Oscillation for a second-order neutral differential equation with impulses

E. M. Bonotto*, L. P. Gimenes† and M. Federson‡

Abstract

We consider a certain type of second-order neutral delay differential systems and we establish two results concerning the oscillation of solutions after the system undergoes controlled abrupt perturbations (called impulses). As a matter of fact, some particular non-impulsive cases of the system are oscillatory already. Thus, we are interested in finding adequate impulse controls under which our system remains oscillatory.

Keywords: neutral differential equations, impulses, oscillation criteria, delays.
2000 Mathematical Subject Classification: 34K40, 34K45, 44K11.

1 Introduction

Because systems subject to impulse effects may undergo unusual phenomena such as "beating", "dying", "merging", "noncontinuation of solutions", etc, and because they are widely used to model real-world problems in science and technology, the theory of impulsive differential systems has been attracting the attention of many mathematicians and the interest in the subject is still growing. In the last years, the action of impulses on functional differential systems has been intensively investigated.

In this paper, we are mainly concerned with oscillating systems which remain oscillating after being perturbed by instantaneous changes of state. We consider a certain type of second-order neutral delay differential system and give sufficient conditions governing the impulse operators acting on the system so that its solutions are oscillatory.

An important application of *second-order differential equations with impulses* appears in impact theory. An impact is an interaction of bodies which happens in a short period of time and can be considered as an impulse. Billiard-type systems, for instance, can be modelled by second-order differential systems with impulses acting on the first derivatives of the solutions. Indeed, the positions of the colliding balls do not change at the moments

*Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto, Universidade de São Paulo, 14040-901 Ribeirão Preto SP, Brazil. E-mail: ebonotto@ffclrp.usp.br.

†Departamento de Matemática, Universidade Estadual de Maringá, 87020-900, Maringá PR, Brazil. E-mail: lpgarantes@uem.br

‡Instituto de Ciências Matemáticas e de Computação, Universidade de São Paulo, Caixa Postal 668, 13560-970 São Carlos SP, Brazil. E-mail: federson@icmc.usp.br. Supported by FAPESP 2008/02879-1.

of impact (impulses), but their velocities gain finite increments. For models describing viscoelastic bodies colliding, *systems with delay and impulses* are more appropriate. See [11].

An application of *second-order neutral delay differential equations* appears, for instance, in problems dealing with vibrating masses attached to an elastic. They also appear, as the Euler equation, in some vibrational problems. See [5, 6, 13, 15, 17, 24], for instance.

In recent years, there has been an increasing interest on the oscillatory behavior of second order nonlinear or quasilinear delay differential equations with impulse action. We refer to the papers [19, 20, 27], for example.

When considering a system subject to impulse effects, one expects that either the impulses act as a control and cease the oscillation of the system, or the impulse operators are somehow "*under control*" so that the system remains oscillating. It is known, for instance, that impulses can make oscillating systems become non-oscillating and, likewise, non-oscillating systems can become oscillating by the imposition of proper impulse controls. In [7], the authors adapt the techniques of [10] and [26] and give conditions so that the solutions of certain second-order delay differential equation oscillates. See also [4, 25, 28, 30, 33, 35, 36].

In the present paper, we consider the second-order neutral delay differential equation

$$\begin{cases} \left[r(t)\left(x(t)+p(t)x(t-\tau)\right)'\right]' + f(t,x(t),x(t-\delta)) = 0, & t \geq t_0, \quad t \neq t_k, \\ x(t_k) = I_k(x(t_k^-)), \qquad x'(t_k) = J_k(x'(t_k^-)), & k = 1,2,\ldots, \\ x(t) = \phi(t), \qquad t_0-\sigma \leq t \leq t_0, \end{cases} \tag{1.1}$$

where $p \in PC^1([t_0, +\infty[, \mathbb{R}_+)$, $r(t)$ is a positive continuous function defined in $[t_0, +\infty[$, δ and τ are non-negative constants, $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim_{k\to+\infty} t_k = +\infty$ and $t_{k+1} - t_k > \sigma$, where $\sigma := \max\{\delta, \tau\}$, and $\phi, \phi' : [t_0 - \sigma, t_0] \to \mathbb{R}$ have at most a finite number of discontinuities of the first kind and are right continuous at these points. Then we state sufficient conditions so that the solutions of system (1.1) are oscillatory.

In fact, it is known that some particular cases of (1.1) oscillate without the presence of impulses. See [16, 25, 35] for instance. Our main results, namely Theorems 2.1 and 2.2, give conditions under which system (1.1) remains oscillating. In order to obtain such result, we employ some ideas from [7] and specially from [16].

We note that when right continuity is replaced by left continuity, the results of the present paper remain valid (with obvious modifications). For left continuous functions and in the absence of impulses, see the results from [16] and [35], for instance. In the absence of delay, see [21] and [31].

In [26], the authors state oscillation results for the impulsive delay differential system

$$\begin{cases} (r(t)(x'(t))^\sigma)' + f(t,x(t),x(t-\delta)) = 0, & t \geq t_0, \quad t \neq t_k, \\ x(t_k) = I_k(x(t_k^-)), \qquad x'(t_k) = J_k(x'(t_k^-)), & k = 1,2,\ldots, \\ x(t_0^+) = x_0, \quad x'(t_0^+) = x'(t_0), \end{cases}$$

where $0 < \sigma = p/q$, with p and q being odd integers. See also [19] and [20].

For neutral differential systems, we mention [32], where the authors consider the non-impulsive system

$$\left[r(t)\left|(x(t)+p(t)x(\sigma(t)))'\right|^{\alpha-1}(x(t)+p(t)x(\sigma(t)))'\right]' + q(t)f(x(\sigma(t))) = 0, \qquad t \geq t_0,$$

where α is a positive constant. An oscillation result is proved for this system. When $\alpha = 1$ and $\sigma(t) = t - \sigma$, where $\sigma := \max\{\delta, \tau\}$ as in (1.1), our results generalize the result from [32].

In the case of neutral difference systems with impulses, we mention [27], where the author states some criteria for the oscillation of the solutions of the discrete system

$$\Delta \left(r_{n-1}|\Delta(x_{n-1} - x_{n-\tau-1})|^{\alpha-1}\Delta(x_{n-1} - x_{n-\tau-1})\right) + f(n, x_n, x_{n-1}) = 0$$

subject to the impulse action

$$\begin{aligned} &r_{n_k}|\Delta(x_{n_k} - x_{n_k-\tau})|^{\alpha-1}\Delta(x_{n_k} - x_{n_k-\tau}) = \\ &= M_k \left(r_{n_k-1}|\Delta(x_{n_k-1} - x_{n_k-\tau-1})|^{\alpha-1}\Delta(x_{n_k-1} - x_{n_k-\tau-1})\right), \end{aligned}$$

where $\Delta x_n = x_{n+1} - x_n$, α is a positive constant and the impulse operator M_k fulfills certain conditions, $k, \tau \in \mathbb{N}$. Thus, up to now, it seems that no result concerning oscillation of solutions for piecewise continuous neutral differential systems subject to impulses have been found yet. Hence our result is a contribution in this direction.

Furthermore, we assume that $p(t)$ in system (1.1) takes any positive value improving the usual assumption that $0 \le p(t) \le 1$.

2 Main results

By $w \in PC^1([T, +\infty[, \mathbb{R}_+)$ we mean the set of functions $w \in C^1([\lambda_k, \lambda_{k+1}[, \mathbb{R}_+)$, for each $k = 0, 1, 2, ...$, where $\{\lambda_k\}_{k\ge1}$ is a sequence of positive real numbers, with $\lambda_0 = T$, and the limits $w(\lambda_k^-)$ and $w'(\lambda_k^-)$ exist, for all $k = 0, 1, 2, \ldots$.

Consider the second-order neutral delay differential equation

$$\begin{cases} [r(t)(x(t) + p(t)x(t-\tau))']' + f(t, x(t), x(t-\delta)) = 0, \qquad t \ge t_0, \quad t \ne t_k, \\ x(t_k) = I_k(x(t_k^-)), \qquad\qquad x'(t_k) = J_k(x'(t_k^-)), \quad k = 1, 2, \ldots, \\ x(t) = \phi(t), \quad t_0 - \sigma \le t \le t_0, \end{cases} \tag{2.1}$$

where δ and τ are positive real numbers, $\sigma := \max\{\delta, \tau\}$, $0 \le t_0 < t_1 < \ldots < t_k < \ldots$ with $\lim\limits_{k\to+\infty} t_k = +\infty$ and $t_{k+1} - t_k > \sigma$, for all $k \in \mathbb{N}$, $p \in PC^1([t_0, +\infty[, \mathbb{R}_+)$ and ϕ, ϕ' : $[t_0 - \sigma, t_0] \to \mathbb{R}$ have at most a finite number of discontinuities of the first kind and are right continuous at these points.

We will state oscillation results for (2.1) in two situations which we will refer to as case A and case B.

