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Limit sets and the Poincaré–Bendixson Theorem in impulsive semidynamical systems

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Abstract

We consider semidynamical systems with impulse effects at variable times and we discuss some properties of the limit sets of orbits of these systems such as invariancy, compactness and connectedness. As a consequence we obtain a version of the Poincaré–Bendixson Theorem for impulsive semidynamical systems.

MSC: 34A37; 37B25; 54H20

Keywords: Impulsive semidynamical systems; Limit sets; Poincaré–Bendixson Theorem

1. Introduction

The theory of impulsive semidynamical systems is an important and modern chapter of the theory of topological dynamical systems. Interesting and important results about this theory have been studied such as “minimality,” “invariancy,” “recurrence,” “periodic orbits,” “stability” and “flows of characteristic 0^+.” For details of this theory, see [2–5,7,8] and [9], for instance.

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In [6], the author presents the theory of Poincaré–Bendixson for non-impulsive two-dimensional semiflows. A natural question that arises is how the Poincaré–Bendixson theory can be described in impulsive semidynamical systems.

In the present paper, we give results about limit sets for impulsive semidynamical systems of type $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$, where X is a metric space, (X, π) is a semidynamical system, Ω is an open set in X, $\mathrm{M} = \partial\Omega$ denotes the impulsive set and $\mathrm{I} : \mathrm{M} \to \Omega$ is the impulse operator. Our goal, however, is to establish the Poincaré–Bendixson Theorem in this setting.

In the next lines, we describe the organization of the paper and the main results.

In the first part of this paper, we present the basis of the theory of impulsive semidynamical systems. In Section 2.1, we give some basic definitions and notations about impulsive semidynamical systems. In Section 2.2, we discuss the continuity of a function which describes the times of meeting impulsive sets. In Section 2.3, we give some additional useful definitions.

The second part of the paper concerns the main results. Section 3.1 deals with various properties about limit sets. An important fact here is that we consider the closure of the trajectories in X rather than in Ω as presented in [8]. Thus, our impulsive system encompasses the one presented in [8]. Indeed, some new phenomena can occur. We begin this section with an example which shows that the impulsive set can "destroy" the invariancy of a limit set in an impulsive semidynamical system. In the sequel, we study the invariancy, compactness and connectedness of limit sets in impulsive semidynamical systems with a finite numbers of impulses. Then we consider the more general case when the system presents infinitely many impulses and we obtain analogous results. In the end of this section, we present an important theorem that will be fundamental in the proofs of the results of Section 3.2. This theorem concerns an impulsive semidynamical systems $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$, where $\overline{\Omega}$ is compact and $x \in \Omega$, and it says that if a trajectory through x has infinitely many impulses, $\{x_n\}_{n\geqslant 1}$, with $x_n \xrightarrow{n\to+\infty} p$, then the limit set of x in $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$, $\widetilde{\mathrm{L}}^+(x)$, is the union of a periodic orbit and the point $\{p\}$.

In Section 3.2, we discuss a version of the Poincaré–Bendixson Theorem for impulsive semidynamical systems. The main result states that given an impulsive semidynamical system $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ and $x \in \Omega$, if we suppose $\overline{\Omega}$ is compact and $\widetilde{\mathrm{L}}^+(x)$ admits neither rest points nor initial points, then $\widetilde{\mathrm{L}}^+(x)$ is a periodic orbit.

2. Preliminaries

For the sake of selfcontainedness of the paper, we present the basic definitions and notation of the theory of impulsive semidynamical systems we need. We also include some fundamental results which are necessary for understanding the theory.

2.1. Basic definitions and terminology

Let X be a metric space and $\mathbb{R}_+$ be the set of non-negative real numbers. The triple $(\mathrm{X}, \pi, \mathbb{R}_+)$ is called a *semidynamical system*, if the function $\pi : \mathrm{X} \times \mathbb{R}_+ \to \mathrm{X}$ is continuous with $\pi(x, 0) = x$ and $\pi(\pi(x, t), s) = \pi(x, t + s)$, for all $x \in \mathrm{X}$ and $t, s \in \mathbb{R}_+$. We denote such a system by $(\mathrm{X}, \pi, \mathbb{R}_+)$ or simply (X, π). When $\mathbb{R}_+$ is replaced by $\mathbb{R}$ in the definition above, the triple $(\mathrm{X}, \pi, \mathbb{R})$ is a *dynamical system*. For every $x \in \mathrm{X}$, we consider the continuous function $\pi_x : \mathbb{R}_+ \to \mathrm{X}$ given by $\pi_x(t) = \pi(x, t)$ and we call it the *motion* of x.

Let (X, π) be a semidynamical system. Given $x \in \mathrm{X}$, the *positive orbit* of x is given by $\mathrm{C}^+(x) = \{\pi(x,t)\colon t \in \mathbb{R}_+\}$ which we also denote by $\pi^+(x)$. For $t \geqslant 0$ and $x \in \mathrm{X}$, we define $\mathrm{F}(x,t) = \{y\colon \pi(y,t) = x\}$ and, for $\Delta \subset [0,+\infty)$ and $D \subset \mathrm{X}$, we define

$$\mathrm{F}(D,\Delta) = \bigcup \big\{\mathrm{F}(x,t)\colon x \in D \text{ and } t \in \Delta\big\}.$$

Then a point $x \in \mathrm{X}$ is called an *initial point*, if $\mathrm{F}(x,t) = \emptyset$ for all $t > 0$.

Now we define semidynamical systems with impulse action. An *impulsive semidynamical system* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ consists of a semidynamical system, (X, π), an open set Ω in X, a non-empty closed subset $\mathrm{M} = \partial\Omega$ of X and a continuous function $\mathrm{I} : \mathrm{M} \to \Omega$ such that for every $x \in \mathrm{M}$, there exists $\varepsilon_x > 0$ such that

$$\mathrm{F}\big(x,(0,\varepsilon_x)\big) \cap \mathrm{M} = \emptyset \quad \text{and} \quad \pi\big(x,(0,\varepsilon_x)\big) \cap \mathrm{M} = \emptyset.$$

The points of M are isolated in every trajectory of the system (X, π). The set M is called the *impulsive set*, the function I is called the *impulse function* and we write $\mathrm{N} = \mathrm{I}(\mathrm{M})$. We also define

$$\mathrm{M}^+(x) = \big(\pi^+(x) \cap \mathrm{M}\big) \setminus \{x\}.$$

Given an impulsive semidynamical systems $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ and $x \in \mathrm{X}$ with $\mathrm{M}^+(x) \neq \emptyset$, it is always possible to find a smallest number s such that the trajectory $\pi_x(t)$ for $0 < t < s$ does not intercept the set M. This result is stated next and a proof of it follows analogously to the proof of Lemma 2.1 from [2].

Lemma 2.1. *Let* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system. Then for every* $x \in \mathrm{X}$, *there is a positive number* s, $0 < s \leqslant +\infty$, *such that* $\pi(x,t) \notin \mathrm{M}$, *whenever* $0 < t < s$, *and* $\pi(x,s) \in \mathrm{M}$ *if* $\mathrm{M}^+(x) \neq \emptyset$.

Note that here, in the present paper, Ω is any open set in X.

Let $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $x \in \mathrm{X}$. By means of Lemma 2.1, it is possible to define a function $\phi : \mathrm{X} \to (0,+\infty]$ in the following manner

$$\phi(x) = \begin{cases} s, & \text{if } \pi(x,s) \in \mathrm{M} \text{ and } \pi(x,t) \notin \mathrm{M} \text{ for } 0 < t < s, \\ +\infty, & \text{if } \mathrm{M}^+(x) = \emptyset. \end{cases}$$

This means that $\phi(x)$ is the least positive time for which the trajectory of x meets M. Thus for each $x \in \mathrm{X}$, we call $\pi(x,\phi(x))$ the *impulsive point* of x.

