0<t_k<u} \frac{\varphi(a_k)}{d_k} p(u)\,du = +\infty, \tag{19}$$

then all solutions of (4),(5) oscillate.

Proof. Let $x(t)$ be a nonoscillatory solution of (4),(5). We can assume that $x(t) > 0$, $t \geq t_0$. By Lemma 3.1, $x'(t) \geq 0$ and $x'(t_k) \geq 0$, $t \in [t_k, t_{k+1})$, where $t_k \geq t_0$. Now let $m(t)$ be defined by (15). Then $m(t_k) \geq 0$ and $m(t) \geq 0$, $t \geq t_0$. By (H_1) and equation (4), we have

$$m'(t) \leq -p(t), \qquad t \geq t_0, \quad t \neq t_k, \quad t_k + \tau.$$

It follows from (H_3), equation (4), $\varphi(ab) \geq \varphi(a)\varphi(b)$ and $\varphi'(x) \geq 0$ that

$$m(t_k) = \frac{x'(t_k)}{\varphi(x(t_k - \tau))} \leq \frac{d_k x'(t_k^-)}{\varphi(x(t_k^- - \tau))} = d_k m(t_k^-) \tag{20}$$

and

$$\begin{aligned} m(t_k + \tau) &= \frac{x'(t_k + \tau)}{\varphi(x(t_k))} \leq \frac{x'(t_k^- + \tau)}{\varphi(a_k x(t_k^-))} \\ &\leq \frac{x'(t_k^- + \tau))}{\varphi(a_k)\varphi(x(t_k^-))} = \frac{1}{\varphi(a_k)} m(t_k^- + \tau). \end{aligned} \tag{21}$$

Then using (20) and (21), by Lemma 2.1, we obtain

$$m(t) \leq m(s) \prod_{s<t_k<t} d_k - \int_s^t \prod_{u<t_k<t} d_k\, p(u)\, du, \qquad t_0 \leq s \leq t. \tag{22}$$

Let $s \to t_0$ and $t \to t_1^-$. It follows from (21) and (22) that

$$m(t_1) \leq d_1 m(t_1^-) \leq d_1 \left[m(t_0) - \int_{t_0}^{t_1} p(u)\, du \right] = d_1 m(t_0) - d_1 \int_{t_0}^{t_1} p(u)\, du.$$

Similarly, knowing that $t_2 - t_1 > \tau$ and using (21) and the above inequality, we get

$$\begin{aligned} m(t_2) \leq d_2 m(t_2^-) &\leq d_2 \left[m(t_1 + \tau) - \int_{t_1+\tau}^{t_2} p(u)\, du \right] \\ &\leq d_2 \left[\frac{1}{\varphi(a_1)} m(t_1^- + \tau) - \int_{t_1+\tau}^{t_2} p(u)\, du \right] \\ &\leq d_2 \left[\frac{1}{\varphi(a_1)} m(t_1) - \int_{t_1}^{t_2} p(u)\, du \right] \\ &\leq \frac{d_2 d_1}{\varphi(a_1)} m(t_0) - \frac{d_2 d_1}{\varphi(a_1)} \int_{t_0}^{t_1} p(u)\, du - d_2 \int_{t_1}^{t_2} p(u)\, du. \end{aligned}$$

Then by induction, we obtain

$$\begin{aligned} m(t_n) \leq{} & \frac{d_1 d_2 \cdots d_n}{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-1})} \left[m(t_0) - \int_{t_0}^{t_1} p(u)\, du - \frac{\varphi(a_1)}{d_1} \int_{t_1}^{t_2} p(u)\, du \right. \\ & \left. - \cdots - \frac{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-2})}{d_1 d_2 \cdots d_{n-2}} \int_{t_{n-2}}^{t_{n-1}} p(u)\, du - \frac{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-1})}{d_1 d_2 \cdots d_{n-1}} \int_{t_{n-1}}^{t_n} p(u)\, du \right] \\ ={} & \frac{d_1 d_2 \cdots d_n}{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-1})} \left[m(t_0) - \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \prod_{t_0<t_k<u} \frac{\varphi(a_k)}{d_k} p(u)\, du \right]. \end{aligned}$$

But in view of (19) and $m(t_n) \geq 0$, we find a contradiction as $n \to +\infty$, and the proof is finished. ∎

Corollary 3.3. *Suppose (H_1) to (H_4) are fulfilled and there exist a positive integer k_0 and a constant $\alpha > 0$ such that $\varphi(a_k)/d_k \geq t_{k+1}^{\alpha}$, for all $k \geq k_0$. If*

$$\int_{t_1}^{+\infty} t^{\alpha} p(t)\, dt = +\infty,$$

then all solutions of (4),(5) oscillate.

The proof of Corollary 3.3 is omitted, since it can be deduced from Theorem 3.2 and it is similar to that of Corollary 3.2.

4. AN EXAMPLE

Consider the impulsive delay differential equation

$$\begin{aligned} x''(t) + x(t-\tau) + \arctan|x'(t)| = 0, \qquad & t \geq 0, \quad t \neq t_k, \\ x(t_k) = \left(\frac{k+1}{k}\right) x(t_k^-), \quad x'(t_k) = x'(t_k^-), \qquad & k = 1, 2, \ldots, \\ x(t) = \phi(t), \qquad & -\tau \leq t \leq 0, \end{aligned} \tag{23}$$

where $t_{k+1} - t_k > \tau$, $k = 1, 2, \ldots$ and $\phi, \phi' : [-\tau, 0] \to \mathbb{R}$ are continuous.

Since $\varphi(v) = v$, $p(t) = 1$, $a_k = b_k = (k+1)/k$ and $c_k = d_k = 1$, $k = 1, 2, \ldots$, hypotheses (H_1) to (H_3) are satisfied. Notice that

$$\begin{aligned} \lim_{t\to+\infty} \int_{t_0}^{t} \prod_{t_0<t_k<s} \frac{c_k}{b_k}\, ds &= \int_{t_0}^{+\infty} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds \\ &= \int_{t_0}^{t_1} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \int_{t_1}^{t_2} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds \\ &\quad + \int_{t_2}^{t_3} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \cdots \\ &= (t_1 - t_0) + \frac{1}{2}(t_2 - t_1) + \frac{1}{3}(t_3 - t_2) + \cdots \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty. \end{aligned}$$

Thus, (H_4) is also satisfied.

Let $k_0 = 1$. Then $a_k \geq 1$ and $d_k = 1$ for all $k \geq 1$. And since

$$\int_{t_0}^{+\infty} p(u)\, du = \int_{0}^{+\infty} du = +\infty,$$

it follows from Corollary 3.1 that all solutions $x(t)$ of (23) oscillate.

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