2.1 Case A

Throughout this section we assume that

(H_1) $f : [t_0 - \sigma, +\infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $uf(t,u,v) > 0$ for all $uv > 0$,

$$\frac{f(t,u,v)}{\varphi(v)} \geq m(t),$$

for all $v \neq 0$, where $m(t)$ is continuous on $[t_0 - \sigma, +\infty[$, $m(t) \geq 0$, and $x\varphi(x) > 0$, for all $x \neq 0$ and $\varphi'(x) \geq 0$;

(H_2) $I_k,\ J_k : \mathbb{R} \to \mathbb{R}$ are continuous, with $I_k(0) = J_k(0) = 0$, and there exist positive numbers a_k, b_k and c_k such that

$$a_k \leq \frac{I_k(x)}{x} \leq b_k, \qquad J_k(x) = c_k x, \qquad x \neq 0, \quad k = 1, 2, \ldots,$$

for all $k \in \mathbb{N}$.

(H_3) r is a positive continuous function on $[t_0, +\infty[$ and

$$\lim_{t \to +\infty} \int_{t_0}^{t} \left(\frac{1}{r(s)} \prod_{t_0 < t_k < s} \frac{c_k}{\max\{b_k,\ c_k\}} \right) ds = +\infty\,.$$

(H_4) $p(t)$ and $p'(t)$ are right continuous on $]t_k,\ t_{k+1}[$ with left lateral limits $p(t_k^-) = \dfrac{1}{c_k} p(t_k)$, and $p'(t_k^-) = \dfrac{1}{c_k} p'(t_k)$, for each $k \in \mathbb{N}$.

We start by presenting a lemma which is borrowed from [14] (see Theorem 1.4.1 there) replacing the left continuity by the right continuity of $g(t)$ and $g'(t)$ at t_k, for all $k \in \mathbb{N}$.

Lemma 2.1 *Suppose*

(i) the sequence $\{t_k\}_{k \in \mathbb{N}}$ *satisfies* $0 \leq t_0 < t_1 < \ldots < t_k < \ldots$, *with* $\lim\limits_{k \to +\infty} t_k = +\infty$.

(ii) $g, g' : \mathbb{R}_+ \to \mathbb{R}$ *are continuous on* $\mathbb{R}_+ \setminus \{t_k : k \in \mathbb{N}\}$, *there exist the lateral limits* $g(t_k^-), g'(t_k^-), g(t_k^+), g'(t_k^+)$ *and* $g(t_k^+) = g(t_k)$, $k = 1, 2, \ldots$.

(iii) for $k = 1, 2, \ldots$ *and* $t \geq t_0$, *we have*

$$g'(t) \leq p(t)\, g(t) + q(t), \qquad t \neq t_k, \tag{2.2}$$

$$g(t_k) \leq \alpha_k g(t_k^-) + \beta_k, \tag{2.3}$$

where $p, q \in C(\mathbb{R}_+, \mathbb{R})$, α_k *and* β_k *are real constants with* $\alpha_k \geq 0$.

Then the following inequality holds

$$\begin{aligned} g(t) \leq\ & g(t_0) \prod_{t_0 < t_k < t} \alpha_k \exp\left(\int_{t_0}^{t} p(s) ds \right) + \int_{t_0}^{t} \prod_{s < t_k < t} \alpha_k \exp\left(\int_{s}^{t} p(u) du \right) q(s) ds \\ & + \sum_{t_0 < t_k < t} \prod_{t_k < t_j < t} \alpha_j \exp\left(\int_{t_k}^{t} p(s) ds \right) \beta_k, \qquad t \geq t_0. \end{aligned} \tag{2.4}$$

Remark 2.1 If the inequalities (2.2) and (2.3) are reversed, then the inequality (2.4) is also reversed.

For the sake of convenient notation, let $z(t) = x(t) + p(t)x(t-\tau)$.

Lemma 2.2 *Suppose (H_1) to (H_4) are fulfilled, $a_k, c_k \geq 1$, $k \in \mathbb{N}$, and there exists $T \geq t_0$ such that $x(t) > 0$ for $t \geq T - \tau - \delta$. Then $z(t) > 0$ on the interval $[T, +\infty[$ and $z'(t) \geq 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \geq T$ and $k \in \mathbb{N}$. Furthermore, $z(t)$ is non-decreasing on $[T, +\infty[$.*

Proof. Suppose $x(t) > 0$, for $t \geq T - \tau - \delta$. Then $x(t-\tau) > 0$ for all $t \geq T - \delta$. In particular $x(t-\tau) > 0$ for all $t \geq T$ and hence

$$z(t) = x(t) + p(t)x(t-\tau) > 0, \quad t \geq T \geq t_0.$$

Now we are going to prove that $z'(t_k^-) \geq 0$, $t_k \geq T$ and $k \in \mathbb{N}$. Suppose the opposite, that is, there exists $t_{j_0} \geq T$ such that $z'(t_{j_0}^-) < 0$. Let $z'(t_{j_0}^-) = -\alpha$, with $\alpha > 0$. Since $t_{k+1} - t_k > \sigma \geq \tau$ for each $k \in \mathbb{N}$, we have

$$t_k < t_{k+1} - \tau < t_{k+1} \tag{2.5}$$

for all $k \in \mathbb{N}$. Thus, from the continuity of x and x' on $[t_{k-1}, t_k[$, inequality (2.5), assumptions (H_2) and (H_4), and equation (2.1), we have

$$\begin{aligned} z'(t_k) &= x'(t_k) + p'(t_k)x(t_k - \tau) + p(t_k)x'(t_k - \tau) \\ &= J_k(x'(t_k^-)) + c_k p'(t_k^-)x(t_k^- - \tau) + c_k p(t_k^-)x'(t_k^- - \tau) \\ &= c_k x'(t_k^-) + c_k p'(t_k^-)x(t_k^- - \tau) + c_k p(t_k^-)x'(t_k^- - \tau) \\ &= c_k z'(t_k^-), \end{aligned}$$

that is, $z'(t_k) = c_k z'(t_k^-)$ for all $k \in \mathbb{N}$.

On the other hand, if $t \in]t_k, t_{k+1}[$, $k \in \mathbb{N}$ and $t_k > T$, it follows by (H_1) that

$$[r(t)z'(t)]' = -f(t, x(t), x(t-\delta)) \leq -m(t)\varphi(x(t-\delta)) \leq 0.$$

Hence $r(t)z'(t)$ is non-increasing on each interval $[t_k, t_{k+1}[$, $k \in \mathbb{N}$, such that $t_k > T$.

We now consider the impulsive differential inequality

$$\begin{aligned} (r(t)z'(t))' &\leq 0, \qquad t > t_{j_0}, \quad t \neq t_k, \quad k = j_0+1, j_0+2, \ldots, \\ z'(t_k) &= c_k z'(t_k^-), \qquad k = j_0+1, j_0+2, \ldots. \end{aligned}$$

Let $g(t) = r(t)z'(t)$. Then

$$\begin{aligned} g'(t) &\leq 0, \qquad t > t_{j_0}, \quad t \neq t_k, \quad k = j_0+1, j_0+2, \ldots, \\ g(t_k) &= c_k g(t_k^-), \qquad k = j_0+1, j_0+2, \ldots. \end{aligned}$$

By Lemma 2.1, we have

$$g(t) \leq g(t_{j_0}^-) \prod_{t_{j_0} < t_k < t} c_k,$$

that is,

$$z'(t) \leq \left(\frac{r(t_{j0})}{r(t)}\right) z'(t_{j0}^{-}) \prod_{t_{j0}<t_k<t} c_k. \tag{2.6}$$

For $k = j_0+1, j_0+2, ...,$ we also have

$$\begin{aligned} z(t_k) &= x(t_k) + p(t_k)x(t_k-\tau) \\ &= I_k(x(t_k^-)) + c_k p(t_k^-)x(t_k^- - \tau) \\ &\leq b_k x(t_k^-) + c_k p(t_k^-)x(t_k^- - \tau) \\ &\leq \max\{b_k,\, c_k\} z(t_k^-). \end{aligned}$$

By (3.2) and since $z(t_k) \leq \max\{b_k,\, c_k\}z(t_k^-)$, $k = j_0+1, j_0+2, \ldots$, it follows from Lemma 2.1 that

$$\begin{aligned} z(t) &\leq z(t_{j0}^-) \prod_{t_{j0}<t_k<t} \max\{b_k,\, c_k\} + \int_{t_{j0}}^{t} \prod_{s<t_k<t} \max\{b_k,\, c_k\} \left[\left(\frac{r(t_{j0})}{r(s)}\right) z'(t_{j0}^-) \prod_{t_{j0}<t_k<s} c_k\right] ds \\ &= \prod_{t_{j0}<t_k<t} \max\{b_k,\, c_k\} \left[z(t_{j0}^-) - \alpha r(t_{j0}) \int_{t_{j0}}^{t} \left(\frac{1}{r(s)} \prod_{t_{j0}<t_k<s} \frac{c_k}{\max\{b_k,\, c_k\}}\right) ds\right]. \end{aligned}$$

And since $z(t) > 0$ for $t \geq T$, the last inequality contradicts (H_3). Therefore $z'(t_k^-) \geq 0$ for all t_k, $t_k \geq T$.