The *impulsive trajectory* of x in $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ is an X-valued function $\tilde{\pi}_x$ defined on the subset $[0,s)$ of $\mathbb{R}_+$ (s may be $+\infty$). The description of such a trajectory follows inductively as described in the following lines.

If $\mathrm{M}^+(x) = \emptyset$, then $\tilde{\pi}_x(t) = \pi(x,t)$, for all $t \in \mathbb{R}_+$, and $\phi(x) = +\infty$. However if $\mathrm{M}^+(x) \neq \emptyset$, it follows from Lemma 2.1 that there is a smallest positive number s_0 such that $\pi(x,s_0) = x_1 \in \mathrm{M}$ and $\pi(x,t) \notin \mathrm{M}$, for $0 < t < s_0$. Then we define $\tilde{\pi}_x$ on $[0,s_0]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x,t), & 0 \leqslant t < s_0, \\ x_1^+, & t = s_0, \end{cases}$$

where $x_1^+ = \mathrm{I}(x_1)$ and $\phi(x) = s_0$.

Since $s_0 < +\infty$, the process now continues from x_1^+ onward. If $\mathrm{M}^+(x_1^+) = \emptyset$, then we define $\tilde{\pi}_x(t) = \pi(x_1^+, t - s_0)$, for $s_0 \leqslant t < +\infty$, and $\phi(x_1^+) = +\infty$. When $\mathrm{M}^+(x_1^+) \neq \emptyset$, it follows again from Lemma 2.1 that there is a smallest positive number s_1 such that $\pi(x_1^+, s_1) = x_2 \in \mathrm{M}$ and $\pi(x_1^+, t - s_0) \notin \mathrm{M}$, for $s_0 < t < s_0 + s_1$. Then we define $\tilde{\pi}_x$ on $[s_0, s_0 + s_1]$ by

$$\tilde{\pi}_x(t) = \begin{cases} \pi(x_1^+, t - s_0), & s_0 \leqslant t < s_0 + s_1, \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

where $x_2^+ = \mathrm{I}(x_2)$ and $\phi(x_1^+) = s_1$, and so on. Hence $\tilde{\pi}_x$ is defined on $[0, t_{n+1}]$.

The process above ends after a finite number of steps, whenever $\mathrm{M}^+(x_n^+) = \emptyset$ for some n. Or it continues infinitely, if $\mathrm{M}^+(x_n^+) \neq \emptyset$, $n = 1, 2, 3, \ldots$, and if $\tilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} s_i$.

It worths noticing as well that given $x \in \mathrm{X}$, one of the three properties hold:

(i) $\mathrm{M}^+(x) = \emptyset$ and hence the trajectory of x has no discontinuities.
(ii) For some $n \geqslant 1$, each x_k^+, $k = 1, 2, \ldots, n$, is defined and $\mathrm{M}^+(x_n^+) = \emptyset$. In this case, the trajectory of x has a finite number of discontinuities.
(iii) For all $k \geqslant 1$, x_k^+ is defined and $\mathrm{M}^+(x_k^+) \neq \emptyset$. In this case, the trajectory of x has infinitely many discontinuities.

Let $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. Given $x \in \mathrm{X}$, the *impulsive positive orbit* of x is defined by the set

$$\widetilde{\mathrm{C}}^+(x) = \{\tilde{\pi}(x, t) \colon t \in \mathbb{R}_+\},$$

which we also denote by $\tilde{\pi}^+(x)$. We denote the closure of $\widetilde{\mathrm{C}}^+(x)$ in X by $\widetilde{\mathrm{K}}^+(x)$.

Analogously to the non-impulsive case, we have standard properties presented in the next proposition whose proof follows straightforward from the definition. See [3, Proposition 2.1].

Proposition 2.1. *Let* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system and* $x \in \mathrm{X}$. *The following properties hold*:

(i) $\tilde{\pi}(x, 0) = x$,
(ii) $\tilde{\pi}(\tilde{\pi}(x, t), s) = \tilde{\pi}(x, t + s)$, *for all* $t, s \in [0, T(x))$ *such that* $t + s \in [0, T(x))$.

2.2. Semicontinuity and continuity of ϕ

The result of this section is borrowed from [4] and concerns the function ϕ defined previously. The function ϕ indicates the moments of impulse action of a trajectory in an impulsive system and the result is applied sometimes intrinsically in the proofs of the main theorems of the next section.

Let (X, π) be a semidynamical system. Any closed set $\mathrm{S} \subset \mathrm{X}$ containing x $(x \in \mathrm{X})$ is called a *section* or a λ*-section* through x, with $\lambda > 0$, if there exists a closed set $\mathrm{L} \subset \mathrm{X}$ such that

(a) $\mathrm{F}(\mathrm{L}, \lambda) = \mathrm{S}$;
(b) $\mathrm{F}(\mathrm{L}, [0, 2\lambda])$ is a neighborhood of x;
(c) $\mathrm{F}(\mathrm{L}, \mu) \cap \mathrm{F}(\mathrm{L}, \nu) = \emptyset$, for $0 \leqslant \mu < \nu \leqslant 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a *λ-tube* and the set L is called a *bar*.

Let (X, π) be a semidynamical system. We now present the conditions (TC) and (STC) for a tube.

Any tube $F(L, [0, 2\lambda])$ given by a section S through $x \in X$ such that $S \subset M \cap F(L, [0, 2\lambda])$ is called *TC-tube* on x. We say that a point $x \in M$ fulfills the *Tube Condition* and we write (TC), if there exists a TC-tube $F(L, [0, 2\lambda])$ through x. In particular, if $S = M \cap F(L, [0, 2\lambda])$ we have an *STC-tube* on x and we say that a point $x \in M$ fulfills the *Strong Tube Condition* (we write (STC)), if there exists an STC-tube $F(L, [0, 2\lambda])$ through x.

The following theorem concerns the continuity of ϕ and it can be proved in a similar way to Theorem 3.8 from [4]. Recall that here Ω is any open set in X.

Theorem 2.1. *Consider an impulsive semidynamical system* $(X, \pi; \Omega, M, I)$. *Assume that no initial point in* (X, π) *belongs to the impulsive set* M *and that each element of* M *satisfies the condition* (TC). *Then* ϕ *is continuous at* x *if and only if* $x \notin M$.

2.3. Additional definitions

Let us consider the metric space X with metric ρ. By $B(x, \delta)$ we mean the open ball with center at $x \in X$ and ratio δ. Let $B(A, \delta) = \{x \in X\colon \rho_A(x) < \delta\}$ and $B[A, \delta] = \{x \in X\colon \rho_A(x) \leqslant \delta\}$, where $\rho_A(x) = \inf\{\rho(x, y)\colon y \in A\}$. Throughout this paper, we use the notation ∂A and $\overline{A}$ to denote respectively the boundary and closure of a set A in X.

In what follows, $(X, \pi; \Omega, M, I)$ is an impulsive semidynamical system and $x \in X$.

We define the *limit set* of x in $(X, \pi; \Omega, M, I)$ by

$$\widetilde{L}^+(x) = \{y \in X\colon \tilde{\pi}(x, t_n) \to y, \text{ for some } t_n \to +\infty\}.$$

The *prolongational limit set* of x in $(X, \pi; \Omega, M, I)$ is given by

$$\widetilde{J}^+(x) = \{y \in X\colon \tilde{\pi}(x_n, t_n) \to y, \text{ for some } x_n \to x \text{ and } t_n \to +\infty\};$$

and the *prolongation set* of x in $(X, \pi; \Omega, M, I)$ is defined by

$$\widetilde{D}^+(x) = \{y \in X\colon \tilde{\pi}(x_n, t_n) \to y, \text{ for some } x_n \to x \text{ and } t_n \in [0, +\infty)\}.$$

For the case of semidynamical systems without impulses, we denote by $L^+(x)$, $J^+(x)$ and $D^+(x)$ respectively the limit set, the prolongational limit set and the prolongation set of a point $x \in X$.

For a set $K \subset X$ we consider $\widetilde{L}^+(K) = \bigcup\{\widetilde{L}^+(x)\colon x \in K\}$.

Let $A \subset X$. We say that A is *minimal* in $(X, \pi; \Omega, M, I)$, whenever $A = \widetilde{K}^+(x)$ for each $x \in A \setminus M$. (This definition is due to S.K. Kaul (see [9]).) If $\tilde{\pi}^+(A) \subset A$, we say that A is $\tilde{\pi}$-*invariant*.

A point $x \in X$ is called:

- a *stationary* or *rest point* with respect to $\tilde{\pi}$, if $\tilde{\pi}(x, t) = x$ for all $t \geqslant 0$,
- a *periodic point* with respect to $\tilde{\pi}$, if $\tilde{\pi}(x, t) = x$ for some $t > 0$ and x is not stationary,
- a *regular point*, if it is neither a rest point nor a periodic point.

For results concerning the stability and invariancy of sets in an impulsive system, the reader may want to consult [2,3,5,8].

3. The main results

We divide this section into two parts. The first part concerns some properties about limit sets. In the second part, we consider a version of the Poincaré–Bendixson Theorem for impulsive systems.

Let $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system. We assume that each element of M satisfies the condition (TC) and that no initial point in (X, π) belongs to the impulsive set M, that is, given $x \in \mathrm{M}$ one always has $y \in \mathrm{X}$ and $t \in \mathbb{R}_+$ such that $\pi(y, t) = x$. Under these conditions, ϕ is always continuous on $\mathrm{X} \setminus \mathrm{M}$ (Theorem 2.1).

3.1. Limit sets in impulsive semidynamical systems

A lot of typical properties of non-impulsive dynamical systems are not present in systems subject to impulse effects. On the other hand, impulsive systems can present interesting and unexpected phenomena as "beating," "dying," "merging," noncontinuation of solutions, etc.

In this section, we discuss some properties of the limit set of an orbit in an impulsive semidynamical systems. We are mainly concerned with semiflows $\tilde{\pi}_x$, $x \in \mathrm{X}$, defined on $[0, +\infty)$.

In [8], the author considers impulsive semidynamical systems $(\mathrm{X}\pi; \Omega, \mathrm{M}, \mathrm{I})$ with the property that $\widetilde{\mathrm{K}}^+(x)$ is the closure of $\widetilde{\mathrm{C}}^+(x)$ in Ω. The author defines

$$\widetilde{\mathrm{L}}^+(x) = \bigl\{y \in \Omega\colon\ \tilde{\pi}(x, t_n) \to y,\ \text{for some } t_n \to +\infty\bigr\},$$
$$\widetilde{\mathrm{J}}^+(x) = \bigl\{y \in \Omega\colon\ \tilde{\pi}(x_n, t_n) \to y,\ \text{for some } x_n \to x \text{ and } t_n \to +\infty\bigr\}$$

and

$$\widetilde{\mathrm{D}}^+(x) = \bigl\{y \in \Omega\colon\ \tilde{\pi}(x_n, t_n) \to y,\ \text{for some } x_n \to x \text{ and } t_n \in [0, +\infty)\bigr\}.$$

Note that the elements of $\widetilde{\mathrm{L}}^+(x)$, $\widetilde{\mathrm{J}}^+(x)$ and $\widetilde{\mathrm{D}}^+(x)$ are taken in Ω. Therefore these sets are closed in Ω.

The definition of impulsive semidynamical system we use here is more general than the one presented in [8], because we consider the closure of $\widetilde{\mathrm{C}}^+(x)$ in X and the elements of $\widetilde{\mathrm{L}}^+(x)$, $\widetilde{\mathrm{J}}^+(x)$ and $\widetilde{\mathrm{D}}^+(x)$ belong to X and not necessarily to Ω. Due to this apparently slight difference in the definitions, a new phenomenon that is not present in the impulsive systems considered by [8] can occur in our impulsive systems. For instance, in [8, Lemma 2.6], the author proves that given an impulsive semidynamical system $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ and $x \in \Omega$, the limit set $\widetilde{\mathrm{L}}^+(x)$ is closed and $\tilde{\pi}$-invariant. But, in our case, $\widetilde{\mathrm{L}}^+(x)$ is not necessarily $\tilde{\pi}$-invariant. The next example clarifies this aspect.

Example 3.1. Consider the impulsive differential system in $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x}_1 = x_1, \\ \dot{x}_2 = 0, \\ \mathrm{I}: \mathrm{M} \to \mathrm{N}, \end{cases} \tag{1}$$

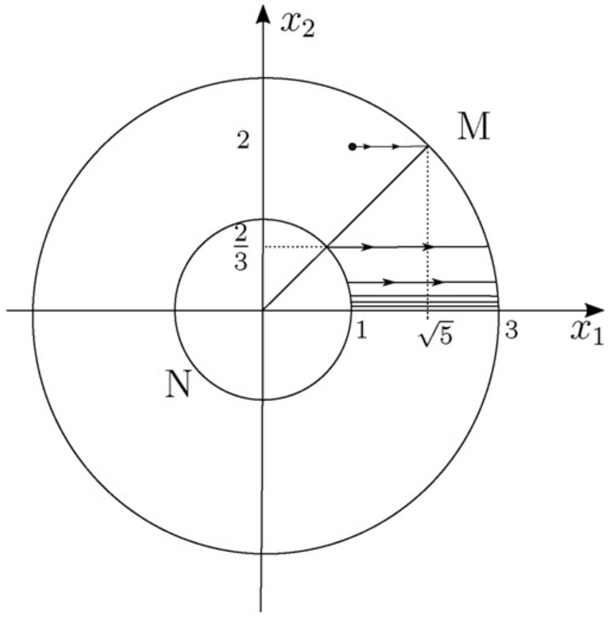

Fig. 1. Trajectories of the impulsive system (1).

where $\mathrm{M} = \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_1^2 + x_2^2 = 9\}$, $\mathrm{N} = \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_1^2 + x_2^2 = 1\}$ and the operator I assigns to every point $x \in \mathrm{M}$ a point $y \in \mathrm{N}$ which is on the ray joining x to the origin in $\mathbb{R}^2$.

The trajectory of the system

$$\begin{cases} \dot{x}_1 = x_1, \\ \dot{x}_2 = 0, \end{cases} \tag{2}$$

with initial condition $(x_1(0), x_2(0)) = (1, 2)$ is $(x_1(t), x_2(t)) = (e^t, 2)$ for all $t \geqslant 0$. At time $t = \ln\sqrt{5}$, the solution of the system (2) meets the circle M. Then the impulse operator I transfers the point $(\sqrt{5}, 2)$ to $\mathrm{I}(\sqrt{5}, 2) = (\frac{\sqrt{5}}{3}, \frac{2}{3})$. Thus the solution of the impulsive system (1), with initial condition $(x_1(0), x_2(0)) = (1, 2)$, on the interval $[0, \ln\sqrt{5}]$ is given by

$$\tilde{x}(t) = \begin{cases} (e^t, 2), & 0 \leqslant t < \ln\sqrt{5}, \\ (\frac{\sqrt{5}}{3}, \frac{2}{3}), & t = \ln\sqrt{5}. \end{cases}$$

Now the solution of system (1) starts at $t = \ln\sqrt{5}$ in $(\frac{\sqrt{5}}{3}, \frac{2}{3})$. Let us consider the system (2) with initial condition $(x_1(0), x_2(0)) = (\frac{\sqrt{5}}{3}, \frac{2}{3})$. Then, the trajectory of the system (2) is $(x_1(t), x_2(t)) = (\frac{\sqrt{5}}{3}e^t, \frac{2}{3})$. At $t = \ln\sqrt{\frac{77}{5}}$, the solution of (2) with initial condition $(\frac{\sqrt{5}}{3}, \frac{2}{3})$ meets the circle M. Again, the operator I transfers the point $(\frac{\sqrt{77}}{3}, \frac{2}{3})$ to the point $(\frac{\sqrt{77}}{9}, \frac{2}{9})$. Thus the solution $\tilde{x}(t)$ of (1) is defined for $\ln\sqrt{5} \leqslant t \leqslant \ln\sqrt{\frac{77}{5}} + \ln\sqrt{5} = \ln\sqrt{77}$ and it is given by

$$\tilde{x}(t) = \begin{cases} (\frac{\sqrt{5}}{3}e^{(t - \ln\sqrt{5})}, \frac{2}{3}), & \ln\sqrt{5} \leqslant t < \ln\sqrt{77}, \\ (\frac{\sqrt{77}}{9}, \frac{2}{9}), & t = \ln\sqrt{77}. \end{cases}$$

The evolution process above continues indefinitely as shown by Fig. 1.