Since $r(t)z'(t)$ is non-increasing on $[t_k, t_{k+1}[$, it is clear that

$$z'(t) \geq \frac{r(t_{k+1}^-)}{r(t)} z'(t_{k+1}^-) \geq 0,$$

for $t \in [t_k, t_{k+1}[$, $t_k \geq T$. Finally, take any t_k, $k \in \mathbb{N}$, such that $t_k > T$. Then

$$\begin{aligned} z(t_k) &= x(t_k) + p(t_k)x(t_k-\tau) \\ &= I_k(x(t_k^-)) + c_k p(t_k^-)x(t_k^- - \tau) \\ &\geq a_k x(t_k^-) + c_k p(t_k^-)x(t_k^- - \tau) \\ &\geq \min\{a_k,\, c_k\} z(t_k^-) \\ &\geq z(t_k^-). \end{aligned}$$

Hence $z(t)$ is non-decreasing on $[T, +\infty[$ and the proof is complete. ■

Remark 2.2 When $x(t)$ is eventually negative and $a_k, c_k \geq 1$, $k \in \mathbb{N}$, then under hypotheses (H_1) to (H_4) one can prove similarly that $z(t) < 0$ on the interval $[T, +\infty[$ and $z'(t) \leq 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \geq T$. In particular, $z(t)$ is non-increasing on $[T, +\infty[$.

Now we present an auxiliary function whose definition is borrowed from [16] and which will be used in the proofs of the following results.

Let $\Phi \in C^2([t_0, +\infty), \mathbb{R}_+)$ be given and define $h \in C([t_0, +\infty[, \mathbb{R})$ by

$$h(t) = -\frac{\Phi'(t)}{2\Phi(t)}.$$

Now, define the function ψ by

$$\psi(t) = \Phi(t)\left\{m(t)[1 - p(t-\delta)] + r(t-\delta)h^2(t)c_k - [r(t-\delta)h(t)c_k]'\right\}$$

for each $t_k \le t < t_{k+1}$, $k = 1, 2, 3, \ldots$.

Proposition 2.1 *Suppose (H_1) to (H_4) are fulfilled, $a_k, c_k \ge 1$, $k \in \mathbb{N}$ and $\varphi(v) = v$ in assumption (H_1). If equation (2.1) is nonoscillatory, then there exist a number $k_0 \in \mathbb{N}$ and a function $w \in PC^1([t_{k_0}, +\infty[, \mathbb{R})$ satisfying*

$$w'(t) + \psi(t) + \frac{w^2(t)}{r(t-\delta)\Phi(t)c_k} \le 0, \quad t_k < t < t_{k+1}, \tag{2.7}$$

for each $k = k_0, k_0+1, k_0+2, \ldots$.

Proof. Let $x(t)$ be a nonoscillatory solution of (2.1). Without loss of generality, we may assume that $x(t) > 0$ on $[T - \tau - \delta, +\infty[$, for some $T \ge t_0$.

Recall that $z(t) = x(t) + p(t)x(t-\tau)$. By Lemma 2.2, $z(t) > 0$, $z'(t) \ge 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \ge T$ and $k \in \mathbb{N}$ and $z(t)$ is non-decreasing on $[T, +\infty[$.

Let $k_0 = \min\{k : t_k \ge T, k = 1, 2, 3, \ldots\}$. By (2.1) and hypothesis (H_1), we obtain

$$[r(t)z'(t)]' = -f(t, x(t), x(t-\delta)) \le -m(t)x(t-\delta) < 0,$$

for every $t \ge T$ and $t \ne t_k$, $k \in \mathbb{N}$. Consequently, $r(t)z'(t)$ is a non-increasing function on each interval $[t_k, t_{k+1}[$, $k = k_0, k_0+1, \ldots$.

Now, we assert that

$$r(t)z'(t) \le c_k r(t-\delta)z'(t-\delta), \tag{2.8}$$

for each $t_k \le t < t_{k+1}$, $k = k_0, k_0+1, \ldots$. Indeed. First, note that

$$\begin{aligned} r(t_k)z'(t_k) &= r(t_k)[x'(t_k) + p'(t_k)x(t_k-\tau) + p(t_k)x'(t_k-\tau)] \\ &= r(t_k^-)[J_k(x'(t_k^-)) + c_k p'(t_k^-)x(t_k^- - \tau) + c_k p(t_k^-)x'(t_k^- - \tau)] \\ &= r(t_k^-)[c_k x'(t_k^-) + c_k p'(t_k^-)x(t_k^- - \tau) + c_k p(t_k^-)x'(t_k^- - \tau)] \\ &= c_k r(t_k^-)z'(t_k^-). \end{aligned}$$

If $t_k + \delta \le t < t_{k+1}$, $k = k_0, k_0+1, \ldots$, we have $t_k \le t - \delta < t_{k+1} - \delta < t_{k+1}$, then

$$r(t)z'(t) \le r(t-\delta)z'(t-\delta) \le c_k r(t-\delta)z'(t-\delta).$$

If $t_k \le t < t_k + \delta$, $k = k_0, k_0+1, \ldots$, we have $t_{k-1} < t_k - \delta \le t - \delta < t_k$, then

$$r(t)z'(t) \le r(t_k)z'(t_k) = c_k r(t_k^-)z'(t_k^-) \le c_k r(t-\delta)z'(t-\delta).$$

Thus, the assertion is proved.

Note that

$$f(t,\, x(t),\, x(t-\delta)) \geq m(t)x(t-\delta) = m(t)[z(t-\delta) - p(t-\delta)x(t-\delta-\tau)],$$

for $t \neq t_k$, $k \in \mathbb{N}$ and $t \geq T$. Then,

$$[r(t)z'(t)]' + m(t)[z(t-\delta) - p(t-\delta)x(t-\tau-\delta)] \leq [r(t)z'(t)]' + f(t,\, x(t),\, x(t-\delta)) = 0,$$

for $t \neq t_k$, $k \in \mathbb{N}$ and $t \geq T$, that is

$$[r(t)z'(t)]' + m(t)[z(t-\delta) - p(t-\delta)x(t-\delta-\tau)] \leq 0.$$

Since $z(t)$ is non-decreasing from Lemma 2.2, we have

$$x(t-\delta-\tau) \leq z(t-\delta-\tau) \leq z(t-\delta), \qquad t \geq T.$$

Then

$$m(t)z(t-\delta)[1-p(t-\delta)] \leq m(t)[z(t-\delta) - p(t-\delta)x(t-\delta-\tau)]$$

and, consequently,

$$[r(t)z'(t)]' + m(t)z(t-\delta)[1-p(t-\delta)] \leq 0,$$

for $t \geq T$, $t \neq t_k$, $k \in \mathbb{N}$.

Now, define

$$w(t) = \Phi(t)\left\{\frac{r(t)z'(t)}{z(t-\delta)} + r(t-\delta)h(t)c_k\right\},$$

for each $t \in [t_k,\, t_{k+1}[$, $k = k_0, k_0+1, \ldots.$ Note that $w \in PC^1([t_{k_0},\, +\infty),\, \mathbb{R})$.

We also have

$$w'(t) \leq -2h(t)w(t) + \Phi(t)\left\{-m(t)[1-p(t-\delta)] - \frac{r(t)z'(t)z'(t-\delta)}{z^2(t-\delta)} + [r(t-\delta)h(t)c_k]'\right\},$$

for each $t_k < t < t_{k+1}$, $k = k_0, k_0+1, \ldots.$

Since $r(t)z'(t) \leq c_k r(t-\delta)z'(t-\delta)$ from (2.8), we have

$$\frac{r(t)z'(t)z'(t-\delta)}{z^2(t-\delta)} \geq \frac{1}{c_k r(t-\delta)}\left(\frac{r(t)z'(t)}{z(t-\delta)}\right)^2.$$

Then

$$w'(t) \leq -2h(t)w(t) + \Phi(t)\left\{-m(t)[1-p(t-\delta)] + \right.$$
$$\left. -\frac{1}{c_k r(t-\delta)}\left(\frac{r(t)z'(t)}{z(t-\delta)}\right)^2 + [r(t-\delta)h(t)c_k]'\right\},$$

for each $t_k < t < t_{k+1}$, $k = k_0, k_0+1, \ldots.$

Since

$$\frac{r(t)z'(t)}{z(t-\delta)} = \frac{w(t)}{\Phi(t)} - r(t-\delta)h(t)c_k,$$

we have

$$w'(t) \leq -\frac{w^2(t)}{c_k\Phi(t)r(t-\delta)} + \Phi(t)\left\{-m(t)[1-p(t-\delta)]+ \right.$$
$$\left. -r(t-\delta)h^2(t)c_k + [r(t-\delta)h(t)c_k]'\right\},$$

Therefore,

$$w'(t) \leq -\psi(t) - \frac{w^2(t)}{r(t-\delta)\Phi(t)c_k}, \qquad t_k < t < t_{k+1},$$

$k = k_0, k_0+1, \ldots$.