Let $P_0 = (x_1(0), x_2(0)) = (1, 2)$ and $P_1 = (1, 0)$. Then the limit set $\widetilde{\mathrm{L}}^+(P_0)$ equals $[1, 3] \times \{0\} = \widetilde{\mathrm{C}}^+(P_1) \cup \{(3, 0)\}$, where $\widetilde{\mathrm{C}}^+(P_1)$ is a periodic orbit of period 2. Note that considering the point $(3, 0) \in \widetilde{\mathrm{L}}^+(P_0)$, we have $\widetilde{\mathrm{C}}^+((3, 0)) = [3, +\infty) \times \{0\}$ which means that $\widetilde{\mathrm{L}}^+(P_0)$ is not $\tilde{\pi}$-invariant.

This example shows that the impulsive set M can "destroy" the invariancy of an orbit if one considers the closure of a set in X rather than in Ω. The next result solves this problem and its proof follows the steps of Lemma 2.4 from [8].

Lemma 3.1. *Given an impulsive semidynamical system* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *and a* $\tilde{\pi}$*-invariant set* $A \subset \Omega$*, then* $\overline{A} \cap \Omega$ *is* $\tilde{\pi}$*-invariant.*

Remark 3.1. Since $\widetilde{\mathrm{L}}^+(x)$, $x \in \Omega$, may not be $\tilde{\pi}$-invariant, it follows that $\widetilde{\mathrm{J}}^+(x)$ and $\widetilde{\mathrm{D}}^+(x)$ are not necessarily $\tilde{\pi}$-invariant.

The following result will be very useful. It is a new version of Lemma 2.5 from [8]. Here we consider X as the phase space and we do not require that the sequence $\{x_n^+\}_{n \geqslant 1}$ converges in Ω. The proof follows as in [8].

Lemma 3.2. *Let* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system. Suppose* $p \in \Omega$ $(p = p_0^+)$, $\mathrm{M}^+(p_n^+) \neq \emptyset$ *for all* $n \in \mathbb{N}$ *and there exists a subsequence of* $\{p_n^+\}_{n \geqslant 1}$ *converging in* Ω*. Then,* $T = \sum_{n=0}^{+\infty} \phi(p_n^+)$ *is infinite.*

In the sequel, we present a series of properties of limit sets in impulsive semidynamical systems. We state conditions so that properties such as invariancy, connectedness and compactness hold.

As we did before in the description of a trajectory of a flow of an impulsive system, given $x \in \Omega$, we write

$$x_n = \pi\left(x_{n-1}^+, \phi\left(x_{n-1}^+\right)\right), \quad \text{for } n = 1, 2, \ldots,$$

where $x = x_0^+$ and $x_n^+ = \mathrm{I}(x_n)$, $n = 1, 2, \ldots$.

Given $x \in \Omega$, we have $\widetilde{\mathrm{C}}^+(x) \cap \mathrm{M} = \emptyset$, because $\mathrm{I}(\mathrm{M}) \subset \Omega$. Thus the flow through x in an impulsive semidynamical system has no points in M. Note as well that if x is not an initial point of an impulsive semidynamical system, then $x \notin \mathrm{M}$. However, if $x \in \mathrm{M}$, then x is an initial point in a given impulsive semidynamical system, since $y \in \mathrm{X}$ and $s \in \mathbb{R}_+$ cannot exist simultaneously satisfying $\tilde{\pi}(y, s) = x$. (Note that $x \in \mathrm{M}$ is an initial point in the "impulsive sense" since no initial point in (X, π) belongs to the impulsive set M, that is, $\mathrm{F}(\mathrm{M}, t) \neq \emptyset$ for all $t \in \mathbb{R}_+$.)

The first result we present says that the convergence of the sequence $\{x_n\}_{n \geqslant 1}$ implies the convergence of $\{x_n^+\}_{n \geqslant 0}$.

Lemma 3.3. *Let* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system and* $x \in \Omega$*. If* $\{x_n\}_{n \geqslant 1}$ *is convergent in* $\mathrm{M} = \partial\Omega$*, then* $\{x_n^+\}_{n \geqslant 0}$ *is convergent in* Ω*.*

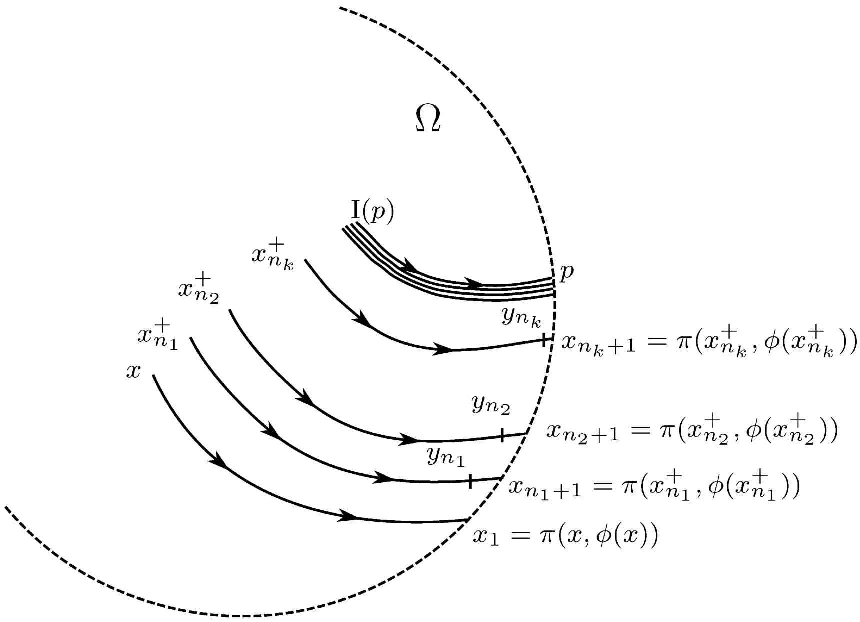

Fig. 2.

Proof. Let $\{x_n\}_{n\geqslant 1}$ be convergent in $\mathrm{M} = \partial\Omega$. Then we can suppose that $x_n \xrightarrow{n\to+\infty} p$, where $p \in \mathrm{M} = \partial\Omega$, because $\{x_n\}_{n\geqslant 1} \subset \mathrm{M}$ and M is closed. Since I is a continuous function, we have

$$\mathrm{I}(x_n) \xrightarrow{n\to+\infty} \mathrm{I}(p).$$

But, since $\mathrm{I}(x_n) = x_n^+$, it follows that

$$x_n^+ \xrightarrow{n\to+\infty} \mathrm{I}(p).$$

Hence $\{x_n^+\}_{n\geqslant 0}$ is convergent in Ω, since $\mathrm{I}(\mathrm{M}) \subset \Omega$. $\square$

The next lemma says that the limit set of the trajectory through x intercepts the set M.