When $x(t)$ is eventually negative, then proof follows analogously. ■

Lemma 2.3 *If $c_k = 1$ and $a_k \geq 1$ for each $k = 1, 2, 3, ...,$ then there is $T \geq t_0$ such that $w(t_k) - w(t_k^-) \leq 0$ for each $k \in \mathbb{N}$ with $t_k > T$.*

PROOF. At first, given t_k for some $k \in \mathbb{N}$, suppose $t_k - \delta - \tau \neq t_{k-1}$. Then,

$$\begin{aligned} w(t_k) &= \Phi(t_k)\left\{\frac{r(t_k)z'(t_k)}{z(t_k-\delta)} + r(t_k-\delta)h(t_k)c_k\right\} \\ &= \Phi(t_k^-)\left\{\frac{r(t_k^-)c_k z'(t_k^-)}{z(t_k^- - \delta)} + r(t_k^- - \delta)h(t_k^-)c_k\right\} \\ &= \Phi(t_k^-)\left\{\frac{r(t_k^-)z'(t_k^-)}{z(t_k^- - \delta)} + r(t_k^- - \delta)h(t_k^-)\right\} \\ &= w(t_k^-), \end{aligned}$$

$k = 1, 2, 3, ...$

Now, we need to consider the case when $t_k - \delta - \tau = t_{k-1}$. Without loss of generality, we may assume that $x(t) > 0$ on $[T - \tau - \delta, +\infty[$, for some $T \geq t_0$. Then

$$\begin{aligned} z(t_k - \delta) - z(t_k^- - \delta) &= p(t_k - \delta)[x(t_k - \delta - \tau) - x(t_k^- - \delta - \tau)] \\ &= p(t_k - \delta)[x(t_{k-1}) - x(t_{k-1}^-)]. \end{aligned}$$

Since, $x(t_{k-1}) \geq a_k x(t_{k-1}^-) \geq x(t_{k-1}^-)$, it follows that

$$z(t_k - \delta) - z(t_k^- - \delta) \geq 0.$$

By Lemma 2.2, $z(t) > 0$ on the interval $[T, +\infty[$ and $z'(t) \geq 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \geq T$ and $k \in \mathbb{N}$. Thus, we can conclude that

$$w(t_k) \leq w(t_k^-),$$

for $t_k > T$.

When $x(t) < 0$ on $[T - \tau - \delta, +\infty[$, for some $T \geq t_0$, the result follows analogously. ■

The following theorem is an extension of Horng-Jaan Li's criteria to oscillation. See [16].

Theorem 2.1 *Suppose* (H_1) *to* (H_4) *are fulfilled,* $a_k \geq 1$*,* $c_k = 1$*,* $k \in \mathbb{N}$*,* $\varphi(v) = v$ *in assumption* (H_1) *and*

$$\sum_{k=n}^{+\infty} \int_{t_k}^{t_{k+1}} \psi(s)ds = +\infty, \tag{2.9}$$

for $n \in \mathbb{N}$*. If there exist sequences* $\{\alpha_n\}_{n\geq 1}$ *and* $\{\xi_n\}_{n\geq 1}$ *of positive real numbers, such that* $\xi_n \in]t_n,\, t_{n+1}[$*,* $n \in \mathbb{N}$*,* $\limsup_{n\to+\infty}(t_{n+1} - \xi_n) > 0$*,* $\sum_{n=1}^{+\infty} \frac{1}{\alpha_n} < +\infty$ *and*

$$\int_{t_k}^{\xi_k} \frac{ds}{r(s-\delta)\Phi(s)} \geq \alpha_k,$$

$k \in \mathbb{N}$*, then system* (2.1) *is oscillatory.*

Proof. Suppose system (2.1) is non-oscillatory. Then it follows from Proposition 2.1 that there exist a number $k_0 \in \mathbb{N}$ and a function $w(t) \in PC^1([t_{k_0},\, +\infty[, \mathbb{R})$ satisfying (2.7) for $t_k < t < t_{k+1}$, $k = k_0, k_0+1, k_0+2, \ldots.$

Integrating (2.7) over $[t_k,\, t_{k+1}]$, $k \in \mathbb{N}$ and $k \geq k_0$, we obtain

$$w(t_{k+1}^-) \leq w(t_k) - \int_{t_k}^{t_{k+1}} \psi(s)ds - \int_{t_k}^{t_{k+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)}ds. \tag{2.10}$$

For $n \in \mathbb{N}$, we have

$$\sum_{k=k_0}^{k_0+n} w(t_{k+1}^-) \leq \sum_{k=k_0}^{k_0+n} w(t_k) - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \psi(s)ds - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)}\, ds.$$

Consequently,

$$\begin{aligned} w(t_{k_0+n+1}^-) \;&\leq\; w(t_{k_0}) + \sum_{k=k_0+1}^{k_0+n} [w(t_k) - w(t_k^-)] - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \psi(s)ds\, + \\ &\quad - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)}\, ds, \end{aligned}$$

where $n \in \mathbb{N}$.

By Lemma 2.3 and equation (2.9), there exists $N_0 > 0$ such that

$$w(t_{k_0}) + \sum_{k=k_0+1}^{k_0+n} [w(t_k) - w(t_k^-)] - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \psi(s)ds < -1, \quad \text{for} \quad n \geq N_0.$$

Thus

$$w(t_{k_0+n+1}^-) \leq -1 - \sum_{k=k_0}^{k_0+n} \int_{t_k}^{t_{k+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)}\, ds, \quad \text{for } \; n \geq N_0.$$

Note that for all $t_{k_0+n} < \xi < t_{k_0+n+1}$, we have

$$w(\xi) \leq -1 - \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{w^2(s)}{r(s-\delta)\Phi(s)}\, ds, \quad \text{for } n \geq N_0. \tag{2.11}$$

Then

$$\begin{aligned}\int_{t_{k_0+n}}^{\xi} \frac{w(s)}{r(s-\delta)\Phi(s)}\, ds \leq -\int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)} + \\ -\int_{t_{k_0+n}}^{\xi} \frac{1}{r(s-\delta)\Phi(s)} \left[\int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{w^2(\mu)}{r(\mu-\delta)\Phi(\mu)}\, d\mu\right] ds,\end{aligned} \tag{2.12}$$

for each $t_{k_0+n} < \xi < t_{k_0+n+1}$ and $n \geq N_0$.

Let us consider $\xi_{k_0+n} \in]t_{k_0+n},\, t_{k_0+n+1}[$, $n \geq N_0$, given by the hypotheses, and define

$$v(\xi) = \int_{t_{k_0+n}}^{\xi} \frac{w(s)}{r(s-\delta)\Phi(s)}\, ds, \quad \xi_{k_0+n} \leq \xi < t_{k_0+n+1}, \quad n \geq N_0.$$

Then the Cauchy-Schwartz inequality implies

$$\int_{t_{k_0+n}}^{s} \frac{w^2(\mu)}{r(\mu-\delta)\Phi(\mu)}\, d\mu \geq v^2(s) \left[\int_{t_{k_0+n}}^{s} \frac{d\mu}{r(\mu-\delta)\Phi(\mu)}\right]^{-1},$$

$\xi_{k_0+n} \leq s < t_{k_0+n+1}$.

Since

$$\int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{w^2(\mu)}{r(\mu-\delta)\Phi(\mu)}\, d\mu \geq \int_{t_{k_0+n}}^{s} \frac{w^2(\mu)}{r(\mu-\delta)\Phi(\mu)} d\mu,$$

$\xi_{k_0+n} \leq s < t_{k_0+n+1}$, then by (2.12), we get

$$\begin{aligned} v(\xi) \quad &\leq \quad -\int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)} + \\ &\quad -\int_{t_{k_0+n}}^{\xi} \frac{v^2(s)}{r(s-\delta)\Phi(s)} \left[\int_{t_{k_0+n}}^{s} \frac{d\mu}{r(\mu-\delta)\Phi(\mu)}\right]^{-1} ds,\end{aligned}$$

where $\xi_{k_0+n} \leq \xi < t_{k_0+n+1}$ and $n \geq N_0$.