Lemma 3.4. *Given an impulsive semidynamical system* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *and* $x \in \Omega$*, suppose* $\mathrm{M}^+(x_n^+) \neq \emptyset$ *for all* $n \in \mathbb{N}$ *and* $\{x_n\}_{n\geqslant 1}$ *admits a convergent subsequence. Then* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} \neq \emptyset$.

Proof. We can suppose $x_{n_k} \xrightarrow{k\to+\infty} p \in \mathrm{M}$ with $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$. We have $\pi(x^+_{n_k}, [0, \phi(x^+_{n_k}))) \subset \widetilde{\mathrm{C}}^+(x)$ for $k = 1, 2, \ldots$. Thus we can choose a sequence of real numbers, $\{\lambda_{n_k}\}_{k\geqslant 1}$, such that $0 < \lambda_{n_k} < \phi(x^+_{n_k})$ for all $k \geqslant 1$ and

$$\rho\big(\lambda_{n_k}, \phi\big(x^+_{n_k}\big)\big) \xrightarrow{k\to+\infty} 0. \tag{3}$$

Now, let us define $y_{n_k} = \pi(x^+_{n_k}, \lambda_{n_k})$ for $k = 1, 2, \ldots$ (see Fig. 2). Note that $y_{n_k} \in \widetilde{\mathrm{C}}^+(x)$ for all $k \in \mathbb{N}$.

Taking $t_{n_k} = \sum_{j=0}^{n_k-1} \phi(x_j^+) + \lambda_{n_k}$ for $k \in \mathbb{N}$, we have $t_{n_k} \xrightarrow{k\to+\infty} +\infty$, because by Lemma 3.2 the series $\sum_{n=0}^{+\infty} \phi(x_n^+)$ is divergent. Since ϕ is continuous on Ω and $x^+_{n_k} \xrightarrow{k\to+\infty} \mathrm{I}(p)$ (because

$\mathrm{I}(x_{n_k}) \xrightarrow{k\to+\infty} \mathrm{I}(p))$, it follows that $\phi(x_{n_k}^+) \xrightarrow{k\to+\infty} \phi(\mathrm{I}(p))$ and, from (3), we have $\lambda_{n_k} \xrightarrow{k\to+\infty} \phi(\mathrm{I}(p))$. It follows from the continuity of π that

$$\tilde{\pi}(x, t_{n_k}) = \pi\left(x_{n_k}^+, \lambda_{n_k}\right) \xrightarrow{k\to+\infty} \pi\left(\mathrm{I}(p), \phi\left(\mathrm{I}(p)\right)\right) \in \mathrm{M}.$$

Hence, $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} \neq \emptyset$. $\square$

Remark 3.2. Note that if we require that $x_n \xrightarrow{n\to+\infty} p$ in Lemma 3.4, then $\pi(\mathrm{I}(p), \phi(\mathrm{I}(p))) = p$ because we have $x_{n+1} = \pi(x_n^+, \phi(x_n^+)) \xrightarrow{n\to+\infty} p$ and $\pi(x_n^+, \phi(x_n^+)) \xrightarrow{n\to+\infty} \pi(\mathrm{I}(p), \phi(\mathrm{I}(p)))$.

If, in Lemma 3.4, we suppose $\{x_n\}_{n\geqslant 1}$ is convergent, then it can be proved that $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M}$ is a singleton, $x \in \Omega$. This fact is stated next.

Lemma 3.5. *Given an impulsive semidynamical system* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *and* $x \in \Omega$, *suppose* $\mathrm{M}^+(x_n^+) \neq \emptyset$ *for all* $n \in \mathbb{N}$ *and* $\{x_n\}_{n\geqslant 1}$ *is convergent. Then* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M}$ *is a singleton.*

Proof. By the previous lemma, $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} \neq \emptyset$, so let $a, b \in \widetilde{\mathrm{L}}^+(x) \cap \mathrm{M}$. Then, there are sequences $\{t_n\}_{n\geqslant 1}, \{\kappa_n\}_{n\geqslant 1} \subset \mathbb{R}_+$, $t_n \xrightarrow{n\to+\infty} +\infty$, $\kappa_n \xrightarrow{n\to+\infty} +\infty$ such that

$$\tilde{\pi}(x, t_n) \xrightarrow{n\to+\infty} a \quad \text{and} \quad \tilde{\pi}(x, \kappa_n) \xrightarrow{n\to+\infty} b.$$

For each $n \in \mathbb{N}$, there exist $k(n), r(n) \in \mathbb{N}^*$, $k(n) \xrightarrow{n\to+\infty} +\infty$ and $r(n) \xrightarrow{n\to+\infty} +\infty$, such that

$$t_n = \sum_{i=0}^{k(n)-1} \phi\left(x_i^+\right) + \bar{t}_n, \quad \text{with } 0 \leqslant \bar{t}_n < \phi\left(x_{k(n)}^+\right),$$

and

$$\kappa_n = \sum_{i=0}^{r(n)-1} \phi\left(x_i^+\right) + \bar{\kappa}_n, \quad \text{with } 0 \leqslant \bar{\kappa}_n < \phi\left(x_{r(n)}^+\right).$$

Consequently,

$$\tilde{\pi}(x, t_n) = \pi\left(x_{k(n)}^+, \bar{t}_n\right) \quad \text{and} \quad \tilde{\pi}(x, \kappa_n) = \pi\left(x_{r(n)}^+, \bar{\kappa}_n\right).$$

Supposing $x_n \xrightarrow{n\to+\infty} p$, it follows that $x_n^+ \xrightarrow{n\to+\infty} \mathrm{I}(p)$, and since ϕ is continuous on Ω, then $\phi(x_n^+) \xrightarrow{n\to+\infty} \phi(\mathrm{I}(p))$. Thus, there are subsequences $\{n_\ell\}$ and $\{n_\jmath\}$ such that

$$\bar{t}_{n_\ell} \xrightarrow{\ell\to+\infty} c \quad \text{and} \quad \bar{\kappa}_{n_\jmath} \xrightarrow{\jmath\to+\infty} d,$$

with $0 \leqslant c \leqslant \phi(\mathrm{I}(p))$ and $0 \leqslant d \leqslant \phi(\mathrm{I}(p))$. Then,

$$\tilde{\pi}(x, t_{n_\ell}) \xrightarrow{\ell\to+\infty} \pi\left(\mathrm{I}(p), c\right) \quad \text{and} \quad \tilde{\pi}(x, \kappa_{n_\jmath}) \xrightarrow{\jmath\to+\infty} \pi\left(\mathrm{I}(p), d\right).$$

Hence, $a = \pi(\mathrm{I}(p), c)$ and $b = \pi(\mathrm{I}(p), d)$. Since $a, b \in \mathrm{M}$, $\mathrm{I}(p) \in \Omega$ and $0 \leqslant c, d \leqslant \phi(\mathrm{I}(p))$, it follows that $c = d = \phi(\mathrm{I}(p))$. Therefore, $a = b$ and the result follows. $\square$

The two next results state conditions for the invariancy, compactness and connectedness of a limit set in an impulsive semidynamical system with a finite number of impulses. Their proofs are simple since we can apply the known results for non-impulsive semidynamical systems. We will also consider the case when the orbit admits infinitely many impulses in the sequel.