Now, we define $H(\xi)$ by

$$\int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)} + \int_{t_{k_0+n}}^{\xi} \frac{v^2(s)}{r(s-\delta)\Phi(s)} \left[\int_{t_{k_0+n}}^{s} \frac{d\mu}{r(\mu-\delta)\Phi(\mu)}\right]^{-1} ds,$$

$\xi_{k_0+n} \leq \xi < t_{k_0+n+1}$ and $n \geq N_0$. Then

$$H'(\xi) = \frac{1}{r(\xi-\delta)\Phi(\xi)} + \frac{v^2(\xi)}{r(\xi-\delta)\Phi(\xi)} \left[\int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)}\right]^{-1}$$

and

$$0 \leq \int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)} \leq H(\xi) \leq |v(\xi)|,$$

for $\xi_{k_0+n} \leq \xi < t_{k_0+n+1}$, $n \geq N_0$. Then

$$\frac{H'(\xi)}{H^2(\xi)} \geq \frac{H'(\xi)}{v^2(\xi)} \geq \frac{1}{r(\xi-\delta)\Phi(\xi)} \left[\int_{t_{k_0+n}}^{\xi} \frac{ds}{r(s-\delta)\Phi(s)}\right]^{-1},$$

$\xi_{k_0+n} \leq \xi < t_{k_0+n+1}$, $n \geq N_0$.

Integrating the above inequality from ξ_{k_0+n} to t_{k_0+n+1}, we have

$$-\frac{1}{H(t_{k_0+n+1}^{-})} + \frac{1}{H(\xi_{k_0+n})} \geq$$

$$\geq \ln \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{ds}{r(s-\delta)\Phi(s)} - \ln \int_{t_{k_0+n}}^{\xi_{k_0+n}} \frac{ds}{r(s-\delta)\Phi(s)}.$$

Thus

$$\frac{1}{H(\xi_{k_0+n})} \geq \ln \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{ds}{r(s-\delta)\Phi(s)} - \ln \int_{t_{k_0+n}}^{\xi_{k_0+n}} \frac{ds}{r(s-\delta)\Phi(s)},$$

$n \geq N_0$.

Since

$$H(\xi_{k_0+n}) \geq \int_{t_{k_0+n}}^{\xi_{k_0+n}} \frac{ds}{r(s-\delta)\Phi(s)} \geq \alpha_{k_0+n},$$

$n \geq N_0$, we have

$$\sum_{n=N_0}^{+\infty} \frac{1}{H(\xi_{k_0+n})} \leq \sum_{n=N_0}^{+\infty} \frac{1}{\alpha_{k_0+n}} < +\infty.$$

Thus,

$$\sum_{n=N_0}^{+\infty} \left[\ln \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{ds}{r(s-\delta)\Phi(s)} - \ln \int_{t_{k_0+n}}^{\xi_{k_0+n}} \frac{ds}{r(s-\delta)\Phi(s)}\right] < +\infty.$$

Then,

$$\lim_{n\to+\infty} \left[\ln \int_{t_{k_0+n}}^{t_{k_0+n+1}} \frac{ds}{r(s-\delta)\Phi(s)} - \ln \int_{t_{k_0+n}}^{\xi_{k_0+n}} \frac{ds}{r(s-\delta)\Phi(s)}\right] = 0,$$

and this is a contradiction, because $\limsup\limits_{n\to+\infty}[t_{k_0+n+1} - \xi_{k_0+n}] > 0$. Hence, we finished the proof. ∎

Consider the following neutral delay differential equation of second-order,

$$\begin{cases} \left(x(t) + \dfrac{1}{t}\, x(t-1)\right)'' + (t^3 - t^2)x(t-1)\arctan(t) = 0, & t \geq 1, \\ \\ x(t) = \phi(t), \qquad -1 \leq t \leq 0, \end{cases} \tag{2.13}$$

where $\phi, \phi' : [-1, 0] \to \mathbb{R}$ are continuous functions. Note that

$$r(t) = 1, \qquad \text{and} \qquad p(t) = \frac{1}{t}.$$

By using the notations from [16], let $q(t) = (t^3 - t^2)\arctan(t)$, $\gamma = 1$ and $f(x) = x$. Choose $\Phi(t) = \dfrac{1}{t^2}$. Then $h(t) = \dfrac{1}{t}$ and we have

$$\psi(t) = \frac{(t^3 - t^2)\arctan(t)}{t^2}\left(\frac{t-2}{t-1}\right) + \frac{2}{t^4}, \quad \text{for} \quad t > 1.$$

Then, by using the software Maple, we obtain

$$\int_t^{+\infty} \frac{ds}{r(s-1)\Phi(s)} = \int_t^{+\infty} \psi(s)ds = +\infty,$$

for all $t \geq 1$. Therefore, from [16], Theorem 2.2, the non-impulsive system (2.13) is oscillatory.

As we did before, we now consider system (2.13) and prove that it remains oscillating after the imposition of proper impulse controls.

Example 2.1 Consider the following second-order neutral delay differential equation

$$\begin{cases} \left(x(t) + \dfrac{1}{t}x(t-1)\right)'' + (t^3 - t^2)x(t-1)\arctan(t) = 0, \quad t \geq 1, \quad t \neq t_k, \\ x(t_k) = \left(\dfrac{k+1}{k}\right)x(t_k^-), \qquad x'(t_k) = x'(t_k^-), \quad k = 1, 2, \ldots, \\ x(t) = \phi(t), \qquad -1 \leq t \leq 0, \end{cases} \tag{2.14}$$

where $\phi, \phi' : [-1, 0] \to \mathbb{R}$ are continuous functions and $t_k = 2k - 1$, $k = 2, 3, 4, \ldots$. Note that $t_{k+1} - t_k = 2 > 1$, for all $k = 2, 3, 4, \ldots$.

We have

$$r(t) = 1, \quad p(t) = \frac{1}{t}, \quad a_k = b_k = \frac{k+1}{k} \quad \text{and} \quad c_k = 1, \; k = 1, 2, \ldots.$$

Let us consider $m(t) = (t^3 - t^2)\arctan(t)$. Then

$$\lim_{t \to +\infty} \int_{t_0}^{t} \left(\frac{1}{r(s)} \prod_{t_0 < t_k < s} \frac{c_k}{\max\{b_k, \, c_k\}}\right) ds =$$

$$
\begin{aligned}
&= \int_{t_0}^{+\infty} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds \\
&= \int_{t_0}^{t_1} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \int_{t_1}^{t_2} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds \\
&\quad + \int_{t_2}^{t_3} \prod_{t_0<t_k<s} \frac{k}{k+1}\, ds + \cdots \\
&= (t_1 - t_0) + \frac{1}{2}(t_2 - t_1) + \frac{1}{3}(t_3 - t_2) + \cdots \\
&\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty.
\end{aligned}
$$

Thus hypotheses (H_1) to (H_4) are satisfied.

Choose $\Phi(t) = \dfrac{1}{t^2}$. Then $h(t) = \dfrac{1}{t}$ and

$$\psi(t) = \frac{m(t)}{t^2}\left(\frac{t-2}{t-1}\right) + \frac{2}{t^4}, \quad \text{for} \quad t > 1.$$

Then

$$\sum_{k=n}^{+\infty} \int_{t_k}^{t_{k+1}} \psi(s) ds = \int_{t_n}^{+\infty} \psi(s)\, ds = +\infty,$$

for each $t_n \geq 1$.

Now, define the sequences $\{\xi_k\}_{k\geq 2}$ and $\{\alpha_k\}_{k\geq 2}$ by

$$\xi_k = 2k \qquad \text{and} \qquad \alpha_k = k^2,$$

for each $k = 2, 3, 4, ...$ Note that

$$t_k < \xi_k < t_{k+1}, \qquad k = 2, 3, 4, ...,$$

$$\limsup_{k\to+\infty} [t_{k+1} - \xi_k] = 1,$$

$$\sum_{k=2}^{+\infty} \frac{1}{\alpha_k} = \sum_{k=2}^{+\infty} \frac{1}{k^2} < +\infty,$$

and

$$
\begin{aligned}
&\int_{t_k}^{\xi_k} \frac{1}{r(s-\delta)\Phi(s)c_k} ds = \int_{t_k}^{\xi_k} \frac{ds}{r(s-\delta)\Phi(s)} = \int_{t_k}^{\xi_k} s^2\, ds = \\
&= \frac{\xi_k^3}{3} - \frac{t_k^3}{3} = \frac{12k^2 - 6k + 1}{3} > k^2 = \alpha_k,
\end{aligned}
$$

for each $k = 2, 3, 4, ...$ Therefore, it follows from Theorem 2.1 that all solutions $x(t)$ of (2.14) are oscillatory.

2.2 Case B

In this section, we establish an oscillation result for (2.1) under the following hypotheses:

(H_1^*) $f : [t_0 - \sigma, +\infty] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $uf(t, u, v) > 0$ for all $uv > 0$,

$$\frac{f(t, u, v)}{u} \geq n(t),$$

for all $u \neq 0$, where $n : [t_0 - \sigma, +\infty[\to \mathbb{R}$ is a positive continuous function.

(H_2^*) $I_k, J_k : \mathbb{R} \to \mathbb{R}$ are continuous, with $I_k(0) = J_k(0) = 0$, and there exists positive number $a_k > 1$ such that

$$I_k(x) = J_k(x) = a_k x,$$

for all $k \in \mathbb{N}$.

We also assume that conditions (H_3) and (H_4) hold, with $c_k = a_k$ and $b_k = a_k$, for each $k \in \mathbb{N}$.

Now, define the function $\psi_1 : [t_0, +\infty[\to \mathbb{R}$ by

$$\psi_1(t) = \Phi(t)\left\{n(t)[1 - p(t)] + r(t)h^2(t)a_k - [r(t)h(t)a_k]'\right\}$$

for each $t_k \leq t < t_{k+1}$, $k = 1, 2, 3, ...$

Remark 2.3 *Lemma 2.2 still holds if we replace hypothesis (H_1) by (H_1^*) and (H_2) by (H_2^*).*

With the new conditions (H_1^*) and $(H_2)^*$, we can rewrite Proposition 2.1 as follows.

Proposition 2.2 *Suppose (H_1^*), (H_2^*), (H_3) and (H_4) are fulfilled. If equation (2.1) is nonoscillatory, then there exist a number $k_0 \in \mathbb{N}$ and a function $w \in PC^1([t_{k_0}, +\infty[, \mathbb{R})$ satisfying*

$$w'(t) + \psi_1(t) + \frac{w^2(t)}{r(t)\Phi(t)a_k} \leq 0, \quad t_k < t < t_{k+1}, \tag{2.15}$$

for each $k = k_0, k_0 + 1, k_0 + 2, ...$

Proof. This proof follows the main ideas of the proof of Proposition 2.1. Let $x(t)$ be a nonoscillatory solution of (2.1). Without loss of generality, we may assume that $x(t) > 0$ on $[T - \tau - \delta, +\infty[$, for some $T \geq t_0$.

Recall that $z(t) = x(t) + p(t)x(t - \tau)$. By Remark 2.3 and Lemma 2.2, $z(t) > 0$, $z'(t) \geq 0$ for $t \in [t_k, t_{k+1}[$, where $t_k \geq T$ and $k \in \mathbb{N}$ and $z(t)$ is non-decreasing on $[T, +\infty[$.

Let $k_0 = \min\{k : t_k \geq T, k = 1, 2, 3, ...\}$. By (2.1) and hypothesis (H_1^*), we obtain

$$[r(t)z'(t)]' = -f(t, x(t), x(t - \delta)) \leq -n(t)x(t) < 0,$$

for every $t \geq T$ and $t \neq t_k$, $k \in \mathbb{N}$. Consequently, $r(t)z'(t)$ is a non-increasing function on each interval $[t_k, t_{k+1}[$, $k = k_0, k_0 + 1, ...$ Since $a_k > 1$, we have

$$r(t)z'(t) \leq a_k r(t)z'(t), \tag{2.16}$$

for each $t_k \le t < t_{k+1}$, $k = k_0, k_0+1, \dots$.

Note that

$$f(t,\, x(t),\, x(t-\delta)) \ge n(t)x(t) = n(t)[z(t) - p(t)x(t-\tau)],$$

for $t \ne t_k$, $k \in \mathbb{N}$ and $t \ge T$. Then,

$$[r(t)z'(t)]' + n(t)[z(t) - p(t)x(t-\tau)] \le [r(t)z'(t)]' + f(t,\, x(t),\, x(t-\delta)) = 0,$$

for $t \ne t_k$, $k \in \mathbb{N}$ and $t \ge T$, that is

$$[r(t)z'(t)]' + n(t)[z(t) - p(t)x(t-\tau)] \le 0.$$

Since $z(t)$ is non-decreasing from Remark 2.3 and Lemma 2.2, we have

$$x(t-\tau) \le z(t-\tau) \le z(t), \qquad t \ge T.$$

Then

$$n(t)z(t)[1 - p(t)] \le n(t)[z(t) - p(t)x(t-\tau)]$$

and, consequently,

$$[r(t)z'(t)]' + n(t)z(t)[1 - p(t)] \le 0,$$

for $t \ge T$, $t \ne t_k$, $k \in \mathbb{N}$.

Now, define

$$w(t) = \Phi(t)\left\{\frac{r(t)z'(t)}{z(t)} + r(t)h(t)a_k\right\},$$

for each $t \in [t_k,\, t_{k+1}[$, $k = k_0, k_0+1, \dots$. Note that $w \in PC^1([t_{k_0},\, +\infty),\, \mathbb{R})$.

We also have

$$w'(t) \le -2h(t)w(t) + \Phi(t)\left\{-n(t)[1-p(t)] - \frac{r(t)z'(t)z'(t)}{z^2(t)} + [r(t)h(t)a_k]'\right\},$$

for each $t_k < t < t_{k+1}$, $k = k_0, k_0+1, \dots$.

Since $r(t)z'(t) \le a_k r(t)z'(t)$ from (2.16), we have

$$\frac{r(t)z'(t)z'(t)}{z^2(t)} \ge \frac{1}{a_k r(t)}\left(\frac{r(t)z'(t)}{z(t)}\right)^2.$$

Then

$$w'(t) \le -2h(t)w(t) + \Phi(t)\left\{-n(t)[1-p(t)] - \frac{1}{a_k r(t)}\left(\frac{r(t)z'(t)}{z(t)}\right)^2 + [r(t)h(t)a_k]'\right\},$$

for each $t_k < t < t_{k+1}$, $k = k_0, k_0+1, \dots$.

Since

$$\frac{r(t)z'(t)}{z(t)} = \frac{w(t)}{\Phi(t)} - r(t)h(t)a_k,$$

we have

$$w'(t) \leq -\frac{w^2(t)}{a_k\Phi(t)r(t)} + \Phi(t)\left\{-n(t)[1-p(t)] - r(t)h^2(t)a_k + [r(t)h(t)a_k]'\right\},$$

Therefore,

$$w'(t) \leq -\psi_1(t) - \frac{w^2(t)}{r(t)\Phi(t)a_k}, \quad t_k < t < t_{k+1},$$

$k = k_0, k_0+1, \ldots$. If $x(t) < 0$ in $[T-\tau-\delta, +\infty[$, for some $T \geq t_0$, the result follows analogously and we complete the proof. ∎

Lemma 2.4 *If* $\sum_{k=1}^{+\infty} r(t_k)h(t_k)(a_k - a_{k-1}) < +\infty$, *then* $\sum_{k=1}^{+\infty}(w(t_k) - w(t_k^-)) < +\infty$.

PROOF. Note that

$$\begin{aligned} w(t_k) &= \Phi(t_k)\left\{\frac{r(t_k)z'(t_k)}{z(t_k)} + r(t_k)h(t_k)a_k\right\} \\ &= \Phi(t_k^-)\left\{\frac{r(t_k^-)a_kz'(t_k^-)}{a_kz(t_k^-)} + r(t_k^-)h(t_k^-)a_k\right\} \\ &= \Phi(t_k^-)\left\{\frac{r(t_k^-)z'(t_k^-)}{z(t_k^-)} + r(t_k^-)h(t_k^-)a_k\right\}, \end{aligned}$$

$k = 1, 2, 3, \ldots$. Since $w(t_k^-) = \Phi(t_k^-)\left\{\frac{r(t_k^-)z'(t_k^-)}{z(t_k^-)} + r(t_k^-)h(t_k^-)a_{k-1}\right\}$, $k = 1, 2, \ldots$, we have

$$w(t_k) - w(t_k^-) = r(t_k)h(t_k)(a_k - a_{k-1}),$$

$k = 1, 2, 3, \ldots$. Therefore, the result is proved. ∎

Next, we establish an oscillation criterium for system (2.1) satisfying hypotheses (H_1^*), (H_2^*), (H_3) and (H_4). The proof follows similarly to the proof of Theorem 2.1 by applying Lemma 2.4 instead of Lemma 2.3.

Theorem 2.2 *Suppose* (H_1^*), (H_2^*), (H_3) *and* (H_4) *are fulfilled,* $\sum_{k=1}^{+\infty} r(t_k)h(t_k)(a_k - a_{k-1}) < +\infty$ *and*

$$\sum_{k=n}^{+\infty}\int_{t_k}^{t_{k+1}} \psi(s)ds = +\infty,$$

for $n \in \mathbb{N}$. *If there exist sequences* $\{\alpha_n\}_{n\geq 1}$ *and* $\{\xi_n\}_{n\geq 1}$ *of positive real numbers, such that* $\xi_n \in]t_n, t_{n+1}[$, $n \in \mathbb{N}$, $\limsup_{n\to+\infty}(t_{n+1} - \xi_n) > 0$, $\sum_{n=1}^{+\infty}\frac{1}{\alpha_n} < +\infty$ *and*

$$\int_{t_k}^{\xi_k} \frac{ds}{r(s)\Phi(s)a_k} \geq \alpha_k,$$

$k \in \mathbb{N}$, *then system* (2.1) *is oscillatory.*

Example 2.2 Consider the following second-order neutral delay differential equation

$$\begin{cases} \left(x(t) + \dfrac{1}{t}\, x(t-1)\right)'' + x(t)t^2 \ln(t-1) = 0, & t \geq 1, \quad t \neq t_k, \\ x(t_k) = \left(\dfrac{k+1}{k}\right) x(t_k^-), \qquad x'(t_k) = \left(\dfrac{k+1}{k}\right) x'(t_k^-), & k = 1, 2, \ldots, \\ x(t) = \phi(t), \qquad -1 \leq t \leq 0, \end{cases} \tag{2.17}$$

where $\phi, \phi' : [-1, 0] \to \mathbb{R}$ are continuous functions and $t_k = 2k-1$, $k = 2, 3, 4, ...$ Note that $t_{k+1} - t_k = 2 > 1$, for all $k = 2, 3, 4, ...$