Theorem 3.1. *Consider an impulsive semidynamical system* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *and* $x \in \Omega$. *If there exists a positive integer* k *such that* $\mathrm{M}^+(x_k^+) = \emptyset$, *then* $\widetilde{\mathrm{L}}^+(x)$ *is closed and* $\tilde{\pi}$*-invariant.*

Proof. Since $\mathrm{M}^+(x_k^+) = \emptyset$, then $\tilde{\pi}(x_k^+, t) = \pi(x_k^+, t)$ for all $t \geqslant 0$. Hence

$$\widetilde{\mathrm{L}}^+(x) = \mathrm{L}^+\big(x_k^+\big).$$

But from the non-impulsive case we know that $\mathrm{L}^+(x_k^+)$ is closed and invariant. Therefore $\widetilde{\mathrm{L}}^+(x)$ is closed and $\tilde{\pi}$-invariant. $\square$

Theorem 3.2. *Let* X *be a locally compact metric space. Given an impulsive semidynamical system* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *and* $x \in \Omega$, *suppose* $\overline{\Omega}$ *is compact and there exists a positive integer* k *such that* $\mathrm{M}^+(x_k^+) = \emptyset$. *Then the following properties hold*:

(a) $\widetilde{\mathrm{L}}^+(x) \neq \emptyset$.
(b) $\widetilde{\mathrm{L}}^+(x)$ *is compact.*
(c) $\widetilde{\mathrm{L}}^+(x)$ *is connected.*

Proof. As in the previous theorem, it follows from the fact that $\mathrm{M}^+(x_k^+) = \emptyset$ that $\tilde{\pi}(x_k^+, t) = \pi(x_k^+, t)$ for all $t \geqslant 0$. Thus

$$\widetilde{\mathrm{L}}^+(x) = \mathrm{L}^+\big(x_k^+\big).$$

Since $\mathrm{C}^+(x_k^+) = \widetilde{\mathrm{C}}^+(x_k^+) \subset \Omega$ and $\overline{\Omega}$ is compact, it follows from the non-impulsive case that $\mathrm{L}^+(x_k^+)$ is non-empty, compact and connected. Therefore $\widetilde{\mathrm{L}}^+(x)$ is non-empty, compact and connected. $\square$

Let us now consider the case when the orbit undergoes infinitely many impulse effects. Clearly, $\widetilde{\mathrm{L}}^+(x)$ is closed for all $x \in \mathrm{X}$ in a given impulsive semidynamical system. See [7, p. 122].

Theorem 3.3. *Let* X *be a locally compact space metric. Given an impulsive semidynamical system* $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *and* $x \in \Omega$, *suppose* $\overline{\Omega}$ *is compact,* $\{x_n\}_{n \geqslant 1}$ *is convergent and* $\mathrm{M}^+(x_n^+) \neq \emptyset$ *for all* $n \in \mathbb{N}$. *Then the following properties hold*:

(a) $\widetilde{\mathrm{L}}^+(x) \cap \Omega \neq \emptyset$.
(b) $\widetilde{\mathrm{L}}^+(x)$ *is compact.*
(c) $\widetilde{\mathrm{L}}^+(x)$ *is connected.*
(d) $\rho(\tilde{\pi}(x, t), \widetilde{\mathrm{L}}^+(x)) \xrightarrow{t \to +\infty} 0$.

Proof. (a) Since $\{x_n\}_{n\geqslant 1}$ is convergent, it follows from Lemma 3.3 that $\{x_n^+\}_{n\geqslant 0}$ is convergent, say

$$x_n^+ \xrightarrow{n\to+\infty} p,$$

for some $p \in \Omega$. Since

$$x_n^+ = \tilde{\pi}(x, t_n),$$

where $t_n = \sum_{j=0}^{n-1} \phi(x_j^+)$ and $t_n \xrightarrow{n\to+\infty} +\infty$ by Lemma 3.2, it follows that $p \in \widetilde{\mathrm{L}}^+(x)$. Thus $\widetilde{\mathrm{L}}^+(x) \cap \Omega \neq \emptyset$.

(b) Since $\overline{\Omega}$ is compact, $\widetilde{\mathrm{K}}^+(x)$ is closed and $\widetilde{\mathrm{K}}^+(x) \subset \overline{\Omega}$. Then $\widetilde{\mathrm{K}}^+(x)$ is compact. But $\widetilde{\mathrm{L}}^+(x) \subset \widetilde{\mathrm{K}}^+(x)$ and $\widetilde{\mathrm{L}}^+(x)$ is closed. Hence $\widetilde{\mathrm{L}}^+(x)$ is compact.

(c) Suppose $\widetilde{\mathrm{L}}^+(x)$ is not connected. Then $\widetilde{\mathrm{L}}^+(x) = A \cup B$, where A and B are non-empty closed sets such that $A \cap B = \emptyset$. It follows from Lemma 3.4 that $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} \neq \emptyset$. Let $p \in \widetilde{\mathrm{L}}^+(x) \cap \mathrm{M}$. We assert that $A \cap \mathrm{M} \neq \emptyset$ and $B \cap \mathrm{M} \neq \emptyset$. Indeed, note that $p \in A$ or $p \in B$. We can consider $p \in B$. If $A \cap \mathrm{M} = \emptyset$, then it follows from the fact that X is locally compact and $\widetilde{\mathrm{L}}^+(x)$ is compact that there exists $\eta > 0$ such that $B[A, \eta] \cap B \cap \mathrm{M} = \emptyset$. Thus taking a point $a \in A$, there exists a sequence $\{t_n\}_{n\geqslant 1} \subset \mathbb{R}_+$, $t_n \xrightarrow{n\to+\infty} +\infty$ such that

$$\tilde{\pi}(x, t_n) \xrightarrow{n\to+\infty} a.$$

We can assume that there exists $m \in \mathbb{N}$ such that

$$\tilde{\pi}(x, t) = \pi\big(\tilde{\pi}(x, t_m), t - t_m\big) \subset B(A, \eta), \quad \text{for all } t > t_m. \tag{4}$$

Otherwise there exists a sequence of positive real numbers, $\{\kappa_n\}_{n\geqslant 1}$, such that $\kappa_n \xrightarrow{n\to+\infty} +\infty$, with $\tilde{\pi}(x, \kappa_n) \in \partial B(A, \eta)$. Then by the fact that $\partial B(A, \eta)$ is compact, we can assume that $\tilde{\pi}(x, \kappa_n) \xrightarrow{n\to+\infty} y \in \partial B(A, \eta)$ and hence $y \in \partial B(A, \eta)$ and $y \in \widetilde{\mathrm{L}}^+(x)$ which is a contradiction. Thus by (4), the trajectory of x has a finite number of discontinuities which is a contradiction, since $\phi(x_n^+) < +\infty$ for all $n \in \mathbb{N}$. Hence $A \cap \mathrm{M} \neq \emptyset$ and $B \cap \mathrm{M} \neq \emptyset$. But this is a contradiction because $\widetilde{\mathrm{L}}^+(x) = A \cup B$, $A \cap B = \emptyset$ and Lemma 3.5 says that $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M}$ is a singleton. Hence $\widetilde{\mathrm{L}}^+(x)$ is connected.

(d) Suppose the assertion does not hold. Then there exists a sequence $\{t_n\}_{n\geqslant 1} \subset \mathbb{R}_+$, with $t_n \xrightarrow{n\to+\infty} +\infty$ and $\eta > 0$, such that

$$\rho\big(\tilde{\pi}(x, t_n), \widetilde{\mathrm{L}}^+(x)\big) \geqslant \eta > 0.$$

But $\tilde{\pi}(x, t_n) \in \widetilde{\mathrm{K}}^+(x)$ and $\widetilde{\mathrm{K}}^+(x)$ is compact because $\widetilde{\mathrm{K}}^+(x) \subset \overline{\Omega}$. Thus we can assume that $\tilde{\pi}(x, t_n) \xrightarrow{n\to+\infty} y \in \widetilde{\mathrm{L}}^+(x)$. Then

$$0 < \eta \leqslant \rho\big(\tilde{\pi}(x, t_n), \widetilde{\mathrm{L}}^+(x)\big) \leqslant \rho\big(\tilde{\pi}(x, t_n), y\big) + \underbrace{\rho\big(y, \widetilde{\mathrm{L}}^+(x)\big)}_{=0}.$$

When $n \to +\infty$, we have

$$0 < \eta \leqslant \rho\big(y, \widetilde{\mathrm{L}}^+(x)\big) \leqslant 0,$$

which is a contradiction. Hence $\rho(\tilde{\pi}(x, t), \widetilde{\mathrm{L}}^+(x)) \xrightarrow{t\to+\infty} 0$. $\square$

Remark 3.3. If we suppose that $\{x_n\}_{n\geqslant 1}$ admits convergent subsequence, then the item (a) of Theorem 3.3 still holds. Items (c) and (d) are independent of the hypothesis on $\{x_n\}_{n\geqslant 1}$.