We have

$$r(t) = 1, \quad p(t) = \frac{1}{t}, \quad a_k = \frac{k+1}{k}, \; k = 1, 2, \ldots .$$

Let us consider $n(t) = t^2 \ln(t-1)$. Then

$$\lim_{t\to+\infty} \int_{t_0}^{t} \left(\frac{1}{r(s)} \prod_{t_0<t_k<s} \frac{c_k}{\max\{b_k,\, c_k\}} \right) ds =$$

$$= \lim_{t\to+\infty} \int_{t_0}^{t} \left(\frac{1}{r(s)} \prod_{t_0<t_k<s} \frac{a_k}{\max\{a_k,\, a_k\}} \right) ds =$$

$$= \lim_{t\to+\infty} \int_{t_0}^{t} \frac{1}{r(s)}\, ds = \lim_{t\to+\infty} \int_{t_0}^{t} 1 ds = +\infty.$$

Thus hypotheses (H_1^*), (H_2^*), (H_3) and (H_4) are satisfied.

Choose $\Phi(t) = \dfrac{1}{t^2}$. Then $h(t) = \dfrac{1}{t}$ and

$$\psi_1(t) = \frac{\ln(t-1)}{t}(t-1) + \frac{2}{t^4} + \frac{2}{kt^4}, \quad \text{for} \quad t_k \leq t < t_{k+1}, \quad t > 1.$$

Then

$$\sum_{k=n}^{+\infty} \int_{t_k}^{t_{k+1}} \psi_1(s) ds = \int_{t_n}^{+\infty} \left(\frac{\ln(t-1)}{t}(t-1) + \frac{2}{t^4} \right) ds + \sum_{k=n}^{+\infty} \int_{t_k}^{t_{k+1}} \frac{2}{kt^4}\, ds = +\infty,$$

for each $t_n \geq 1$.

As before, let us define the sequences $\{\xi_k\}_{k\geq 2}$ and $\{\alpha_k\}_{k\geq 2}$ by $\xi_k = 2k$ and $\alpha_k = k^2$, for each $k = 2, 3, 4, ...$ Then

$$t_k < \xi_k < t_{k+1}, \qquad k = 2, 3, 4, ...,$$

$$\limsup_{k\to+\infty} [t_{k+1} - \xi_k] = 1,$$

$$\sum_{k=2}^{+\infty} \frac{1}{\alpha_k} = \sum_{k=2}^{+\infty} \frac{1}{k^2} < +\infty,$$

and

$$\int_{t_k}^{\xi_k} \frac{1}{r(s)\Phi(s)a_k} ds = \frac{k}{k+1} \int_{t_k}^{\xi_k} s^2 \, ds =$$

$$= \left(\frac{k}{k+1}\right)\left(\frac{12k^2-6k+1}{3}\right) > k^2 = \alpha_k,$$

for each $k = 2, 3, 4, \ldots$. We also have

$$\sum_{k=2}^{+\infty} r(t_k)h(t_k)(a_k - a_{k-1}) = \sum_{k=2}^{+\infty} \frac{-1}{(2k-1)k(k-1)} < +\infty.$$

Therefore, it follows from Theorem 2.2 that all solutions $x(t)$ of (3.4) are oscillatory.

3 Application to the Emden-Fowler Equation

The Emden-Fowler equation is very important in mathematical physics, theoretical physics and chemical physics, and it has being attracting much attention over the years. This equation has the following general form

$$Y'' + p(X)Y' + q(X)Y = r(X)Y^n, \tag{3.1}$$

where n is a real number.

By using a Kummer-Liouville transformation ([12] and [18]), the equation (3.1) can be transformed into standard form,

$$y'' = f(x)y^n, \tag{3.2}$$

see [23]. Equation (3.2) becomes increasingly important as it arises in the modelling of many physical systems. Perhaps its occurrence is best known as the quintessential equation in the study of the case shear-free spherically symmetric perfect fluid motion in cosmology when $n = 2$ ([8], [9], [22] and [29]).

The particular Emden-Fowler equation (Lane-Emden equation):

$$y'' + \frac{2}{t}y' + y^n = 0,$$

is used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. Its equation arises in astrophysics [3].

For special forms of $g(y)$, the well-known Lane-Emden equation

$$y'' + \frac{2}{t}y' + g(y) = 0,$$

is used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents.

On the other hand, the second order neutral delay differential equation is used in many fields such as vibrating masses attached to an elastic bar and some variational problems, etc. Recently, the results of Atkinson [1] and Belohorec [2] for the Emden-Fowler equation

$$y''(t) + q(t)|y(t)|^{\gamma-1}y(t) = 0,$$

where $q \in C([t_0, +\infty), \mathbb{R})$ and $\gamma > 0$, have been extended to the second order neutral delay differential equation

$$[y(t) + p(t)y(t-\tau)]'' + q(t)f(y(t-\delta)) = 0 \tag{3.3}$$

by Wong [34] under the assumption that the nonlinear function f satisfies the sublinear condition

$$0 < \int_{0^+}^{\varepsilon} \frac{du}{f(u)}, \int_{0^-}^{-\varepsilon} \frac{du}{f(u)} < \infty \quad \text{for all} \quad \varepsilon > 0,$$

as well as the superlinear condition

$$0 < \int_{\varepsilon}^{\infty} \frac{du}{f(u)}, \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Now, we are going to show that the solution of the extended Emden-Fowler equation (3.3) is oscillatory under small perturbations (impulses).

Consider the following second-order neutral delay differential equation with impulse

$$\begin{cases} [y(t) + p(t)y(t-\tau)]'' + q(t)f(y(t-\delta)) = 0, \quad t \geq t_0, \quad t \neq t_k, \\ x(t_k) = \left(\dfrac{k+1}{k}\right) x(t_k^-), \qquad x'(t_k) = x'(t_k^-), \quad k = 1, 2, \ldots, \\ x(t) = \phi(t), \qquad -\sigma \leq t \leq 0, \end{cases} \tag{3.4}$$

where:

1. $q \in C([t_0, +\infty], \mathbb{R}_+)$ and $f \in C^1(\mathbb{R}, \mathbb{R})$;
2. $\dfrac{f(y)}{y} \geq \gamma > 0$ if $y \neq 0$;
3. $0 \leq p(t) \leq 1$, $t \geq t_0$, is continuous;
4. $\tau > 0$ and $\delta > 0$ are constants, and $\sigma = \max\{\delta, \tau\}$;
5. $\phi, \phi' : [t_0 - \sigma, t_0] \to \mathbb{R}$ are continuous functions;
6. $t_k = 2k - 1$, $k = 2, 3, 4, \ldots.$
7. $\displaystyle\int_t^{+\infty} \frac{q(s)(1 - p(s-\delta))}{s^2} ds = +\infty$ for every $t \geq t_0$.

Note that

$r(t) = 1$, $a_k = b_k = \dfrac{k+1}{k}$, $k = 1, 2, \ldots$ and $t_{k+1} - t_k = 2 > 1$, for all $k = 2, 3, 4, \ldots.$

Let us verify that the hypothesis $(H_1) - (H_4)$ (from Case A) are satisfied:

(H_1) Let $F : [t_0 - \sigma, +\infty] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $F(t, u, v) = q(t)f(v)$ which is continuous. By assumption, $vf(v) > 0$ if $v \neq 0$. Then

$$uF(t, u, v) = uq(t)f(v) = q(t)\frac{f(v)}{v}(uv) > 0,$$

whenever $uv > 0$.

Let us consider $m(t) := \gamma q(t)$. Then

$$\frac{F(t, u, v)}{v} = \frac{q(t)f(v)}{v} \geq \gamma q(t) = m(t).$$

(H_2) By assumptions, we have $I_k(x) = \dfrac{k+1}{k}x$ and $J_k(x) = x$.

(H_3) We also have,

$$\lim_{t\to+\infty} \int_{t_0}^{t} \left(\frac{1}{r(s)} \prod_{t_0<t_k<s} \frac{c_k}{\max\{b_k,\, c_k\}} \right) ds =$$

$$= \lim_{t\to+\infty} \int_{t_0}^{t} \left(\frac{1}{r(s)} \prod_{t_0<t_k<s} \frac{1}{\max\{b_k,\, 1\}} \right) ds =$$

$$= \lim_{t\to+\infty} \int_{t_0}^{t} \left(\prod_{t_0<t_k<s} \frac{k}{k+1} \right) ds =$$

$$= (t_1 - t_0) + \frac{1}{2}(t_2 - t_1) + \frac{1}{3}(t_3 - t_2) + \cdots$$

$$\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = +\infty.$$

(H_4) By assumptions, $p(t)$ and $p'(t)$ are continuous.

Thus conditions (H_1), (H_2), (H_3) and (H_4) of Theorem 2.1 are satisfied.

Choose $\Phi(t) = \dfrac{1}{t^2}$. Then $h(t) = \dfrac{1}{t}$ and

$$\psi(t) = \frac{m(t)(1 - p(t - \delta))}{t^2} + \frac{2}{t^4}, \quad \text{for} \quad t > 1.$$

Thus

$$\sum_{k=n}^{+\infty} \int_{t_k}^{t_{k+1}} \psi(s)ds = \int_{t_n}^{+\infty} \left(\frac{\gamma q(s)(1 - p(s - \delta))}{s^2} + \frac{2}{s^4} \right) ds = +\infty,$$

for each $t_n \geq t_0$.

Now, let us define the sequences $\{\xi_k\}_{k\geq 2}$ and $\{\alpha_k\}_{k\geq 2}$ by $\xi_k = 2k$ and $\alpha_k = k^2$, for each $k = 2, 3, 4, ...$ Then

$$t_k < \xi_k < t_{k+1}, \qquad k = 2, 3, 4, ...,$$

$$\limsup_{k\to+\infty} [t_{k+1} - \xi_k] = 1,$$

$$\sum_{k=2}^{+\infty} \frac{1}{\alpha_k} = \sum_{k=2}^{+\infty} \frac{1}{k^2} < +\infty,$$

and

$$\int_{t_k}^{\xi_k} \frac{1}{r(s-\delta)\Phi(s)c_k} ds = \int_{t_k}^{\xi_k} s^2\, ds = \frac{\xi^3}{3} - \frac{t_k^3}{3} = \frac{12k^2-6k+1}{3} > k^2 = \alpha_k,$$

for each $k = 2, 3, 4, ...$

Therefore, it follows from Theorem 2.1 that all solutions $y(t)$ of (3.4) are oscillatory.

4 Final comments and an open problem

It worths mentioning that in [28], the authors give a counter-example to a result from [35] (namely, Lemma 1) for the non-impulsive case, when the function p in (1.1) takes negative values (in $[\alpha, 0]$, with $\alpha > -1$). As a consequence, counter-examples to results from [4] and [33] appear naturally, since these papers use Lemma 1 from [35]. As a matter of fact, when $-1 < \alpha \leq p(t) \leq 0$, under the conditions of [4], [33] or [35], the solutions of the systems considered in these papers may be non-oscillatory. In view of this, a question arises: is it possible to find adequate impulse operators which, in the case where the function p takes negative values, the system (2.1) is oscillatory?

References

[1] F. V. Atkinson, On second order nonlinear oscillation, Pacific J. Math. 5 (1955) 643-647.

[2] S. Belohorec, Oscillatory solution of certain nonlinear differential equations of the second order, Math. Fyz. Casopis Sloven. Akad. Vied. 11 (1961) 250-255.

[3] S. Chandrasekhar, Principles of stellar dynamics (University of Chicago Press, Chicago, 1942).

[4] M. Chen, Z. Xu, Interval oscillation of second-order Emden-Fowler neutral delay differential equations. Electron. J. Differential Equations, 58 (2007), pp. 1-9.

[5] J. Diblk, Z. Svoboda, Positive solutions of p-type retarded functional differential equations. Nonlinear Anal. 64 (2006), pp. 1831-1848.

[6] R. D. Driver, A mixed neutral system, Nonlinear Anal., 8 (1984), pp. 155-158.

[7] L. P. Gimenes, M. Federson, Oscillation by impulses for a second-order delay differential equation. Comput. Math. Appl., 52 (2006), pp. 819-828.

[8] K. S. Govinder, P. G. L. Leach and S. D. Maharaj, Integrability analysis of a conformal equation arising in general relativity, International Journal of Theoretical Physics 34 (1994), 625-639.

[9] P. Havas, Shear-free spherically symmetric perfect fluid solutions with conformal symmetry, General Relativity and Gravitation, 24 (1992), 599-615.

[10] Z. He, W. Ge, Oscillations of second-order nonlinear impulsive ordinary differential equations. J. Comput. Appl. Math., 158 (2003), pp. 397-406.

[11] V. V. Koslov, D. V. Treshcheëv, Billiards - A Genetic Introduction to the Dynamics of Systems with Impacts, Amer. Math. Soc., Providence, Rhode Isle, 1991.

[12] E. E. Kummer, De generali quadam æquatione differentiali tertii ordinis, Journal fr die Reine und Angewandte Mathematik, 100 (1887), 1-9 (reprinted from the Programm des evangelischen Knigl und Stadtgymnasiums in Liegnitz for 1834).

[13] G. S. Ladde, V. Lakshmikantham, B. G. Zhang, Oscillation theory of differential equations with deviating arguments, Marcel Dekker, New York, 1987.

[14] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov, Theory of impulsive differential equations, World Scientific, Singapore, 1989.

[15] W. T. Li, Interval oscillation criteria of second-order half-linear functional differential equations. Appl. Math. Comput., 155 (2004), pp. 451-468.

[16] Horng-Jaan, Li. Oscillation of solutions of second-order neutral delay differential equations with integrable coefficients. Math. Comput. Modelling, 25 (1997), no. 3, pp. 69-79.

[17] X. Lin, Oscillation of second-order nonlinear neutral differential equations. J. Math. Anal. Appl., 309 (2005), pp. 442-452.

[18] J. Liouville, Sur le développement des functions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre contenant un paramètre variable, Journal de Mathématiques pures et appliquées, II (1837), 16-35.

[19] J. Luo, Second-order quasilinear oscillation with impulses. Comput. Math. Appl., 46 (2003), pp. 279-291.

[20] W. Luo, J. Luo, L. Debnath, Oscillation of second order quasilinear delay differential equations with impulses. J. Appl. Math. and Computing, 13 (2003), pp. 165-182.

[21] Z. Luo, J. Shen, Oscillation of second order linear differential equations with impulses. Appl. Math. Lett., 20 (2007), no. 1, pp. 75-81.

[22] F. M. Mahomed and P. G. L. Leach, Symmetry Lie algebras of nth order ordinary differential equations, Journal of Mathematical Analysis and Applications, bf 151 (1990), 80-107.

[23] C. M. Mellin, F. M. Mahomed and P. G. L. Leach, Solution of generalized Emden-Fowler equations with two symmetries, International Journal of Nonlinear Mechanics, 29 (1994), 529-538.

[24] F. Meng, J. Wang, Oscillation criteria for second order quasi-linear neutral delay differential equations. J. Indones. Math. Soc (MIHMI), 10 (2004), pp. 61-75.

[25] Q. Meng, J. Yan, Bounded oscillation for second order non-linear neutral delay differential equations in critical and non-critical cases. Nonlinear Anal., 64 (2006), pp. 1543-1561.

[26] M. Peng, W. Ge, Oscillation criteria for second-order nonlinear differential equations with impulses. Comput. Math. Appl., 39 (2000), pp. 217-225.

[27] M. Peng, Oscillation theorems of second-order nonlinear neutral delay difference equations with impulses. Comput. Math. Appl., 44 (2002), no. 5-6, pp. 741-748.

[28] H. Qin, N. Shang, Y. Lu, A note on oscillation criteria of second-order nonlinear neutral differential equations, to appear.

[29] D. C. Srivastava, Exact solutions for shear-free motion of spherically symmetric perfect fluid distributions in general relativity, Classical and Quantum Gravity 4 (1987), 1093-1117.

[30] Y. G. Sun, S. H. Saker, Oscillation for second-order nonlinear neutral delay difference equations. Appl. Math. Comput., 163 (2005), pp. 909-918.

[31] Wu Xiu-li, Chen Si-Yang, Hong Ji, Oscillation of a class of second-order nonlinear ODE with impulses. Appl. Math. Comput., 138 (2003), no. 2-3, pp. 181-188.

[32] R. Xu, F. Meng, Oscillation criteria for second order quasi-linear neutral delay differential equations. J. Comput. Appl. Math. Comput., 192 (2007), pp. 216-222.

[33] Z. Xu, X. Liu, Philos-type oscillation criteria for Emden-Fowler neutral delay differential equations. J. Comput. Appl. Math., 206 (2007), no. 2, pp. 1116-1126.

[34] J. S. W. Wong, Necessary and sufficient conditions for oscillation for second order neutral differential equations, J. Math. Anal. Appl. 252 (2000) 342-352.

[35] Q. Yang, L. Yang, S. Zhu, Interval criteria for oscillation of second-order nonlinear neutral differential equations. Comput. Math. Appl., 46 (2003), pp. 903-918.

[36] B. G. Zhang, Zhu Shanliang, Oscillation of second-order nonlinear delay dynamic equations on time scales. Comput. Math. Appl., 49 (2005), pp. 599-609.