By Lemma 3.4, when a trajectory undergoes infinitely many impulses and its limit set is nonempty, then there is a point of M which belongs to the limit set. The following result states that when the limit set of a trajectory and M are disjoint, then the trajectory has a finite number of discontinuities.

Lemma 3.6. *Let* X *be a locally compact metric space. Given an impulsive semidynamical system* $(\mathrm{X},\pi;\Omega,\mathrm{M},\mathrm{I})$ *and* $x\in\Omega$, *suppose* $\widetilde{\mathrm{L}}^+(x)\cap\mathrm{M}=\emptyset$ *and* $\overline{\Omega}$ *is compact. Then there exists* $\ell\in\mathbb{N}$ *such that each* x_k^+, $k=1,\dots,\ell$, *is defined and* $\mathrm{M}^+(x_\ell^+)=\emptyset$.

Proof. Suppose that $\mathrm{M}^+(x_\ell^+)\neq\emptyset$ for all $\ell\in\mathbb{N}$. Since $\overline{\Omega}$ is compact so $\partial\Omega$ is compact, consequently the sequence $\{x_n\}_{n\geqslant 1}=\{\pi(x_{n-1}^+,\phi(x_{n-1}^+))\}_{n\geqslant 1}\subset\partial\Omega$, $x_0^+=x$, admits a convergent subsequence. By Lemma 3.4, $\widetilde{\mathrm{L}}^+(x)\cap\mathrm{M}\neq\emptyset$ and it is a contradiction. Therefore, the proof is complete. □

Before we say something about the Poincaré–Bendixson Theorem for impulsive systems, we shall prove a result that will be fundamental in the proofs of the results of the next section. We need the following lemma from [8] (see Lemma 2.3 there).

Lemma 3.7. *Suppose* $\{z_n\}_{n\geqslant 1}$ *is a sequence in* Ω *that converges to a point* $y\in\Omega$. *Then for any* $t\in[0,T(y))$, *there exists a sequence of real numbers*, $\{\varepsilon_n\}_{n\geqslant 1}$, $\varepsilon_n\xrightarrow{n\to+\infty}0$, *such that* $t+\varepsilon_n<T(z_n)$ *and* $\tilde{\pi}(z_n,t+\varepsilon_n)\xrightarrow{n\to+\infty}\tilde{\pi}(y,t)$.

Theorem 3.4 below states conditions so that the trajectory through x tends to the union of a periodic orbit and an point on M.

Theorem 3.4. *Let* $(\mathrm{X},\pi;\Omega,\mathrm{M},\mathrm{I})$ *be an impulsive semidynamical system,* X *be a locally compact metric space,* $\overline{\Omega}$ *be compact and* $x\in\Omega$. *Suppose* $\mathrm{M}^+(x_n^+)\neq\emptyset$ *for all* $n\in\mathbb{N}$, $x_n\xrightarrow{n\to+\infty}p$. *Then* $\widetilde{\mathrm{L}}^+(x)$ *is the union of a periodic orbit* $\widetilde{\mathrm{C}}^+(\mathrm{I}(p))$ *and the point* $\{p\}$.

Proof. Since $x_n\xrightarrow{n\to+\infty}p$, then $x_n^+\xrightarrow{n\to+\infty}\mathrm{I}(p)$. By Lemma 3.4 and Remark 3.2, $p\in\widetilde{\mathrm{L}}^+(x)$ and it is easy to see that $\mathrm{I}(p)\in\widetilde{\mathrm{L}}^+(x)$. Taking $0<t<\phi(\mathrm{I}(p))$, then from Lemma 3.7, there exists a sequence of real numbers, $\{\varepsilon_n\}_{n\geqslant 1}$, $\varepsilon_n\xrightarrow{n\to+\infty}0$, such that

$$\tilde{\pi}\big(x_n^+,t+\varepsilon_n\big)\xrightarrow{n\to+\infty}\tilde{\pi}\big(\mathrm{I}(p),t\big).$$

But $\tilde{\pi}(x_n^+,t+\varepsilon_n)=\tilde{\pi}(x,\sum_{j=0}^{n-1}\phi(x_j^+)+t+\varepsilon_n)$ and $(\sum_{j=0}^{n-1}\phi(x_j^+)+t+\varepsilon_n)\xrightarrow{n\to+\infty}+\infty$ by Lemma 3.2. Thus, $\tilde{\pi}(\mathrm{I}(p),t)\in\widetilde{\mathrm{L}}^+(x)$ for $0<t<\phi(\mathrm{I}(p))$. Hence,

$$\big\{\tilde{\pi}\big(\mathrm{I}(p),t\big)\colon 0\leqslant t<\phi\big(\mathrm{I}(p)\big)\big\}\cup\{p\}\subset\widetilde{\mathrm{L}}^+(x).$$

Then $\widetilde{\mathrm{C}}^+(\mathrm{I}(p))\cup\{p\}\subset\widetilde{\mathrm{L}}^+(x)$, where $\widetilde{\mathrm{C}}^+(\mathrm{I}(p))$ is a periodic orbit with period $\phi(\mathrm{I}(p))$.

We shall prove that $\widetilde{\mathrm{L}}^+(x)=\widetilde{\mathrm{C}}^+(\mathrm{I}(p))\cup\{p\}$.

Let $w \in \widetilde{\mathrm{L}}^+(x)$ be arbitrary. Thus there exists a sequence of positive real numbers, $\{\tau_k\}_{k\geqslant 1}$, such that $\tau_k \xrightarrow{k\to+\infty} +\infty$ and

$$\tilde{\pi}(x, \tau_k) \xrightarrow{k\to+\infty} w.$$

Since $\tau_k \xrightarrow{k\to+\infty} +\infty$, we can choose a subsequence $\{\tau_{k_n}\}_{n\geqslant 1}$ and, for each n, also a positive integer $m(k_n)$ such that

$$\tau_{k_n} = \sum_{j=0}^{m(k_n)-1} \phi\big(x_j^+\big) + T_{k_n} \quad \text{and} \quad \lim_{n\to+\infty} m(k_n) = +\infty,$$

with $0 \leqslant T_{k_n} < \phi(x_{m(k_n)}^+)$. Since $\phi(x_n^+) \xrightarrow{k\to+\infty} \phi(\mathrm{I}(p))$, we can assume without loss of generality that $T_{k_n} \xrightarrow{n\to+\infty} T$ and $0 \leqslant T \leqslant \phi(\mathrm{I}(p))$.

Note that

$$\tilde{\pi}(x, \tau_{k_n}) = \pi\big(x_{m(k_n)}^+, T_{k_n}\big) \xrightarrow{n\to+\infty} \pi\big(\mathrm{I}(p), T\big).$$

By the uniqueness of the limit, $w = \pi(\mathrm{I}(p), T)$. Recall that $\pi(\mathrm{I}(p), \phi(\mathrm{I}(p))) = p$ because we have $x_n = \pi(x_{n-1}^+, \phi(x_{n-1}^+)) \xrightarrow{n\to+\infty} p$ and $\pi(x_{n-1}^+, \phi(x_{n-1}^+)) \xrightarrow{n\to+\infty} \pi(\mathrm{I}(p), \phi(\mathrm{I}(p)))$. Since $0 \leqslant T \leqslant \phi(\mathrm{I}(p))$, it follows that $w \in \widetilde{\mathrm{C}}^+(\mathrm{I}(p)) \cup \{p\}$. Therefore,

$$\widetilde{\mathrm{L}}^+(x) = \widetilde{\mathrm{C}}^+\big(\mathrm{I}(p)\big) \cup \{p\}$$

and the proof is complete. □

3.2. The Poincaré–Bendixson Theorem for impulsive systems

Our aim in this section is to obtain a version of the Poincaré–Bendixson Theorem for impulsive systems. We shall consider an impulsive semidynamical system $(\mathrm{X}, \pi; \Omega, \mathrm{M}, \mathrm{I})$, where $\mathrm{X} = \mathbb{R}^2$. Thus the system $(\mathbb{R}^2, \pi)$ does not have any initial point (see [1, Theorem 11.8]). In this section, by an initial point we mean an initial point in the impulsive system $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$.

The next results are the corresponding impulsive formulations of classical results. For the latter see, for instance, [6].

Lemma 3.8. *Let $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $x \in \Omega$. Suppose that $\widetilde{\mathrm{L}}^+(x)$ contains a periodic orbit Γ. Then $\Gamma \cap \mathrm{M} = \emptyset$.*

Proof. The set M consists of initial points in the impulsive system $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ and an initial point cannot be in a periodic orbit Γ, because if $z \in \Gamma \cap \mathrm{M}$ then we cannot have $y \in \mathbb{R}^2$ and $s > 0$ simultaneously satisfying $\tilde{\pi}(y, s) = z$. Hence, $\Gamma \cap \mathrm{M} = \emptyset$. □

Theorem 3.5. *Let $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $x \in \Omega$. Suppose $\overline{\Omega}$ is compact and $\widetilde{\mathrm{L}}^+(x)$ contains a periodic orbit Γ. Then $\widetilde{\mathrm{L}}^+(x) = \Gamma$ if and only if there exists an integer $\ell > 0$ such that $\mathrm{M}^+(x_\ell^+) = \emptyset$.*

Proof. Let us prove the sufficient condition. Suppose $\mathrm{M}^+(x_n^+) \neq \emptyset$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}_{n \geqslant 1}$ is infinite and since $\overline{\Omega}$ is compact, we can assume that $x_{n_k} \xrightarrow{k \to +\infty} p \in \mathrm{M}$. By Lemma 3.4, it follows that $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} \neq \emptyset$. But this is a contradiction because by hypothesis $\widetilde{\mathrm{L}}^+(x) = \Gamma$ and Γ is a periodic orbit, see Lemma 3.8.

Now we will prove the necessary condition. Suppose there exists an $\ell > 0$ such that $\mathrm{M}^+(x_\ell^+) = \emptyset$. Then

$$\tilde{\pi}\big(\tilde{\pi}(x, t_\ell), t\big) = \tilde{\pi}\big(x_\ell^+, t\big) = \pi\big(x_\ell^+, t\big), \quad \text{for all } t \geqslant 0,$$

where $t_\ell = \sum_{j=0}^{\ell-1} \phi(x_j^+)$. Thus

$$\widetilde{\mathrm{C}}^+\big(x_\ell^+\big) = \mathrm{C}^+\big(x_\ell^+\big) \quad \text{and} \quad \mathrm{L}^+\big(x_\ell^+\big) = \widetilde{\mathrm{L}}^+(x).$$

It follows from the non-impulsive case (see [6, Lemma 5.1]) that $\mathrm{L}^+(x_\ell^+) = \Gamma$, that is, $\widetilde{\mathrm{L}}^+(x) = \Gamma$. $\square$

The analogue of Theorem 3.5 for the case in which $\mathrm{M}^+(x_j^+) \neq \emptyset$ for all $j \in \mathbb{N}$ is presented in Theorem 3.6 and follows straightforwardly from Theorem 3.4.

Theorem 3.6. *Let $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $x \in \Omega$. Suppose $\overline{\Omega}$ is compact, $\{x_n\}_{n \geqslant 1}$ is convergent and $\widetilde{\mathrm{L}}^+(x)$ contains a periodic orbit Γ. If $\mathrm{M}^+(x_j^+) \neq \emptyset$ for all $j \in \mathbb{N}$, then $\widetilde{\mathrm{L}}^+(x) = \Gamma \cup \{p\}$ where $p = \lim_{n \to +\infty} x_n$.*

Theorem 3.7. *Let $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $x \in \Omega$. Suppose $\overline{\Omega}$ is compact and $\widetilde{\mathrm{L}}^+(x)$ does not contain any rest point. Then $\widetilde{\mathrm{L}}^+(x)$ is equal to a periodic orbit Γ if and only if there exists an integer $\ell > 0$ such that $\mathrm{M}^+(x_\ell^+) = \emptyset$.*

Proof. The sufficient condition follows from Theorem 3.5. Let us prove the necessary condition. By hypothesis, $\mathrm{M}^+(x_\ell^+) = \emptyset$ for a positive integer ℓ. Then

$$\tilde{\pi}\big(x_\ell^+, t\big) = \pi\big(x_\ell^+, t\big), \quad \text{for all } t \geqslant 0.$$

Since $\mathrm{L}^+(x_\ell^+) = \widetilde{\mathrm{L}}^+(x)$, it follows from the case without impulses (see [6, Theorem 5.3]) that

$$\mathrm{L}^+\big(x_\ell^+\big) \quad \text{is a single periodic orbit } \Gamma$$

and the result follows. $\square$

The analogue of Theorem 3.7 for the case in which $\mathrm{M}^+(x_j^+) \neq \emptyset$ for all $j \in \mathbb{N}$, is presented below. Its proof also follows straightforwardly from Theorem 3.4.

Theorem 3.8. *Let $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ be an impulsive semidynamical system and $x \in \Omega$. Suppose $\overline{\Omega}$ is compact and $\{x_n\}_{n \geqslant 1}$ is convergent. If $\widetilde{\mathrm{L}}^+(x)$ contains an orbit Γ such that Γ admits no initial points and $\mathrm{M}^+(x_j^+) \neq \emptyset$ for all $j \in \mathbb{N}$, then Γ is periodic and $\widetilde{\mathrm{L}}^+(x) = \Gamma \cup \{p\}$, where $p = \lim_{n \to +\infty} x_n$.*

Finally, we shall present a version of Poincaré–Bendixson Theorem for impulsive systems.

Theorem 3.9. *Let* $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system and* $x \in \Omega$. *Suppose* $\overline{\Omega}$ *is compact and* $\widetilde{\mathrm{L}}^+(x)$ *admits neither rest points nor initial points. Then* $\widetilde{\mathrm{L}}^+(x)$ *is a periodic orbit.*

Proof. The set of initial points is equal to M, so $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$ and according to Lemma 3.6, there exists an $\ell > 0$ such that $\mathrm{M}^+(x_\ell^+) = \emptyset$. By Theorem 3.7, it follows that $\widetilde{\mathrm{L}}^+(x)$ is a periodic orbit. □

Corollary 3.1. *Let* $(\mathbb{R}^2, \pi; \Omega, \mathrm{M}, \mathrm{I})$ *be an impulsive semidynamical system. If* C *is a non-empty closed subset of* Ω, $\tilde{\pi}$*-invariant,* $\overline{C}$ *is compact,* $C \cap \Omega$ *does not contain initial points and* $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$ *for all* $x \in C$, *then* C *contains a rest point or a periodic orbit.*

Proof. Since C is closed in Ω, $\tilde{\pi}$-invariant and $\widetilde{\mathrm{L}}^+(x) \cap \mathrm{M} = \emptyset$ for all $x \in C$ it follows that $\widetilde{\mathrm{L}}^+(x) \subset C$ for all $x \in C$. If there are not rest points in C, then $\widetilde{\mathrm{L}}^+(x)$ admits neither rest points nor initial points. By the Poincaré–Bendixson Theorem, $\widetilde{\mathrm{L}}^+(x) \subset C$ is a periodic orbit. □

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